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A Study on the Geometry of Pairs of Positive Linear Forms, Algebraic Transition Probability and Geometrical Phase over Non-Commutative Operator Algebras (I)

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Some aspects in the geometry of pairs of positive linear forms on unital C^*-algebras are considered. Especially, the geometrical relations among the vector representatives of the forms of such a pair within a representation, where both forms can be realized as vectors simultaneously, are studied and discussed in detail. The results obtained in this part extend early results of H. Araki and are intimately related to such functors like the Bures distance and the algebraic transition probability considered by A. Uhlmann and others. The results will be used to discuss and to investigate some extensions of geometrical concepts, which have been found to be of interest recently in Mathematical Physics in context of the problems of the so-called geometrical phase.

Key words: *Functional Analysis, C^*-Algebras, vN-Algebras, Non-Commutative Probability, Non-Commutative Geometry*

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0. Introduction

Let H be a complex Hilbert space, with scalar product $\langle \cdot,\cdot \rangle$. In all what follows we suppose that $\langle \cdot,\cdot \rangle : H \times H \ni \{\xi,\eta\} \longmapsto \langle \xi,\eta \rangle$ behaves linearly with respect to ξ and anti-linearly with respect to η. The C^*-algebra of all bounded linear operators on H will be denoted by $B(H)$. Let ω be a *vector state* on $B(H)$, i.e. ω is a positive linear form on $B(H)$, normalized to one, such that there exists a unit vector $\xi \in H$ with $\omega(x) = \langle x\xi, \xi \rangle$ for all $x \in B(H)$. The set of all vectors ξ which obey the latter will be denoted by $S(\omega)$. Note that, for any given $\xi \in S(\omega)$, we have $S(\omega) = \{ \lambda\xi : \lambda \in \mathbb{C}, |\lambda| = 1 \}$.

In quantum mechanics vector states are the basic objects for the mathematical description of physical states (roughly speaking, in elementary quantum mechanics it is supposed that physical states are in one-to-one correspondence to the vector states on $B(H)$ for an appropriately chosen H). In this context $S(\omega)$ is referred to as the *unit ray* of the vector state ω, and each $\xi \in S(\omega)$, which is a representing vector or a representative for ω, corresponds to a state vector or wave function of the quantum mechanical system in question. Suppose now, ω and σ are vector states. For given vectors $\xi \in S(\omega)$ and $\eta \in S(\sigma)$ the number $P(\omega,\sigma) = |\langle \xi,\eta \rangle|^2$ depends only on the pair $\{\omega,\sigma\}$. Also this number plays a distinguished role in context of quantum mechanics. It is the quantum mechanical *transition probability* between the vector states ω and σ. From the mathematical point of view, the importance of the transition probability, if seen as a functional on the pairs of vector states or unit rays, is mainly due to a famous theorem of E. P. Wigner on symmetry operations. Wigner's theorem asserts that any *symmetry*, i.e. a map Ψ acting from the set of all vector states onto itself which preserves the transition probability: $P(\Psi(\omega), \Psi(\sigma)) = P(\omega, \sigma)$ for any pair $\{\omega, \sigma\}$ of vector states, can be implemented by some *unitary* or *anti-unitary* operator u on H. This means that we can associate with Ψ some unitary or anti-unitary operator u, with $u\xi \in S(\Psi(\omega))$ for any $\xi \in S(\omega)$ and each vector state ω. This appears to be one of the basic mathematical results on the foundations of axiomatic (algebraic) quantum theory (cf. the lectures of R. Jost [18], or [11] e.g.).

Let us now consider a pair of vector states $\{\omega, \sigma\}$ and state vectors ξ, η being representatives of them, $\xi \in S(\omega)$ and $\eta \in S(\sigma)$. Assume the given vector states are not mutually orthogonal, i.e. $P(\omega, \sigma) \neq 0$. Then, $\langle \xi, \eta \rangle = e^{i\vartheta} P(\omega, \sigma)^{1/2}$ for some uniquely determined complex number $\delta(\eta, \xi) = e^{-i\vartheta}$ of modulus one. We call $\delta(\eta, \xi)$ the *relative phase* between the representatives η and ξ, whereas $\xi' = \delta(\eta, \xi)\xi \in S(\omega)$ is referred to as the η*-relative representative* of the vector state ω. Obviously, for the given pair of vector states $\{\omega, \sigma\}$ the η- relative representative of the vector state ω is the unique state vector ψ of the unit ray $S(\omega)$ such that $\langle \psi, \eta \rangle > 0$. For some given number $n \in \mathbb{N}$, let $\gamma = \{\omega_0, \omega_1, \dots, \omega_{n+1}\}$ be a sequence of vector states such that $\omega_j \neq \omega_{j+1}$ and ω_j is *not* orthogonal with ω_{j+1}, for $j = 0, 1, \dots, n$, and $\omega_{n+1} = \omega_0$. We call such a sequence *loop* of vector states (at ω_0). Assume $\varphi \in S(\omega_0)$ is fixed. Then, we construct another state vector $\varphi(\gamma) \in S(\omega_0)$ by the following recursive rule. For $j = 0$ we define $\varphi_j = \varphi$. For $j \geq 1$ we define φ_j as the φ_{j-1}-relative representative of ω_j. Finally we put $\varphi(\gamma) = \varphi_{n+1}$. In practice, however, each of the vector states $\omega_0, \omega_1, \dots, \omega_n$ is given by some particular state vector ξ_j taken from the unit ray $S(\omega_j)$. We take $\xi_0 = \xi_{n+1} = \varphi$. Then, having in mind the above characterization of the relative representative, we easily see that $\varphi_j = \delta(\xi_j, \xi_{j-1})\,\delta(\xi_{j-1}, \xi_{j-2}) \cdots \delta(\xi_1, \xi_0)\xi_j$ for any j. Thus, we have $\varphi(\gamma) = \delta(\xi_0, \xi_n) \cdots \delta(\xi_1, \xi_0)\varphi$. Although the factors $\delta(\xi_j, \xi_{j-1})$ heavily depend on the choice of the sequence $\{\xi_0, \dots, \xi_n\}$ of representatives $\xi_j \in S(\omega_j)$, the product of these factors $\delta(\xi_0, \xi_n) \cdots \delta(\xi_1, \xi_0)$ does not depend on this choice. In fact, we have $\delta(\xi_j, \xi_{j-1}) = \langle \xi_j, \xi_{j-1} \rangle |\langle \xi_j, \xi_{j-1} \rangle|^{-1}$, for any j. Since $\langle \xi_0, \xi_n \rangle \cdots \langle \xi_1, \xi_0 \rangle$ is invariant under the replacement $\xi_j \longrightarrow z_j \xi_j$, for any $z_j \in \mathbb{C}$ of modulus $|z_j| = 1$, this fact also applies to $\Delta(\gamma) = \Delta(\omega_0, \dots, \omega_n) = \delta(\xi_0, \xi_n) \cdots \delta(\xi_1, \xi_0)$. Note that $\Delta(\gamma)$ is a complex number of modulus one. We shall refer to this number as the *global phase* of the loop γ. For $n \geq 2$ the global phase can be non-trivial, i.e. there are loops γ with $n \geq 2$ such that $\Delta(\gamma) \neq 1$. We remark that the complex unitary invariant $\Delta(\gamma)$ has been considered and discussed (at least in case of $n = 2$) by V. Bargmann within his treatise on Wigner's theorem in [8]. We also note that Δ has been used there in order to decide between both the possibilities of the unitary or anti-unitary implementability of a given symmetry Ψ. In fact, following Wigner's theorem, for a symmetry Ψ there are only two principal possibilities. Either $\Delta(\Psi(\gamma)) = \Delta(\gamma)$ for any loop γ (in using the notations from above, $\gamma' = \Psi(\gamma)$ is the loop of the vector states $\Psi(\omega_j)$), or we find that $\Delta(\Psi(\gamma)) = \overline{\Delta(\gamma)}$ for any loop γ. In the first case Ψ can be unitarily implemented, in the second case only an anti-unitary implementation is possible. In case of dim $H \geq 2$, due to the existence of loops γ with $\operatorname{Im} \Delta(\gamma) \neq 0$, both cases occur and describe an intrinsic property of the symmetry Ψ.

Besides this interesting meaning of $\Delta(\gamma)$ we discussed in context of the problem of the implementability of symmetries there is yet another aspect under which the global phase of a loop γ can be considered. A loop γ, as we defined this term, can also be considered as a caricature of a closed curve in the set of all vector states (with the restriction that neighbouring states are mutually non-orthogonal). We can imagine a variable ω, with values in the set of all vector states, which we will force to drive successively through the states of the curve γ, starting with a state ω_0, and finally ending up with the same state ω_0. Now, let us consider as a *local law of transportation* (or *law of conditional choice*) for representatives $\xi(\omega) \in S(\omega)$ of vector states from one vector state ω to some next neighbouring (non-orthogonal) state ω' the following rule: *if* $\varphi = \xi(\omega) \in S(\omega)$ is given, then $\xi(\omega') \in S(\omega')$ is defined to be the *φ-relative representative of ω'*. Note that in our above discussions around $\Delta(\gamma)$ the sequence of vectors $\{\varphi_j\}$ was constructed from the starting vector φ by repeatedly applying this rule. Let now $\xi(\omega_0) = \varphi$ be chosen in $S(\omega_0)$. Due to the fact that $\Delta(\gamma)$ is nontrivial in many cases of loops, i.e. $\Delta(\gamma) \neq 1$ or even Im $\Delta(\gamma) \neq 0$, transporting $\xi(\omega)$ according to the given law around the loop γ globally will cause an effect of anholonomy. In other words this means the phenomenon that if our driving parameter ω has rounded the closed curve γ and $\omega = \omega_0$ is reached again we will in general have that $\xi(\omega) = \varphi(\gamma) = \Delta(\gamma)\varphi \neq \varphi$.

For a survey around problems of anholonomy, especially in quantum theory, we refer the reader to M. Berry in [10]. Morever, concepts like relative phase and global phase in more or less specific situations in physics are sometimes referred to as *Pancharatnam phase*, *Aharonov-Anandan phase* or *Berry phase* (cf. [22], [1] and [9]; for the mathematician [21] provides a nice short introduction to the phenomenon and gives indications of its physical relevance).

The aim of this paper will be to provide and to investigate some extensions of geometrical concepts like *relative phase*, *relative representative* and *global phase* from the context of considering vector states on $B(H)$ into the more general context of unital C^*-algebras and their states, or even more generally, to positive linear forms. This change into some wider algebraic context mathematically corresponds to a change in the physical concepts from a pure quantum mechanics frame into that of quantum statistical mechanics. Recently such generalizations have been proposed by A. Uhlmann in [26], [30], and have been considered in the set of density operators (cf. [27]-[29]).

The generalizations we are aiming at will now be indicated. For this sake, let us suppose $\{\omega, \sigma\}$ to be a pair of positive linear forms on a given unital C^*-algebra A. Assume $\{\pi, H\}$ is a unital *-representation of A on a Hilbert space H such that vectors $\varphi, \psi \in H$ exist with $\omega(\cdot) = \langle \pi(\cdot)\varphi, \varphi \rangle$ and $\sigma(\cdot) = \langle \pi(\cdot)\psi, \psi \rangle$, i.e. φ is a vector representative (w.r.t. π) of ω, and ψ is a vector representative (w.r.t. π) of σ. Such representations always exist. Moreover, one can show that with respect to $\{\pi, H\}$ the representatives φ and ψ of ω and σ can always be chosen in such a way that the linear form $h^{\pi}_{\varphi,\psi}$ defined by $h^{\pi}_{\varphi,\psi}(\cdot) = \langle (\cdot)\psi, \varphi \rangle$ on the vN-algebra $\pi(A)'$ (the *commutant* of $\pi(A)$) is positive. This is a remarkable fact on its own.

But even more than this can happen. It can happen that $\psi' = \psi$ is the unique vector representing σ within $\{\pi, H\}$ such that $h^{\pi}_{\varphi,\psi'} \geq 0$ and, at the same time, $\varphi' = \varphi$ is also the unique representative of ω within $\{\pi, H\}$ such that $h^{\pi}_{\varphi',\psi} \geq 0$. As the analysis shows, if this case occurs for a pair $\{\varphi, \psi\}$ of representatives for $\{\omega, \sigma\}$ within $\{\pi, H\}$ such that $h^{\pi}_{\varphi,\psi} \geq 0$, then the same fact also happens to be true for any other pair $\{\varphi', \psi'\}$ of representatives for $\{\omega, \sigma\}$ within $\{\pi, H\}$ such that $h^{\pi}_{\varphi',\psi'} \geq 0$. With other words, this case occurs if the positivity requirement for $h^{\pi}_{\varphi,\psi}$ fixes the relative position in H of two representatives φ and ψ to each other uniquely for the given pair $\{\omega, \sigma\}$. Note also that in this case one finds that to each representative ξ of ω within $\{\pi, H\}$ one can find a representative η of σ such that $h^{\pi}_{\xi,\eta} \geq 0$, and vice versa. Moreover, one can prove that, if this occurs for a pair of positive linear forms $\{\omega, \sigma\}$, this fact is independent of the special choice of the representation $\{\pi, H\}$ provided both linear forms can be represented w.r.t. $\{\pi, H\}$ simultaneously by vectors. Hence, the occurence of this fact reflects an intrinsic geometrical property a pair of positive linear forms can possess. A pair of positive linear forms $\{\omega, \sigma\}$ with the previously described property fulfilled will be referred to as a «-*minimal pair*.

For the vector representatives φ and ψ of the forms ω and σ of a «-minimal pair $\{\omega, \sigma\}$ in a representation $\{\pi, H\}$ the notion of *relative representatives* can now be introduced as follows. The representative ψ will be referred to as the *φ-relative representative of σ (with respect to π)* if $\psi' = \psi$ is the unique vector representing σ w.r.t. π such that $h^{\pi}_{\varphi,\psi'} \geq 0$. In this situation we will write $\psi \parallel_{\pi} \varphi$. By definition, $\parallel_{\pi}$ is symmetric, i.e. if ψ is the φ-relative representative of σ (with respect to π), then also φ is the ψ-relative representative of ω (with respect to π). Note that, by definition of the term «-minimal pair, to each representative φ of ω w.r.t. $\{\pi, H\}$ there is exactly one representative ψ of σ w.r.t. $\{\pi, H\}$ such that $\psi \parallel_{\pi} \varphi$. In this situation, let us assume ψ' is another representative of σ w.r.t. $\{\pi, H\}$. Then, we find a uniquely determined partial isometry $v \in \pi(A)'$ such that $\psi = v^{*}\psi'$ and $p = vv^{*}$ is the smallest orthoprojection in $\pi(A)'$ such that $p\psi' = \psi'$. The partial isometry $v^{*} = \delta_{\pi}(\varphi, \psi')$ will be referred to as *relative phase* between the representatives φ of ω and ψ' of σ w.r.t. $\{\pi, H\}$. It is evident that both the notions of the relative representative and the isometry δ_{π} in a natural way extend the notions of the relative representative and relative phase from the special context of considering pairs of non-orthogonal vector states on $B(H)$ w.r.t. the trivial representation $\{id, H\}$ (note that $B(H)' = \mathbb{C}e$ in this case) into a C^{*}-algebraic context of considering pairs of «-minimal positive linear forms of some unital C^{*}-algebra A w.r.t. a representation $\{\pi, H\}$, where both forms can be realized as vectors simultaneously.

As mentioned above, for an *arbitrary* pair $\{\omega, \sigma\}$ of positive linear forms on a unital C^{*}-algebra A and given representation $\{\pi, H\}$ such that both forms can be realized as vectors simultaneously, we have the remarkable fact to hold that representatives φ and ψ of ω and σ can always be chosen in such a way that the functional $h^{\pi}_{\varphi,\psi}$ on $\pi(A)'$ is positive, but possibly the relative position among the representing vectors is not uniquely determined by this condition of positivity. Assume $h^{\pi'}_{\varphi',\psi'} \geq 0$, where φ' and ψ' are representatives of the same pair of linear forms w.r.t. another representation $\{\pi', H'\}$. Then one observes that $h^{\pi}_{\varphi,\psi}(e) = h^{\pi'}_{\varphi',\psi'}(e)$, i.e. the value taken by the form $h^{\pi}_{\varphi,\psi}$ on the unity e is independent of the special representation π chosen provided $h^{\pi}_{\varphi,\psi}$ is positive. The real $P_A(\omega, \sigma) = h^{\pi}_{\varphi,\psi}(e)^2$, which exists for any pair $\{\omega, \sigma\}$ of positive linear forms, characterizes some aspects of the relative geometry of the components of the pair $\{\omega, \sigma\}$. This number $P_A(\omega, \sigma)$ can also be calculated without any reference to some particular representation. One can prove e.g. $P_A(\omega, \sigma) = \inf \omega(x)\sigma(x^{-1})$, with the infimum extending on all invertible, positive elements x of A. Note that in the special case of some pair of *vector states* $\{\omega, \sigma\}$ on $B(H)$, and granting the above mentioned observation to be true, $P_{B(H)}(\omega, \sigma) = |\langle \varphi, \psi \rangle|^2$ had to be valid for any other two vector representatives $\{\varphi, \psi\}$ of the pair of vector states in question. Therefore, in this case $P_A(\omega, \sigma)$ is intimately related to the distinguished transition probability between vector states. By considering $P_A(\omega, \sigma)$ in a rather general situation we are led to one of the concepts of a **-algebraic transition probability* between general positive linear forms.

By means of this concept we will be able to give the following characterization of *«-minimality* of a pair $\{\omega, \sigma\}$ of positive linear forms on a unital C^*-algebra A: $\{\omega, \sigma\}$ is «-minimal if, and only if, for any pair $\{\nu, \mu\}$ of positive linear forms on A such that $0 \le \nu \le \omega$ and $0 \le \mu \le \sigma$ the condition $P_A(\omega, \sigma) = P_A(\nu, \mu)$ always implies $\nu = \omega$ and $\mu = \sigma$ to be true. As mentioned above, for such pairs concepts like (*mutually*) *relative representatives* and *relative phases* can be developped and extended consistently with the corresponding notions for (non-orthogonal) vector states. An essential part of this paper is devoted to investigations of the properties and the geometry of the «-minimal pairs of positive linear forms. Among other things we will show that for a «-minimal pair $\{\omega, \sigma\}$ there exists a unique linear form g on A such that $g(e) = P_A(\omega, \sigma)^{1/2}$ and $|g(y^*x)|^2 \le \omega(y^*y)\sigma(x^*x)$, for all $y, x \in A$, are fulfilled. The structure of this linear form reflects some aspects of the *non-commutativity* in the pair $\{\omega, \sigma\}$. The main properties of g are investigated together with the basic properties of the algebraic transition probability .

Within Sections 1 $\div$ 7, which constitute the first part of the paper, one can find the proofs of all the facts previously indicated. Whereas in Sections 1 - 3 we will be concerned with the general C^*-algebraic context, from Section 4 on the underlying algebras in the investigation will be vN-algebras and the linear forms considered will be supposed to be positive normal linear forms. It is only for convenience that the vN-algebras considered are supposed to act in standard form on some Hilbert space. This will meet the main cases of applications we will have in mind. Especially, in Section 5 a representation theorem for «-minimal pairs of normal positive linear forms on such an algebra is proved which reads in terms of the unique solution g. Also several other characterizations of «-minimality are given in this context. The problems around the definition of the relative representatives and relative phases will be considered in Section 6. To simplify the notations, also in this section we shall mainly restrict our considerations to vN-algebras and normal positive linear forms. In Section 7 we then show how the results of Sections 4 - 6 can be extended from the case of vN-algebras and normal positive linear forms to arbitrary unital C^*-algebras and their positive linear forms.

For convenience of the reader, some common notations and technical facts that belong, more or less, to the mathematical folclore in the field of vN-algebras and which we will make use of repeatedly throughout the investigations are explained and derived and collected in the Appendix (Section A). Some of these results and tools (especially A.2 and A.7) are mainly due, respective closely related to some of the ideas developped by H. Araki in [6].

The second part of the paper contains Sections 8 - 11. The results of the first part mentioned will be used extensively for generalizing and analyzing some phenomena of more algebraic-geometrical type in the normal state space of a vN-algebra. The concepts of the *global phase*, the *phase group* and *holonomy group* of a *normal state* of a vN-algebra will be introduced and discussed. The concept of the global phase is the generalization of our above considerations relating the invariant Δ on (discrete) paths of vector states on $B(H)$. Whereas in Section 8 the discrete case is considered in detail, Section 9 is devoted to the continuous case. The latter case is largely based on the results of the discrete case. The above mentioned groups both are isomorphic respective anti-isomophic to the full group of certain equivalence classes of closed (discrete or continuous) paths in the normal state space arising from and ending in the state in question. The global phases are, roughly speaking, the members of the phase group. The paths considered in the continuous case have - among other things - the following property : two states belonging to such a path form a «-minimal pair if they are lying *close enough* to each other on this path. Hence, both the geometrical and topological properties of these paths to a large extend follow from the geometry of pairs of positive linear forms which form a «-minimal pair. The second part of the paper is finished with Section 11 where first (essentially finite-dimensional) examples and effects of the theory presented are illustrated. Further examples and applications (to *AFD*-algebras, mainly) are under investigation and will be published elsewhere.

1. Generalities on pairs of positive linear forms

Following the line of investigations as indicated in Section 0, in this section we are going to introduce the mathematical concept of an *algebraic transition probability*. Let A be a C^*-algebra with unit e and topological dual space A^*. Let ω and σ be positive linear forms on A, i.e. $\omega, \sigma \in A^*_+$. Suppose $\{\pi, H\}$ is some unital *-representation of A on the Hilbert space H, with inner product $\langle \cdot , \cdot \rangle$.

For a positive linear form τ on A we define a subset $S(\pi, \tau) \subset H$ by $S(\pi, \tau) = \{\Phi \in H: \tau(x) = \langle \pi(x)\Phi, \Phi\rangle, \forall x \in A\}$. The representation $\{\pi, H\}$ is said to be ω, σ - *admissible* provided both sets $S(\pi, \omega)$ and $S(\pi, \sigma)$ are non-void.

In [12] D. Bures introduced and investigated his 'distance' function $d_A(\omega, \sigma)$ in the case of normal states on a W^*-algeba. The definition extends in an obvious manner to unital C^*-algebras and pairs of positive linear forms. Henceforth we shall refer to this trivial extension. To give the definition, assume $\varphi, \psi \in H$ to be vectors of a Hilbert space H. Let f_φ and f_ψ be the positive linear forms on the space $B(H)$ of bounded linear operators on H which are generated by φ and ψ, respectively, i.e. $f_\varphi(x) = \langle x\varphi, \varphi\rangle$ and $f_\psi(x) = \langle x\psi, \psi\rangle$, for any $x \in B(H)$. Then, the *distance* $d(f_\varphi, f_\psi)$ between f_φ, f_ψ is defined as $d(f_\varphi, f_\psi) = \| f_\varphi - f_\psi \|_1$, with $\| \cdot \|_1$ denoting the functional norm in $B(H)^*$. The *Bures distance* $d_A(\omega, \sigma)$ between two forms $\omega, \sigma \in A^*_+$ is defined as the following infimum:

$$d_A(\omega, \sigma) = \inf \{d(f_\varphi, f_\psi): \varphi \in S(\pi, \omega), \psi \in S(\pi, \sigma)\}, \tag{1.1}$$

where the infimum also extends on all ω, σ-admissible $\{\pi, H\}$. It is easy to see that in a Hilbert space H for any two vectors φ, ψ the following relation is valid :

$$|\langle \varphi, \psi\rangle|^2 = \tfrac{1}{4}\left\{\left(\| \varphi \|^2 + \| \psi \|^2\right)^2 - d(f_\varphi, f_\psi)^2\right\}.$$

For given $\omega, \sigma \in A^*_+$ let us define the positive real $P_A(\omega, \sigma)$ as follows:

$$P_A(\omega, \sigma) = \sup\left\{|\langle \varphi, \psi\rangle|^2: \varphi \in S(\pi, \omega), \psi \in S(\pi, \sigma), \forall\ \omega,\sigma\text{-admissible } \{\pi, H\}\right\}.$$

Using (1.1) we see that $d_A(\omega, \sigma)$ and $P_A(\omega, \sigma)$ are connected by the relation

$$P_A(\omega, \sigma) = \tfrac{1}{4}\left\{\left(\| \omega \|_1 + \| \sigma \|_1\right)^2 - d_A(\omega, \sigma)^2\right\}. \tag{1.2}$$

In case of states, i.e. positive linear forms normalized to one, the expression $P_A(\omega,\sigma)$ had been taken by *A. Uhlmann* in [24] (cf. also [25]) as a definition of a *C^*-algebraic transition probability*. We tacitely shall refer to this notion also in case that ω and σ are not normalized, necessarily.

Both P_A and d_A are mathematically well-investigated functors and many properties of them are known. Due to (1.2), between both functors and their properties there are intimate correspondences. Especially, for the positive linear forms f_φ and f_ψ on the C^*-algebra $A = B(H)$ one finds $P_A(f_\varphi, f_\psi) = |\langle \varphi, \psi\rangle|^2$ and $d_A(f_\varphi, f_\psi) = d(f_\varphi, f_\psi)$, i.e. P_A and d_A reduce to the *transition probability* between the vectors considered or the *functional distance* between the *pure states* generated by these vectors, respectively. For a survey on *C^*-algebraic transition probabilities* and problems related to them the reader is referred to [4] and the references there. Both the notions P and d have been used in investigations of certain geometrical properties of algebraic state spaces. For instance, in [6] one can find some results which later have been used extensively for deriving crucial properties of the functor P (cf. also our Theorem 4.4 and Remark 4.5).

