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On a General Inequality with Applications

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A general set-valued inequality is proved in two analogical forms. As applications we obtain some simple inequalities for convex, concave, subadditive and superadditive functions. We also point out that some classical inequalities (e.g., those by Minkowski and by Beckenbach and Dresher) and some fairly new results (e.g. by Pečaric and Beesack [7] and Peetre and Persson [8]) are special cases of our results.

Key words: *Inequalities, Minkowski inequality, Beckenbach-Dresher inequality, convex functions, subadditive functions*

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1. Introduction

In this paper we denote by D an additive Abelian semigroup and by I a subset of R^n. We consider vectors $\bar{u} = (u_1, u_2, \dots, u_n), \bar{v} = (v_1, v_2, \dots, v_n) \in R^n$ and we will write $\bar{u} \le \bar{v}$ if $u_1 \le v_1, u_2 \le v_2, \dots, u_n \le v_n$. We say that a function $f : D \to R^n$ is *subadditive* if

$$f(a + b) \le f(a) + f(b) \tag{1.1}$$

for all $a,b \in D$. If (1.1) holds in the reversed direction, then we say that f is *superadditive*. If equality holds in (1.1), then we say that f is *affine*. The function $F: I \to R$ is *non-decreasing* (*non-increasing*) if the inequality $\bar{u} \le \bar{v}$ implies that $F(\bar{u}) \le F(\bar{v})$ ($F(\bar{u}) \ge F(\bar{v})$, respectively).

For definitions and basic facts about classical inequalities we refer to the books [2] and [5]. Moreover, some recent advances about inequalities can be found in the books [3] and [6] and the references given there. In this paper we prove some new inequalities for convex, concave, subadditive and superadditive functions, e.g., the following ones.

Proposition 1.1 : *Let* $F: I \to R$, $g: D \to R_+$ *and* $f: D \to I$ *be given functions.*

(a) *Assume that* F *is convex and that one of the following conditions holds*:

$(a)_1$ f *is affine*

$(a)_2$ F *is non-increasing and* f *is superadditive*

$(a)_3$ F *is non–decreasing and* f *is subadditive.*
If g *is affine or if* g *is superadditive and* $F(0) \le 0$, *then*

$$g(x+y)F\left(\frac{f(x+y)}{g(x+y)}\right) \le g(x)F\left(\frac{f(x)}{g(x)}\right) + g(y)F\left(\frac{f(y)}{g(y)}\right).$$

(b) *Suppose that* F *is concave and that one of the following conditions holds:*
$(a)_1$ f *is affine*
$(a)_4$ F *is non–increasing and* f *is subadditive*
$(a)_5$ F *is non–decreasing and* f *is superadditive.*
If g *is affine or if* g *is superadditive and* $F(0) \ge 0$, *then*

$$g(x+y)F\left(\frac{f(x+y)}{g(x+y)}\right) \ge g(x)F\left(\frac{f(x)}{g(x)}\right) + g(y)F\left(\frac{f(y)}{g(y)}\right).$$

We remark that some classical inequalities (e.g., those by Minkowski and by Beckenbach and Dresher) and some fairly new results (e.g., by Pečarić and Beesack [7] and by Peetre and Persson [8]) are special cases of Proposition 1.1 (see Section 3 and compare also with [8]). In Section 2 we present and prove Proposition 1.1 in a somewhat more general "set-valued" setting (see Theorems 2.1 and 2.2). One reason for that is that we want to incooperate also some recent inequalities by Peetre and Persson [8] and another reason is that we get more possibilities to obtain new applications. In Section 3 we present some examples and concluding remarks.

2. A general inequality in two analogical forms

Let $P(\Omega)$ denote the power set of the set Ω, i.e., the set of subsets of Ω. We state and prove the following "set–valued" versions of the inequalities in Proposition 1.1.

