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On the Boundedness and Periodicity of a Certain Differential Equation of Fifth Order

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There are given sufficient conditions for the ultimate boundedness of Solutions and for the existence of periodic solutions of a certain ordinary differential equation of fifth order.

Key words: *Ultimate boundedness, periodic solutions*

AMS subject classification: 34 C, 34 D

1. Introduction and statement of results

We consider the real non-linear non-autonomous ordinary differential equation of fifth order

$$x^{(5)} + f_1(\dddot{x})x^{(4)} + f_2(\dddot{x}) + f_3(\ddot{x}) + f_4(\dot{x}) + f_5(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}), \tag{1.1}$$

in which the functions $f_1, \dots, f_5$ and p depend at most on the arguments shown, are continuous for all values of their respective arguments and the derivatives f_4' and f_5' exist and are continuous. The boundedness result obtained here generalizes that of Chukwu [4], where $f_1(\dddot{x}) = a$ and $f_3(\ddot{x}) = c\ddot{x}$ for some constants a and c and reads as follows.

Theorem 1: *In addition of the basic assumptions on the functions $f_1, \dots, f_5$ and p suppose the existence of arbitrary constants $\alpha_1, \dots, \alpha_5$ and of sufficiently small positive constants $\varepsilon, \varepsilon_0, \varepsilon_1, \dots, \varepsilon_5$ such that the following conditions are satisfyed:*

(i) $\alpha_1 > 0,\ \alpha_1\alpha_2 - \alpha_3 > 0,\ (\alpha_1\alpha_2 - \alpha_3)\alpha_3 - (\alpha_1\alpha_4 - \alpha_5)\alpha_1 > 0$ (1.2)

$$\delta_0 := (\alpha_4\alpha_3 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3) - (\alpha_1\alpha_4 - \alpha_5)^2 > 0,\ \alpha_5 > 0$$

$$\Delta_1 := \frac{(\alpha_4\alpha_3 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - (\alpha_1 f_4'(y) - \alpha_5) > 2\varepsilon\alpha_2 \quad \textit{for all } y$$

$$\Delta_2 := \frac{\alpha_4\alpha_3 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} - \frac{\gamma(\alpha_1\alpha_4 - \alpha_5)}{\alpha_4(\alpha_1\alpha_2 - \alpha_3)} - \frac{\varepsilon}{\alpha_1} > 0 \ \text{ where } \gamma = \begin{cases} f_4(y)/y & \textit{for } y \neq 0 \\ f_4'(y) & \textit{for } y = 0. \end{cases}$$

(ii) $\varepsilon_0 \le f_1(w) - \alpha_1 \le \varepsilon_1$ *for all* w.

(iii) $f_2(0) = 0$ *and* $0 \le f_2(w)/w - \alpha_2 \le \varepsilon_2$ *for all* $w \neq 0$.

(iv) $f_3(0) = 0$ *and* $0 \le f_3(z)/z - \alpha_3 \le \varepsilon_3$ *for all* $z \neq 0$.

(v) $f_4(0) = 0$ *and* $f_4(y)/y \ge \alpha_4$ *for all* $y \neq 0$, $|\alpha_4 - f_4'(y)| \le \varepsilon_4$ for all y,

$$f_4'(y) - \frac{f_4(y)}{y} \le \frac{\alpha_5\delta_0}{\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)} \quad \textit{for all } y \neq 0.$$

(vi) $f_5(0) = 0$ *and* $f_5(x)\operatorname{sgn} x > 0$ *for all* $x \neq 0$, $\int_0^x f_5(\xi)\,d\xi \to \infty$ *as* $|x| \to \infty$

$$0 \le \alpha_5 - f_5'(x) \le \varepsilon_5 \text{ for all } x.$$

(vii) $|p(t,x,y,z,w,u)| \le \Delta < \infty$ *for all* (t,x,y,z,w,u), *where* Δ *is some constant.*

Then there exists a constant D_0 *depending on* Δ *and* $f_1,\dots,f_5$ *only such that every solution* $x = x(t)$ *of equation* (1.1) *satisfies*

$$|x(t)|, |\dot{x}(t)|, |\ddot{x}(t)|, |\dddot{x}(t)|, |x^{(4)}(t)| \le D_0 \tag{1.3}$$

for all sufficiently large t.