The starting point of the investigations of this paper will be a result proved in [2: Theorem 1 and Corollary 1] in case of states ω, σ on a unital C^*-algebra A. This result persists to hold for non-normalized positive linear forms. For convenience we include also a proof. The result reads as follows.

Lemma 1.1 : *Let $Q_A(\omega,\sigma) \subset A^*$ be defined by $Q_A(\omega,\sigma) = \left\{ f \in A^* : |f(y^*x)|^2 \leq \omega(y^*y)\,\sigma(x^*x),\ \forall\ x, y \in A \right\}$ and assume $\{\pi, H\}$ is ω, σ-admissible. We suppose $\varphi \in S(\pi,\omega)$ and $\psi \in S(\pi,\sigma)$. Let $h \in \pi(A)'_*$ be given by $h(x) = \langle x\psi, \varphi\rangle$,* for all $x \in \pi(A)'$. *Then, $Q_A(\omega,\sigma)$ is non-void and the following assertions hold:*

(i) $\|h\|_1 = P_A(\omega,\sigma)^{1/2}$;

(ii) $|f(e)| \leq P_A(\omega,\sigma)^{1/2}$ *for any* $f \in Q_A(\omega,\sigma)$;

(iii) $f(e) = P_A(\omega,\sigma)^{1/2}$ *for some* $f \in Q_A(\omega,\sigma)$.

Proof : What will be shown first is the existence of $f \in A^*$ which fulfils the relations $f(e) = P_A(\omega,\sigma)^{1/2}$ and $|f(y^*x)| \leq \omega(y^*y)^{1/2}\sigma(x^*x)^{1/2}$, for all $x, y \in A$. Let $\varepsilon > 0$. Let $\{\pi_\varepsilon, H_\varepsilon\}$ be *ω, σ-admissible* such that $\varphi_\varepsilon \in S(\pi_\varepsilon,\omega)$ and $\psi_\varepsilon \in S(\pi_\varepsilon,\sigma)$ exist with $P_A(\omega,\sigma)^{1/2} - \langle \psi_\varepsilon, \varphi_\varepsilon\rangle \leq \varepsilon$. This is always possible to arrange. Let us define $f_\varepsilon(\cdot) = \langle \pi_\varepsilon(\cdot)\psi_\varepsilon, \varphi_\varepsilon\rangle$. Then, $f_\varepsilon \in Q_A(\omega,\sigma)$. Since $Q_A(\omega,\sigma)$ obviously is *w^*-compact* by its very definition, the net $\{f_\varepsilon\}_{\varepsilon>0}$ has a *w^*-accumulation point* $f \in Q_A(\omega,\sigma)$, and $f(e) = P_A(\omega,\sigma)^{1/2}$ follows. Let $\{\pi, H\}$ be any *ω, σ-admissible* representation of A, and be $\varphi \in S(\pi,\omega)$ and $\psi \in S(\pi,\sigma)$. Assume now we have given $g \in A^*$, such that, for any $x, y \in A$, $|g(y^*x)| \leq \omega(y^*y)^{1/2}\,\sigma(x^*x)^{1/2}$ is fulfilled. Working within the *-representation $\{\pi, H\}$, we see that $|g(y^*x)| \leq \|\pi(y)\varphi\|\,\|\pi(x)\psi\|$, for any $x, y \in A$. Hence, the map γ given by $\gamma : \{\pi(y)\varphi, \pi(x)\psi\} \longmapsto g(y^*x) \in \mathbb{C}$, for all $x, y \in A$ is well defined as a bounded by one *sesquilinear form* from a dense linear subspace of $p'(\varphi)H \times p'(\psi)H$ into $\mathbb{C}$. We extend the form γ by continuity to a sesquilinear form, bounded by one, on the whole $p'(\varphi)H \times p'(\psi)H$ which, for simplicity, will also be denoted by γ. Note that for $\xi \in H$ we define $p'(\xi)$ as the orthoprojection of M' that projects onto $\overline{[\pi(A)\xi]}$. There is some operator $K \in B(H)$, with $K = p'(\varphi)K = Kp'(\psi)$ and $\|K\| \leq 1$, such that $\gamma(\xi,\eta) = \langle K\xi, \eta\rangle$, for any $\{\xi, \eta\} \in p'(\varphi)H \times p'(\psi)H$. Especially, in chosing $\xi = \pi(x)\psi$ and $\eta = \pi(y)\varphi$ for x, y

$\in A$ we see that $\gamma(\xi, \eta) = g(y^*x) = \langle K\pi(x)\psi, \pi(y)\varphi\rangle$. Since $g((yz)^*x) = g(z^*y^*x)$ is fulfilled for all $x,y,z \in A$, and since π is a representation of A, the latter implies that $\langle \pi(y^*)K\pi(x)\psi, \pi(z)\varphi\rangle = \langle K\pi(y^*)\pi(x)\psi, \pi(z)\varphi\rangle$ holds for $x,y,z \in A$. Hence, for any $y \in A$ we get

$$p'(\varphi)\pi(y^*)Kp'(\psi) = p'(\varphi)K\pi(y^*)p'(\psi).$$

Hence, we realize that $\pi(y)K = K\pi(y)$ for any $y \in A$. Taking together all these facts we arrive at the existence of $K \in M'$, $\|K\| \le 1$, such that $g(x) = \langle K\pi(x)\psi, \varphi\rangle$, for all $x \in A$. From this $|g(e)| = |\langle K\psi, \varphi\rangle|$ follows. By the known result of B. Russo and H.A. Dye [14] we have $\{J \in M' : \|J\| \le 1\} = \overline{\mathrm{conv}\, \mathcal{U}(M')}$ (uniform closure). Hence we have $|g(e)| = |\langle K\psi, \varphi\rangle| \le \sup\{|\langle u\psi, \varphi\rangle| : u \in \mathcal{U}(M')\}$. Since for any unitary $u \in \mathcal{U}(M')$ and $\psi \in S(\pi, \sigma)$ also $u\psi \in S(\pi, \sigma)$ holds, $|\langle u\psi, \varphi\rangle| \le P_A(\omega,\sigma)^{1/2}$ follows. Thus $|g(e)| = |\langle K\psi, \varphi\rangle| \le \sup\{|\langle u\psi, \varphi\rangle| : u \in \mathcal{U}(M')\} \le P_A(\omega, \sigma)^{1/2}$ is seen, i.e. (ii) follows. Hence, if we chose $g = f$, with the linear form f defined above, then the relation $f(e) = P_A(\omega, \sigma)^{1/2}$ implies $\sup\{|\langle u\psi, \varphi\rangle| : u \in \mathcal{U}(M')\} = P_A(\omega, \sigma)^{1/2}$. But then, in defining $h \in M'_*$ by $h(\cdot) = \langle (\cdot)\psi, \varphi\rangle$ and arguing by the theorem of B. Russo and H. A. Dye again, we can conclude as follows :

$$\begin{aligned}\|h\|_1 &= \sup\{|\langle J\psi, \varphi\rangle| : J \in M', \|J\| \le 1\}\\ &= \sup\{|\langle u\psi, \varphi\rangle| : u \in \mathcal{U}(M')\} = P_A(\omega, \sigma)^{1/2}.\end{aligned}$$

Hence $\|h\|_1 = P_A(\omega, \sigma)^{1/2}$. This proves (i) ∎

In general, to a given pair $\{\omega, \sigma\} \in A^*_+ \times A^*_+$ there exists more than one linear form $f \in Q_A(\omega, \sigma)$ with $f(e) = P_A(\omega, \sigma)^{1/2}$. To give an example, remind the argumentation in the case of $A = B(H)$ and $\omega = f_\varphi$, $\sigma = f_\psi$ for vectors $\varphi, \psi \in H$ with $\varphi \perp \psi$. In this case both $f(\cdot) = \langle (\cdot)\psi, \varphi\rangle$ and the 0-form belong to $Q_A(\omega, \sigma)$. There are, however, special cases where uniqueness of $f \in Q_A(\omega, \sigma)$ with $f(e) = P_A(\omega, \sigma)^{1/2}$ can be proved. To indicate the solution of this problem, let us suppose $\{\omega, \sigma\} \in A^*_+ \times A^*_+$ has the property that for positive linear forms ω' and σ', with $\omega' \le \omega$, $\sigma' \le \sigma$ and $P_A(\omega, \sigma) = P_A(\omega', \sigma')$, always $\omega = \omega'$ and $\sigma = \sigma'$ follows. Then, $\{\omega, \sigma\}$ will be referred to as a *«-minimal pair* , and the set of all such pairs of positive linear forms will be denoted by $\Gamma(A)^*$ (for a discussion see Section 2). Note that on $B(H)$ any pair of non-orthogonal pure normal states yields an example of a «-minimal pair. As we shall prove subsequently in case of «-minimal pairs uniqueness occurs.

2. «-minimal pairs

In this section a *partial order* « on the set of ordered pairs of positive linear forms on A will be introduced. Let $\{\nu, \mu\}$ and $\{\omega, \sigma\}$ be pairs of positive linear forms. Then, by definition $\{\nu, \mu\} \ll \{\omega, \sigma\}$ if and only if $\nu \le \omega$, $\mu \le \sigma$ and $P_A(\nu, \mu) = P_A(\omega, \sigma)$.

Lemma 2.1: *For given $\{\omega, \sigma\} \in A^*_+ \times A^*_+$ there exists exactly one «-minimal pair $\{\nu, \mu\}$ with $\{\nu, \mu\} \ll \{\omega, \sigma\}$.*

Proof : Let us define a set $\Gamma(\omega,\sigma)$ as $\Gamma(\omega,\sigma)=\{(\nu,\mu)\in A^*_+\times A^*_+ : (\nu,\mu)\ll(\omega,\sigma)\}$. By definition of ≪ the assertion is valid if, and only if the existence of a ≪-least pair in $\Gamma(\omega,\sigma)$ can be established. This will be done. Let $\{\pi,H\}$ be an ω,σ-admissible unital *-representation of A, and assume $\varphi\in S(\pi,\omega)$ and $\psi\in S(\pi,\sigma)$. Let us define the vN-algebra M as $M=\pi(A)''$(the bar, double bar, means the commutant, double commutant, respectively). Suppose first $(\nu,\mu)\ll(\omega,\sigma)$. Then, because of $\nu\le\omega$, $\mu\le\sigma$, by standard constructions it is known that $t,s\in M'_+$ can be found, with both $\|t\|$ and $\|s\|$ smaller than one, such that $\nu(x)=\langle t\pi(x)\varphi,\varphi\rangle$, $\mu(x)=\langle s\pi(x)\psi,\psi\rangle$, for all $x\in A$. We define a normal linear form h on M' by $h(y)=\langle y\psi,\varphi\rangle$, for any $y\in M'$. Let $h=R_v|h|=|h|((\cdot)v)$ be the polar decomposition of h. By Lemma 1.1/(i) and known properties of the polar decomposition we have

$$\||h|\|_1=\|h\|_1=P_A(\omega,\sigma)^{1/2}\,. \tag{2.1}$$

We note that $t^{1/2}\varphi\in S(\pi,\nu)$ and $s^{1/2}\psi\in S(\pi,\mu)$. Hence, by the assumptions on the pair (ν,μ) and (2.1) and Lemma 1.1/(i) we can apply Appendix 7 to these particular representatives of ν, μ with the result that partial isometries v_1 and v_2 exist in M' such that

$$\||h|\|_1=P_A(\omega,\sigma)^{1/2}=P_A(\nu,\mu)^{1/2}=\langle t^{1/2}v_1^*v_2s^{1/2}\psi,\varphi\rangle=|h|(t^{1/2}v_1^*v_2s^{1/2}v).$$

Therefore, and since $|h|$ is a positive linear form on M', by the Cauchy-Schwarz Inequality for positive linear forms and since t,s,v_1,v_2,v all belong to the unit ball of M' we can conclude that

$$\begin{aligned}\||h|\|_1^2&=|h|(t^{1/2}v_1^*v_2s^{1/2}v)^2\\ &\le|h|(t)\,|h|(v^*s^{1/2}v_2^*v_1\,v_1^*v_2s^{1/2}v)\le|h|(t)\,\||h|\|_1\le\||h|\|_1^2.\end{aligned}$$

From this $|h|(t)=\||h|\|_1$ has to be followed. Analogously $|h|(v^*sv)=\||h|\|_1$ can be derived. Let $s(|h|)$ denote the support of $|h|$. Applying Appendix 1 to the situation at hand yields that $t=s(|h|)+m$ and $v^*sv=s(|h|)+m'$, with $m,m'\in s(|h|)^\perp M's(|h|)^\perp$. Note that $v^*v=s(|h|)$ implies $m'=0$. Thus $v^*sv=s(|h|)$ has to hold. The latter means that $(e-s)^{1/2}v=0$, from which $(e-s)vv^*=0$ is obtained. Since $vv^*=s(|h^*|)$ we finally see that

$$s(|h|)=t\,s(|h|),\quad s(|h^*|)=s\;s(|h^*|). \tag{2.2}$$

We remind that h^* is defined by $h^*(x)=\overline{h(x^*)}$, for any $x\in M'$, where $\overline{z}$ means the complex conjugate of the number $z\in\mathbb{C}$. Let us define $\varphi',\psi'\in H$ and $\omega_0,\sigma_0\in A^*_+$ as follows:

$$\varphi'=s(|h|)\varphi,\ \psi'=v^*\psi\ \text{ and }\ \omega_0(x)=\langle\pi(x)\varphi',\varphi'\rangle,\ \sigma_0(x)=\langle\pi(x)\psi',\psi'\rangle,\ \forall\ x\in A. \tag{2.3}$$

Using (2.2), for any $x\in A$ we see that

$$\begin{aligned}\omega_0(x^*x)&=\langle\pi(x^*x)\varphi',\varphi'\rangle=\langle s(|h|)\pi(x^*x)\varphi,\varphi\rangle\\ &=\langle t^{1/2}s(|h|)t^{1/2}\pi(x^*x)\varphi,\varphi\rangle\\ &=\langle s(|h|)\pi(x)t^{1/2}\varphi,\ s(|h|)\pi(x)t^{1/2}\varphi\rangle\\ &\le\langle\pi(x)t^{1/2}\varphi,\ \pi(x)t^{1/2}\varphi\rangle=\nu(x^*x),\end{aligned}$$

and analogously

$$\begin{aligned}\sigma_0(x^*x) &= \langle \pi(x^*x)\psi', \psi' \rangle = \langle v\pi(x^*x)v^*\psi, \psi \rangle \\ &= \langle s(|h^*|)\pi(x^*x)\psi, \psi \rangle \\ &= \langle s^{1/2}s(|h^*|)s^{1/2}\pi(x^*x)\psi, \psi \rangle \\ &= \langle s(|h^*|)\pi(x)s^{1/2}\psi, s(|h^*|)\pi(x)s^{1/2}\psi \rangle \\ &\leq \langle \pi(x)s^{1/2}\psi, \pi(x)s^{1/2}\psi \rangle = \mu(x^*x).\end{aligned}$$

Hence $\omega_0 \leq \nu$ and $\sigma_0 \leq \mu$ is fulfilled. Let us define now $g \in M'_*$ by $g(x) = \langle x\psi', \varphi' \rangle$, for all $x \in M'$. By Lemma 1.1, (2.3), and due to the definition of h we infer $P_A(\omega_0, \sigma_0)^{1/2} = \| g \|_1 = \| h(s(|h|)(\cdot)v^*) \| \leq \| h \|_1 = P_A(\omega, \sigma)^{1/2}$. On the other hand, $P_A(\omega_0, \sigma_0)^{1/2} = \| g \|_1 \geq |g(e)| = |\langle \psi', \varphi' \rangle| = |\langle v^*\psi, \varphi \rangle| = |h(v^*)| = |h|(e) = \| |h| \|_1 = \| h \|_1 = P_A(\omega, \sigma)^{1/2}$. Hence, by the assumptions on $\{\nu, \mu\}$, in using these inequalities we get $P_A(\omega_0, \sigma_0) = P_A(\omega, \sigma) = P_A(\nu, \mu)$. Since also $\omega_0 \leq \nu \leq \omega$ and $\sigma_0 \leq \mu \leq \sigma$ are true, we conclude to $\{\omega_0, \sigma_0\} \ll \{\nu, \mu\} \ll \{\omega, \sigma\}$. By the construction (2.3) both linear forms ω_0 and σ_0 do not refer to the particular $\{\nu, \mu\}$ chosen provided the latter belonged to $\Gamma(\omega, \sigma)$. Hence, $\{\omega_0, \sigma_0\} \ll \{\nu, \mu\}$ for any $\{\nu, \mu\} \in \Gamma(\omega, \sigma)$ ∎

Remark 2.2: The proof yields an explicite way (cf.(2.3)) for constructing the $\ll$-least pair in $\Gamma(\omega, \sigma)$. In fact, if $\{\pi, H\}$ is any ω, σ -admissible representation of A, and if we have $\varphi \in S(\pi, \omega)$ *and* $\psi \in S(\pi, \sigma)$, then we have to determine the support projections of the moduli $|h|$ and $|h^*|$ of the linear forms h and h^*, respectively, with h defined on $\pi(A)'$ by $h(y) = \langle y\psi, \varphi \rangle$, for any $y \in \pi(A)'$. Then, because of $v \in \pi(A)'$ and since $vv^* = s(|h^*|)$ holds, (2.3) shows that the $\ll$-minimal pair $\{\omega_0, \sigma_0\}$ can be given also by

$$\omega_0(x) = \langle s(|h|)\pi(x)\varphi, \varphi \rangle, \quad \sigma_0(x) = \langle s(|h^*|)\pi(x)\psi, \psi \rangle, \ \forall x \in A. \tag{2.4}$$

Evidently, $\{\omega, \sigma\}$ is $\ll$-minimal if and only if $\Gamma(\omega, \sigma)$ contains only *one* pair, i.e. $\omega_0 = \omega$ and $\sigma_0 = \sigma$ have to be fulfilled. By (2.4) this is the case if and only if the inclusions $s(|h|)H \supset \overline{[\pi(A)\varphi]}$ and $s(|h^*|)H \supset \overline{[\pi(A)\psi]}$ are valid (by $\overline{[G]}$ we mean the closed linear span of $G \subset H$).

3. A uniqueness result

In this section we are going to derive a basic result for all what follows (for a preliminary form of the result cf. [S]).

Theorem 3.1: *Let A be a unital C^*-algebra. For each pair $\{\omega, \sigma\}$ of positive linear forms on A there exists exactly one linear form $f \in A^*$ such that*

(i) $f(e) = P_A(\omega, \sigma)^{1/2}$

(ii) $|f(y^*x)| \leq \mu(y^*y)^{1/2}\nu(x^*x)^{1/2}$, $\forall\, x, y \in A$, $\forall \{\mu, \nu\}$ *with* $\{\mu, \nu\} \ll \{\omega, \sigma\}$.

Proof : In accordance with Lemma 2.1 let $\{\omega_0, \sigma_0\}$ be the uniquely determined $\ll$-minimal pair such that $\{\omega_0, \sigma_0\} \ll \{\omega, \sigma\}$. What will be shown now is the existence and uniqueness of $f \in A^*$ fulfilling condition (i) and $|f(y^*x)| \leq \omega_0(y^*y)^{1/2}\sigma_0(x^*x)^{1/2}$ for any $x, y \in A$. Since $\{\omega_0, \sigma_0\}$ is the $\ll$-least pair in $\Gamma(\omega, \sigma)$ (cf. the proof of Lemma 2.1), the linear form f also has to satisfy condition (ii).

Let $\{\pi, H\}$ be ω, σ*-admissible*. Let $M = \pi(A)''$. By Remark 2.2 we know that ω_0 and σ_0 are given by $\omega_0(x) = \langle s(|h|)\pi(x)\varphi, \varphi\rangle$ and $\sigma_0(x) = \langle s(|h^*|)\pi(x)\psi, \psi\rangle$, for any $x \in A$, with $\varphi \in S(\pi, \omega)$ and $\psi \in S(\pi, \sigma)$, and h defined on M' by $h(y) = \langle y\psi, \varphi\rangle$ for all $y \in M'$. Let $h = R_v|h|$ be the polar decomposition of $h \in M'_*$. Then, we define a linear form f on A by $f(x) = \langle \pi(x)v^*\psi, \varphi\rangle$, for $x \in A$. Because of $v^*v = s(|h|) \in M'$, $vv^* = s(|h^*|) \in M'$, and using the Cauchy-Schwarz Inequality we conclude for any $x, y \in A$ as follows:

$$
\begin{aligned}
|f(y^*x)|^2 &\le |\langle \pi(x)v^*\psi, \pi(y)\varphi\rangle|^2 \\
&= |\langle \pi(x)v^*\psi, s(|h|)\,\pi(y)\varphi\rangle|^2 \\
&\le \|s(|h|)\pi(y)\varphi\|^2\, \|\pi(x)v^*\psi\|^2 \\
&= \langle s(|h|)\pi(y^*y)\varphi, \varphi\rangle \langle s(|h^*|)\pi(x^*x)\psi, \psi\rangle \\
&= \omega_0(y^*y)\,\sigma_0(x^*x).
\end{aligned}
$$

Moreover, applying Lema 1.1/(i) one obtains that $f(e) = \langle v^*\psi, \varphi\rangle = h(v^*) = |h|(v^*v) = = |h|(s(|h|)) = \||h|\|_1 = \|h\|_1 = P_A(\omega, \sigma)^{1/2}$. Hence, there is at least one f obeying (i),(ii).

Assume now we have given $g \in A^*$ with $g(e) = P_A(\omega, \sigma)^{1/2}$ and suppose $|g(y^*x)| \le \omega_0(y^*y)^{1/2}\sigma_0(x^*x)^{1/2}$ is fulfilled, for any $x, y \in A$. Working within the *-representation $\{\pi, H\}$, and adopting the notations from above, we see that $|g(y^*x)| \le \|\pi(y)s(|h|)\varphi\| \times \|\pi(x)v^*\psi\|$ for any $x, y \in A$. Hence, the map γ given by

$$
\gamma:\ \{\pi(y)s(|h|)\varphi,\ \pi(x)v^*\psi\} \longmapsto g(y^*x) \in \mathbb{C}\ ,\quad \forall\ x, y \in A,
$$

is well defined as a sesquilinear form, bounded by one, from a dense linear subspace of $s(|h|)H \times s(|h|)H$ into $\mathbb{C}$. Concerning the latter, note that $s(|h|)H \subset \overline{[\pi(A)\varphi]}$ and $s(|h^*|)H \subset \overline{[\pi(A)\psi]}$ holds. From the first of these inclusions $s(|h|)H = \overline{[\pi(A)s(|h|)\varphi]}$ is obtained, whereas the latter inclusion, due to $v^*v = s(|h|)$ and $vv^* = s(|h^*|)$, yields

$$
\begin{aligned}
s(|h|)H &= v^*s(|h^*|)vH \subset v^*\overline{[\pi(A)\psi]} = \overline{[\pi(A)v^*\psi]} \\
&= \overline{[\pi(A)s(|h|)v^*\psi]} = s(|h|)\,\overline{[\pi(A)v^*\psi]} \subset s(|h|)H.
\end{aligned}
$$

Hence, $s(|h|)H = \overline{[\pi(A)v^*\psi]}$ in this case, too. We extend the form γ by continuity to a bounded by one sesquilinear form $\gamma^\wedge$ on whole of $s(|h|)H \times s(|h|)H$. There is a unique $k \in B(s(|h|)H)$ with $\|k\| \le 1$ and $\gamma^\wedge(\xi, \eta) = \langle k\xi, \eta\rangle$, for all vectors $\xi, \eta \in s(|h|)H$. Especially, in chosing $\xi = \pi(x)v^*\psi$ and $\eta = \pi(y)s(|h|)\varphi$ for $x, y \in A$ we see that

$$
\gamma^\wedge(\xi, \eta) = \gamma(\xi, \eta) = g(y^*x) = \langle k\pi(x)v^*\psi, \pi(y)s(|h|)\varphi\rangle\ . \tag{3.1}
$$

Because of $g((yz)^*x) = g(z^*y^*x)$, for $x, y, z \in A$, and since π is a representation of A, (3.1) implies the relation

$$
\langle \pi(y^*)k\pi(x)v^*\psi, \pi(z)s(|h|)\varphi\rangle = \langle k\pi(y^*)\pi(x)v^*\psi, \pi(z)s(|h|)\varphi\rangle\ ,\ \forall\ x, y, z \in A.
$$

Therefore, and since $s(|h|)H$ is invariant under the action of $\pi(A)$, for any $y \in A$ we get

$$
s(|h|)\pi(y^*)k\,s(|h|) = s(|h|)k\,\pi(y^*)s(|h|). \tag{3.2}
$$

We define a bounded linear operator K on H by $K = s(|h|)k\,s(|h|)$. Due to $s(|h|) \in M'$ and (3.2) we realize that $\pi(y)K = K\pi(y)$ for any $y \in A$. Taking together all these facts we have arrived at the existence of $K \in M'$, $\|K\| \leq 1$, such that

$$s(|h|)K = Ks(|h|) = K,\ \ g(x) = \langle K\pi(x)v^{*}\psi, \varphi\rangle, \text{ for any } x \in A. \tag{3.3}$$

By assumption $g(e) = P_A(\omega, \sigma)^{1/2} = \|h\|_1$ is fulfilled. In view to this and using the special representation of g from (3.3) we see that

$$g(e) = \langle Kv^{*}\psi, \varphi\rangle = h(Kv^{*}) = |h|(Kv^{*}v) = |h|(K) = \|\,|h|\,\|_1 = \|h\|_1 .$$

By (3.3) we have $K \in M'$, $\|K\| \leq 1$, with $s(|h|)K = Ks(|h|) = K$. Hence, Appendix 1 can be applied to this situation with the result that $K = s(|h|)$. Hence, it follows that $g(x) = \langle \pi(x)v^{*}\psi, \varphi\rangle$, for any $x \in A$, i.e. $g = f$, with the f from the beginning of the proof. Since g has been chosen arbitrarily from the set of all linear forms obeying conditions (i) and (ii), the proof of the uniqueness statement is complete ∎

Remark 3.2: In the above proof the functional f obeying Theorem 3.1 is explicitely given in terms of an arbitrary ω, σ-admissible *-representation of A. Of course, the linear form f does *not* depend on any particular ω, σ-admissible *-representation of A. Hence, what we also have proved is a fact which we should keep in mind and which reads as follows:

Let both (π, H) and (π', H') be unital *-representations of A. Suppose

$$\varphi \in S(\pi, \omega),\quad \varphi' \in S(\pi', \omega) \text{ and } \psi \in S(\pi, \sigma),\quad \psi' \in S(\pi', \sigma).$$

Let v and v' be the partial isometries in the polar decompositions $h = R_v|h|$ and $h' = R_{v'}|h'|$ of h and h', respectively, where h and h' are given on M' by $h(y) = \langle y\psi, \varphi\rangle$ and $h'(y) = \langle y\psi', \varphi'\rangle$, for all $y \in M'$, respectively. Then, for any $x \in A$ one has

$$\langle \pi(x)v^{*}\psi, \varphi\rangle = \langle \pi'(x)v'^{*}\psi', \varphi'\rangle . \tag{3.4}$$

The unique f determined by Theorem 3.1/(i),(ii) will be denoted by $I(\omega, \sigma)$ and will be referred to as the ω, σ-*skew form* on A henceforth.