Theorem 2.1 : *Let the function* $F: I \to R$ *be convex and let* $G: D \to P(R_+)$ *and* $f: D \to I$ *be arbitrary functions. Then the function*

$$f_1(x) = \inf_{a \in G(x)} a\,F\left(\frac{f(x)}{a}\right), \quad x \in D,$$

is subadditive if one of the conditions $(a)_1$, $(a)_2$ *or* $(a)_3$ *holds and if, for all* $a \in G(x)$ *and* $b \in G(y)$, $a+b \in G(x+y)$ *or if there exists* $c \ge a+b$ *such that* $c \in G(x+y)$ *and* $F(0) \le 0$.

Theorem 2.2 : *Let the function* $F: I \to R$ *be concave and let* $G: D \to P(R_+)$ *and* $f: D \to I$ *be arbitrary functions. Then the function*

$$f_2(x) = \sup_{a \in G(x)} a\,F\left(\frac{f(x)}{a}\right), \quad x \in D,$$

is superadditive if one of the conditions $(a)_1$, $(a)_4$ *or* $(a)_5$ *holds and if, for all* $a \in G(x)$ *and* $b \in G(y)$, $a+b \in G(x+y)$ *or if there exists* $c \geq a+b$ *such that* $c \in G(x+y)$ *and* $F(0) \geq 0$.

Remark : Theorem 2.1 may be seen as a further generalization of results in [8, Theorem 2.1] and [10, Theorem 1]. Moreover, Theorem 2.2 generalizes Theorem 2 in [10] in a similar way (compare also with Theorem 2.2 in [8]).

Proof of Theorem 2.1 : First we assume that $F(0) \leq 0$, F is non–decreasing, f is subadditive and, for all $a \in G(x)$ and $b \in G(y)$, there exists $c \geq a + b$ such that $c \in G(x+y)$. Consider $a \in G(x)$ and $b \in G(y)$. We note that the function $H(t) = F(tf(x))$, $t \geq 0$, is convex and (since also $F(0) \leq 0$) we conclude that the function $H(t)/t$ is non–decreasing. Therefore, by also using the assumptions that f is subadditive and F is convex and non–decreasing , we obtain that, for some $c \geq a + b$ such that $c \in G(x+y)$,

$$c\,F\left(\frac{f(x+y)}{c}\right) \leq (a+b)\,F\left(\frac{f(x+y)}{a+b}\right) \leq (a+b)F\left(\frac{f(x)+f(y)}{a+b}\right) \leq a\,F\left(\frac{f(x)}{a}\right) + b\,F\left(\frac{f(y)}{b}\right).$$

Therefore, for any ε , $0 < \varepsilon < 1/2$, there exists $c \in G(x+y)$ such that

$$c\,F\left(\frac{f(x+y)}{c}\right) \leq (1+\varepsilon)\, f_1(x) + (1+\varepsilon)\, f_1(y)\,.$$

By taking infimum once more and letting $\varepsilon \to 0$ we obtain

$$f_1(x+y) \leq f_1(x) + f_1(y).$$

The proofs of the remaining cases only consist of making obvious modifications of the proof above so we omit the details ∎

Proof of Theorem 2.2 : Suppose that f is superadditive, $F(0) \geq 0$, F is non–decreasing and, for all $a \in G(x)$ and $b \in G(y)$, there exists $c \geq a + b$ such that $c \in G(x+y)$. Then, in particular, we find that $H(t) = F(tf(x))$, $t \geq 0$, is a concave function and, thus, that the function $H(t)/t$ is non–increasing. Hence, by arguing in a similar way as in the proof of Theorem 2.1, we find that, for any ε , $0 < \varepsilon < 1/2$, and some $c \in G(x+y)$,

$$(1-\varepsilon)\, f_2(x) + (1-\varepsilon)\, f_2(y) \leq c\,F\left(\frac{f(x+y)}{c}\right)$$

and by taking supremum once more and letting $\varepsilon \to 0$ we find that the function f_2 is superadditive. The proofs of the other cases are similar ∎