Theorem 2: *If, in addition to the conditions of Theorem 1, the function p satisfies for some* $\omega \in \mathbb{R}$ *the condition*

(viii) $p(t+\omega,x,y,z,w,u) = p(t,x,y,z,w,u)$ *for all* (t,x,y,z,w,u),

then there exists at least one ω*-periodic solution* $x = x(t)$ *of equation* (1.1).

Notation: In what follows we shall use the letter D for positive constants whose magnitudes depend on $\alpha_1,\dots,\alpha_5$, Δ and $f_1,\dots,f_5$. No two D's are ever the same unless they are numbered, but all the D's: $D_1, D_2, \dots$ with suffixes attached retain their identities throughout the sequel.

2. The function $V(x,y,z,w,u)$

In what follows it will be convenient to use the equivalent differential system

$$\begin{aligned} &\dot{x} = y,\ \dot{y} = z,\ \dot{z} = w,\ \dot{w} = u \\ &\dot{u} = -f_1(w)u - f_2(w) - f_3(z) - f_4(y) - f_5(x) + p(t,x,y,z,w,u) \end{aligned} \tag{2.1}$$

which is obtained from (1.1) by setting $\dot{x} = y$, $\ddot{x} = z$, $\dddot{x} = w$ and $x^{(4)} = u$. The actual proof of Theorem 1 will rest mainly on certain properties of a piecewise continuously differentiable function $V = V(x,y,z,w,u)$ defined by $V = V_1 + V_2$, where

$$\begin{aligned} 2V_1(x,y,z,w,u) = {} & u^2 + 2\alpha_1 uw + \frac{2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} uz + 2\delta yu + 2\int_0^w f_2(\omega)\,d\omega \\ & + \left\{\alpha_1^2 + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_5)}{\alpha_1\alpha_4 - \alpha_5}\right\} w^2 + 2\left\{\alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_5)}{\alpha_1\alpha_4 - \alpha_5} - \delta\right\} wz \\ & + 2\alpha_1\delta wy + 2wf_4(y) + 2wf_5(x) + 2\alpha_1\int_0^z f_3(\zeta)\,d\zeta \\ & + \left\{\frac{\alpha_2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \alpha_4 - \alpha_1\delta\right\} z^2 + 2\delta\alpha_2 yz + 2\alpha_1 z f_4(y) \\ & - 2\alpha_5 zy + 2\alpha_1 z f_5(x) + \frac{2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}\int_0^y f_4(\eta)\,d\eta \\ & + (\delta\alpha_3 - \alpha_5\alpha_1)y^2 + \frac{2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} y f_5(x) + 2\delta\int_0^x f_5(\xi)\,d\xi, \end{aligned} \tag{2.2}$$

with $\delta = \varepsilon + \alpha_5(\alpha_1\alpha_2 - \alpha_3)/(\alpha_1\alpha_4 - \alpha_5)$, and

$$V_2(x, u) = \begin{cases} x \operatorname{sgn} u & \text{if } |u| \geq |x| \\ u \operatorname{sgn} x & \text{if } |u| \leq |x| \end{cases}. \tag{2.3}$$

The first property of V is stated in the following

Lemma 1: *Subject to the assumptions* (i) - (vi) *of Theorem* 1, *there is a constant D_5 such that*

$$V(x,y,z,w,u) \geq -D_5 \text{ for all } (x,y,z,w,u) \quad \text{and} \quad V(x,y,z,w,u) \to \infty \text{ as } x^2 + \ldots + u^2 \to \infty. \tag{2.4}$$

Proof: From (2.3) we obtain $|V_2(x, u)| \leq |u|$ for all x and u. It follows that

$V_2(x, u) \geq -|u|$ for all x and u.

Now, V_1 here is the same as the function V defined in [2]. The estimate for V there gives

$$2V_1(x,y,z,w,u) \geq 2\varepsilon\int_0^x f_5(\xi)\,d\xi + D_1 y^2 + D_2 z^2 + D_3 w^2 + D_4 u^2.$$

From these estimates for V_1 and V_2 we get the estimate for V

$$\begin{aligned} 2V(x,y,z,w,u) &\geq 2\varepsilon\int_0^x f_5(\xi)\,d\xi + D_1 y^2 + D_2 z^2 + D_3 w^2 + D_4 u^2 - 2|u| \\ &= 2\varepsilon\int_0^x f_5(\xi)\,d\xi + D_1 y^2 + D_2 z^2 + D_3 w^2 + D_4(|u| - D_4^{-1})^2 - D_4^{-1}. \end{aligned}$$