Suppose now A is a W^{*}-algebra, and $\omega, \sigma \in A_{*+}$, i.e. both ω and σ are assumed to be normal positive linear forms on A. Due to Theorem 3.1/(ii) the ω, σ-skew form $I(\omega, \sigma)$ is a normal linear form, too (note that the latter remains true provided at least one of the forms ω, σ is normal). Hence, there is the uniquely determined polar decomposition $I(\omega, \sigma) = R_u|I(\omega, \sigma)|$. The partial isometry $u(\omega, \sigma) = u^{*}$ will be referred to as the ω, σ-*skew phase* and the normal positive linear form $|I(\omega, \sigma)|$ will be called ω, σ-*skew modulus.* In case of a C^{*}-algebra, let $\{\Pi, K\}$ be the universal *-representation of A. Then, $A^{**} = \Pi(A)''$ is the universal envelopping vN-algebra of A, and for each $g \in A^{*}$ there is a unique $g^{**} \in (A^{**})_{*}$ such that $g^{**}\Pi = g$. Since $P_{A^{**}}(\omega^{**}, \sigma^{**}) = P_A(\omega, \sigma)$ for any two positive linear forms ω, σ on A, it is easily recognized that we have $I(\omega, \sigma)^{**} = I(\omega^{**}, \sigma^{**})$ and $I(\omega, \sigma) = I(\omega, \sigma)^{**}\Pi$ is fulfilled. We then have a polar decomposition $I(\omega, \sigma)^{**} = R_u|I(\omega^{**}, \sigma^{**})|$ within $(A^{**})_{*}$, and $u^{*} = u(\omega^{**}, \sigma^{**})$. Therefore, also in the C^{*}-algebraic case with general positive linear forms one could be tempted to associate both a positive linear form and a partial isometry to the pair $\{\omega, \sigma\}$ in a unique way by defining $|I(\omega, \sigma)| = |I(\omega, \sigma)^{**}|\Pi$ and $u(\omega, \sigma) = u(\omega^{**}, \sigma^{**})$. Whereas $|I(\omega, \sigma)|$ is a positive linear form on A, the ω, σ-*skew phase* defined in this way belongs to A^{**}. In the case of a W^{*}-algebra A and normal positive linear forms on A there is a simple relation between these two settings $u(\omega, \sigma)$ and $u(\omega^{**}, \sigma^{**})$ which are slighly differing from each other:

$$u(\omega^{**}, \sigma^{**}) = \Pi(u(\omega,\sigma))\, s(|I(\omega^{**},\sigma^{**})^{*}|) = s(|I(\omega^{**},\sigma^{**})|)\, \Pi(u(\omega,\sigma)) .$$

Subsequently, we shall be dealing with the case of W^*-algebras and pairs of normal positive linear forms almost exclusively, and the terms *skew modulus* and *skew phase* then refer to the original W^*-algebraic definition.

4. Properties of C^*-algebraic transition probability

Unless stated otherwise, throughout this section we suppose $A = M$ is a vN-algebra acting on some Hilbert space H, and the positive linear forms considered are normal ones, exclusively. Furthermore, let us assume that M acts in standard form on H, with a cyclic and separating vector Ω and associated to $\{M,\Omega\}$ natural positive cone P_Ω. The unitary group of M will be denoted by $U(M)$. For $f \in M^*$ and $a \in M$ the notation f^a is an abbreviation for the linear form $f(a(\cdot)a^*)$.

Lemma 4.1: *Suppose $\omega,\sigma \in M_{*+}$. The skew form $I(\omega,\sigma)$ has the following properties:*

(i) $I(\omega,\sigma)(e) = P_A(\omega,\sigma)^{1/2}$

(ii) $I(\omega,\sigma) = I(\sigma,\omega)^*$, $u(\omega,\sigma) = u(\sigma,\omega)^*$

(iii) $I(\omega,\sigma) = I(\mu,\nu)$ *and* $u(\omega,\sigma) = u(\mu,\nu)$ *whenever* $\{\omega,\sigma\} \ll \{\mu,\nu\}$

(iv) $I(\omega,\omega) = \omega$

(v) $I(\omega^u,\sigma^u) = I(\omega,\sigma)^u$, $u(\omega^u,\sigma^u) = u^* u(\omega,\sigma) u$ *for any* $u \in U(M)$

(vi) $I(\omega,\sigma) = 0$ *if, and only if* $\omega \perp \sigma$.

Proof : (i) is valid by definition of $I(\omega,\sigma)$, and (iii) follows from Lemma 2.1 and Theorem 3.1. Hence, in order to see the properties (ii), (iv), (v) we can suppose that $\{\omega,\sigma\}$ is $\ll$-minimal. We also note that $\{\omega^u,\sigma^u\}$ is $\ll$-minimal for any unitary u of M if, and only if, $\{\omega,\sigma\}$ is $\ll$-minimal, and $\{\omega,\omega\}$ is $\ll$-minimal in any case. But then, in using known properties of the polar decomposition, (ii) and (iv), (v) follow at once from the definition of the skew form and the proven uniqueness. To see (vi), let $\varphi,\psi \in P_\Omega$ be the vector representatives of ω and σ in P_Ω. Let $h \in M'_*$ be given by $h(x) = \langle x\psi,\varphi\rangle$, and let $h = R_v|h|$ be the polar decomposition of h. As we know from Remark 3.2, for $y \in M$ we find that $I(\omega,\sigma)(y) = \langle yv^*\psi,\varphi\rangle$. Hence, $I(\omega,\sigma) = 0$ holds if, and only if $0 = \langle x^*yv^*\psi,\varphi\rangle = \langle yv^*\psi, x\varphi\rangle$, for any $x,y \in M$. The latter is equivalent with $p'(\varphi)H \perp v^*p'(\psi)H$, where, for $\xi \in H$, $p'(\xi) \in M'$ is the orthoprojection onto the subspace $\overline{[M\xi]}$, and $p(\xi) \in M$ is the orthoprojection onto the subspace $\overline{[M'\xi]}$. Now, on the one hand , $p'(\psi) \geq s(|h^*|)$ and $p'(\varphi) \geq s(|h|)$ holds true. On the other hand, from $v^*v = s(|h|)$ and $vv^* = s(|h^*|)$ we get $v^*p'(\psi)H = v^*H = s(|h|)H$. Hence, $I(\omega,\sigma) = 0$ if, and only if $p'(\varphi) \perp s(|h|)$, with $p'(\varphi) \geq s(|h|)$. Therefore, $I(\omega,\sigma) = 0$ is equivalent with $s(|h|) = 0$. This occurs if, and only if $h = 0$, i.e. $0 = \langle y\psi, x\varphi\rangle = h(x^*y)$, for all $x,y \in M'$. Therefore, $p(\psi) \perp p(\varphi)$ is a necessary and sufficient condition for $I(\omega,\sigma) = 0$. For the supports $s(\omega)$ and $s(\sigma)$ of ω and σ we have $s(\omega) = p(\varphi)$ and $s(\sigma) = p(\psi)$. This proves (vi) ∎

Lemma 4.2 : *Let $\omega,\sigma \in M_{*+}$, and assume $\{\omega_n\},\{\sigma_n\} \subset M_{*+}$ to be sequences such that* $\lim_n \omega_n = \omega$ *and* $\lim_n \sigma_n = \sigma$ *with respect to the norm topology on M_{*+}. Then, we also have* $\lim_n P_M(\omega_n,\sigma_n) = P_M(\omega,\sigma)$.

Proof : Let $\varphi_n, \psi_n \in P_\Omega$ and $\varphi, \psi \in P_\Omega$, respectively, be the uniquely determined vector representatives in P_Ω of $\omega_n, \sigma_n, \omega, \sigma \in M_{*+}$. Since the correspondence between M_{*+} and P_Ω is a homeomorphism, our assumptions imply that $\varphi_n \longrightarrow \varphi$ and $\psi_n \longrightarrow \psi$ in H. Hence, for $h_n, h \in M'_*$ defined by $h_n(\cdot) = \langle (\cdot)\psi_n, \varphi_n \rangle$ and $h(\cdot) = \langle (\cdot)\psi, \varphi \rangle$ we see $h_n \longrightarrow h$. The latter implies $\| h_n \|_1 \longrightarrow \| h \|_1$. By Lemma 1.1/(i) the assertion follows ∎

Remark 4.3 : For the representatives $\Phi, \Psi \in P_\Omega$ of two normal positive linear forms ω, σ on M it is well known that $\|\Phi - \Psi\|^2 \le \|\omega - \sigma\|_1 \le \|\Phi - \Psi\| \, \|\Phi + \Psi\|$. This can be used to give a more quantitative estimation of the convergence behaviour, cf. [4] (cf. also another derivation of this estimation in context of Lemma 6.1).The result is

$$|P_M(\omega,\sigma)^{1/2} - P_M(\mu,\nu)^{1/2}| \le \|\sigma\|_1^{1/2} \|\omega - \mu\|_1^{1/2} + \|\mu\|_1^{1/2} \|\sigma - \nu\|_1^{1/2}.$$

By Lemma 4.1/(i) and (iv) we know that $P_M(\rho,\rho)^{1/2} = \rho(e) = \|\rho\|_1$, for any $\rho \in M_{*+}$. Hence, as a special case of the preceding estimation we get

$$|P_M(\omega,\sigma)^{1/2} - \|\sigma\|_1| \le \|\sigma\|_1^{1/2} \|\omega - \sigma\|_1^{1/2} \text{ and } |\|\nu\|_1 - P_M(\mu,\nu)^{1/2}| \le \|\nu\|_1^{1/2} \|\nu - \mu\|_1^{1/2}.$$

We can compose these estimates to see

$$\begin{aligned} &|P_M(\omega,\sigma)^{1/2} - P_M(\mu,\nu)^{1/2}| \\ &\quad \le |P_M(\omega,\sigma)^{1/2} - \|\sigma\|_1| + |\|\sigma\|_1 - \|\nu\|_1| + |\|\nu\|_1 - P_M(\mu,\nu)^{1/2}| \\ &\quad \le \|\sigma\|_1^{1/2} \|\omega - \sigma\|_1^{1/2} + \|\nu\|_1^{1/2} \|\nu - \mu\|_1^{1/2} + |\|\sigma\|_1 - \|\nu\|_1|, \end{aligned}$$

which sometimes is also of use.

Let $\omega, \sigma \in M_{*+}$, and suppose ε is a real, $\varepsilon > 0$. We define a set $M_\varepsilon(\omega,\sigma) \subset M_+$ by setting

$$M_\varepsilon(\omega,\sigma) = \{ x \in M: x \ge 0, \textit{ invertible}, \omega(x), \sigma(x^{-1}) \le \varepsilon + P_M(\omega,\sigma)^{1/2} \}.$$

The following result is true.

Theorem 4.4 : *For any* $\varepsilon > 0$ *one has*

$$\begin{aligned} P_M(\omega,\sigma) &= \inf\{\omega(x)\sigma(x^{-1}) : x \in M_\varepsilon(\omega,\sigma)\} \\ P_M(\omega,\sigma) &= \inf\{\omega(x)\sigma(x^{-1}) : x \in M, x \textit{ invertible}\}. \end{aligned} \tag{4.1}$$

Proof : In the case that one of the linear forms vanishes the assertion is obviously valid. Also, by the very definition of $P_M(\omega,\sigma)$, in case of $\omega \neq 0$ and $\sigma \neq 0$ one has $P_M(\omega,\sigma) = \omega(e)\sigma(e)P_M(\mu,\nu)$, with the normalized to one positive linear forms $\mu = \omega(e)^{-1}\omega$, $\nu = \sigma(e)^{-1}\sigma$. From this $[\sigma(e)\omega(e)^{-1}]^{1/2} M_\varepsilon(\mu,\nu) = M_{\varepsilon'}(\omega,\sigma)$ can easily be seen, with $\varepsilon' = [\omega(e)\sigma(e)]^{1/2}\varepsilon$. Hence it suffices to prove the assertion in case of normal states ω and σ. We are going to do this. By Theorem 3.1/(i) we see for any invertible $x \in M_+$ that $P_M(\omega,\sigma) = I(\omega,\sigma)(e)^2 = |I(\omega,\sigma)(x^{1/2}x^{-1/2})|^2 \le \omega(x)\sigma(x^{-1})$. This implies

$$\begin{aligned} P_M(\omega,\sigma) &\le \inf\{\omega(x)\sigma(x^{-1}) : x \in M_+, x \textit{ invertible}\} \\ &\le \inf\{\omega(x)\sigma(x^{-1}) : x \in M_\varepsilon(\omega,\sigma)\}, \end{aligned} \tag{4.2}$$

provided the sets $M_\varepsilon(\omega,\sigma)$ are non-void (this will be shown below). Let us define

$$\omega_n = (1 - \tfrac{1}{n})\omega + \tfrac{1}{n}\sigma \quad \text{and} \quad \sigma_n = (1 - \tfrac{1}{n})\sigma + \tfrac{1}{n}\omega, \text{ for any } n \in \mathbb{N}.$$

Then, $\{\omega_n\}$ and $\{\sigma_n\}$ are sequences of normal states with *equal* supports and $\omega_n \longrightarrow \omega$, $\sigma_n \longrightarrow \sigma$. Then, if we were able to derive (4.1) in case of normal states with equal supports the validity of the assertion in the general case could be followed by means of (4.2) and Lemma 4.2. In line with this, let us assume that ω, σ have equal supports in M. Since in a vN-algebra in standard form both $S(id, \omega)$ and $S(id, \sigma)$ are non-void (id stands for the identical representation of M), by Appendix 7 we can also find that $\varphi \in S(id, \omega)$ and $\psi \in S(id, \sigma)$ such that

$$P_M(\omega, \sigma)^{1/2} = \langle \psi, \varphi \rangle. \tag{4.3}$$

Let $h \in M'_*$ be defined by $h(\cdot) = \langle (\cdot)\psi, \varphi \rangle$. Then, $\|h\|_1 = P_M(\omega, \sigma)^{1/2}$ due to Lemma 1.1/(i), and $\|h\|_1 = h(e)$ can be followed from (4.3). But then, applying a well-known characterization of the positivity of a linear form to the situation at hand yields that h has to be positive. Hence, Appendix 2 can be applied with $p(\varphi) = p(\psi)$ (we supposed ω and σ to have equal supports). We see that there exists a densely defined, positive, selfadjoint linear operator F, affiliated with M and invertible, such that $F\varphi = \psi$, $F^{-1}\psi = \varphi$. Hence, by (4.3) we see

$$P_M(\omega, \sigma)^{1/2} = \langle F\varphi, \varphi \rangle \text{ and } P_M(\omega, \sigma)^{1/2} = \langle F^{-1}\psi, \psi \rangle. \tag{4.4}$$

Let $F_n = F + \frac{1}{n}e$. Then, $F_n^{-1} \in M_+$ and $F_n^{-1} \le F^{-1}$ on the domain of definition of F^{-1}. This implies

$$\sigma(F_n^{-1}) = \langle F_n^{-1}\psi, \psi \rangle \le \langle F^{-1}\psi, \psi \rangle = P_M(\omega, \sigma)^{1/2} \ . \tag{4.5}$$

Let $\{E(\lambda)\} \subset M$ be the resolution of the identity of F, and let $e_n = E([0, n))$. We define a sequence $\{G_n\}$ of linear operators by setting $G_n = e_n F_n + n e_n^{\perp}$. Then, $G_n^{-1} = e_n F_n^{-1} + n^{-1} e_n^{\perp}$, and both G_n and G_n^{-1} belong to M_+. Moreover, using (4.5) we see

$$\sigma(G_n^{-1}) \le \sigma(e_n F_n^{-1}) + n^{-1} \le P_M(\omega, \sigma)^{1/2} + n^{-1} \ . \tag{4.6}$$

On the other hand, on the domain of definition of F we have $G_n \le F_n$. Thus, from (4.4) we get

$$\omega(G_n) \le \langle F\varphi, \varphi \rangle + n^{-1} \le P_M(\omega, \sigma)^{1/2} + n^{-1}. \tag{4.7}$$

Note that both (4.6) and (4.7) show that, with $\delta = n^{-1}$, $M_\delta(\omega, \sigma)$ is non-void, for any $n \in \mathbb{N}$. Combining (4.6) with (4.7) yields

$$\omega(G_n)\, \sigma(G_n^{-1}) \le P_M(\omega, \sigma) + n^{-1}[2 P_M(\omega, \sigma)^{1/2} + n^{-1}].$$

Hence, $\limsup_n \omega(G_n)\, \sigma(G_n^{-1}) \le P_M(\omega, \sigma)$ is evident. In view to (4.2), and since each G_n is a positive invertible element of M, we now conclude that

$$P_M(\omega, \sigma) = \inf\{\omega(x)\, \sigma(x^{-1}) : x \in M_+,\ x \textit{ invertible}\} = \inf\{\omega(x)\, \sigma(x^{-1}) : x \in M_\varepsilon(\omega, \sigma)\}$$

holds for any $\varepsilon > 0$ under the assumption that ω, σ have the same supports. Finally, in case that ω, σ have *not* the same supports, to given $\varepsilon > 0$ we can approximate both states by states ω_n, σ_n with equal supports (cf. the construction from the beginning of the proof) such that (by repeating the arguments which have led us to (4.6) and (4.7)) for some invertible $H_n \in M_+$ we have

$$\sigma_n(H_n^{-1}) \le P_M(\omega_n,\sigma_n)^{1/2}+n^{-1}, \ \ \omega_n(H_n)\le P_M(\omega_n,\sigma_n)^{1/2}+n^{-1}.$$

By the construction of the approximating sequences and since all functionals involved are positively defined, we also see that

$$\sigma(H_n^{-1}) \le (1-n^{-1})\Big(P_M(\omega_n,\sigma_n)^{1/2}+n^{-1}\Big),\ \omega(H_n)\le(1-n^{-1})\Big(P_M(\omega_n,\sigma_n)^{1/2}+n^{-1}\Big).$$

For $n\longrightarrow\infty$, by Lemma 4.2 the expressions on the right-hand sides of the preceding inequalities tend to $P_M(\omega,\sigma)^{1/2}$. Hence, $H_n\in M_\varepsilon(\omega,\sigma)$ for all $n\in\mathbb{N}$ sufficiently large. Knowing this, and respecting that in the case of states with equal supports (4.1) has been proved yet, we get the general case from (4.2) by continuity (cf. Lemma 4.2) as explained at the beginning ∎

The result just proved is of importance in a context of concepts and questions referred to as *noncommutative probability*. We are going to describe and discuss some of these immediate consequences of the theorem. First we remark that for an unital, positive linear mapping Θ acting from an unital C^*-algebra N into another one M we have $\Theta(a^{-1})\ge\Theta(a)^{-1}$ for any invertible positive element $a\in N$. This is a well-known result of M.-D. Choi [13]. Assume now the algebras M, N are standard vN-algebras and ω,σ are normal positive linear forms on M and the mapping Θ to be a normal map. Suppose $\varepsilon>0$. According to Theorem 4.4 there exists an invertible element $a\in M_+$ such that $P_N(\omega\Theta,\sigma\Theta)+\varepsilon\ge\omega(\Theta(a))\,\sigma(\Theta(a^{-1}))$. Using $\Theta(a^{-1})\ge\Theta(a)^{-1}$ and arguing by Theorem 4.4 once more again we infer that

$$P_N(\omega\Theta,\sigma\Theta)+\varepsilon\ge\omega(\Theta(a))\,\sigma(\Theta(a^{-1}))\ge\omega(\Theta(a))\,\sigma(\Theta(a)^{-1})\ge P_M(\omega,\sigma),$$

i.e. we get $P_N(\omega\Theta,\sigma\Theta)+\varepsilon\ge P_M(\omega,\sigma)$. The latter has to hold for any $\varepsilon>0$. Hence

$$P_N(\omega\Theta,\sigma\Theta)\ge P_M(\omega,\sigma) \tag{4.8}$$

has to be valid for any unital, normal positive linear map Θ acting from N into M (cf. [4], [3] and the references quoted there).

Another application of Theorem 4.4. reads as follows. Suppose first both ω,σ are faithful positive linear forms on our vN-algebra M. Suppose $\varepsilon>0$. Because of Theorem 4.4 there exists an invertible element $a\in M_+$ such that $P_M(\omega,\sigma)+\varepsilon\ge\omega(a)\,\sigma(a^{-1})$. Now, by continuity we may even assume a such that its spectrum is a finite set $\{\lambda_1,\ldots,\lambda_n\}\subset\mathbb{R}_+\backslash\{0\}$, i.e. $a=\sum_{j=1}^n\lambda_j p_j$, for some decomposition $\{p_j\}$ of the unity e into mutually orthogonal orthoprojections p_j. We get

$$P_M(\omega,\sigma)+\varepsilon\ge\sum_j\omega(p_j)\sigma(p_j)+\sum_{j\neq k}\lambda_j\lambda_k^{-1}\omega(p_j)\sigma(p_k).$$

Now, the function $f(t)=t\,\omega(p_j)\sigma(p_k)+t^{-1}\omega(p_k)\sigma(p)$ takes for some $t>0$ its infimum. A simple calculation shows that $f(t_{inf})=2\,\{\omega(p_k)\sigma(p_k)\}^{1/2}\,\{\omega(p_j)\sigma(p_j)\}^{1/2}$. Therefore

$$P_M(\omega,\sigma)+\varepsilon\ge\sum_j\omega(p_j)\sigma(p_j)+\sum_{j\neq k}\lambda_j\lambda_k^{-1}\omega(p_j)\sigma(p_k)\ge\Big(\sum_j\omega(p_j)^{1/2}\sigma(p_j)^{1/2}\Big)^2.$$

Hence, we see $\{P_M(\omega,\sigma)+\varepsilon\}^{1/2}\ge\sum_j\omega(p_j)^{1/2}\sigma(p_j)^{1/2}$. On the other hand, in defining $b=\sum_j\omega(q_j)^{-1/2}\sigma(q_j)^{1/2}q_j$ for *any* decomposition $\{q_j\}$ of the unity e into mutually orthogonal orthoprojections q_j, we have $\omega(b)=\sigma(b^{-1})=\sum_j\omega(q_j)^{1/2}\sigma(q_j)^{1/2}$. Since $b\in M_+$, Theorem 4.4 applies and shows that

$$P_M(\omega,\sigma)^{1/2} \le \{\omega(b)\,\sigma(b^{-1})\}^{1/2} = \sum_j \omega(q_j)^{1/2}\sigma(q_j)^{1/2}$$

holds for any orthogonal decomposition $\{q_j\}$ of the unity. Hence

$$P_M(\omega,\sigma)^{1/2} \le \inf \sum_j \omega(q_j)^{1/2}\sigma(q_j)^{1/2},$$

where the infimum extends over all such systems $\{q_j\}$. By our discussion from above, to any $\varepsilon > 0$ we find such a system $\{p_j\}$ with $\{P_M(\omega,\sigma)+\varepsilon\}^{1/2} \ge \sum_j \omega(p_j)^{1/2}\sigma(p_j)^{1/2}$. Therefore, we finally get that

$$P_M(\omega,\sigma)^{1/2} = \inf \sum_j \omega(q_j)^{1/2}\sigma(q_j)^{1/2}. \tag{4.9}$$

By a continuity argument (cf. Lemma 4.2 and the analoguous remarks in the proof of Theorem 4.4) we easily see that the assertion remains valid for arbitrary pairs of normal positive linear forms. The result is due to S. Gudder [15] (cf. also [16] and [17] for the appropriate context). H. Araki and G.A. Raggio [7] proved that there exists a projection valued measure E belonging to M such that $P_M(\omega,\sigma)$ is the quadratic mean

$$P_M(\omega,\sigma)^{1/2} = \int dQM(\mu_{\omega,E}, \mu_{\sigma,E}) \tag{4.10}$$

of the induced measures $\mu_{\omega,E}(\cdot) = \omega(E(\cdot))$ and $\mu_{\sigma,E}(\cdot) = \sigma(E(\cdot))$ on the σ-algebra of Borel sets of R. Obviously, (4.9) yields the approximation of $P_M(\omega,\sigma)^{1/2}$ by quadratic means in the sense of the right-hand side of (4.10) referring to simple projection valued measures with values in M. Note that in case of faithful forms ω,σ the E in (4.10) exactly corresponds to the spectral resolution of the affiliated with M operator F we have been using in the proof of Theorem 4.4 (cf.(4.4)). The representation (4.10) can be obtained from (4.4) by applying essentially the same kind of arguments we have been using in the derivation of (4.9).Therefore, both (4.10) and (4.9) are equivalent with Theorem 4.4.