Proof of Proposition 1.1 : A proof of Proposition 1.1 follows by applying Theorems 2.1 and 2.2 with $G(x) = \{g(x)\}$ (the singleton case) ▮

3. Concluding remarks and examples

We apply Proposition 1.1 with $F(u) = u^p$, $p = \alpha/(\alpha-\beta)$, $f(x) = (\int_\Omega x^\alpha \, d\mu)^{1/\alpha}$, $g(x) = (\int_\Omega x^\beta \, d\mu)^{1/\beta}$ and obtain

Example 3.1 (Beckenbach-Dresher's inequality, see [1,2,4,8]): Let $x,y > 0$ a.e. on Ω. If $0 \le \alpha \le 1 \le \beta$ or if $0 \le \beta \le 1 \le \alpha$, $\alpha \ne \beta$, then

$$\left(\frac{\int_\Omega (x+y)^\alpha d\mu}{\int_\Omega (x+y)^\beta d\mu} \right)^{\frac{1}{\alpha-\beta}} \le \left(\frac{\int_\Omega x^\alpha d\mu}{\int_\Omega x^\beta d\mu} \right)^{\frac{1}{\alpha-\beta}} + \left(\frac{\int_\Omega y^\alpha d\mu}{\int_\Omega y^\beta d\mu} \right)^{\frac{1}{\alpha-\beta}} . \tag{3.1}$$

If $\beta \le 0 \le \alpha \le 1$ or if $\alpha \le 0 \le \beta \le 1$, $\alpha \ne \beta$, then (3.1) holds in the reversed direction.

In view of our discussion above it is obvious that Example 3.1 easily can be generalized in various directions. Here we only mention the following examples of such generalizations/complements :

1. By using a general isotone linear functional $A(x)$ instead of the special cases $A(x) = \int_\Omega x \, d\mu$ we obtain (generalized forms of) some versions of the Beckenbach-Dresher inequality previously proved by Pečarić and Beesack [7] and by Peetre and Persson [8] (see also [9,10]).

2. The inequality (3.1), in its turn, is a subadditivity condition and the reversed inequality is a superadditivity condition. Therefore, we can use Proposition 1.1 and iterate the procedure. After the first step we obtain the following generalization of Example 3.1: If $0 \le \beta \le 1 \le \alpha$, $\gamma \le 0 \le \delta \le 1$, $\alpha \ne \beta$, $\gamma \ne \delta$, $\alpha-\beta-\gamma+\delta \ge 0$, then

$$\left(\frac{\int_\Omega (x+y)^\alpha d\mu \int_\Omega (x+y)^\delta d\mu}{\int_\Omega (x+y)^\beta d\mu \int_\Omega (x+y)^\gamma d\mu} \right)^{\frac{1}{\alpha-\beta-\gamma+\delta}} \le \left(\frac{\int_\Omega x^\alpha d\mu \int_\Omega x^\delta d\mu}{\int_\Omega x^\beta d\mu \int_\Omega x^\gamma d\mu} \right)^{\frac{1}{\alpha-\beta-\gamma+\delta}} + \left(\frac{\int_\Omega y^\alpha d\mu \int_\Omega y^\delta d\mu}{\int_\Omega y^\beta d\mu \int_\Omega y^\gamma d\mu} \right)^{\frac{1}{\alpha-\beta-\gamma+\delta}} .$$

Moreover, if $\beta \le 0 \le \alpha \le 1$, $\gamma \le 0 \le \delta \le 1$, $\alpha \ne \beta$, $\gamma \ne \delta$, $\alpha-\beta-\delta+\gamma \ge 0$, then

$$\left(\frac{\int_\Omega (x+y)^\alpha d\mu \int_\Omega (x+y)^\gamma d\mu}{\int_\Omega (x+y)^\beta d\mu \int_\Omega (x+y)^\delta d\mu} \right)^{\frac{1}{\alpha-\beta-\delta+\gamma}} \ge \left(\frac{\int_\Omega x^\alpha d\mu \int_\Omega x^\gamma d\mu}{\int_\Omega x^\beta d\mu \int_\Omega x^\delta d\mu} \right)^{\frac{1}{\alpha-\beta-\delta+\gamma}} + \left(\frac{\int_\Omega y^\alpha d\mu \int_\Omega y^\gamma d\mu}{\int_\Omega y^\beta d\mu \int_\Omega y^\delta d\mu} \right)^{\frac{1}{\alpha-\beta-\delta+\gamma}} .$$