By using (vi) we deduce that the integral on the right-hand side here is non-negative and tends to infinity when $|x|$ do so. It is evident that (2.4) is verified, where $D_5 = D_4^{-1}$ ∎

The next property of the function V is connected with its total time derivative and is contained in the following

Lemma 2: *Let (x,y,z,w,u) be any solution of the differential system* (2.1) *and the function $v = v(t)$ be defined by $v(t) = V(x(t),y(t),z(t),w(t),u(t))$. Then the limit $\dot{v}^+(t) = \limsup_{h \to +0} (v(t+h) - v(t))/h$ exists and there is a constant D_6 such that*

$$\dot{v}^+(t) \leq -1 \text{ whenever } x^2(t) + \ldots + u^2(t) \geq D_6. \tag{2.5}$$

Proof: In accordance with the representation $V = V_1 + V_2$ we have an representation $v = v_1 + v_2$. The existence of $\dot{v}^+$ is quite immediate, since v_1 has continuous first partial derivatives and v_2 is easily shown to be locally Lipschitzian in x and u so that the composite function $v = v_1 + v_2$ is at least locally Lipschitzian in x,y,z,w and u. An easy calculation from (2.1) and (2.2) shows that, for $y,z,w \neq 0$, we have

$$\begin{aligned} \dot{v}_1 = &-u^2(f_1(w) - \alpha_1) - w^2\left[\alpha_1\frac{f_2(w)}{w} - \left\{\alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \delta\right\}\right] \\ &-z^2\left\{\frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}\frac{f_3(z)}{z} - (\delta\alpha_2 + \alpha_1 f_4'(y) - \alpha_5)\right\} \end{aligned}$$

$$-y^2\left\{\delta\frac{f_4(y)}{y}-\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}f_5'(x)\right\}-\alpha_1 uw(f_1(w)-\alpha_1)$$

$$-uz\left\{\frac{f_3(z)}{z}-\alpha_3\right\}-\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}uz(f_1(w)-\alpha_1)$$

$$-\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}wz\left\{\frac{f_2(w)}{w}-\alpha_2\right\}-\delta yu(f_1(w)-\alpha_1)$$

$$-wz(\alpha_4-f_4'(y))-yw(\alpha_5-f_5'(x))-\delta yz\left\{\frac{f_3(z)}{z}-\alpha_3\right\}$$

$$-\delta yw\left\{\frac{f_2(w)}{w}-\alpha_2\right\}-\alpha_1 yz(\alpha_5-f_5'(x))$$

$$+\left\{u+\alpha_1 w+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z+\delta y\right\}p(t,x,y,z,w,u).$$

As is shown in [2] we have

$$\dot v_1 \le -\tfrac{1}{2}(\varepsilon_0 u^2+\varepsilon w^2+\varepsilon\alpha_2 z^2+\varepsilon\alpha_4 y^2)+\left\{u+\alpha_1 w+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z+\delta y\right\}p(t,x,y,z,w,u).$$

The case $(y,z,w)=0$ is trivially dealth with. From (vii) we find

$$\dot v_1 \le -\tfrac{1}{2}(\varepsilon_0 u^2+\varepsilon w^2+\varepsilon\alpha_2 z^2+\varepsilon\alpha_4 y^2)+D_7(|u|+|w|+|z|+|y|),$$

where $D_7=\Delta\max\{1,\alpha_1,\alpha_4(\alpha_1\alpha_2-\alpha_3)/(\alpha_1\alpha_4-\alpha_5),\delta\}$. A straightforward calculation from (2.1) and (2.3) gives by (ii) - (v) and (vii) that

$$\dot v_2^+ = \begin{cases} y\,\mathrm{sgn}\,u & \text{if } |u|\ge|x| \\ -f_5(x)\,\mathrm{sgn}\,x-(uf_1(w)+f_2(w)+f_3(z)+f_4(y)-p(t,x,y,z,w,u))\,\mathrm{sgn}\,x & \text{if } |u|\le|x| \end{cases}$$

$$\le \begin{cases} |y| & \text{if } |u|\ge|x| \\ -f_5(x)\,\mathrm{sgn}\,x+D_8(|u|+|w|+|z|+|y|+1) & \text{if } |u|\le|x|, \end{cases}$$