Remark 4.5 : Let A be a unital C^*-algebra, and $\omega,\sigma \in A^*_+$. Then, as has been yet mentioned at the end of Section 2, $P_{A^{**}}(\omega^{**},\sigma^{**}) = P_A(\omega,\sigma)$. It is also easy to understand that

$$P_{A^{**}}(\omega^{**},\sigma^{**}) = P_B(\omega^{**}|_B, \sigma^{**}|_B),$$

with $B = sA^{**}s$, where s denotes the support projection of $\omega^{**}+\sigma^{**}$ within A^{**}. B is W^*-*isomorphic* to a vN-algebra in standard form, and the C^*-algebraic transition probability P is *invariant* with respect to *-isomorphisms, i.e. if β is a *-isomorphism of B then $P_{\beta(B)}(\nu,\mu) = P_B(\nu\beta,\mu\beta)$, for all $\nu,\mu \in \beta(B)^*$. All these facts, together with an standard application of Kaplansky's Density Theorem, imply that the assertion of Theorem 4.4 can be extended to hold on an arbitrary unital C^*-algebra A for any pair of positive linear forms ω, σ on A. Especially, this also implies that the assertion of (4.8) remains true for any pairs of unital C^*-algebras M and N, positive linear forms ω, σ on M and an arbitrary unital positive linear map $\Theta : N \longrightarrow M$. Note that by a similar reasoning norm continuity of P_A on $A^*_+ \times A^*_+$ for an arbitrary unital C^*-algebra can be followed from Lemma 4.2 .

Let us now come back to the case of a vN-algebra M in standard form. Let $\Gamma_* \subset M_{*+} \times M_{*+}$ be the set of all «-minimal pairs of normal positive linear forms on M. We consider Γ_* equipped with the product norm topology.

Theorem 4.6: *The maps* $I: \Gamma_* \ni \{\omega,\sigma\} \mapsto I(\omega,\sigma) \in M_*$ *and* $|I|: \Gamma_* \ni \{\omega,\sigma\} \mapsto |I(\omega,\sigma)| \in M_{*+}$ *are norm continuous mappings from* Γ_* *into* M_* *and* M_{*+}, *respectively.*

Proof: Let $\{\omega_n,\sigma_n\} \in \Gamma_*$, for any $n \in \mathbb{N}$, and be $\{\omega,\sigma\}$ «-minimal. Assume $\omega_n \longrightarrow \omega$ and $\sigma_n \longrightarrow \sigma$ in norm. We have to show that $I(\omega_n,\sigma_n) \longrightarrow I(\omega,\sigma)$ with respect to the $\|\cdot\|_1$-norm. From this, according to Appendix 8 also $|I(\omega_n,\sigma_n)| \longrightarrow |I(\omega,\sigma)|$ can be followed. Let $\varphi_n, \psi_n, \varphi, \psi \in P_\Omega$ be the vectors representing $\omega_n, \sigma_n, \omega, \sigma$ within P_Ω. With the same reasoning as in the proof of Lemma 4.2, $\varphi_n \longrightarrow \varphi$, $\psi_n \longrightarrow \psi$ can be followed. Let $h, h_n \in M'_*$ have the same meaning as in the proof of Lemma 4.2, and let $v, v_n \in M'$ be the partial isometries of the polar decomposition of h, h_n, respectively. By Theorem 3.1 and Remark 3.2 we know that $I(\omega_n,\sigma_n)(\cdot) = \langle(\cdot)v_n^*\psi_n, \varphi_n\rangle$ and $I(\omega,\sigma)(\cdot) = \langle(\cdot)v^*\psi, \varphi\rangle$ on M. Our assumptions imply $h_n \longrightarrow h$, from which by Appendix 8 also $|h_n| \longrightarrow |h|$ follows. Let $\{v_{n(\lambda)}s(|h|): \lambda \in \Lambda\}$ be a universal subnet of the sequence $\{v_n s(|h|)\}$. Then, due to the weak compactness of the unit ball of M', we have w-$\lim_\lambda \{v_{n(\lambda)}s(|h|)\}^* = w$ for some $w \in M'$, with $\|w\| \le 1$. Hence, we have

$$|h|(y) = |h|(ys(|h|)) = \lim_\lambda |h_{n(\lambda)}|(y\,s(|h|))$$
$$= \lim_\lambda \langle y\,s(|h|)\,v_{n(\lambda)}^*\psi_{n(\lambda)}, \varphi_{n(\lambda)}\rangle = \langle yw\psi, \varphi\rangle$$

for any $y \in M'$. The latter means that $|h| = \mathrm{R}_w h$, with $w \in M'$, $\|w\| \le 1$. An application of Appendix 6 yields now that $w^* s(|h|) = v$. Thus, what we have shown is that

$$\text{w-}\lim_\lambda v_{n(\lambda)}s(|h|) = v$$

occurs for any universal subnet $\{v_{n(\lambda)}s(|h|): \lambda \in \Lambda\}$ of the sequence $\{v_n s(|h|)\}$. From this, and since the unit ball of M' is weakly compact, we then infer that even

$$\text{w-}\lim_n v_n s(|h|) = v \qquad (4.11)$$

has to hold. Because of $v_n v_n^* = s(|h_n^*|)$ and $vv^* = s(|h^*|)$ we see that

$$\|v_n^*\psi_n - v^*\psi\|^2 = \|s(|h_n^*|)\psi_n\|^2 + \|s(|h^*|)\psi\|^2 - \langle v_n^* v\psi, \psi_n\rangle - \langle vv_n^*\psi_n, \psi\rangle. \qquad (4.12)$$

By the supposed «-minimality of $\{\omega_n,\sigma_n\}$ and $\{\omega,\sigma\}$, and following Remark 2.2, we can be assured of the validity of

$$\|s(|h_n^*|)\psi_n\|^2 = \|\psi_n\|^2, \quad \|s(|h^*|)\psi\|^2 = \|\psi\|^2, \text{ and } \|s(|h|)\varphi\|^2 = \|\varphi\|^2.$$

Since $v^* = s(|h|)v^*$, from (4.10) w-$\lim_n v_n v^* = vv^* = s(|h^*|)$ is obtained. Hence, taking the limit for $n \longrightarrow \infty$ in (4.12) results in $\lim_n \|v_n^*\psi_n - v^*\psi\|^2 = 0$. We now consider the following estimation:

$$|I(\omega_n,\sigma_n)(x) - I(\omega,\sigma)(x)|$$
$$= |\langle xv_n^*\psi_n, \varphi_n\rangle - \langle xv^*\psi, \varphi\rangle|$$
$$\le |\langle xv_n^*\psi_n, \varphi_n\rangle - \langle xv_n^*\psi_n, \varphi\rangle| + |\langle xv_n^*\psi_n, \varphi\rangle - \langle xv^*\psi, \varphi\rangle|$$
$$\le \{\|\varphi-\varphi_n\|\,\|\psi_n\| + \|v_n^*\psi_n - v^*\psi\|\,\|\varphi\|\}\,\|x\|,$$

for any $x \in M$. The latter says that $\|I(\omega_n,\sigma_n) - I(\omega,\sigma)\|_1 \le \|\varphi-\varphi_n\|\,\|\psi_n\| + \|v_n^*\psi_n - v^*\psi\|\,\|\varphi\|$. According to our preceding considerations $I(\omega_n,\sigma_n)$ has to tend to $I(\omega,\sigma)$ in norm ∎

5. Geometry of representatives of «-minimal pairs

The main result of this section will be a representation theorem for «-minimal pairs of normal positive linear forms on a vN-algebra M acting standardly on a Hilbert space H. A preliminary form of this result has been derived in [5]. In all what follows the notations and conventions of Section 4 and the appendix (Section A) will be adopted and tacitly used. For a positive linear form ω the set $S(id,\omega)$ of all vector representatives of ω in H will be abbreviated as $S(\omega)$.

Definition 5.0: Let $\varphi, \psi \in H$. The vector ψ is said to be *φ-associated (over M)* if there exists $u \in M$ such that

$$uu^* = p(\psi),\ \ u^*u = p(\varphi),\ \ p'(\varphi) = p'(\psi) = p'(u\psi) \tag{5.1}$$

$$u\psi \in \overline{M_+ u\varphi}\ ,\ \ u^*\psi \in \overline{M'_+\varphi}\,. \tag{5.2}$$

In the case that ψ is φ-associated the notation $\psi \parallel \varphi$ will be in use.

Theorem 5.1: *Let ω and σ be normal positive linear forms on M.*

(i) *$\{\omega,\sigma\}$ is «-minimal if and only if there exist vectors $\varphi \in S(\omega)$ and $\psi \in S(\sigma)$ such that $\psi \parallel \varphi$.*

(ii) *Suppose vectors $\varphi \in S(\omega)$ and $\psi \in S(\sigma)$ are given such that $\psi \parallel \varphi$, and let u be a partial isometry obeying (5.1) and (5.2). Then, the skew form $I(\omega,\sigma)$ and the skew phase $u(\omega,\sigma)$ are given by*

$$I(\omega,\sigma)(\cdot) = \langle (\cdot)\psi, \varphi\rangle \text{ and } u(\omega,\sigma) = u^*. \tag{5.3}$$

(iii) *Assume $\varphi, \varphi' \in S(\omega)$ and $\psi, \psi' \in S(\sigma)$ are given. Suppose $\psi \parallel \varphi$. Then, $\psi' \parallel \varphi'$ if and only if there is $w \in M'$ with $\psi = w\psi'$, $\varphi = w\varphi'$ and $ww^* = p'(\varphi)$, $w^*w = p'(\psi)$.*

(iv) *Let $\{\omega,\sigma\}$ be «-minimal, and suppose $\varphi \in S(\omega)$ is given. There is a unique vector $\psi \in S(\sigma)$ with $\psi \parallel \varphi$.*

Proof: Suppose first that $\{\omega,\sigma\}$ is «-minimal. According to Appendix 7 and Lemma 1.1/(i) there exist vectors $\varphi \in S(\omega)$ and $\psi \in S(\sigma)$ with $\langle \psi, \varphi\rangle = P_M(\omega,\sigma)^{1/2} = \|h\|_1$, where $h \in M'_*$ is defined by $h(\cdot) = \langle (\cdot)\psi, \varphi\rangle$. Hence, we have $h(e) = \|h\|_1$. This implies h to be *positive* on M. The latter means that $s(h) = s(|h|) = s(|h^*|)$. Since $\{\omega,\sigma\}$ is «-minimal, Remark 2.2 (with $\pi = id$) shows that $s(h) \geq p'(\varphi)$ and $s(h) \geq p'(\psi)$ in this situation. On the other hand, by the definition of h, we have $s(h) \leq p'(\varphi)$ and $s(h) \leq p'(\psi)$, obviously. Thus, we have the equality

$$s(h) = p'(\varphi) = p'(\psi). \tag{5.4}$$

We define $f \in M_*$ by $f(\cdot) = \langle (\cdot)\psi, \varphi\rangle$. Let $f = R_u|f|$ be the polar decomposition of f. Then we have $|f|(\cdot) = \langle (\cdot)u^*\psi, \varphi\rangle$. According to (5.4) the conditions for an application of Appendix 4 are given. The result is

$$p'(u^*\psi) = p'(\psi),\ uu^* = p(\psi),\ u^*u = p(\varphi) = s(|f|). \tag{5.5}$$

The first relation of (5.5) and $|f| \geq 0$ make that Appendix 3 can be applied (with the replacements $M \longrightarrow M'$, $p \longrightarrow p'$ etc. in the formulation of Appendix 3) with the result

$$s(|f|) = p(\varphi) = p(u^*\psi). \tag{5.6}$$

The second relation of (5.5) gives that $h(\cdot) = \langle(\cdot)\psi, \varphi\rangle = \langle(\cdot)uu^*\psi, \varphi\rangle = \langle(\cdot)u^*\psi, u^*\varphi\rangle$. Positivity of h then implies $s(h) \leq p'(u^*\varphi)$. From this and the obvious fact $p'(u^*\varphi) \leq p'(\varphi)$ together with (5.4) and (5.5) we conclude that

$$s(h) = p'(u^*\varphi) = p'(\varphi) = p'(\psi) = p'(u^*\psi). \tag{5.7}$$

On the other hand, starting from the third relation of (5.5) we arrive at another representation of h, namely $h(\cdot) = \langle(\cdot)\psi, \varphi\rangle = \langle(\cdot)\psi, u^*u\varphi\rangle = \langle(\cdot)u\psi, u\varphi\rangle$. Positivity of h implies $s(h) \leq p'(u\varphi) \leq p'(\varphi)$ and $s(h) \leq p'(u\psi) \leq p'(\psi)$. Once more argueing with (5.4) we see that in fact equality holds, i.e. $s(h) = p'(u\varphi) = p'(\varphi) = p'(u\psi) = p'(\psi)$. Therefore, (5.7) can be extended to the sequence of equalities

$$s(h) = p'(\varphi) = p'(\psi) = p'(u\varphi) = p'(u\psi) = p'(u^*\varphi) = p'(u^*\psi). \tag{5.8}$$

By (5.5) we see that $p(u\varphi) = p(\psi)$ *and* $p(u\psi) \leq p(\psi)$. Hence $p(u\psi) \leq p(u\varphi)$, and since $h(\cdot) = \langle(\cdot)u\psi, u\varphi\rangle$ is positive over M' we can apply Appendix 2 to obtain that $u\psi \in \overline{M_+ u\varphi}$, i.e. the first part of (5.2) is seen. Since $|f|(\cdot) = \langle(\cdot)u^*\psi, \varphi\rangle$ is a positive linear form over M, and because of $p'(\varphi) = p'(u^*\psi)$, Appendix 2 applies (with the replacements $M \longrightarrow M'$, $p \longrightarrow p'$ etc. in the formulation of Appendix 2) and gives $u^*\psi \in \overline{M'_+\varphi}$, i.e. the second part of (5.2) is shown. In view to (5.5) and (5.8) $\psi \| \varphi$ follows. Hence, the one direction of (i) is shown. Note that from $h(\cdot) = \langle(\cdot)\psi, \varphi\rangle = \langle(\cdot)u\psi, u\varphi\rangle \geq 0$ over M' by Appendix 2 also follows that

$$p(\varphi)\psi \in \overline{M_+\varphi} \quad \textit{is equivalent with} \quad u\psi \in \overline{M_+ u\varphi}. \tag{5.9}$$

To see the other direction of (i), assume $\varphi \in S(\omega)$ and $\psi \in S(\sigma)$ with $\psi \| \varphi$ are given. Suppose u is a partial isometry of M such that (5.1) and (5.2) are satisfied. From (5.1) we see that $p(u\varphi) = p(\psi) \geq p(u\psi)$. By (5.2) positivity of $h \in M'_*$ with $h(\cdot) = \langle(\cdot)\psi, \varphi\rangle$ on M' follows. Due to (5.1) we also see that $h(\cdot) = \langle(\cdot)u\psi, u\varphi\rangle$. Appendix 3 can be applied to the vectors $u\psi$, $u\varphi$ in this situation to see that $s(h) = p'(u\psi) \leq p'(u\varphi)$. Since by the second relation of (5.1) $p'(u\varphi) = p'(\varphi)$, we get that $s(h) = p'(u\psi) \leq p'(\varphi)$. From this and the third relation of (5.1) we conclude that

$$s(h) = p'(\psi) = p'(\varphi). \tag{5.10}$$

According to Remark 2.2, and since, due to $h \geq 0$, $s(h) = s(|h|) = s(|h^*|)$ holds, the $\{\omega, \sigma\}$ corresponding $\ll$-minimal pair $\{\omega_0, \sigma_0\}$ is given by $\omega_0(\cdot) = \langle s(h)(\cdot)\varphi, \varphi\rangle$ and $\sigma_0(\cdot) = \langle s(h)(\cdot)\psi, \psi\rangle$. Relation (5.10) then shows that $\{\omega_0, \sigma_0\} = \{\omega, \sigma\}$, i.e. $\{\omega, \sigma\}$ is $\ll$-minimal, and the proof of (i) is complete.

To see (ii), we can continue the preceding considerations by defining $f, g \in M_*$ by $f(\cdot) = \langle(\cdot)\psi, \varphi\rangle$ and $g(\cdot) = \langle(\cdot)u^*\psi, \varphi\rangle$, respectively. By (5.2) we can be assured of the positivity of $g = R_{u^*} f$. Since also $f = R_u g$ by (5.1), certainly $\|f\|_1 = \|g\|_1$. An application of Appndix 5 shows that $g = |f|$, and Appendix 6 yields $us(|f|) = w$, with $f = R_w |f|$ being the polar decomposition of f. Now, due to (5.10) and $uu^* = p(\psi)$ we have $p'(u^*\psi) = p'(\psi) = p'(\varphi)$. Hence Appendix 4 applies with the result that $s(|f|) = p(\varphi)$. Therefore, in view to the second relation of (5.1) we have to conclude that $w = us(|f|) = up(\varphi) = u$. This, however, means that $I(\omega,\sigma)=f$ and $u(\omega,\sigma)=u^*$ for the pair $\{\omega,\sigma\}$, which is «-minimal by (i). This completes the proof of (ii).

To prove (iii), let us assume that $\varphi,\varphi'\in S(\omega)$ and $\psi,\psi'\in S(\sigma)$ are given with $\psi \parallel \varphi$ and $\psi' \parallel \varphi'$. Then, by (i) we know that $\{\omega,\sigma\}$ is «-minimal. By standard arguments we can be sure to find $w\in M'$ such that $\varphi=w\varphi'$, $ww^*=p'(\varphi)$, $w^*w=p'(\varphi')$. By (ii) we infer that $I(\omega,\sigma)(y)=\langle y\psi,\varphi\rangle=\langle y\psi',\varphi'\rangle$, for any $y\in M$. Hence, $\langle (w^*\psi-\psi'),x\varphi'\rangle=0$ for any $x\in M$. This implies $p'(\varphi')(w^*\psi-\psi')=0$. In view to $p'(\varphi')w^*=w^*$, and since $p'(\varphi')=p'(\psi')$ by our assumptions on ψ',φ' (cf. (5.1)), $\psi'=w^*\psi$ has to be followed. By the same reasoning the assumptions on ψ,φ give $p'(\varphi)=p'(\psi)$, and $w\psi'=ww^*\psi=p'(\varphi)\psi=p'(\psi)\psi=\psi$ is clear.

To go the other way around, assume $\psi\parallel\varphi$ and $\varphi\in S(\omega)$ and $\psi\in S(\sigma)$. Suppose $v\in M'$ is given such that $v^*v=p'(\varphi)=p'(\psi)$. Note that $p'(\varphi)=p'(\psi)$ is a consequence of (i). Let us define vectors $\varphi'=v\varphi$ and $\psi'=v\psi$. Then, $\varphi'\in S(\omega)$ and $\psi'\in S(\sigma)$. By the assumptions, (S.1) holds, i.e. $p'(\varphi)=p'(\psi)=p'(u\psi)$, with u being the partial isometry figuring within (S.1), (S.2). Hence, also $vv^*=p'(\varphi')=p'(\psi')=p'(u\psi')$. Since also $uu^*=p(\psi)=p(\psi')$, $u^*u=p(\varphi)=p(\varphi')$ holds, we have seen that (S.1) remains valid with ψ',φ' in place of ψ,φ, respectively. Let us consider the funtionals h,h' over M' given by $h(\cdot)=\langle(\cdot)u\psi,u\varphi\rangle$, $h'(\cdot)=\langle(\cdot)u\psi',u\varphi'\rangle$. By means of the definitions of ψ' and φ' we realize that $h'(\cdot)=h(v^*(\cdot)v)$. By assumptions (S.2) has to be fulfilled, i.e. $h\geq 0$. Therefore, h' is *positive*, too. Due to $uu^*=p(\psi)=p(\psi')$, $u^*u=p(\varphi)=p(\varphi')$ we get $p(u\varphi')=p(\psi')\geq p(u\psi')$. Thus, the conditions for an application of Appendix 2 are given. The result is

$$p(u\varphi')u\psi'=u\psi',\ \text{and}\quad u\psi'\in\overline{M_+u\varphi'}.$$

Let $f,f'\in M_*$ be given as $f(\cdot)=\langle(\cdot)u^*\psi,\varphi\rangle$ and $f'(\cdot)=\langle(\cdot)u^*\psi',\varphi'\rangle$. By the definition of φ',ψ' we realize that $f=f'$ holds. Due to the assumption $\psi\parallel\varphi$ and (S.2) f is positive, i.e. also f' is positive over M. We apply Appendix 2 to the situation at hand and see that

$$p'(\varphi')u^*\psi'\in\overline{M'_+\varphi'}.$$

Since $p'(\varphi')=p'(\psi')$, the relation $p'(\varphi')u^*\psi'=u^*\psi'$ follows. Finally, where we have arrived at is the following:

$$uu^*=p(\psi'),\ u^*u=p(\varphi'),\ p'(\varphi')=p'(\psi')=p'(u\psi'),\ u\psi'\in\overline{M_+u\varphi'},\ u^*\psi'\in\overline{M'_+\varphi'}\ .$$

Hence, $\psi'\parallel\varphi'$ is seen, and the proof of (iii) is complete.

Suppose now $\{\omega,\sigma\}$ is a «-minimal pair, and $\varphi\in S(\omega)$. By (i) we know that there are vectors $\varphi'\in S(\omega)$ and $\psi'\in S(\sigma)$ with $\psi'\parallel\varphi'$. There is a partial isometry $w\in M'$ with $\varphi=w\varphi'$, $w^*w=p'(\varphi')$, $ww^*=p'(\varphi)$. Since $\psi'\parallel\varphi'$ requires $p'(\psi')=p'(\varphi')$, we have $\psi=w\psi'\in S(\sigma)$. By (iii) we conclude that $\psi\parallel\varphi$. Assume there is another $\psi''\in S(\sigma)$, with $\psi''\parallel\varphi$. By (ii) we have, for any $x\in M$, $I(\omega,\sigma)(x)=\langle x\psi,\varphi\rangle=\langle x\psi'',\varphi\rangle$. Therefore $\langle(\psi-\psi''),y\varphi\rangle=0$, for all $y\in M$, and $p'(\varphi)(\psi-\psi'')=0$ has to be followed. Now, $\psi\parallel\varphi$ and $\psi''\parallel\varphi$ require $p'(\psi)=p'(\varphi)=p'(\psi'')$. Thus, $\psi=\psi''$ has to hold, i.e. (iv) is true ∎

Remark S.2 : The preceding theorem has some interesting consequences.

(1) Since $\{\omega,\sigma\}$ is «-minimal if and only if $\{\sigma,\omega\}$ is «-minimal, by (i) we follow that $\parallel$ is *symmetric, i.e.* $\psi\parallel\varphi$ if, and only if $\varphi\parallel\psi$. Note that (ii) also shows that the partial isometry u in Definition S.0 is uniquely determined. Due to Lemma 4.1/(ii), the changes $\varphi\longrightarrow\psi$ and $\psi\longrightarrow\varphi$ within (S.1), (S.2) require the replacement $u\longrightarrow u^*$ in order to remain valid.

(2) Especially, (iv) shows the following. Let $\{\omega,\sigma\}$ be $\ll$-minimal and $\varphi\in S(\omega)$ given. There exists $\psi\in S(\sigma)$ such that $P_M(\omega,\sigma)^{1/2}=\langle\psi,\varphi\rangle$. In fact, this is a consequence of (ii), (iv) and Lemma 4.1/(i), and is a sharpening of Appendix 7 for $\ll$-minimal pairs. The uniquely determined $\psi\in S(\sigma)$ with $\psi\parallel\varphi$ is referred to as the *φ-relative representative* of σ (cf. Section 0).

(3) By (i) and (iv) and (5.9) we realize that in the definition of $\psi\parallel\varphi$ the condition (5.2) can be replaced with the requirements $p(\varphi)\psi\in\overline{M_+\varphi}$, $u^*\psi\in\overline{M_+'\varphi}$.

(4) Let $\{\omega,\sigma\}$ be a $\ll$-minimal pair of normal positive linear forms over M. Suppose $\varphi\in S(\omega)$, and be ψ the (as we now know unique) vector of $S(\sigma)$ with $\psi\parallel\varphi$. Then we have the estimation $\|\psi-\varphi\|^2\le\|\omega-\sigma\|_1$. In fact, for the uniquely determined representing vectors $\Phi,\Psi\in P_\Omega$ of ω,σ in the natural positive cone P_Ω of $\{M,\Omega\}$ (cf. the suppositions of Section 4) we have $0\le\langle\Psi,\Phi\rangle\le P_M(\omega,\sigma)^{1/2}$, whereas in (2) we remarked that $P_M(\omega,\sigma)^{1/2}=\langle\psi,\varphi\rangle$. Hence,

$$\|\psi-\varphi\|^2=\|\psi\|^2+\|\varphi\|^2-2\langle\psi,\varphi\rangle\le\|\Psi\|^2+\|\Phi\|^2-2\langle\Psi,\Phi\rangle=\|\Psi-\Phi\|^2.$$

The result now follows from the well-known inequality $\|\Psi-\Phi\|^2\le\|\omega-\sigma\|_1$ for the representing vectors $\Phi,\Psi\in P_\Omega$ yet mentioned in Remark 4.3.