3. By using the continuity property of generalized Gini means (see [8]) we obtain the inequalities corresponding to the exceptional cases in Example 3.1

(and the inequalities in case 2 above), e.g., for the extremal case $\alpha = \beta = 1$ the inequality (3.1) reads

$$\exp\left(\frac{\int\limits_\Omega (x+y)\ln(x+y)d\mu}{\int\limits_\Omega (x+y)d\mu}\right) \leq \exp\left(\frac{\int\limits_\Omega x\ln x d\mu}{\int\limits_\Omega x d\mu}\right) + \exp\left(\frac{\int\limits_\Omega y\ln y d\mu}{\int\limits_\Omega y d\mu}\right)$$

and the corresponding inequality for the other limiting case $\alpha = \beta = 0$ reads

$$\exp\left(\frac{1}{\mu(\Omega)}\int\limits_\Omega \ln(x+y)\, d\mu\right) \geq \exp\left(\frac{1}{\mu(\Omega)}\int\limits_\Omega \ln x\, d\mu\right) + \exp\left(\frac{1}{\mu(\Omega)}\int\limits_\Omega \ln y\, d\mu\right).$$

A special case of this inequality is the following well-known inequality for positive sequences (see, e.g., [2,p. 26]):

$$\left(\prod_1^n (x_k + y_k)\right)^{1/n} \geq \left(\prod_1^n x_k\right)^{1/n} + \left(\prod_1^n y_k\right)^{1/n}.$$

So far we have only given applications of our general theorems for one extremal case, namely the single–valued case presented in Proposition 1.1. We also present an application for another extremal case namely when $G(x) = R_+$ for all $x \in D$.

Example 3.2 : Let $D = R^n$, $f(\bar{x}) = \bar{x} = (x_1, x_2, \ldots, x_n)$, $G(\bar{x}) = R_+$ for all $\bar{x} \in D$ and consider the (Amemiya) norm

$$\|x\|_\Phi = \inf_{a \in R_+} a\left(1 + \sum_{k=1}^n \Phi\left(\frac{x_k}{a}\right)\right),$$

where the function $\Phi: R_+ \to R_+$ is convex. By applying Theorem 2.1 with

$$F(\bar{u}) = 1 + \sum_1^n \Phi(|u_k|)$$

we find that

$$\|\bar{x} + \bar{y}\|_\Phi \leq \|\bar{x}\|_\Phi + \|\bar{y}\|_\Phi,$$

and, thus, we have obtained another proof of Minkowski's inequality for Orlicz sequence spaces. Moreover, by using Theorem 2.2 in a similar way we find that the inequality

$$\|\bar{x} + \bar{y}\|_\Psi \geq \|\bar{x}\|_\Psi + \|\bar{y}\|_\Psi$$

holds, where the function $\Psi : R_+ \to R_+$ is concave and

$$\|x\|_\Psi = \sup_{a \in R_+} a\left(1 + \sum_{k=1}^n \Psi\left(\frac{x_k}{a}\right)\right).$$

Finally we remark that the Beckenbach-Dresher inequality (Example 3.1) means that (integral forms of) the classical *Gini means* are subadditive or superadditive with certain restrictions on the parameters involved. Some new results concerning generalized Gini means have recently been obtained in [8] and [11]. These results can be useful to investigate the "intermediate" cases in Theorems 2.1 and 2.2 but this possibility is not fully investigated yet.

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