where $D_8=\max\{\alpha_1+\varepsilon_1,\dots,\alpha_4+\varepsilon_4,\Delta\}$. From these estimates for $\dot v_1$ and $\dot v_2^+$ it can be shown that $\dot v^+=\dot v_1+\dot v_2^+$ necessarily satisfies the estimate

$$\dot v^+\le\begin{cases} -\tfrac{1}{2}(\varepsilon_0 u^2+\varepsilon w^2+\varepsilon\alpha_2 z^2+\varepsilon\alpha_4 y^2)+D_9(|u|+|w|+|z|+|y|), & \text{if } |u|\ge|x| \\ \textbf{or} & \\ -\tfrac{1}{2}(\varepsilon_0 u^2+\varepsilon w^2+\varepsilon\alpha_2 z^2+\varepsilon\alpha_4 y^2)+D_{10}(|u|+|w|+|z|+|y|+1)-f_5(x)\mathrm{sgn}\,x & \text{if } |u|\le|x|, \end{cases} \tag{2.6}$$

where $D_9=2D_7$ and $D_{10}=D_7+D_8$. Thus, it will be clear from either one of these two estimates that

$$\dot v^+\le -1, \text{ provided that } y^2+\dots+u^2 \text{ is large enough, say } y^2+\dots+u^2\ge D_{11}^2. \tag{2.7}$$

If, however, $y^2+\dots+u^2\le D_{11}$ and $|x|\ge D_{11}$, then we have $|x|\ge|u|$. Hence $\dot v^+$ satisfies the second estimate of (2.6). Since $y^2+\dots+u^2\le D_{11}^2$, it is clear here that $\dot v^+\le -f_5(x)\mathrm{sgn}\,x+D_{12}\le -1$, provided that $|x|\ge D_{11}$ is sufficiently large, say $|x|\ge D_{13}\ (\ge D_{11})$, by (vi). Therefore $\dot v^+\le -1$ if $y^2+\dots+u^2\le D_{11}^2$ but $|x|\ge D_{13}$. This result combined with (2.7) clearly shows that $\dot v^+\le -1$ if $x^2+y^2+\dots+u^2\ge D_{11}^2+D_{13}^2=D_6$, which verifies (2.5) ∎

Proof of Theorem 1: The usual Yoshizawa-type argument, a simple extension of Theorem 1′ in [3], applied to (2.4) and (2.5) would then show that, for any solution (x,y,z,w,u) of the differential system (2.1), we have $|x(t)|, |y(t)|, |z(t)|, |w(t)|, |u(t)| \le D_0$ for all sufficiently large t, which is equivalent to (1.3) ∎

3. Proof of the periodicity theorem

Our method of proof here will be based on an adaptation of a procedure which has been used (see, for example, [6,1]) for some third and fourth order differential equations.

Consider the parameter μ-dependent ($0 \le \mu \le 1$) ordinary differential equation of fifth order

$$x^{(5)}+(1-\mu)(\alpha_1 x^{(4)}+\ldots+\alpha_4\dot{x}+\alpha_5 x)=\mu(p(t,x,\dot{x},\ldots,x^{(4)})-f_1(\dddot{x})x^{(4)}-f_2(\dddot{x})\cdots-f_4(\dot{x})-f_5(x)). \quad (3.1)$$

This equation reduces to the linear one $x^{(5)}+\alpha_1 x^{(4)}+\alpha_2\dddot{x}+\alpha_3\ddot{x}+\alpha_4\dot{x}+\alpha_5 x=0$, when $\mu=0$. And (1.2) are the Routh-Hurwitz conditions for the asymptotic stability in the large of its trivial solution. Replace (3.1) by the equivalent system

$$\begin{aligned}&\dot{x}=y,\ \dot{y}=z,\ \dot{z}=w,\ \dot{w}=u,\\ &\dot{u}=-(1-\mu)(\alpha_1 u+\alpha_2 w+\alpha_3 z+\alpha_4 y+\alpha_5 x)+\mu(p-uf_1-f_2-f_3-f_4-f_5),\end{aligned} \quad (3.2)$$

which is obtained from (3.1) by setting $\dot{x}=y$, $\ddot{x}=z$, $\dddot{x}=w$ and $x^{(4)}=u$. It is more convenient now to consider the system (3.2) in the vector form

$$\dot{X}=AX+\mu E(t,X) \quad (3.3)$$

where

$$X=\begin{bmatrix}x\\y\\z\\w\\u\end{bmatrix},\ A=\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\-\alpha_5&-\alpha_4&-\alpha_3&-\alpha_2&-\alpha_1\end{bmatrix},\ E(t,X)=\begin{bmatrix}0\\0\\0\\0\\p-(f_1-\alpha_1)u-(f_2-\alpha_2 w)-\ldots-(f_5-\alpha_5 x)\end{bmatrix}. \quad (3.4)$$