Up to now we tried to give characterizations of $\ll$-minimal pairs of positive linear forms. Whereas Theorem 3.1 gives an abstract characterization in form of a uniqueness result, in case of normal linear forms over the vN-algebra M by Theorem 5.1 a characterization is given which is of different kind. It reads in terms of the relative geometry of the vector representatives (of the normal linear forms in question) with respect to the action of M on the underlying Hilbert space H. However, there is yet another interesting question to be worth to be considered in case of a general pair $\{\omega,\sigma\}$ of positive linear forms which is not $\ll$-minimal: *What can be said relating the structure of those parts of ω and σ that prevent the pair $\{\omega,\sigma\}$ from being $\ll$-minimal?* We are going to give an answer for pairs of normal positive linear forms in the vN-algebra case. To this sake, let us suppose that $\{\omega,\sigma\}$ is a pair of normal positive linear forms on our vN-algebra M. We want to characterize first the *geometrical* relations between the pair $\{\omega,\sigma\}$ and some pair $\{\omega',\sigma'\}$ such that $\{\omega',\sigma'\}\ll\{\omega,\sigma\}$. By definition of $\ll$, both $\omega_1=\omega-\omega'$ and $\sigma_1=\sigma-\sigma'$ are normal positive linear forms on our vN-algebra M. From Theorem 4.4 and obvious properties of the infimum we infer that

$$P_M(\omega,\sigma)\ge P_M(\omega',\sigma')+P_M(\omega',\sigma_1)+P_M(\omega_1,\sigma')+P_M(\omega_1,\sigma_1)\qquad(5.11)$$

By definition of $\ll$ also $P_M(\omega,\sigma)=P_M(\omega',\sigma')$ is fulfilled. Hence, from the inequality (5.11) together with the non-negativity of P_M we have to follow that $P_M(\omega',\sigma_1)=P_M(\omega_1,\sigma')=P_M(\omega_1,\sigma_1)=0$. Now, for two given positive linear forms ν, μ and representing vectors φ, ψ of them, according to Lemma 1.1/(i), we know that for a form h defined over M' by $h(\cdot)=\langle(\cdot)\psi,\varphi\rangle$, $\|h\|_1=P_M(\nu,\mu)^{1/2}$ has to hold. Hence, in case of $P_M(\nu,\mu)=0$ we get $h=0$, and vice versa. The latter is equivalent to $|h|=0$. We can now conclude as at the end of the proof of Lemma 4.1/(vi) and see that $P_M(\nu,\mu)=0$ is equivalent with $\nu\perp\mu$. In application to our situation with $P_M(\omega',\sigma_1)=P_M(\omega_1,\sigma')=P_M(\omega_1,\sigma_1)=0$ we thus get that $\omega'\perp\sigma_1$, $\omega_1\perp\sigma'$, $\omega_1\perp\sigma_1$. Since $\omega=\omega'+\omega_1$ and $\sigma=\sigma'+\sigma_1$ hold, we can also follow that $\omega\perp\sigma_1$, $\sigma\perp\omega_1$. Especially, the latter holds of course in case that $\omega'=\omega_0$ and $\sigma'=\sigma_0$, with the $\{\omega,\sigma\}$ corresponding $\ll$-minimal pair $\{\omega_0,\sigma_0\}$. Let us suppose now that $\{\omega,\sigma\}$ is a pair of normal positive linear forms, and let $\{\omega_1,\sigma_1\}$ be another pair such that $\omega\ge\omega_1$ and $\sigma\ge\sigma_1$ and $\omega\perp\sigma_1$, $\sigma\perp\omega_1$. Let $\{\omega_0,\sigma_0\}$ be the $\{\omega,\sigma\}$ corresponding $\ll$-minimal pair. Assume $\varphi \in S(\omega)$ and $\psi \in S(\sigma)$ are chosen in such a way that $\langle \psi, \varphi\rangle = P_M(\omega,\sigma)^{1/2}$. According to Appendix 7 this choice of representatives is always possible. Let us define $h \in M'_*$ as $h(\cdot) = \langle(\cdot)\psi,\varphi\rangle$. By Lemma 1.1/(i) we see that $h(e) = \|h\|_1 = P_M(\omega,\sigma)^{1/2}$. Hence, h has to be a positive linear form on M'. Let $t, t' \in M'_+$ be given such that $\|t\| \le 1$, $\|t'\| \le 1$ and $\varphi' = t\varphi \in S(\omega_1)$, $\psi' = t'\psi \in S(\sigma_1)$ are fulfilled. The existence of t, t' with the properties indicated follows by standard conclusions from our assumptions $\omega \ge \omega_1$ and $\sigma \ge \sigma_1$. On the other hand, from the assumptions $\omega \perp \sigma_1$ and $\sigma \perp \omega_1$ the conditions $p(\varphi') \perp p(\psi)$ and $p(\psi') \perp p(\varphi)$ can be derived. Hence, both the linear forms f, g defined over M' by $f(\cdot) = \langle(\cdot)\psi',\varphi\rangle$ and $g(\cdot) = \langle(\cdot)\psi,\varphi'\rangle$ have to vanish. Let $s(t)$ and $s(t')$ be the supports of t and t', respectively. For $n \in \mathbb{N}$ we define $t_n = t + n^{-1}e$ and $t'_n = t' + n^{-1}e$. Then $\{tt_n^{-1}\}$ and $\{t'^{-1}_n t'\}$ are increasingly directed systems of positive operators of M' such that last upper bound $tt_n^{-1} = s(t)$ and last upper bound $t'^{-1}_n t' = s(t')$. Thus, since h is positive and normal and since f and g are the zero-form on M', we get

$$h(s(t)) = \lim_n h(tt_n^{-1}) = \lim_n \langle tt_n^{-1}\psi,\varphi\rangle$$
$$= \lim_n \langle tt_n^{-1}\psi,\varphi\rangle = \lim_n \langle t_n^{-1}\psi, t\varphi\rangle = \lim_n g(t_n^{-1}) = 0,$$

and analogously, $h(s(t')) = \lim_n f(t_n^{-1}) = 0$. This means that both the relations $s(t) \le s(h)^\perp$ and $s(t') \le s(h)^\perp$ have to hold. Since the norms of t and t' are smaller than one, the relations $t^2 \le s(h)^\perp$ and $t'^2 \le s(h)^\perp$ can be followed. According to the latter, and due to Remark 2.2 once more again, we infer that in our situation for any $x \in M$

$$\omega_1(x^*x) = \langle xt\varphi, xt\varphi\rangle = \langle t^2 x\varphi, x\varphi\rangle \le \langle s(h)^\perp x\varphi, x\varphi\rangle$$
$$= \omega(x^*x) - \langle s(h)x\varphi, x\varphi\rangle = (\omega - \omega_0)(x^*x).$$

This together with an analoguous argumentation in case of σ_1 leads us to

$$\omega_1 \le \omega - \omega_0 \le \omega\,, \quad \sigma_1 \le \sigma - \sigma_0 \le \sigma \;. \tag{5.12}$$

Hence, $\omega \ge \omega - \omega_1 \ge \omega_0$ and $\sigma \ge \sigma - \sigma_1 \ge \sigma_0$. In view to Theoem 4.4 we may conclude that $P_M(\omega,\sigma) \ge P_M(\omega-\omega_1, \sigma-\sigma_1) \ge P_M(\omega_0,\sigma_0)$. Now, by definition of $\{\omega_0,\sigma_0\}$ we know that $P_M(\omega,\sigma) = P_M(\omega_0,\sigma_0)$. Hence also $P_M(\omega,\sigma) = P_M(\omega-\omega_1,\sigma-\sigma_1) = P_M(\omega_0,\sigma_0)$. This tells us that $\{\omega_0,\sigma_0\} \ll \{\omega-\omega_1, \sigma-\sigma_1\} \ll \{\omega,\sigma\}$ provided that $\omega \ge \omega_1$ and $\sigma \ge \sigma_1$ and $\omega \perp \sigma_1$, $\sigma \perp \omega_1$. On the other hand, by our considerations following (5.11) we know that for any pair $\{\omega',\sigma'\}$ with $\{\omega',\sigma'\} \ll \{\omega,\sigma\}$ we have that $\omega_1 = \omega - \omega'$ and $\sigma_1 = \sigma - \sigma'$ obey the relations $\omega \perp \sigma_1$ and $\sigma \perp \omega_1$. With regard to (5.12) we can now summarize all the derived facts into the following

Theorem 5.3 : *Let M be a vN-algebra. Suppose $\{\omega,\sigma\}$ and $\{\omega',\sigma'\}$ are pairs of normal positive linear forms over M, and be $\{\omega_0,\sigma_0\}$ the $\{\omega,\sigma\}$ corresponding $\ll$-minimal pair. Let functionals ω_1 and σ_1 be defined as $\omega_1 = \omega - \omega'$ and $\sigma_1 = \sigma - \sigma'$. Then, the following assertions are valid:*

(1) *$\{\omega',\sigma'\} \ll \{\omega,\sigma\}$ if, and only if, $\omega \ge \omega_1 \ge 0$, $\sigma \ge \sigma_1 \ge 0$ and $\omega \perp \sigma_1$, $\sigma \perp \omega_1$.*

(2) *ω_0 is the smallest positive linear form ν with $\nu \le \omega$ such that $\omega - \nu \perp \sigma$ and σ_0 is the smallest positive linear form μ with $\mu \le \sigma$ such that $\sigma - \mu \perp \omega$.*

(3) *$\omega - \omega_0$ is the largest positive linear form ν with $\nu \le \omega$ such that $\nu \perp \sigma$ and $\sigma - \sigma_0$ is the largest positive linear form μ with $\mu \le \sigma$ such that $\mu \perp \omega$.*

We remark that, for given two positive linear forms ω and σ, the existence of largest positive linear forms ν and μ with $\nu \leq \omega$ and $\mu \leq \sigma$ such that $\nu \perp \sigma$ and $\mu \perp \omega$, respectively, has been proved first by H. Araki [6]. He refers to this result, which is equivalent with Theorem 5.3 (3) essentially, as an extension of the non-commutative Radon- Nikodym theorem of S. Sakai (cf. [19]). We also note that the general supposition of this section saying that M should be acting in standard form over H is of no relevance for the validity of Theorem 5.3 (in this point we anticipate the reasoning in Section 7) and thus we have decided to give the general formulation at once.

6. Perturbations of «-minimal pairs. Examples

In this section some important special cases of «-minimal pairs of normal positive linear forms will be given. The suppositions and notations of the preceding section will be adopted and tacitely used, i.e. we are working in a vN-algebra M acting standardly on a Hilbert space H. Then, the identical representation $\{id, H\}$ is ω,σ-admissible for any pair of normal positive linear forms $\{\omega,\sigma\}$. Especially, Remark 2.2 applies with $\pi = id$ and says that such a pair $\{\omega,\sigma\}$ is «-minimal if, and only if $s(|h|) \geq p'(\varphi)$, $s(|h^*|) \geq p'(\psi)$, where $\varphi \in S(\omega)$, $\psi \in S(\sigma)$ and $h \in M'_*$ is given by $h(\cdot) = \langle (\cdot)\psi, \varphi \rangle$. Assume $h = R_v|h|$ is the polar decomposition of h. Then, $|h|(\cdot) = \langle (\cdot) v^*\psi, \varphi \rangle$ and $|h^*|(\cdot) = \langle (\cdot)\psi, v\varphi \rangle$, from which $s(|h|) \leq p'(\varphi)$ and $s(|h^*|) \leq p'(\psi)$ is evident. Hence, $\{\omega,\sigma\}$ is «-minimal if, and only if $s(|h|) = p'(\varphi)$, $s(|h^*|) = p'(\psi)$, for a pair (and so for any pair) of vectors $\varphi \in S(\omega)$, $\psi \in S(\sigma)$, with $h \in M'_*$ given by $h(\cdot) = \langle (\cdot)\psi, \varphi \rangle$.

We want to apply this criterion in a very specific situation. Suppose, $s(\omega) \leq s(\sigma)$ for a pair $\{\omega,\sigma\}$ of normal positive linear forms. Then $p(\varphi) \leq p(\psi)$ for any two vectors $\varphi \in S(\omega)$, $\psi \in S(\sigma)$. By Appendix 7 and Lemma 1.1/(i) there exist vectors $\varphi \in S(\omega)$, $\psi \in S(\sigma)$ such that $\langle \psi, \varphi \rangle = P_M(\omega,\sigma)^{1/2} = \| h \|_1$, where h is defined on M' by $h(\cdot) = \langle (\cdot)\psi, \varphi \rangle$. Due to $h(e) = \| h \|_1$, h has to be positive. Thus, Appendix 3 applies with the result that $s(h) = s(|h|) = s(|h^*|) = p'(\varphi) \leq p'(\psi)$. Moreover, Appendix 3 tells us that $p(\varphi) = p(\psi)$ implies $s(h) = s(|h|) = s(|h^*|) = p'(\varphi) = p'(\psi)$. With regard to the above criterion we can take for established the first parts of the following assertions.

Lemma 6.1: *Let $\{\omega,\sigma\}$ be a pair of normal positive linear forms. Suppose $\{\omega_0,\sigma_0\}$ is the «-minimal pair such that $\{\omega_0,\sigma_0\}$ « $\{\omega,\sigma\}$. Suppose the supports $s(\omega)$ and $s(\sigma)$ of ω and σ fulfil $s(\omega) \leq s(\sigma)$ (resp. $s(\omega) \geq s(\sigma)$). Then, $\omega = \omega_0$ (resp. $\sigma = \sigma_0$), and $s(\omega) = s(\sigma)$ implies $\{\omega,\sigma\}$ to be «-minimal. Suppose $\{\omega,\sigma\}$ is «-minimal and $s(\omega) \in M \cap M'$. Then, $s(\sigma) \leq s(\omega)$ and $s(\sigma) \sim s(\omega)$.*

For the last assertion of Lemma 6.1 we note that for a «-minimal pair $\{\omega,\sigma\}$ according to Definition 5.0 (cf.(5.1)) and Theorem 5.1/(i), $s(\omega) \sim s(\sigma)$ has to hold. The skew phase of the pair is a partial isometry of M accomplishing the transformation from the initial projection $s(\omega)$ into the final projection $s(\sigma)$. Since the inital projection is a central projection the final projection has to be majorized by this central projection, i.e. $s(\sigma) \leq s(\omega)$ has to be fulfilled. We remark, however, that all these results could be seen also as an immediate consequence of Theoem 5.3. As an application we can now include a quite short proof of the estimation in Remark 4.3. Let $\omega, \sigma, \mu, \nu \in M_{*+}$. We have to prove that

$$|P_M(\omega,\sigma)^{1/2} - P_M(\mu,\nu)^{1/2}| \le \|\sigma\|_1^{1/2}\|\omega-\mu\|_1^{1/2} + \|\mu\|_1^{1/2}\|\sigma-\nu\|_1^{1/2}$$

First of all note that due to Lemma 4.2 it is sufficient to derive this estimation for normal positive linear forms with equal supports. In fact, we might define sequences $\{\omega_k\}, \{\sigma_k\}, \{\mu_k\}$ and $\{\nu_k\}$, with

$$\omega_k = \omega + \frac{1}{k}(\sigma+\mu+\nu),\ \sigma_k = \sigma + \frac{1}{k}(\omega+\mu+\nu),\ \mu_k = \mu + \frac{1}{k}(\sigma+\omega+\nu),\ \nu_k = \nu + \frac{1}{k}(\sigma+\mu+\omega).$$

Then $\omega_k \longrightarrow \omega$, $\sigma_k \longrightarrow \sigma$, $\mu_k \longrightarrow \mu$ and $\nu_k \longrightarrow \nu$ in norm, and the above estimation can be obtained as the limit of the estimations

$$|P_M(\omega_k,\sigma_k)^{1/2} - P_M(\mu_k,\nu_k)^{1/2}| \le \|\sigma_k\|_1^{1/2}\|\omega_k-\mu_k\|_1^{1/2} + \|\mu_k\|_1^{1/2}\|\sigma_k-\nu_k\|_1^{1/2}$$

provided these can be shown to hold for normal positive linear forms with equal supports ($\omega_k, \sigma_k, \mu_k$ and ν_k have equal supports). In line with this, suppose $\omega, \sigma, \mu, \nu \in M_{*+}$ all have the the same support. According to a special case of Lemma 6.1 each of the pairs $\{\omega,\sigma\}, \{\mu,\omega\}, \{\mu,\nu\}, \{\sigma,\nu\}, \ldots$ is «-minimal. Assume that $|P_M(\omega,\sigma)^{1/2} - P_M(\mu,\sigma)^{1/2}| = P_M(\omega,\sigma)^{1/2} - P_M(\mu,\sigma)^{1/2}$, e.g. Suppose $\psi \in S(\sigma)$ is given. According to Theorem 5.1 /(iv) and Remark 5.2/(2) we may choose φ as the ψ-relative representative of ω, and ξ can be chosen as the φ-relative representative of μ. Then, according to Remark 5.2/(2) and (4) we have

$$\begin{aligned} &P_M(\omega,\sigma)^{1/2} - P_M(\mu,\sigma)^{1/2} \\ &\quad = \langle\varphi,\psi\rangle - P_M(\mu,\sigma)^{1/2} \le \langle\varphi,\psi\rangle - |\langle\xi,\psi\rangle| \le |\langle\varphi,\psi\rangle - \langle\xi,\psi\rangle| \\ &\quad = |\langle\varphi-\xi,\psi\rangle| \le \|\psi\|\,\|\varphi-\xi\| \le \|\sigma\|_1^{1/2}\|\omega-\mu\|_1^{1/2}, \end{aligned}$$

where we used that $P_M(\mu,\sigma)^{1/2} \ge |\langle\xi,\psi\rangle|$. Therefore, we have arrived at

$$|P_M(\omega,\sigma)^{1/2} - P_M(\mu,\sigma)^{1/2}| \le \|\sigma\|_1^{1/2}\|\omega-\mu\|_1^{1/2}.$$

Analogously we can show that

$$|P_M(\mu,\sigma)^{1/2} - P_M(\mu,\nu)^{1/2}| \le \|\mu\|_1^{1/2}\|\sigma-\nu\|_1^{1/2}.$$

Hence, we can follow that

$$\begin{aligned} &|P_M(\omega,\sigma)^{1/2} - P_M(\mu,\nu)^{1/2}| \\ &\qquad \le |P_M(\omega,\sigma)^{1/2} - P_M(\mu,\sigma)^{1/2}| + |P_M(\mu,\sigma)^{1/2} - P_M(\mu,\nu)^{1/2}| \\ &\qquad \le \|\sigma\|_1^{1/2}\|\omega-\mu\|_1^{1/2} + \|\mu\|_1^{1/2}\|\sigma-\nu\|_1^{1/2}. \end{aligned}$$

Therefore, by our discussion from above Remark 4.3 can be taken for proven.

The representation Theorem 5.1 gives also some idea for an important class of «-minimal pairs.

Example 6.2 : (1) Let ω be a normal positive linear form on M. Let $\sigma = \omega^a$, with $a \ge 0$, $a \in M$. Then, $I(\omega,\sigma) = R_a\,\omega$, and $\sigma = \sigma_0$, where $\{\omega_0, \sigma_0\}$ denotes the «-minimal pair which is majorized by $\{\omega,\sigma\}$. For $a \in M_+$ with invertible a the pair $\{\omega,\sigma\}$ is «-minimal. Assume $\{\omega,\sigma\}$ is «-minimal. In this case, for given $\varphi \in S(\omega)$ the vector $\psi = a\varphi$ is the uniquely determined vector $\psi \in S(\sigma)$ with $\psi \parallel \varphi$.

(2) In case M possesses tracial normal positive linear forms, for a tracial $\tau \in M_{*+}$ and for $\omega \in M_{*+}$ the pair $\{\omega,\tau\}$ is «-minimal if, and only if, $s(\omega) = s(\tau)$.

Proof : We start with (2). Note first that for a tracial τ we have $s(\tau) \in M \cap M'$. By the last part of Lema 6.1 $s(\omega) \le s(\tau)$ and $s(\omega) \sim s(\tau)$ provided $\{\omega,\tau\}$ has been supposed to be «-minimal. The vN-algebra $N = Ms(\tau)$ acting on $s(\tau)H$ is of finite type since $\tau_{|N}$ yields a faithful normal tracial positive linear form on N. Since $s(\omega) \sim s(\tau)$ also with respect to N and N is finite, $s(\omega) = s(\tau)$ follows. The other direction follows from the last part of Lemma 6.1 which then closes the proof of (2).

To prove (1), we will follow step by step the line of the proof of Theorem 5.1. Suppose $\varphi \in S(\omega)$. Then $\psi = a\varphi \in S(\sigma)$. For h defined on M' by $h(\cdot) = \langle (\cdot)\psi, \varphi\rangle$ we have $h(y^*y) = \|ya^{1/2}\varphi\|^2 \ge 0$, for any $y \in M'$. Hence, h is positive, with $s(h) = p'(a^{1/2}\varphi)$. Owing to $p'(\psi) = p'(a\varphi) \le p'(a^{1/2}\varphi) \le p'(\varphi)$ we see $p'(\psi) \le s(h) \le p'(\varphi)$, and because $p'(\psi) \ge s(h)$ obviously holds, $p'(\psi) = s(h) \le p'(\varphi)$ follows. From this $\sigma = \sigma_0$ is evident. By our criterion from the beginning of this section, $\{\omega,\sigma\}$ is «-minimal if, and only if $p'(\psi) = s(h) = p'(\varphi)$. Since, for an invertible $a \in M_+$, $p'(a\varphi) = p'(\varphi)$ holds, the conditions of our criterion of minimallity are fulfilled, i.e. $\{\omega,\sigma\}$ is «-minimal in that case. Note that $v = s(h) = p'(a\varphi)$ is the partial isometry of the polar decomposition of h. According to the argumentation in the proof of Theorem 3.1 in our special situation

$$I(\omega,\sigma)(\cdot) = \langle (\cdot)v^*a\varphi, \varphi\rangle = \langle (\cdot)a\varphi, \varphi\rangle = (R_a\omega)(\cdot)$$

has to hold. Finally, suppose $\{\omega,\sigma\}$ is «-minimal, and $\varphi \in S(\omega)$ is given, and $\psi = a\varphi$. Then, as we know from above $p'(\psi) = s(h) = p'(\varphi)$. For $y \in M'$ we have $\sigma'(y^*y) = \|y\psi\|^2 = \|ya\varphi\|^2 = \|ay\varphi\|^2 \le \|a\|^2\ \|y\varphi\|^2 = \|a\|^2\omega'(y^*y)$, with $\omega', \sigma' \in M'_{*}$ defined by $\omega'(\cdot) = \langle (\cdot)\varphi, \varphi\rangle$ and $\sigma'(\cdot) = \langle (\cdot)\psi, \psi\rangle$ on M', respectively. We also note that $s(\sigma') = s(\omega') = p'(\varphi) = p'(\psi)$ by the assumptions on φ and ψ. By the Radon-Nikodym-Theorem of S. Sakai [19] (cf. also [20]) there exists a unique $t \in M'_+$ with support $\mathrm{supp}(t) = s(\sigma') = s(\omega')$, such that $\sigma'(\cdot) = \omega'(t(\cdot)t)$. Thus $\|y\psi\|^2 = \|yt\varphi\|^2$, for any $y \in M'$, from which condition the existence of a partial isometry $u \in M$ follows, with $u^*\psi = t\varphi$ and $uu^* = p(\psi) = s(\sigma)$, $u^*u = p(t\varphi)$. Now, $p(t\varphi) \le p(\varphi)$, and since $p'(u^*\psi) = p'(t\varphi) = s(\sigma') = s(\omega') = p'(\psi) = p'(\varphi)$ and the linear form $f(\cdot) = \langle (\cdot)\varphi, u^*\psi\rangle = \langle (\cdot)\varphi, t\varphi\rangle$ on M is positive, Appendix 3 applies (with $M \to M'$, $p \to p'$) and shows that $s(f) = p(\varphi) = p(t\varphi)$. Since $t \ge 0$, there exists $u \in M$, with

$$u^*\psi = t\varphi \in \overline{M'_+\varphi}\,,\ uu^* = p(\psi) = s(\sigma),\ u^*u = p(t\varphi) = p(\varphi) = s(\omega). \tag{6.1}$$

Now, $u\psi = ua\varphi = uau^*u\varphi$ due to $u^*u = p(\varphi)$, i.e. we also have

$$u\psi \in \overline{M_+u\varphi}\,. \tag{6.2}$$

Finally, h defined on M' by $h(\cdot) = \langle (\cdot)u\psi, u\varphi\rangle = \langle (\cdot)a\varphi, \varphi\rangle$ is positive on M'. Since, on the one hand $p'(u\psi) \ge s(h)$, and $p'(u\psi) = p'(ua\varphi) \le p'(a\varphi) = p'(\psi) = s(h) = p'(\varphi)$ by our assumptions, we conclude that $p'(u\psi) = p'(\psi) = p'(\varphi)$. This together with (6.1) and (6.2) shows that $\psi \parallel \varphi$ (see the definition of $\parallel$ in Definition 5.0). By our assumptions and Theorem 5.1/(iv) the uniqueness of ψ is clear ∎

Suppose now, $\omega \in M_{*+}$ and $a \in M_+$ are given. Assume as above $\sigma = \omega^a$. As we have seen in Example 6.2 $I(\omega,\sigma) = R_a\omega$, and $\sigma = \sigma_0$, where $\{\omega_0, \sigma_0\}$ denotes the «-minimal pair which is majorized by $\{\omega, \sigma\}$. We are going to find an interpretation of the skew phase $u(\omega,\sigma)$ in this case. Let $\varphi \in S(\omega)$. Then, $\psi = a\varphi \in S(\sigma) = S(\sigma_0)$. By Remark 2.2 we have $\varphi' = s(|h|)\varphi \in S(\omega_0)$, with h defined on M' by $h(\cdot) = \langle(\cdot)\psi, \varphi\rangle$. As we explained above, $h \geq 0$, and $s(|h|) = s(h) = p'(\psi)$. Similarly as in the preceeding proof, by the Radon-Nikodym-Theorem of S. Sakai there exists a unique $t \in M'_+$ with support $\mathrm{supp}(t) = p'(\psi)$, such that $\|y\psi\|^2 = \|yt\varphi\|^2$, for $y \in M'$, from which relation once more again the existence of a partial isometry $u \in M$ follows, with $u^*\psi = t\varphi$ and $uu^* = p(\psi) = s(\sigma) = s(\sigma_0)$, $u^*u = p(t\varphi) \leq p(\varphi) = s(\omega)$. Now, $s(\omega_0) = p(s(|h|)\varphi) = p(p'(\psi)\varphi) = p(t\varphi)$, where we used that $\mathrm{supp}(t) = p'(\psi)$. From this we also get that $t\varphi = tp'(\psi)\varphi = ts(h)\varphi = t\varphi'$. But then, the last part of the proof of Examples 6.2 (with $\varphi', \omega_0, \sigma_0$ in place of φ, ω, σ) together with the equation $u^*\psi = t\varphi = t\varphi'$ says that $\psi \parallel \varphi'$, necessarily. In view to Theorem 5.1/(ii) $u(\omega_0, \sigma_0) = u^*$ has to be followed. Since $I(\omega,\sigma) = I(\omega_0,\sigma_0)$, $u^* = u(\omega,\sigma)$ is seen. Hence, what we have proved is the following

Lemma 6.3 : *Suppose $\omega \in M_{*+}$ and $a \in M_+$ are given. Assume $\sigma = \omega^a$. Let $\varphi \in S(\omega)$, and let $t \in M'_+$ be the uniquely determined S. Sakai's Radon-Nikodym operator which obeys $\langle ya\varphi, a\varphi\rangle = \langle yt\varphi, t\varphi\rangle$, for all $y \in M'$. Then, the equations*

$$w^*w = s(\sigma), \quad wa\varphi = t\varphi$$

have as the unique solution in M the skew phase $w = u(\omega,\sigma)$.

Let us assume now that $\{\omega,\sigma\}$ is «-minimal. Suppose $\varphi \in S(\omega)$. By Theorm 5.1 there exists $\psi \in S(\sigma)$ with $\psi \parallel \varphi$. By (5.2) and (5.9) we have $s(\omega)\psi \in \overline{M_+\varphi}$. Let $\{a_n\} \subset M_+$ be a sequence of positive elements such that $a_n\varphi \longrightarrow s(\omega)\psi$. We define

$$b_n = s(\omega)a_n s(\omega) + \tfrac{1}{n}s(\omega) + n\,s(\omega)^\perp \text{ and } \varphi_n = \varphi + \tfrac{1}{n}s(\omega)^\perp\psi, \text{ for any } n \in \mathrm{N}.$$

Then $\varphi_n \longrightarrow \varphi$. In setting $\psi_n = b_n\varphi_n = s(\omega)a_n\varphi + \frac{1}{n}\varphi + s(\omega)^\perp\psi$ we see that $\psi_n \longrightarrow \psi$. Hence, for $\omega_n(\cdot) = \langle(\cdot)\varphi_n, \varphi_n\rangle$ and $\sigma_n(\cdot) = \langle(\cdot)\psi_n, \psi_n\rangle$ we realize that $\omega_n \longrightarrow \omega$ and $\sigma_n \longrightarrow \sigma$ in norm. Since $\sigma_n = \omega_n(b_n(\cdot)b_n)$ and $b_n \geq 0$ is in M and is invertible, Example 6.2 says that $\{\omega_n, \sigma_n\}$ is «-minimal for any $n \in \mathrm{N}$. Thus we have arrived at the following

Theorem 6.4 : *The set $\{(\omega,\sigma): \omega \in M_{*+}$, there is $a \in M_+$, invertible, with $\sigma = \omega^a\}$ is a dense subset of the set Γ_* of all «-minimal pairs in $M_{*+} \times M_{*+}$.*

As another application of Theorm 5.1. we find the following

Proposition 6.5 : *$I(\omega,\sigma)$ is a hermitian form if and only if $I(\omega,\sigma)$ is positive.*

Proof : Assume $I(\omega,\sigma)$ is a hermitian form. Then, by Lemma 4.1/(ii) $I(\omega,\sigma) = I(\sigma,\omega)$. By Lemma 4.1/(iii) we can suppose that $\{\omega,\sigma\}$ is «-minimal. Let $\varphi \in S(\omega)$ and $\psi \in S(\sigma)$ be chosen such that $\psi \parallel \varphi$. By our assumptions we get $\langle(\cdot)\psi, \varphi\rangle = \langle(\cdot)\varphi, \psi\rangle$. The latter also means that $I(\omega,\sigma) = I(\sigma,\omega) = \langle(\cdot)s(\omega)\psi, \varphi\rangle = \langle(\cdot)\varphi, s(\omega)\psi\rangle$ (we note $s(\omega) = p(\varphi)$). By (5.2) and (5.9) we have $s(\omega)\psi \in \overline{M_+\varphi}$. By Appendix 2, a positive, selfadjoint linear operator F, which is affiliated with M, exists and fulfils $s(\omega)\psi = F\varphi$. Let $F = \int_0^\infty \lambda E(\mathrm{d}\lambda)$

be the spectral representation of F. We define $E = E([0,\infty))$, $E_n = E([0,n))$, and $F_n = FE_n$, for any $n \in \mathbb{N}$. Then, all the operators defined belong to M, and st-$\lim_n E_n = E$ and $Es(\omega)\psi = F\varphi = s(\omega)\psi$. By assumptions, $\langle y\varphi, F\varphi\rangle = \langle yF\varphi, \varphi\rangle$ for all $y \in M$. Especially, for $y \in E_nME_n$ we get $\langle y\varphi, F_n\varphi\rangle = \langle yF_n\varphi, \varphi\rangle$. Since yF_n, $F_ny \in E_nME_n$, by the same reasons we are obtaining that $\langle yF_n^2\varphi, \varphi\rangle = \langle yF_n\varphi, F_n\varphi\rangle = \langle F_nyF_n\varphi, \varphi\rangle = \langle F_ny\varphi, F_n\varphi\rangle = \langle y\varphi, F_n^2\varphi\rangle$, for any $y \in E_nME_n$. Concluding further in this way shows that $\langle y\varphi, F_n^k\varphi\rangle = \langle yF_n^k\varphi, \varphi\rangle$, for any $k \in \mathbb{N}$ and $y \in E_nME_n$. Our conclusion is that $\langle y\varphi, B\varphi\rangle = \langle yB\varphi, \varphi\rangle$ for any B taken from the C^*-algebra that is generated by e and F_n. Especially, for $B = F_n^{1/2}$ we get in this way that $\langle yF_n\varphi, \varphi\rangle = \langle yBB\varphi, \varphi\rangle = \langle yB\varphi, B\varphi\rangle$, for any $y \in E_nME_n$. Hence, for $z \in M$ we see that

$$I(\omega,\sigma)(E_nz^*zE_n) = \langle E_nz^*zE_nB\varphi, B\varphi\rangle = \|zE_nB\varphi\|^2 \geq 0.$$

The inequality $I(\omega,\sigma)(E_nz^*zE_n) \geq 0$ has to hold for any $n \in \mathbb{N}$ and any $z \in M$. Since $I(\omega,\sigma)$ belongs to M_* and st-$\lim_n E_nz^*zE_n = Ez^*zE$, $I(\omega,\sigma)(Ez^*zE) \geq 0$ has to hold for any $z \in M$. Because of $Es(\omega)\psi = s(\omega)\psi$ and $\langle(\cdot)s(\omega)\psi, \varphi\rangle = \langle(\cdot)\varphi, s(\omega)\psi\rangle$ we can conclude as follows:

$$\begin{aligned} I(\omega,\sigma)(z^*z) &= \langle z^*z\,s(\omega)\psi, \varphi\rangle = \langle z^*zE\,s(\omega)\psi, \varphi\rangle = \langle z^*zE\varphi, s(\omega)\psi\rangle \\ &= \langle z^*zE\varphi, Es(\omega)\psi\rangle = \langle Ez^*zE\varphi, s(\omega)\psi\rangle = \langle Ez^*zEs(\omega)\psi, \varphi\rangle \\ &= I(\omega,\sigma)(Ez^*zE) \geq 0, \end{aligned}$$

for any $z \in M$. This proves positivity of the skew form provided hermiticity has been supposed. The other direction is trivial ∎

Remark 6.6: Note that Example 6.2, among other things yields that $P_M(\omega,\sigma) = I(\omega,\sigma)(e)^2 = \langle a\varphi, \varphi\rangle^2 = \omega(a)^2$ in the case that $\sigma = \omega^a$ for some $a \in M_+$. This is a well-known fact, cf. [2], [24]. By standard conclusions the result persists to hold true also in the case of a unital C^*-algebra A and $\omega \in A^*_+$, $a \in A$.

Let us suppose now we have given two vectors $\psi', \varphi' \in H$ such that $\psi' \parallel \varphi'$. Then, it is quite interesting to analyze under what kind of perturbations $\delta\psi', \delta\varphi' \in H$ of the vectors $\psi', \varphi' \in H$ the relation $\parallel$ behaves stable, i.e. $\psi \parallel \varphi$ persists to hold with $\psi = \psi' + \delta\psi'$ and $\varphi = \varphi' + \delta\varphi'$. Our Example 6.2 yields a special case of this problem. In fact, if we define $\psi' = \varphi' = \varphi$, $\delta\varphi' = 0$, then Example 6.2 tells us that $\psi \parallel \varphi$ persists for any perturbation of ψ' of the form $\delta\psi' = b\varphi$, with $b \in M_h$, spec$(b) \subset (-1,\infty)$. Note that in the proof of Example 6.2 we have been following step by step the line of argumentation proposed by the proof of Theorem 5.1 . This made sense because we were also aiming at an interpretation of the skew phase in the context considered there (cf. Lemma 6.3). In applications like the mentioned question on the behavior of $\parallel$ under perturbations, however, in most cases we have in mind we will not care about the behavior of the skew phases. In line with this, it is of interest for us to have to our dispose a simple criterion in order to decide whether or not for two given individual vectors $\psi, \varphi \in H$ the relation $\psi \parallel \varphi$ takes place. Such a criterion, which has its motivation in the above problem and proves quite useful in such a context, reads as follows.

Proposition 6.7: *Suppose $\varphi, \psi \in H$ are given. Let $h \in M'_*$ be defined by $h(\cdot) = \langle(\cdot)\psi, \varphi\rangle$. The following assertions are mutually equivalent*:

(a) *h is positive on M' and $p'(\varphi) = p'(\psi) = s(h)$.*
(b) *$p(\varphi)\psi \in M_+\varphi$ and $p'(\varphi) = p'(\psi) = s(h)$.*
(c) *for $\sigma \in M_{*+}$ with $\psi \in S(\sigma)$, ψ is the unique vector within $S(\sigma)$ such that $\psi \| \varphi$.*

Proof : The equivalence between (a) and (b) is part of the assertion of Appendix 2. Suppose (a) holds, and let $\omega, \sigma \in M_{*+}$ be such that $\varphi \in S(\omega)$, $\psi \in S(\sigma)$. We have $p'(\varphi) = p'(\psi) = s(|h|) = s(|h^*|) = s(h)$. With regard to Remark 2.2 we then realize that $\{\omega, \sigma\}$ is « - minimal. Moreover, (a) together with Lemma 1.1/(i) implies that $\varphi \in S(\omega)$ and $\psi \in S(\sigma)$ are vectors obeying $\langle \psi, \varphi \rangle = h(e) = \| h \|_1 = P_M(\omega, \sigma)^{1/2}$. Thus, φ and ψ are representing vectors for the components of a «-minimal pair $\{\omega, \sigma\}$ fulfilling assumptions as in accordance with those we have met in the first part of the proof of Theorem 5.1. In conclusion of this part of the proof we deduced that $\psi \| \varphi$. Hence also $\psi \| \varphi$ in our case at hand. The remaining assertion of (c) follows from Theorem 5.1/(iv). Suppose now that $\psi \| \varphi$. By Theorem 5.1/(iv) ψ is the unique vector in $S(\sigma)$ with $\psi \| \varphi$. By Theorem 5.1 /(i) $\{\omega, \sigma\}$ is «-minimal. In view of (5.2) and (5.9) then (b) follows ∎

Remark 6.8 : (1) The preceding result could be taken as an invitation for a comprehensive redefinition of $\|$ which reads as follows: for $\psi, \varphi \in H$ let $\psi \| \varphi$ in case that $h(\cdot) = \langle (\cdot)\psi, \varphi \rangle$ is positive on M' and $p'(\varphi) = p'(\psi) = s(h)$.

(2) A pair of normal positive linear forms $\{\omega, \sigma\}$ is «-minimal if, and only if for some $\varphi \in S(\omega)$, $\psi \in S(\sigma)$ such that $h_{\varphi,\psi}(\cdot) = \langle (\cdot)\psi, \varphi \rangle$ is positive on M', $\psi' = \psi$ is the unique $\psi' \in S(\sigma)$ such that $h_{\varphi,\psi'} \geq 0$ and, at the same time, $\varphi' = \varphi$ is the unique $\varphi' \in S(\omega)$ such that $h_{\varphi',\psi} \geq 0$.

In order to see this, suppose the pair $\{\omega, \sigma\}$ is given and, according to Appendix 7, $\varphi \in S(\omega)$ and $\psi \in S(\sigma)$ have been chosen such that $\langle \psi, \varphi \rangle = h(e) = \| h \|_1 = P_M(\omega, \sigma)^{1/2}$, with the linear form $h(\cdot) = \langle (\cdot)\psi, \varphi \rangle$ on M'. As we know $h \geq 0$. Then, either Proposition 6.7/(a) is fulfilled, and then the pair is «-minimal in the sense introduced in Section 1 , or (a) does not hold. Since h is positive, in the latter case necessarily $p'(\varphi) > s(h)$ or $p'(\psi) > s(h)$ have to occur. Put this case and suppose $p(\psi) > s(h)$. Let us define

$$\psi(\Theta) = s(h)\psi + \exp i\Theta\,\{p'(\psi) - s(h)\}\psi, \text{ for } \Theta \in \mathbb{R}.$$

Then, $\psi(\Theta) \in S(\sigma)$, for any $\Theta \in \mathbb{R}$. Moreover we have $h^{\Theta}(\cdot) = \langle (\cdot)\psi(\Theta), \varphi \rangle = h((\cdot)s(h)) + h((\cdot)\{p'(\psi) - s(h)\}) \exp i\Theta$, for any $\Theta \in \mathbb{R}$. Since $h((\cdot)s(h)) = h$ and $h((\cdot)\{p'(\psi) - s(h)\}) = 0$ are fulfilled, we get $h^{\Theta} = h$ for any $\Theta \in \mathbb{R}$. Since by assumption of this case we have $\{p'(\psi) - s(h)\}\psi \neq 0$, we see that $\psi' = \psi$ is definitely *not* the unique $\psi' \in S(\sigma)$ such that $h_{\varphi,\psi'}(\cdot) = \langle (\cdot)\psi', \varphi \rangle$ is *positive*. Every choice of $\Theta \in [0, 2\pi)$ yields another $\psi' = \psi(\Theta) \in S(\sigma)$ with $h_{\varphi,\psi'} = h^{\Theta} = h \geq 0$. Analogous arguments work in case that $p'(\varphi) > s(h)$ occurs. In this case the conclusion is that $\varphi' = \varphi$ is not the unique φ' such that $h_{\varphi',\psi} \geq 0$. Hence, if $\{\omega, \sigma\}$ is not «-minimal in the sense of Section 1, the positivity requiremnt for $h_{\varphi,\psi}$ does *not* fix uniquely the relative position in H of the representatives φ and ψ to each other . On the other hand, let us suppose now there exist $\psi' \neq \psi$, with $\psi', \psi \in S(\sigma)$, such that both $h_{\varphi,\psi} \geq 0$ and $h_{\varphi,\psi'} \geq 0$ are satisfied. As we know from Lemma 1.1 we have

$$\| h_{\varphi,\psi} \|_1 = \| h_{\varphi,\psi'} \|_1 = P_M(\omega, \sigma)^{1/2}.$$

Let $w \in M'$ be the partial isometry with $w^* w = p'(\psi)$ and $\psi' = w\psi$. We then have $h_{\varphi,\psi'} = R_w h_{\varphi,\psi} = h_{\varphi,\psi}((\cdot)w)$. Since both $h_{\varphi,\psi}$ and $h_{\varphi,\psi'}$ are positive and $\|w\| \leq 1$ holds, we can apply a result of [20: 1.24.1] saying that in our situation at hand $h_{\varphi,\psi'} \leq h_{\varphi,\psi}$ has to be followed. Since both functionals are positive and of the same norm, necessarily $h_{\varphi,\psi'} = h_{\varphi,\psi}$ has to hold. But then, by the uniqueness of the polar decomposition we may conclude that $w\, s(h_{\varphi,\psi}) = s(h_{\varphi,\psi})$ holds. Assuming $p'(\psi) = s(h_{\varphi,\psi})$ we have to follow that $w = p'(\psi)$, i.e. $\psi' = \psi$. This is in contradiction to our suppositions. Hence $p'(\psi) >$

$s(h_{\varphi,\psi})$ has to be true. But then, according to Theorem 5.1/(i) and Proposition 6.7/(a), (c), $\{\omega,\sigma\}$ can not be «-minimal in the sense of the definition given in Section 1. An analogous conclusion has to be drawn in case of $\varphi' \neq \varphi$, $\varphi' \in S(\omega)$, with $h_{\varphi',\psi} \geq 0$. Hence, what we have proved now is that $\{\omega,\sigma\}$ is «-minimal if and only if for some $\varphi \in S(\omega)$, $\psi \in S(\sigma)$ such that $h_{\varphi,\psi}(\cdot) = \langle (\cdot)\psi, \varphi \rangle$ is positive on M', $\psi' = \psi$ is the unique $\psi' \in S(\sigma)$ such that $h_{\varphi,\psi'} \geq 0$ and, at the same time, $\varphi' = \varphi$ is the unique $\varphi' \in S(\omega)$ such that $h_{\varphi',\psi} \geq 0$.

(3) Suppose $\{\omega,\sigma\}$ is «-minimal, and $\varphi \in S(\omega)$, $\psi \in S(\sigma)$ have been chosen such that $\langle \psi,\varphi \rangle = P_M(\omega,\sigma)^{1/2}$. Then, we have $\psi \parallel \varphi$. With other words, in case we know that ω and σ form a «-minimal pair, for any given $\varphi \in S(\omega)$ there exists exactly one $\psi' \in S(\sigma)$ such that $\langle \psi', \varphi \rangle = P_M(\omega,\sigma)^{1/2}$. This unique element $\psi' = \psi$ of $S(\sigma)$ is the *φ-relative representative of σ*.

In fact, this follows from (2) since $\psi' = \psi$ is uniquely determined by the conditions $\psi' \in S(\sigma)$ and $h_{\varphi,\psi'} \geq 0$. According to Lemma 1.1/(i) $h_{\varphi,\psi'} \geq 0$ is equivalent with $\langle \psi',\varphi \rangle = h_{\varphi,\psi'}(e) = \| h_{\varphi,\psi'} \|_1 = P_M(\omega,\sigma)^{1/2}$. The rest follows with view to Remark 5.2/(2).

Note that by (2) the usage of the term «-*minimality* of a pair $\{\omega,\sigma\}$ as it was mentioned in the introduction (cf. Section 0) is now verified. This says that the characterization of «-minimality we have been using in the introduction amounts to be equivalent with the definition given at the end of Section 1. The difficulties with Proposiion 6.7 or the seemingly sound redefinition of $\parallel$ in Remark 6.8/(1) arise from the fact that for arbitrarily given vectors $\psi, \varphi \in H$ it is not easy to verify whether h is positive or, if this is the case, whether $p'(\varphi) = p'(\psi) = s(h)$ occurs. But the criterion (and therefore also Remark 6.8/(2) and (3)) becomes easily applicable in the context of perturbations of an existing relation $\psi \parallel \varphi$.

Example 6.9: Let $\psi, \varphi \in H\backslash\{0\}$ be vectors such that $\psi \parallel \varphi$. For $\alpha,\beta \in \mathbf{R}_+$ let $\varphi(\alpha,\beta) \in H$ be defined as $\varphi(\alpha,\beta) = \alpha\psi + \beta\varphi$. Then, $\varphi(\alpha,\beta) \parallel \varphi(\alpha',\beta')$, for any $\alpha,\beta,\alpha',\beta' \in \mathbf{R}_+$.

Proof : Suppose $\alpha,\beta,\alpha',\beta' \in \mathbf{R}_+$ are chosen according to our assumptions. Let us look on the linear functionals h', h, ω', σ' defined on M' by $h'(\cdot) = \langle (\cdot)\varphi(\alpha,\beta), \varphi(\alpha',\beta') \rangle$, $h(\cdot) = \langle (\cdot)\psi, \varphi \rangle$, $\omega'(\cdot) = \langle (\cdot)\varphi, \varphi \rangle$, $\sigma'(\cdot) = \langle (\cdot)\psi, \psi \rangle$, respectively. Then,

$$h' = \alpha\alpha'\sigma' + \beta\beta'\omega' + (\alpha\beta' + \beta\alpha')h ,$$

where we used that $h^* = h \geq 0$, due to $\psi \parallel \varphi$ and Poposition 6.7. Since $\alpha\alpha', \beta\beta', (\alpha\beta' + \beta\alpha') \in \mathbf{R}_+$ and $h, \omega', \sigma' \in M'_{*+}$ we deduce $h' \geq 0$. By our assumptions and Proposition 6.7, $p'(\varphi) = p'(\psi) = s(h)$ holds, and since $s(\omega') = p'(\varphi)$ and $s(\sigma') = p'(\psi)$, we have $s(\omega') = s(\sigma') = s(h)$. Thus $s(h') = s(h)$ has to be valid. Due to $p'(\varphi) = p'(\psi) = s(h) = s(h')$ once more again, and by the construction of the vectors $\varphi(\alpha,\beta)$ and $\varphi(\alpha',\beta')$, we see that $s(h')\varphi(\alpha,\beta) = \varphi(\alpha,\beta)$ and $s(h')\varphi(\alpha',\beta') = \varphi(\alpha',\beta')$. Hence $s(h') \geq p'(\varphi(\alpha,\beta))$ and $s(h') \geq p'(\varphi(\alpha',\beta'))$. Since in our case $s(h') = s(|h'|) = s(|h'^*|)$, and because $s(|h'|) \leq p'(\varphi(\alpha,\beta))$ and $s(|h'^*|) \leq p'(\varphi(\alpha',\beta'))$ follow as usually from the construction of h', we finally can summarize that $s(h') = p'(\varphi(\alpha,\beta)) = p'(\varphi(\alpha',\beta'))$, with h' being positive. Finally, an application of our Proposition 6.7 now yields $\varphi(\alpha,\beta) \parallel \varphi(\alpha',\beta')$ ∎

Another quite useful application of our criterion is in the following situation.

Example 6.10: Let $\psi, \varphi \in H$ be vectors such that $\psi \parallel \varphi$.

(1) Suppose $a \in M$ is invertible. Then, $a^{-1}\psi \parallel a^*\varphi$.

(2) Suppose $u \in M$ is a partial isometry such that $u^* u \geq p(\psi)$ or $u^* u \geq p(\varphi)$ holds. Then, $u\psi \parallel u\varphi$.

(3) Suppose $b \in M'$. Then, $b\psi \parallel b\varphi$.

Proof: Let $\psi' = a^{-1}\psi$ and $\varphi' = a^*\varphi$. Then, since $a \in M$ is invertible, $p'(\psi') = p'(\psi)$ and $p'(\varphi') = p'(\varphi)$ have to hold. Let h and h' be defined on M' by $h(\cdot) = \langle (\cdot)\psi, \varphi \rangle$ and $h'(\cdot) = \langle (\cdot)\psi', \varphi' \rangle$, respectively. Then, for any $x \in M'$, $h'(x) = \langle axa^{-1}\psi, \varphi \rangle = \langle x\psi, \varphi \rangle = h(x)$, i.e. $h' = h$. From $\psi \parallel \varphi$ it follows by Proposition 6.7 that $h \geq 0$ and $s(h) = p'(\varphi) = p'(\psi)$. Hence $s(h') = p'(\varphi') = p'(\psi')$ has to be fulfilled. By Proposition 6.7 $\psi' \parallel \varphi'$ follows. The second part is obtained by an analogous argumentation. To see (3), let $\psi' = b\psi$, $\varphi' = b\varphi$ and let h and h' be defined as in the proof of (1). Let q be the orthoprojection onto the *range* of $bs(h)$, i.e. q is the left support of $bs(h)$. Since $s(h) = p'(\varphi) = p'(\psi)$ holds, we have $p'(\psi') = p'(b\psi) = p'(bs(h)\psi) = q = p'(bs(h)\varphi) = p'(b\varphi) = p'(\psi')$. By the meaning of q and since $h'(\cdot) = h(b^*(\cdot)b)$ holds, we see $h' \geq 0$ and $h'(q) = h(b^*q\,b) = h(b^*b) = h'(e)$, i.e. $s(h') \leq q$. On the other hand, for $z = q - s(h')$ we have $0 = h'(z) = h(s(h)b^*z\,bs(h))$. Hence, we have $s(h)b^*z\,bs(h) = 0$. The latter implies $z\,bs(h) = 0$. From this and from the meaning of q we infer that $bs(h) = s(h')bs(h)$. Hence, $s(h') \geq q$. Taking together these facts gives $p'(\psi') = p'(\psi') = s(h') = q$. An application of Proposition 6.7 now yields the result ∎

Remark 6.11: The assertion of Example 6.10/(1) remains valid if in the assumptions a is supposed to be a densely defined, closed, invertible operator on H which is affiliated *with* M, and $\psi \in D(a^{-1})$, $\varphi \in D(a^*)$. In fact, under these assumtions $a^{-1}x \supseteq xa^{-1}$ for any $x \in M'$. Hence, $a^{-1}x\psi = xa^{-1}\psi$, and thus $h'(x) = \langle xa^{-1}\psi, a^*\varphi \rangle = \langle a^{-1}x\psi, a^*\varphi \rangle = \langle x\psi, \varphi \rangle = h(x)$ for any $x \in M'$. Now, in the polar decomposition $a = u|a|$ of a we have $u \in U(M)$ and $|a|$ is a selfadjoint, invertible and positive operator affiliated with M. Thus, also the conclusion that $p'(a^{-1}\psi) = p'(\psi)$ and $p'(a^*\varphi) = p'(\varphi)$ hold remains valid. Also it is easy to see that the assertion of Example 6.10/(3) remains true for a densely defined, closed operator b on H which is affiliated with M' and φ and ψ with $\varphi, \psi \in D(b)$.