The relationship between the equation (3.1) and the original equation (1.1) is that (3.1) reduces to (1.1) for $\mu=1$. Thus, since (3.1) is equivalent to (3.3), it is enough, in order to prove Theorem 2, to show that (3.3) has at least one periodic solution for each value of μ in the range $0 \le \mu \le 1$.

Observe that since $f_1,\ldots,f_5$ and p are assumed to be continuous and p to be periodic then $E(t,X)$ is continuous in X and t and is ω-periodic in t. Furthermore, there exist positive constants c_1 and c_2 such that

$$\|e^{(t-\tau)A}\| \le c_1 e^{-c_2(t-\tau)} \text{ for all } t \ge \tau, \quad (3.5)$$

since the characteristic roots of A have negative real parts from (1.2). The norm here denotes the sum of the absolute values of the entries of the matrix.

Here we denote by S the normed linear space of all continuous and periodic 5-vector functions $X=(x,y,z,w,u)$ which are of period ω, with the norm $\|X\|_S$ of X defined by

$$\|X\|_S = \sup_{0\le t\le\omega}\{|x(t)|+|y(t)|+|z(t)|+|w(t)|+|u(t)|\}. \quad (3.6)$$

We are now in a position to define the operator T. It is given for any $X \in S$ by

$$TX = \int_{-\infty}^{t} e^{(t-\tau)A}E(\tau,X(\tau))\,d\tau,$$

where A and E are given by (3.4). The infinite integral on the right-hand side here exists and is differentiable in t, since $e^{(t-\tau)A}$ satisfies (3.5) and since $X(t) \in S \to E(t,X) \in S$. Next, since the integral may be reset in the form $\int_0^{\infty} e^{\tau A} E(t-\tau, X(t-\tau))\,d\tau$, it is also clear that TX is periodic for arbitrary $X \in S$. The continuity of TX for arbitrary $X \in S$ follows readily from the differentiability of TX, so that T maps S into itself. The relationship between the operator T and our existence problem is the subject of the next

Lemma 3: *Any $X \in S$ satisfying the functional equation $X - \mu TX = 0$ $(0 \le \mu \le 1)$ is necessarily a solution of the vector equation* (3.3).

Proof: If $X \in S$, then dTX/dt exists. From the definition of T, $dTX/dt = ATX + E(t,X(t))$ follows. If further $X - \mu TX = 0$, then $\dot{X} = \mu\, dTX/dt = \mu ATX + \mu E(t,X(t)) = AX + \mu E(t,X(t))$ ∎

In view of the above lemma the existence of a ω-periodic solution $X = X(t)$ of (3.3) will therefore be assured, once it can be shown that the functional equation $X - \mu TX = 0$ has at least one solution $X \in S$ for each value μ in the range $[0,1]$.

The Schäffer's theorem [5] guarantees the existence of an $X \in S$ satisfying $X - \mu TX = 0$ for each $\mu \in [0,1]$ provided that

(i) $T: S \to S$ is completely continuous and

(ii) $\|X\|_S \le D$ for every $X \in S$ satisfying $X - \mu TX = 0$.

where D is a constant independent of μ. The proof of (i) proceeds as in [1,6]. Indeed, let $\{X_i\} \subset S$ be an infinite sequence satisfying $\|X_i\|_S \le D$ for all i. It is a simple matter to verify that the sequences $\{TX_i\}$ and $\{dTX_i/dt\}$ are uniformly bounded and this implies the equi-continuity of the sequence $\{TX_i\}$. Thus by the Arzela-Ascoli theorem, this sequence is compact, and so T is completely continuous.