Let us suppose now that $\{\omega, \sigma\}$ is «-minimal, and assume $\varphi \in S(\omega)$ and $\psi \in S(\sigma)$ are given. Let h be defined on M' by $h(\cdot) = \langle (\cdot)\psi, \varphi \rangle$, and let $h = R_v|h|$ be the polar decomposition of h. Then $v \in M'$ is a partial isometry with $v^* v = s(|h|)$, $v v^* = s(|h^*|)$. We define $\psi' = v^*\psi$. By Remark 2.2, (2.3) and by our assumptions we have $s(|h|)\varphi = \varphi$ and $\psi' \in S(\sigma)$. Hence, $p'(\psi') \leq s(|h|)$ and $p'(\varphi) \leq s(|h|)$. Because $|h|(\cdot) = \langle (\cdot)\psi', \varphi \rangle$ is positive, on the other hand, $p'(\psi') \geq s(|h|)$ and $p'(\varphi) \geq s(|h|)$ have to be valid. Let be defined $h'(\cdot) = \langle (\cdot)\psi', \varphi \rangle$ on M'. Then $h' = |h|$ is positive on M' and $p'(\psi') = s(h') = p'(\varphi)$ is fulfilled. An application of Proposition 6.7 then yields the following

Proposition 6.12: *Let $\{\omega, \sigma\}$ be «-minimal, and assume $\varphi \in S(\omega)$ and $\psi \in S(\sigma)$. Let h be defined on M' by $h(\cdot) = \langle (\cdot)\psi, \varphi \rangle$, and let $h = R_v|h|$ be the polar decomposition of h. Then, $v^*\psi \parallel \varphi$ holds.*

Note that this result has been proved implicitely already in course of the proof of Theoem 5.1 (cf. also Remark 5.2/(2)). We shall refer to the partial isometry $\delta(\varphi, \psi) = v^*$ as the *relative phase* between the representatives $\varphi \in S(\omega)$ and $\psi \in S(\sigma)$. According to the definition, we find that the φ-relative representative ψ' of σ (cf. the definition of this term in Remark 5.2/(2)) is obtained from ψ by means of the formula $\psi' = \delta(\varphi,\psi)\psi$. This shows how the algebraic scheme we discussed in Section 0 can be accomplished in the case of a (standard form) vN-algebra.

Let $w, w' \in M'$ be partial isometries with $w^* w = p'(\varphi)$ and $w'^* w' = p'(\psi)$, respectively. We define $\varphi' = w\varphi$ and $\psi' = w'\psi$. Because of $w^* w = p'(\varphi)$ the operator w is the uniquely determined partial isometry r in M' with $r^* r = p'(\varphi)$ and $\varphi' = r\varphi$. Hence, by Theorem 5.1/(iii) from $v^*\psi \parallel \varphi$ we obtain $wv^*\psi \parallel \varphi'$. Since $\psi = w'^*\psi'$ holds, we get $wv^*w'^*\psi' \parallel \varphi'$. Because of $w'^* w' = p'(\psi)$, $w' w'^* = p'(\psi')$, $vv^* = p'(\psi)$, $v^* v = p'(\varphi)$ and $w^* w = p'(\varphi)$, $ww^* = p'(\varphi')$, $wv^*w'^*$ is a partial isometry of M' with initial projection $p'(\psi')$ and final projection $p'(\varphi')$. Since $wv^*w'^*\psi' \in S(\sigma)$, by Theorem 5.1/(iv) and Proposition 6.7 we have that $g \in M'_*$ defined by $g(\cdot) = \langle (\cdot) wv^*w'^*\psi', \varphi' \rangle$ is positive and $s(g) = p'(\varphi')$. Let $h' \in M'_*$ be given by $h'(\cdot) = \langle (\cdot)\psi', \varphi' \rangle$. Let $h' = R_{v'} | h' |$ be the polar decomposition of h'. Then, by Proposition 6.12 we have $v'^*\psi' \parallel \varphi'$. From Theorem 5.1/(iv) we follow that $v'^*\psi' = wv^*w'^*\psi'$. Hence $g = | h' |$, i.e. $v'^* v' = s(| h' |) = s(g) = p'(\varphi')$. On the other hand, since $\{\omega, \sigma\}$ is $\ll$-minimal, by the criterion from the beginning of this section we also know that $s(| h'^* |) = v' v'^* = p'(\psi')$. Therefore, both the partial isometries v'^* and $wv^*w'^*$ have the same initial projection $p'(\psi')$ and fulfil $v'^*\psi' = wv^*w'^*\psi'$. From this $v'^* = wv^*w'^*$ has to be followed. Summarizing we get the following

Lemma 6.13: *Let $\{\omega, \sigma\}$ be $\ll$-minimal, and assume $\varphi, \varphi' \in S(\omega)$ and $\psi, \psi' \in S(\sigma)$. Let $w, w' \in M'$ be partial isometries with $w^* w = p'(\varphi)$, $w'^* w' = p'(\psi)$ and $\varphi' = w\varphi$, $\psi' = w'\psi$. The relative phases between φ, ψ and φ', ψ', respectively, transform into each other by the law*

$$\delta(\varphi', \psi') = \delta(w\varphi, w'\psi) = w\delta(\varphi, \psi) w'^*. \tag{6.3}$$

We want to comment once more on the relative geometry of $\ll$-minimal pairs. From Definiton 5.0 and Theorem 5.1/(i) we know that for a $\ll$-minimal pair $\{\omega, \sigma\}$ the relation $s(\omega) \sim s(\sigma)$ holds. We will give an example showing that the relations $s(\sigma) < s(\omega)$ (i.e. $s(\sigma) \leq s(\omega)$ and $s(\sigma) \neq s(\omega)$) and $s(\omega) \sim s(\sigma)$ can occure for a $\ll$-minimal pair. This then proves that even in case of $s(\sigma) \leq s(\omega)$ (cf. the corresponding parts of the assertion of Lemma 6.1) equality $s(\sigma) = s(\omega)$ is not necessary in order to assure that $\{\omega, \sigma\}$ be $\ll$-minimal (but compare this to the special situation described in Example 6.2/(2)). On the other hand, by Theorem 5.3 one can easily provide examples of pairs $\{\omega, \sigma\}$ with $s(\sigma) < s(\omega)$ and $s(\omega) \sim s(\sigma)$ fulfilled, which are not $\ll$-minimal (note that this requires M to be an infinite vN-algebra). In fact, suppose H to be separable, and M to be infinite. Then, there are orthoprojections p, q such that $q < p$ and $q \sim p$, and $\sigma, \mu \in M_{*+}$ with $s(\sigma) = q$, $s(\mu) = p - q$. We define $\omega = \sigma + \mu$. By Theorem 5.3/(1) we have $\{\sigma, \sigma\} \ll \{\omega, \sigma\}$, with $\omega \neq \sigma$. Hence, $\{\omega, \sigma\}$ is not $\ll$-minimal.

Example 6.14 : Assume $M \simeq B(H)$, with separable, infinite dimensional H, and be $\{\omega_k\}$ a maximal family of mutually orthogonal, pure normal states on M. Let p be a minimal projection in M such that $\omega_k(p) \neq 0$, for any k. Let $\{\varepsilon_k\}$ be an arbitrarily chosen sequence of strictly positive reals converging to zero. With $\beta = \sum_{k=1}^{\infty} \varepsilon_k \omega_k(p)$, let us define $\lambda_k = \varepsilon_k \beta^{-1} \omega_k(p)$. Then, $\omega = \sum_{k=1}^{\infty} \lambda_k \omega_k$ is a faithful normal state. Moreover, $\{\omega, \sigma\}$ is $\ll$-minimal for any normal state σ with $s(\sigma) = p^{\perp}$.

Proof: Let σ be a normal state with $s(\sigma)=p^{\perp}$. Let μ be the uniquely determined pure normal state with $s(\mu)=p$. Since $s(\omega)=e$, we have $s(\sigma)\leq s(\omega)$. According to Lemma 6.1 we have $\sigma=\sigma_0$ provided $\{\omega_0,\sigma_0\}$ is the «-minimal pair which is uniquely associated with $\{\omega,\sigma\}$. Let $\omega_1=\omega-\omega_0$. By Theorem 5.3/(3) we have $\omega_1\perp\sigma$. Hence, $s(\omega_1)$ is zero (and then $\omega=\omega_0$) or $s(\omega_1)=p$. In the latter case we had $\omega_1=\lambda\mu$ for some positive λ, and $\omega_0=\omega-\lambda\mu$. Let p_k be the support of ω_k. Then, we have $\omega_0(p_k)=\omega(p_k)-\lambda\mu(p_k)=\lambda_k-\lambda\mu(pp_kp)$. Since p, p_k are minimal projections, we have $pp_kp=t(pp_kp)p$ and $p_kpp_k=t(p_kpp_k)p_k$, where t is the canonical trace on M. Therefore, $\mu(p_k)=\mu(pp_kp)=t(pp_kp)=t(p_kpp_k)=\omega_k(p_kpp_k)=\omega_k(p)$. Hence $\omega_0(p_k)=\lambda_k-\lambda\omega_k(p)=\lambda_k-\lambda\lambda_k\beta\ \varepsilon_k^{-1}=\lambda_k(1-\lambda\beta\,\varepsilon_k^{-1})$. Since $\varepsilon_k\longrightarrow 0$ and ω_0 has to be positive, $\lambda_k(1-\lambda\beta\,\varepsilon_k^{-1})\geq 0$ has to be valid for any k. The latter implies $\lambda=0$. This is in contradiction with the suppositions of the case in question. Hence, only the case $s(\omega_1)=0$ is possible, i.e. $\omega=\omega_0$ ∎

As mentioned in the introduction, the structure of the skew form $I(\omega,\sigma)$ reflects some aspects of the non-commutativity in the pair $\{\omega,\sigma\}$. According to Definition 5.0 and Theorem 5.1 this is evident at least in case of a «-minimal pair. In this case one has the feeling that the skew phase $u(\omega,\sigma)$ provides a quantity which estimates how far from mutually "commuting" the components of $\{\omega,\sigma\}$ are. We have to explain first what *commutativity* among positive linear forms should be. We will say that the positive linear form ω *commutes* with another positive linear form σ if $I(\omega,\sigma)$ is symmetric, i.e. $I(\omega,\sigma)=I(\sigma,\omega)$. In terms of the geometry of the representatives we have the following

Theorem 6.15: *Suppose $\{\omega,\sigma\}$ is a «-minimal pair of positive normal linear forms on M. ω commutes with σ if, and only if, for any $\varphi\in S(\omega)$ the set*

$$S(\sigma)\cap\overline{M_+\varphi}\cap\overline{M'_+\varphi} \tag{6.4}$$

is non-void.

Proof: Assume the set of (6.4) is non-void. Let $\psi\in S(\sigma)$ with $\psi\in\overline{M_+\varphi}\cap\overline{M'_+\varphi}$. Then, $p'(\psi)\leq p'(\varphi)$ and $p(\psi)\leq p(\varphi)$ follows. We define $h(\cdot)=\langle(\cdot)\psi,\varphi\rangle$ on M'. Because $\psi\in\overline{M_+\varphi}$ holds, by means of Appendix 2 we see that $h\geq 0$. This together with $p(\psi)\leq p(\varphi)$ yields $s(h)=p'(\varphi)\leq p'(\psi)$ (cf. Appendix 3). From $p'(\psi)\leq p'(\varphi)$ then follows $s(h)=p'(\varphi)=p'(\psi)$. Applying Proposition 6.7/(a) in this situation shows that $\psi\parallel\varphi$. From Theorem 5.1/(ii) we see that on M we have $I(\omega,\sigma)=\langle(\cdot)\psi,\varphi\rangle$. Because of $\psi\in\overline{M'_+\varphi}$ and according to Appendix 2 (in application to M') we see that $I(\omega,\sigma)\geq 0$, hence $I(\omega,\sigma)$ is *hermitian*. From Lemma 4.1/(ii) we then conclude to $I(\omega,\sigma)=I(\sigma,\omega)$, i.e. ω commutes with σ. Suppose ω commutes with σ. From Lemma 4.1/(ii) we then conclude that $I(\omega,\sigma)$ is hermitian. Moreover, by Proposition 6.5 even $I(\omega,\sigma)\geq 0$ can be followed. Let $\varphi\in S(\omega)$ be given. Since $\{\omega,\sigma\}$ is «-minimal, according to Theorem 5.1/(iv) there is a unique $\psi\in S(\sigma)$ such that $\psi\parallel\varphi$. Hence, by Theorem 5.1/(ii) we have $I(\omega,\sigma)=\langle(\cdot)\psi,\varphi\rangle$. Note also that $I(\omega,\sigma)\geq 0$ implies that $u(\omega,\sigma)$ has to be an orthoprojection: $u(\omega,\sigma)=p$. By Definition 5.0, (5.1), from $\psi\parallel\varphi$ we then have to follow that $p=p(\psi)=p(\varphi)$ holds. By Definition 5.0, (5.2), $\psi\in\overline{M'_+\varphi}$ and $\psi\in\overline{M_+\varphi}$ follows ∎

Note also that as a consequence of the proof we have that $\#\left\{S(\sigma)\cap\overline{M_+\varphi}\cap\overline{M'_+\varphi}\right\}=1$ for any $\varphi\in S(\omega)$ in case of a commuting «-minimal pair $\{\omega,\sigma\}$.

We are going to give an example illustrating the notion of commutativity.

Example 6.16: Let $\tau \in M_{*+}$ be tracial. Suppose $a, b \in M_+$ with $s(a) = s(b)$, and let $\omega, \sigma \in M_{*+}$ be defined as $\omega(\cdot) = \tau(a(\cdot))$ and $\sigma(\cdot) = \tau(b(\cdot))$, respectively. Then, ω commutes with σ if and only if $(ab - ba)s(\tau) = 0$.

Proof: Since $s(\tau) \in M \cap M'$ holds, without loss of generality we may contend with supposing that $s(\tau) \geq s(a) = s(b)$ is fulfilled. Then, $s(\omega) = s(\sigma) = s(a) = s(b) = p$ is valid. By Lemma 6.1 we know that $\{\omega, \sigma\}$ is a $\ll$-minimal pair. For $\varepsilon > 0$ let us define $a_\varepsilon = a + \varepsilon p$ and $\omega_\varepsilon(\cdot) = \tau(a_\varepsilon(\cdot))$. One easily sees that

$$\sigma = \omega_\varepsilon^{c_\varepsilon}, \text{ with } c_\varepsilon \text{ defined by } c_\varepsilon = a_\varepsilon^{-1/2}(a_\varepsilon^{1/2} b\, a_\varepsilon^{1/2})^{1/2} a_\varepsilon^{-1/2},$$

where the inverse a_ε^{-1} is taken with respect to pMp. By Example 6.2 we have

$$I(\omega_\varepsilon, \sigma)(\cdot) = \omega_\varepsilon(c_\varepsilon(\cdot)) = \tau(a_\varepsilon^{1/2}(a_\varepsilon^{1/2} b\, a_\varepsilon^{1/2})^{1/2} a_\varepsilon^{-1/2}(\cdot)).$$

Since $\{\omega_\varepsilon, \sigma\}$ is $\ll$-minimal (cf. Lema 6.1), Theorem 4.6 can be applied with the result that

$$\begin{aligned} I(\omega, \sigma)(a(\cdot)) &= \lim_\varepsilon I(\omega_\varepsilon, \sigma)(a_\varepsilon(\cdot)) \\ &= \lim_\varepsilon \tau(a_\varepsilon^{1/2}(a_\varepsilon^{1/2} b\, a_\varepsilon^{1/2})^{1/2} a_\varepsilon^{1/2}(\cdot)) \\ &= \tau(a^{1/2}(a^{1/2} b\, a^{1/2})^{1/2} a^{1/2}(\cdot)). \end{aligned}$$

Hence, $I(\omega, \sigma)(a(\cdot))$ is a positive linear form.

Suppose now that ω commutes with σ. Then, following Lemma 4.1/(ii) and Proposition 6.5, $I(\omega, \sigma)$ has to be positive. The positivity of the form $I(\omega, \sigma)(a(\cdot))$ then implies that we have $I(\omega, \sigma)((\cdot)a) = I(\omega, \sigma)(a(\cdot))$. Hence also $I(\omega, \sigma)(a(\cdot)) = I(\omega, \sigma)(a^{1/2}(\cdot)a^{1/2})$. Note that, due to $u(\omega, \sigma) = p = s(\omega) = s(\sigma) = s(a) = s(b)$ (cf. Definition 5.0 and Theorem 5.1), and since $I(\omega, \sigma)$ and τ are normal, we have for any $x \in M$

$$\begin{aligned} I(\omega, \sigma)(x) &= I(\omega, \sigma)(s(a)x\,s(a)) \\ &= \lim_\varepsilon I(\omega, \sigma)(a^{1/2} a_\varepsilon^{-1/2} x\, a_\varepsilon^{-1/2} a^{1/2}) \\ &= \lim_\varepsilon \tau\left(a_\varepsilon^{-1/2} a^{1/2}(a^{1/2} b\, a^{1/2})^{1/2} a^{1/2} a_\varepsilon^{-1/2} x\right) \\ &= \tau((a^{1/2} b\, a^{1/2})^{1/2} x), \end{aligned}$$

where we used that $0 \leq a^{1/2} a_\varepsilon^{-1/2} \leq s(a)$ and st-$\lim_\varepsilon a^{1/2} a_\varepsilon^{-1/2} = s(a)$ holds. Therefore we obtain

$$I(\omega, \sigma)(\cdot) = \tau((a^{1/2} b\, a^{1/2})^{1/2}(\cdot)). \tag{6.5}$$

As mentioned above $I(\omega, \sigma)(a(\cdot)) = I(\omega, \sigma)((\cdot)a)$. Now, according to (6.5) this implies that $a(a^{1/2} b\, a^{1/2})^{1/2} = (a^{1/2} b\, a^{1/2})^{1/2} a$. Hence, also $a(a^{1/2} b\, a^{1/2}) = (a^{1/2} b\, a^{1/2})a$ is true. Thus, we arrived at the relation $a^{1/2}(ab - ba)a^{1/2} = 0$. Since $0 = a_\varepsilon^{-1/2} a^{1/2}(ab - ba)a^{1/2} a_\varepsilon^{-1/2}$ holds, in taking the limit $\varepsilon \longrightarrow 0$ and respecting st-$\lim_\varepsilon a^{1/2} a_\varepsilon^{-1/2} = s(a) =$ st-$\lim_\varepsilon a_\varepsilon^{-1/2} a^{1/2}$ we finally get $ab - ba = 0$. On the other hand, if we suppose that $ab - ba = 0$ holds, for c_ε we find $c_\varepsilon = b^{1/2} a_\varepsilon^{-1/2}$. From this then $I(\omega_\varepsilon, \sigma)(\cdot) = \omega_\varepsilon(c_\varepsilon(\cdot)) = \tau(b^{1/2} a_\varepsilon^{1/2}(\cdot))$ follows. Hence, in taking the limit and argueing by Theorem 4.6 we see $I(\omega, \sigma)(\cdot) = \tau(b^{1/2} a^{1/2}(\cdot))$. Since $b^{1/2} a^{1/2} \geq 0$ and τ is *tracial*, $I(\omega, \sigma) \geq 0$ has to be fulfilled. But then $I(\omega, \sigma) = I(\sigma, \omega)$, i.e. ω commutes with σ ∎

7. Some auxiliary results

In this section we shall show how we have to deal with the general C^*-algebraic case. Suppose A is a unital C^*-algebra, and $\omega, \sigma \in A^*$. Let $\{\pi, H_\Omega, \Omega\}$ be the $(\omega+\sigma)$-*GNS*-representation of A. Then, $\{\pi, H_\Omega\}$ is ω,σ-admissible. Let p be the orthoprojection of $\pi(A)''$ with $pH_\Omega = [\pi(A)'\Omega]$, and define $M = p\pi(A)''p$, $H = [\pi(A)'\Omega]$. Then, M is a *vN*-algebra with a cyclic and separating vector on the Hilbert space H. Let $\varphi \in S(\pi,\omega)$ and $\psi \in S(\pi,\sigma)$, and suppose ω_π, σ_π to be defined over M by

$$\omega_\pi(\cdot) = \langle (\cdot)\varphi, \varphi\rangle, \quad \sigma_\pi(\cdot) = \langle (\cdot)\psi, \psi\rangle,$$

respectively. By Lemma 1.1/(i) and in using Appendix 7 we find partial isometries $v, w \in M' = \pi(A)'p$ such that $v^*v = p'(\varphi)p$ and $w^*w = p'(\psi)p$ and

$$P_M(\omega_\pi, \sigma_\pi)^{1/2} = \langle w\psi, v\varphi\rangle = \sup\left\{|\langle z\psi, \varphi\rangle| : z \in M', \|z\| \le 1\right\}.$$

Now, obviously we have $v\varphi \in S(\omega_\pi)$ and $w\psi \in S(\sigma_\pi)$ fulfilled. In applying Lemma 1.1/(i) and Appendix 7 once more again we realize that we are allowed to suppose that $v', w' \in \pi(A)'$ exist with $v'^*v' = p'(v\varphi)$ and $w'^*w' = p'(w\psi)$ and

$$\begin{aligned} P_A(\omega,\sigma)^{1/2} &= \langle w'w\psi, v'v\varphi\rangle = \sup\left\{|\langle zw\psi, v\varphi\rangle| : z \in \pi(A)', \|z\| \le 1\right\} \\ &\ge |\langle w\psi, v\varphi\rangle| = \langle w\psi, v\varphi\rangle = P_M(\omega_\pi, \sigma_\pi)^{1/2}, \end{aligned}$$

i.e. $P_A(\omega,\sigma)^{1/2} \ge P_M(\omega_\pi,\sigma_\pi)^{1/2}$ holds. On the other hand, $m = v^*v'^*w'w \in \pi(A)'p = M'$, with $\|m\| \le 1$, i.e.

$$P_A(\omega,\sigma)^{1/2} = \langle w'w\psi, v'v\varphi\rangle = \langle m\psi, \varphi\rangle \le P_M(\omega_\pi,\sigma_\pi)^{1/2}$$

by Lemma 1.1/(i). Therefore we conclude that equality has to occur:

$$P_A(\omega,\sigma) = P_M(\omega_\pi,\sigma_\pi). \tag{7.1}$$

We might continue our conclusion in this way (making use of Remarks 2.2, 3.2, 4.5 and (7.1)) to see in addition that

$$I(\omega_\pi,\sigma_\pi)\,p\,\pi(\cdot)p = I(\omega,\sigma), \tag{7.2}$$

$$\{\omega,\sigma\} \text{ is } \ll\text{-minimal over } A \text{ if and only if } \{\omega_\pi,\sigma_\pi\} \text{ is } \ll\text{-minimal over } M. \tag{7.3}$$

Hence, many of the results of the Sections 4 - 6 which have been derived in the special case of a *vN*-algebra with cyclic and separating vector persist to be valid (with obvious modifications) in the general case of a unital C^*-algebra and arbitrary pairs of positive linear forms and «-minimal pairs of them, respectively. We omit detailed formulations of these results which are heavily based on (7.1) - (7.3).

APPENDIXES

In this section we collect and prove some auxiliary technical facts and results we have need for throughout this paper. Suppose f is a positive normal linear form over some *vN*-algebra on H. Then, the support of f is named by $s(f)$. Suppose now M is a

vN-algebra over some Hilbert space H. Let M' denote the commutant of M. For a vector $\psi \in H$, by $p(\psi)$ and $p'(\psi)$ the orthoprojections onto the closed linear subspaces

$$\overline{[M'\psi]} = \overline{\{x\psi : x \in M'\}}\ ,\quad \overline{[M\psi]} = \overline{\{x\psi : x \in M\}}$$

of H will be meant, respectively. Remind that $p(\psi) = s(f)$ and $p'(\psi) = s(g)$, with $f \in M_{*+}$ and $g \in M'_{*+}$ given by $f(x) = \langle x\psi, \psi\rangle$, for any $x \in M$, and $g(y) = \langle y\psi, \psi\rangle$, for any $y \in M'$.

Suppose now f is a linear form on a vN-algebra M over H, and $a \in M$. Let $R_a f$ be the linear form $R_a f(\cdot) = f((\cdot)a)$ over M. In case that f belongs to M_* we have the polar decomposition theorem for f. We remind that by this theorem two assertions are established. Firstly, there are a partial isometry $u \in M$ and a positive normal linear form g over M such that $f = R_u g$ and, secondly, both u and $g \geq 0$ are uniquely determined by the condition $u^*u = s(g)$. In the latter situation $g = |f|$ is called *modulus* of f, and $\|f\|_1 = \||f|\|_1$ holds. The decomposition $f = R_u|f|$ is referred to as the polar decomposition of f. Note that in this case also $uu^* = s(|f^*|)$ is valid, where the adjoint f^* of f is defined by $f^*(x) = \overline{f(x^*)}$ for all $x \in M$ (in this case the bar indicates the complex conjugation of a complex number). For further details and generalities on vN- and C^*-algebras we shall make use of throughout we want to refer to [20], [23] (cf. also [11]).