It is difficult to prove (ii) directly because of the nature of the conditions on the functions $f_1,\dots,f_5$ and p. We observe that, since every $X \in S$ satisfying $X - \mu TX = 0$ is necessarily a solution of (3.3) and since (3.3) is equivalent to (3.2) it suffices to show that every solution (x, y, z, w, u) of (3.2) ultimately satisfies

$$|x(t)| + |y(t)| + |z(t)| + |w(t)| + |u(t)| \le D. \tag{3.7}$$

For if this condition is fulfilled, then (ii) would follow from (3.6) and the periodicity of S. Now to prove (3.7) it suffices to show that there exits a constant D independent of μ, $0 \le \mu \le 1$, such that every solution (x,y,z,w,u) of (3.2) ultimately satisfies the condition

$$|x(t)|, |y(t)|, |z(t)|, |w(t)|, |u(t)| \le D. \tag{3.8}$$

Rewrite (3.2) as follows:

$$\dot{x} = y,\ \dot{y} = z,\ \dot{z} = w,\ \dot{w} = u,\ \dot{u} = P - uF_1 - F_2 - F_3 - F_4 - F_5$$

where

$$F_1(w) = \alpha_1(1-\mu) \quad + \mu f_1(w)\ , \qquad F_2(w) = \alpha_2(1-\mu)w + \mu f_2(w)$$

$$F_3(z) = \alpha_3(1-\mu)z + \mu f_3(z)\ , \qquad F_4(y) = \alpha_4(1-\mu)y + \mu f_4(y)$$

$$F_5(x) = \alpha_5(1-\mu)x + \mu f_5(x)\ , \qquad P(t,x,y,z,w,u) = \mu p(t,x,y,z,w,u).$$

The procedure for this is by the Yoshizawa technique, using the same function V defined by (2.2), (2.3) but with $F_1,\dots,F_5$ and P instead of $f_1,\dots,f_5$ and p, respectively. To prove (3.8) we only need to verify that if the functions $f_1,\dots,f_5$ and p satisfy hypotheses (i) - (vii) of Theorem 1, so also do the functions $F_1,\dots,F_5$ and P:

(i)' $\Delta_1 = \dfrac{(\alpha_4\alpha_3 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - (\alpha_1 F_4'(y) - \alpha_5)$

$$\geq \frac{(\alpha_4\alpha_3 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \mu\left\{\frac{(\alpha_4\alpha_3 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - 2\varepsilon\alpha_2\right\} - (1-\mu)(\alpha_1\alpha_4 - \alpha_5)$$

$\geq 2\varepsilon\alpha_2(1-\mu) + 2\mu\varepsilon\alpha_2 = 2\varepsilon\alpha_2$, by (i).

$\Delta_2 = \dfrac{\alpha_4\alpha_3 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} - \dfrac{\gamma^*(\alpha_1\alpha_4 - \alpha_5)}{\alpha_4(\alpha_1\alpha_2 - \alpha_3)} - \dfrac{\varepsilon}{\alpha_1}$, where $\gamma^*(y) = \begin{cases} F_4(y)/y & \text{for } y \neq 0 \\ F_4'(0) & \text{for } y = 0. \end{cases}$ Then

$$\Delta_2 = \frac{\alpha_4\alpha_3 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} - \mu\frac{\gamma(\alpha_1\alpha_4 - \alpha_5)}{\alpha_4(\alpha_1\alpha_2 - \alpha_3)} - \frac{(1-\mu)(\alpha_1\alpha_4 - \alpha_5)}{\alpha_1\alpha_2 - \alpha_3} - \frac{\varepsilon}{\alpha_1}$$

$$\geq (1-\mu)\left\{\frac{\alpha_4\alpha_3 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} - \frac{\gamma(\alpha_1\alpha_4 - \alpha_5)}{\alpha_4(\alpha_1\alpha_2 - \alpha_3)} - \frac{\varepsilon}{\alpha_1}\right\} \geq 0, \text{ by (i).}$$

(ii)' $F_1(w) - \alpha_1 = \mu(f_1(w) - \alpha_1)$. Then $\varepsilon_0 \leq F_1(w) - \alpha_1 \leq \varepsilon_1$ for all w.

(iii)' $F_2(0) = 0, \dfrac{F_2(w)}{w} - \alpha_2 = \mu\left\{\dfrac{f_2(w)}{w} - \alpha_2\right\}$, therefore $0 \leq \dfrac{F_2(w)}{w} - \alpha_2 \leq \varepsilon_2$ for all $w \neq 0$.

(iv)' $F_3(0) = 0, \dfrac{F_3(z)}{z} - \alpha_3 = \mu\left\{\dfrac{f_3(z)}{z} - \alpha_3\right\}$, therefore $0 \leq \dfrac{F_3(z)}{z} - \alpha_3 \leq \varepsilon_3$ for all $z \neq 0$.

(v)' $F_4(0) = 0, \dfrac{F_4(y)}{y} = \alpha_4(1-\mu) + \mu\dfrac{f_4(y)}{y} \geq \alpha_4$ for all $y \neq 0$.

$$F_4'(y) - \frac{F_4(y)}{y} = \mu\left\{f_4'(y) - \frac{f_4(y)}{y}\right\} \leq \frac{\alpha_5\delta_0}{\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)} \quad \textit{for all } y \neq 0.$$

$\alpha_4 - F_4'(y) = \mu(\alpha_4 - f_4'(y))$, therefore $|\alpha_4 - F_4'(y)| \leq \varepsilon_4$ for all y.

(vi)' $F_5(0) = 0, \alpha_5 - F_5'(x) = \mu(\alpha_5 - f_5'(x))$, therefore $0 \leq \alpha_5 - F_5'(x) \leq \varepsilon_5$ for all x. By considering the cases $0 \leq \mu \leq 1/2$ and $1/2 \leq \mu \leq 1$ separately, it is clear that $F_5(x)\operatorname{sgn}x \geq \tfrac12 \times \min\{f_5(x)\operatorname{sgn}x, \alpha_5|x|\}$, so that $F_5(x)\operatorname{sgn}x > 0$ for all $x \neq 0$. Now

$$\int_0^x F_5(\xi)\,d\xi = \tfrac12\alpha_5(1-\mu)x^2 + \mu\int_0^x f_5(\xi)\,d\xi, \text{ therefore } \int_0^x F_5(\xi)\,d\xi \to \infty \text{ as } |x| \to \infty.$$

(vii)' $|P(t,x,y,z,w,u)| = |\mu\, p(t,x,y,z,w,u)| < \Delta < \infty$.

Now (3.8) is verified which completes the proof ∎

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Book review

H. BAUM, TH. FRIEDRICH, R. GRUNEWALD and I. KATH: **Twistors and Killing Spinors on Riemannian Manifolds** (Teubner-Texte zur Mathematik: Vol. 124). Leipzig - Stuttgart: B.G. Teubner Verlagsges. 1991, 180 pp.

A spin manifold, that is a Riemannian manifold (M^n, g) carrying a global spin structure, admits two natural differential operators: the well-known Dirac operator D and the twistor operator $\mathcal{D}$. By definition, $\mathcal{D}\varphi$ is the projection of the spinor derivative $\nabla\varphi$ to the kernel of the Clifford multiplication $(X,\varphi) \to X\cdot\varphi$ by a vector field X. Among the twistor spinors, i.e. the solutions of $\mathcal{D}\varphi = 0$, the Killing spinors are particularly interesting. They obey $\nabla_X \varphi = \beta X\cdot\varphi$ with some complex number β, called the Killing number to φ. Physical motivations come from twistor theory and supergravity; a mathematical motivation comes from the eigenvalue problem of the Dirac operator D on a compact manifold (M^n, g).

The present book develops the theory of twistor spinors and Killing spinors in a systematic and self-consistent way. It brings together the various results which the group of authors from the Humboldt University (cf. the 28 citations of their papers in the book) and other authors achieved over years. It serves both as an introduction into the field and as a resource book.

A manifold which admits a Killing spinor $\varphi \neq 0$ is Einstein with constant scalar curvature $R = 4n(n-1)\beta^2$ (Theorem 8 of Subsection 1.5); φ is called real, imaginary or parallel if β is real, imaginary or zero, respectively. The methods differ in the three cases: The manifold is compact or non compact if φ is real or imaginary, respectively (Theorem 9 of Subsection 1.5). The methods also depend of the parity of the dimension n: For even n, certain almost complex structures give rise to Killing spinors (Chapters 5 - 6), while for odd n, Einstein-Sasaki structures generate Killing spinors (Chapter 4). Thus families of manifolds of any dimension are constructed. Moreover, low dimensions $n \leq 7$ are treated very detailed. A new characterization of the manifolds of constant curvature and other results deserve attention.

An ultimate goal is the full classification of Killing spinors. This is reached in the present book for the imaginary case: A complete manifold which admits an imaginary Killing spinor $\varphi \neq 0$ is isometric to some warped product of a special spin manifold and the real line. The real case is solved in a paper which appeared after the reviewed book (Ch. Bär: *Real Killing spinors and holonomy.* Preprint. Univ. Bonn 1991).

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