Appendix 1: *Let M be a vN-algebra over H. Suppose $f \in M_{*+}$, and assume $f(x) = \|f\|_1$ holds for some $x \in M$, $\|x\| \leq 1$. Then, $x = s(f) + m$ for some $m \in s(f)^{\perp} M s(f)^{\perp}$.*

Proof: Let $y = s(f)xs(f)$. By our assumptions $f(y) = f(x) = \|f\|_1 = f(e) = f(y^*) = f(x^*)$ for the positive linear form f, where e means the identity operator over H. Hence, we have the following estimates :

$$\|f\|_1^2 = |f(ex)|^2 \leq f(x^*x)\,\|f\|_1\,,\qquad \|f\|_1^2 = |f(ey)|^2 \leq f(y^*y)\,\|f\|_1$$

$$\|f\|_1^2 = |f(ex^*)|^2 \leq f(xx^*)\,\|f\|_1\,,\qquad \|f\|_1^2 = |f(ey^*)|^2 \leq f(yy^*)\,\|f\|_1\,.$$

Since both x and y are in the unit ball of M, we follow that $f(s(f)x^*xs(f)) = \|f\|_1 = f(s(f)xx^*s(f))$ and $f(y^*y) = \|f\|_1 = f(yy^*)$ hold, from which we conclude that

$$s(f)x^*xs(f) = y^*y = s(f) = yy^* = s(f)xx^*s(f).$$

According to the definition of y the latter also shows that $s(f)x^*s(f)^{\perp}xs(f) = 0 = s(f)xs(f)^{\perp}x^*s(f)$. Hence $s(f)^{\perp}xs(f) = 0 = s(f)xs(f)^{\perp}$, which also means that $xs(f) = s(f)xs(f) = s(f)x = y$. Let $x = a + ib$, with selfadjoint operators a, b ("i" means the imaginary unit of the complex numbers $\mathbb{C}$). Since both x and x^* commute with $s(f)$, and since we have $a = \frac{1}{2}(x + x^*)$ and $b = \frac{1}{2i}(x - x^*)$, both a and b commute with $s(f)$, too. Hence $s(f) = \frac{1}{2}s(f)(x^*x + xx^*) = s(f)(a^2 + b^2) = (as(f))^2 + (bs(f))^2$, where we used that $x^*x + xx^* = 2(a^2 + b^2)$ holds. The elements $as(f)$ and $bs(f)$ are hermitian with $f(as(f)) = \frac{1}{2}(f(x) + f(x^*)) = \|f\|_1 = f(e) = f(a)$. Since a is in the unit ball, we get the estimation $f(e)^2 = f(a)^2 \leq f(a^2)\,f(e) \leq f(e)^2$, i.e. $f(a^2) = f((as(f))^2) = f(e)$ follows. Since $(as(f))^2$ is positive, we have $(as(f))^2 = s(f)$, and $s(f) = (as(f))^2 + (bs(f))^2$ then implies $bs(f) = 0$. Thus, the partial isometry y has the form $y = xs(f) = as(f)$. Especially, and according to the above mentioned, the latter means that y is hermitian with $y^*y = y^2 = s(f)$ and $f(y) = f(as(f)) = f(e) = f(s(f))$. But then, $s(f) - y \geq 0$ and

$f(s(f)-y)=0$ have to be fulfilled. Since f is positive and $s(f)-y \in s(f)Ms(f)$ we get $s(f)=y$. This means $s(f)=xs(f)=s(f)x$. We finally get the desired decomposition of x as $x=xs(f)+xs(f)^{\perp}=s(f)+m$, with $m=xs(f)^{\perp}=s(f)^{\perp}x \in s(f)^{\perp}Ms(f)^{\perp}$ ∎

Appendix 2: *Let M be a vN-algebra over the Hilbert space H. Let ψ, φ be vectors taken from H. Suppose a functional h is defined over M' by $h(\cdot)=\langle(\cdot)\varphi,\psi\rangle$. The following conditions are mutually equivalent:*

(i) *h is a positive linear form.*

(ii) *there exists a densely defined, positive, selfadjoint linear operator F which is affiliated with M and such that $p(\varphi)\psi=F\varphi$.*

(iii) $p(\varphi)\psi \in \overline{M_+\varphi}$.

Moreover, in case that $p(\varphi)=p(\psi)$ the F in (ii) can be chosen to be invertible.

Proof: We are going to show the net of implications (ii)⟶(iii)⟶(i)⟶(ii). Assume we have given F as described in (ii). Suppose $F=\int_0^{\infty}\lambda\, e(d\lambda)$, where $\{e(\lambda)\}\subset M$ is the F corresponding resolution of the identity. We define $F_n=\int_0^{n}\lambda\, e(d\lambda)$. Then, $F_n \in M_+$ and $p(\varphi)\psi=F\varphi=\lim_n F_n\varphi$, i.e. (iii) follows. Suppose (iii) to be valid. Let $\{a_n\}\subset M_+$ be a sequence of non-negative elements of M such that $p(\varphi)\psi=\lim_n a_n\varphi$. For any $y \in M'$ our conclusion is

$$h(y^*y)=\langle y\varphi,y\psi\rangle=\langle p(\varphi)y\varphi,y\psi\rangle=\langle p(\varphi)y\varphi,y\psi\rangle=\langle y\varphi,yp(\varphi)\psi\rangle$$
$$=\lim_n\langle y\varphi,ya_n\varphi\rangle=\lim_n\langle ya_n^{1/2}\varphi,ya_n^{1/2}\varphi\rangle=\lim_n\|ya_n^{1/2}\varphi\|^2\geq 0\,.$$

Hence (i) is seen to be true. Suppose (i) to hold, i.e. h is a positive normal linear form over the vN-algebra M'. By definition of h we see $h(p'(\varphi))=h(p'(\psi))=h(s(h))=h(e)=\|h\|_1$. Hence $p'(\varphi)\geq s(h)$ and $p'(\psi)\geq s(h)$. Let the orthoprojection z be defined as $z=p'(\psi)-s(h)$. Then, for any $x,y \in M'$ we have

$$h(zx^*y)=h(s(h)(zx^*y))=0=\langle y\varphi,xz\psi\rangle.$$

From this we have to conclude that $\overline{[M'z\psi]}\subset p(\varphi)H$. This says that $p(z\psi)\leq p(\varphi)$. We define a linear subspace D of H by $D=[M'\varphi]\oplus p(\varphi)H$. Since $p(\varphi)$ projects onto $[M'\varphi]$ we see that D is dense in H. Assume $\delta \in p(\varphi)^{\perp}H$ and $x \in M'$. Then, $x\varphi+\delta=0$ if, and only if, $x\varphi=0$ and $\delta=0$. Note that $x\varphi=0$ is equivalent with $xp'(\varphi)=0$. Due to $p'(\varphi)\geq s(h)$ also $xs(h)=0$ has to be valid. According to this we see that $p(\varphi)x\psi=p(\varphi)xp'(\psi)\psi=p(\varphi)xs(h)\psi+p(\varphi)xz\psi=p(\varphi)xz\psi$. Since $xz\psi \in p(\varphi)^{\perp}H$, $p(\varphi)x\psi=0$ can be followed. To summarize, what we have shown is that $x\varphi=0$ always implies $p(\varphi)x\psi=0$. This proves that F_0 given by

$$F_0 : D \ni y\varphi+\delta=\vartheta \longmapsto F_0\vartheta=p(\varphi)y\psi+\delta \in H$$

for any $y \in M'$ and any $\delta \in p(\varphi)^{\perp}H$ yields a well-defined linear operator acting from the dense domain D into H. Let $u \in U(M')$ be an element of the unitary group of M'. Then, $uD\subset D$, and for $\vartheta=y\varphi+\delta$ with $y \in M'$ and $\delta \in p(\varphi)^{\perp}H$ we see that

$$F_0u\vartheta=F_0(uy\varphi+p(\varphi)^{\perp}u\delta)=p(\varphi)uy\varphi+p(\varphi)^{\perp}u\delta=u(p(\varphi)y\varphi+\delta)=uF_0\vartheta.$$

Hence $F_0\subset u^*F_0u$, for any $u \in U(M')$. Moreover, if $\vartheta=y\varphi+\delta$ with $y \in M'$ and $\delta \in p(\varphi)^{\perp}H$, we see that

$$\langle F_0\vartheta,\vartheta\rangle = \langle p(\varphi)y\psi+\delta, y\varphi+\delta\rangle = \langle p(\varphi)y\psi, y\varphi\rangle + \langle\delta,\delta\rangle$$
$$= \langle y\psi, p(\varphi)y\varphi\rangle + \|\delta\|^2 = \langle y\psi, yp(\varphi)\varphi\rangle + \|\delta\|^2 = h(y^*y) + \|\delta\|^2 \geq 0,$$

due to the supposed positivity of h. To summarize, we have proved that F_0 is positive, and thus also symmetric, on the dense domain D, with $F_0 = u^*F_0 u_{|D}$, for any $u \in U(M')$. Two conclusions can be drawn from these facts. Firstly, the Friedrich's extension F of F_0 exists as a densely defined (say on the domain of definition D_F), positive, selfadjoint linear operator over H. Secondly, due to the uniqueness of this extension, from $F_0 = u^*F_0 u_{|D}$, for any $u \in U(M')$, in fact $F = u^*Fu$ has to be followed on D_F, for any $u \in U(M')$ (and thus also $uD_F \subset D_F$ has to hold, i.e. D_F is invariant under the action of the unitaries of M'). This means that F is affiliated with M. Moreover, by the definition of F_0 we get $p(\varphi)\psi = F_0\varphi = F\varphi$. This completes the proof of (i). In order to see the validity of the last assertion we remark that $p(\varphi) = p(\psi)$ implies that

$$\text{range}(F) = F(D_F) \supset F_0(D) = p(\varphi)[M'\psi] \oplus p(\varphi)^\perp H = [M'\psi] \oplus p(\psi)^\perp H.$$

Since $H = \overline{[M'\psi] \oplus p(\psi)^\perp H}$, range$(F)$ is dense within H. But then, by standard conclusions, one can be assured that F^{-1} exists. Since the properties of being positive, selfadjoint and affiliated with M are hereditary ones, the assertion follows ∎

Appendix 3 : *Let M be a vN-algebra over a Hilbert space H. Assume $\varphi, \psi \in H$ are vectors with $p(\varphi) \leq p(\psi)$. Suppose a linear form h on M' is given as $h(\cdot) = \langle(\cdot)\psi, \varphi\rangle$. In the case that h is positive we find $s(h) = p'(\varphi) \leq p'(\psi)$, and $p'(\varphi) = p'(\psi)$ follows in the case of $p(\varphi) = p(\psi)$.*

Proof: Since h is positive on M', Appendix 2 applies and gives $p(\psi)\varphi \in \overline{M_+\psi}$. The assumption $p(\varphi) \leq p(\psi)$ implies $p(\psi)\varphi = \varphi$. Thus, $\varphi \in \overline{M_+\psi}$ has to be valid. Hence, as consequence of this we conclude that $p'(\varphi) \leq p'(\psi)$. Since, by definition of h, $s(h) \leq p'(\psi)$ and $s(h) \leq p'(\varphi)$, $s(h) \leq p'(\varphi) \leq p'(\psi)$ follows. In defining $z = p'(\varphi) - s(h)$ one easily sees that $0 = h(zx) = \langle x\psi, z\varphi\rangle$, for all $x \in M'$. The conclusion is that $z\varphi \in p(\psi)^\perp H \subset p(\varphi)^\perp H$. On the other hand, since $z \in M'$ we have $z\varphi \in p(\varphi)H$. This implies $z\varphi = 0$. Since $z \leq p'(\varphi)$ within M', $z\varphi = 0$ yields $z = 0$. This proves $p'(\varphi) = s(h)$, and $s(h) = p'(\varphi) \leq p'(\psi)$ is seen. Note that, as a consequence of the assumption $h \geq 0$, h is hermitian. Therefore also $h(\cdot) = \langle(\cdot)\psi, \varphi\rangle = \langle(\cdot)\varphi, \psi\rangle$. Hence, in case of $p(\varphi) = p(\psi)$, by interchanging the roles of φ and ψ we have to conclude as above that, in addition to the yet proven $s(h) = p'(\varphi) \leq p'(\psi)$, also $s(h) = p'(\psi) \leq p'(\varphi)$ happens to be true. Thus, $s(h) = p'(\psi) = p'(\varphi)$ holds in case of $h \geq 0$ and $p(\varphi) = p(\psi)$ ∎

Appendix 4 : *Let M be a vN-algebra over the Hilbert space H, and suppose $\varphi, \psi \in H$ with $p'(\varphi) = p'(\psi)$. Let $f = R_u|f|$ be the polar decomposition of the linear form f over M given by $f(\cdot) = \langle(\cdot)\psi, \varphi\rangle$. Then, $p'(u^*\psi) = p'(\psi)$, $uu^* = s(|f^*|) = p(\psi)$, and $u^*u = s(|f|) = p(\varphi)$ are fulfilled.*

Proof: We have $f^*(\cdot) = \langle(\cdot)\varphi, \psi\rangle$. Let $f^* = R_v|f^*|$ be the polar decomposition of f^*. Then, $|f^*|(\cdot) = \langle(\cdot)v^*\varphi, \psi\rangle$, and $s(|f^*|) \leq p(\psi)$ is clear. Due to $|f|(\cdot) = \langle(\cdot)u^*\psi, \varphi\rangle$, and since $uu^* = s(|f^*|)$, we see that

$$\langle x\psi, \varphi\rangle = f(x) = R_u|f|(x) = \langle xuu^*\psi, \varphi\rangle = \langle xs(|f^*|)\psi, \varphi\rangle,$$

for any $x \in M$. The latter means that $\langle s(|f^*|)^\perp\psi, y\varphi\rangle = 0$ for any $y \in M$. Hence, we have $s(|f^*|)^\perp\psi \in p'(\varphi)^\perp H$. On the other hand we have

$$p'(\psi)s(|f^*|)^\perp\psi = s(|f^*|)^\perp p'(\psi)\psi = s(|f^*|)^\perp\psi, \text{ i.e. } s(|f^*|)^\perp\psi \in p'(\varphi)H.$$

Taking together these facts yields $s(|f^*|)^\perp\psi = 0$, and we get $\psi = s(|f^*|)\psi$. This shows two facts to hold. Firstly, we must have that $s(|f^*|) \geq p(\psi)$, and, in view to the above mentioned, $s(|f^*|) = p(\psi)$ has to be valid. Secondly, $\psi = s(|f^*|)\psi = uu^*\psi$ implies that $p'(\psi) \leq p'(u^*\psi)$. Since $p'(u^*\psi) \leq p'(\psi)$ holds by triviality, $p'(\psi) = p'(u^*\psi)$ has to be valid. Together with our assumptions we finally see that $p'(\varphi) = p'(\psi) = p'(u^*\psi)$. Let us look now on the normal positive linear form $|f|(\cdot) = \langle(\cdot)u^*\psi, \varphi\rangle$ over the commutant N' of the vN-algebra $N = M'$ over the Hilbert space H. The equation $p'(\varphi) = p'(u^*\psi)$ with respect to M reads now with respect to N as $p(\varphi) = p(u^*\psi)$. Thus, Appendix 3 can be applied with respect to the vN-algebra N. The result is that, with respect to N, $s(|f|) = p'(\varphi) = p'(u^*\psi)$. With respect to the vN-algebra M the latter reads as $s(|f|) = p(\varphi) = p(u^*\psi)$. This proves our assertion ∎

Appendix 5 : *Let M be a vN-algebra over the Hilbert space H. Assume $g \in M_*$, and be $m \in M$, $\|m\| \leq 1$. Suppose $R_{m^*}g$ is positive. Then we have $|g| \geq R_{m^*}g$, and $|g| = R_{m^*}g$ occurs if and only if $\|g\|_1 = \|R_{m^*}g\|_1$.*

Proof: Let $g = R_u|g|$ be the polar decomposition of g, and define f by $f = R_{m^*}g$. Then $f = R_{m^*u}|g|$. f being positive especially means that f is hermitian, hence we can apply a technical fact (cf. [20: 1.24.1]) to the situation given by $f = R_{m^*u}|g|$. The result says that $f \leq \|m^*u\|\,|g|$. Since both m and u are in the unit ball of M, $f \leq |g|$ follows. Supposing that f and g have the same norm gives $f(e) = |g|(e)$, due to $\|g\|_1 = \||g|\|_1$. Hence, $f \leq |g|$ implies $f = |g|$. The other direction is trivial ∎

Appendix 6 : *Let M be a vN-algebra over the Hilbert space H. Assume $g \in M_*$, and let $g = R_u|g|$ be the polar decomposition of g. Suppose $R_{m^*}g = |g|$ for some $m \in M$, $\|m\| \leq 1$. Then we have $ms(|g|) = u$.*

Proof: We define $f = |g|$ and $x = m^*u$. f is positive with $f = R_x f$. Since $\|x\| \leq 1$ and $\|f\|_1 = f(e) = f(x)$, Appendix 1 applies to this and proves that $xs(f) = s(f)xs(f) = s(f)$. Because of $u^*u = s(f)$ we may conclude as follows:

$$x = m^*u = m^*us(f) = s(f) = s(f) = s(f)m^*u.$$

Hence $u^* = s(f)u^* = s(f)m^*s(|g^*|)$, where we used that $uu^* = s(|g^*|)$. This means that $s(f) = u^*u = s(f)m^*s(|g^*|)ms(f) \leq s(f)m^*ms(f) \leq s(f)$, i.e.

$$s(f)m^*s(|g^*|)ms(f) = s(f)m^*ms(f).$$

Hence $s(|g^*|)^\perp ms(f) = 0$. This implies $u = s(|g^*|)ms(f) = ms(f) = ms(|g|)$ ∎

Appendix 7 : *Let M be a vN-algebra over the Hilbert space H. For given vectors $\varphi, \psi \in H$ let β be defined as $\beta = \sup \{ |\langle K\psi, \varphi\rangle| : K \in M', \|K\| \leq 1 \}$. There are vectors $\varphi', \psi' \in H$ such that $\langle x\varphi, \varphi\rangle = \langle x\varphi', \varphi'\rangle$, $\langle x\psi, \psi\rangle = \langle x\psi', \psi'\rangle$, for all $x \in M$, and $\beta = \langle \psi', \varphi'\rangle$.*

Proof: We define a normal linear form h over M' by setting $h(x) = \langle x\psi, \varphi\rangle$, $x \in M'$. Then, $\beta = \|h\|_1$. Suppose $h = R_u|h|$ is the polar decomposition of h. Because of $|h| = R_{u^*}h$ and $\|h\|_1 = \||h|\|_1$ we see that $\beta = \langle u^*\psi, \varphi\rangle$. We are going to construct partial isometries $v, w \in M'$ such that

$$v^*v \geq p'(\varphi), \; w^*w \geq p'(\psi) \text{ and } \langle u^*\psi, \varphi\rangle = \langle v^*w\psi, \varphi\rangle.$$

Then, $\varphi' = v\varphi$ and $\psi' = w\psi$ can be taken to meet our demands. Let z be the central projection of M such that $M'z$ is finite, and $M'z^\perp$ is properly infinite. To be non-trivial, let us assume $0 < z < e$. Since $s(|h|) = u^*u$ and $s(|h^*|) = uu^*$, with $u \in M'$, we have $s(|h|) \sim s(|h^*|)$ in the vN-algebra M'. Hence $zs(|h|) \sim zs(|h^*|)$ within the finite vN-algebra $M'z$. This implies $zs(|h|)^\perp \sim zs(|h^*|)^\perp$ in $M'z$, too. Now, let m be a partial isometry of M' with $m^*m = zs(|h|)^\perp$, $mm^* = zs(|h^*|)^\perp$. Then $v_3 = uz + m$ is unitary in $M'z$, and since $s(|h|)m^* = 0$, we are allowed to conclude that

$$\begin{aligned}\langle v_3^*\psi, \varphi\rangle &= h(v_3^*) = |h|(v_3^*u) = |h|(s(|h|)v_3^*us(|h|)) \\ &= |h|(s(|h|)(zu^* + m^*)us(|h|)) \\ &= |h|(s(|h|)zu^*us(|h|)) = |h|(z) = \langle zu^*\psi, \varphi\rangle.\end{aligned}$$

Thus, what we have proved is the relation $\langle v_3^*\psi, \varphi\rangle = \langle zu^*\psi, \varphi\rangle$. Since $M'z^\perp$ is properly infinite, we can find mutually orthogonal orthoprojections p_1 and p_2 in $M'z^\perp$ such that $p_1 \sim p_2 \sim z^\perp$ and $p_1 + p_2 = z^\perp$. We find partial isometries $v_1, v_2 \in M'z^\perp$ with

$$v_1^*v_1 = (p'(\varphi) - s(|h|))z^\perp, \; v_1v_1^* \leq p_1, \; v_2^*v_2 = p'(\psi)z^\perp, \; v_2v_2^* \leq p_2,$$

where we used that $p'(\varphi) \geq s(|h|)$. Let us define $v = v_1 + v_2u + v_3$ and $w = v_2 + z$. According to the definitions of v_1, v_2, v_3, and since $v_1^*v_2 = 0 = v_2^*v_1$ holds, one easily infers that

$$\begin{aligned}v^*v &= v_1^*v_1 + u^*v_2^*v_2u^* + v_3^*v_3 = (p'(\varphi) - s(|h|))z^\perp + u^*uz^\perp + z \\ &= (p'(\varphi) - s(|h|))z^\perp + s(|h|)z^\perp + z = p'(\varphi)z^\perp + z\end{aligned}$$

and $w^*w = v_2^*v_2 + z = p'(\psi)z^\perp + z$ are fulfilled. Hence, v and w are partial isometries of M' such that $v^*v \geq p'(\varphi)$ and $w^*w \geq p'(\psi)$. Moreover, since also $p'(\psi) \geq s(|h^*|)$ holds, we get that $v^*w = u^*v_2v_2^* + v_3^* = u^*p'(\psi)z^\perp + v_3^* = u^*z^\perp + v_3^*$. Hence

$$\langle v^*w\psi, \varphi\rangle = \langle u^*z^\perp\psi, \varphi\rangle + \langle v_3^*\psi, \varphi\rangle = \langle u^*z^\perp\psi, \varphi\rangle + \langle u^*z\psi, \varphi\rangle = \langle u^*\psi, \varphi\rangle,$$

due to the previously derived relation $\langle v_3^*\psi, \varphi\rangle = \langle u^*z\psi, \varphi\rangle$. This closes the proof in case of M' with non-trivial finite and properly infinite parts existing. Note that the proof has been organized in such a way that makes evident how to deal with the remaining "pure" cases of a finite or properly infinite M'. Hence, we can take our assertion as verified ∎

Appendix 8 : *Let $\{f_n\}$ be a sequence of normal linear forms over a vN-algebra M such that $f_n \longrightarrow f$ in norm. Then also $|f_n| \longrightarrow |f|$ in norm.*

For a **proof** the reader is referred to [23 : III.4.10], e.g. ∎

REFERENCES

[1] AHARONOV,Y., and J. ANANDAN: *Phase change during a cyclic quantum evolution.* Phys. Rev. Lett. **58** (1987),1593 - 1596.

[2] ALBERTI, P. M.: *A note on the transition probability over C^*-algebras.* Lett. Math. Phys. **7** (1983), 25 - 32.

[3] ALBERTI, P. M., and A. UHLMANN: *Stochastic linear maps and transition probability.* Lett. Math. Phys. **7** (1983),107 - 112.

[4] ALBERTI, P.M., and A. UHLMANN: *Transition probabilities on C^*- and W^*-algebras.* In: Proc. Int. Conf. Oper. Algebras, Ideals, and Appl. Theor. Phys. (eds.: H. Baumgärtel et al). Teubner-Texte zur Mathematik: Vol. 67. Leipzig: B. G.Teubner Verlagsges. 1984, pp. 5 - 11.

[5] ALBERTI, P. M.: *On the geometry of pairs of positive linear forms (I).* Wiss. Z. Univ. Leipz., Math.-Naturwiss. Reihe **39** (1990), 579 - 597.

[6] ARAKI, H.: *Bures distance function and a generalization of Sakai's non-commutative Radon-Nikodym theorem.* Publ. RIMS Kyoto Univ. **8** (1972), 335 - 362.

[7] ARAKI, H., and G. A. RAGGIO: *A remark ontransition probability.* Lett. Math. Phys. **6** (1982), 237 - 240.

[8] BARGMANN, V.: *Note on Wigner's theorem on symmetry operations.* J. Math. Phys. **5** (1964), 862 - 868.

[9] BERRY, M.: *Quantal phase factors accompanying adiabatic changes.* Proc. R. Soc. Lond. **A 392** (1984), 45 - 57.

[10] BERRY, M.: *Quantum adiabatic anholonomy.* In: Anomalies, Phases, Defects. Proc. Ferrara School Theor. Phys. 1989 (eds.: M.Bregola et al). Bibliopolis, Naples, 1990, pp. 1 - 53.

[11] BRATTELI, O., and D.W. ROBINSON: *Operator Algebras and Quantum Statistical Mechanics,* I. Berlin-Heidelberg-NewYork: Springer-Verlag 1979.

[12] BURES, D.: *An extension of Kakutani's theorem on infinite product measures to the tensor product of semifinite W^*-algebras.* Trans. Amer. Math. Soc. **135** (1969), 199 - 212.

[13] CHOI, M.-D.: *Positive Linear Mappings.* Lecture AMS Summer Institut. Kingston, 1980.

[14] DYE, H. A., and B. RUSSO: *A note on unitary operators in C^*-algebras.* Duke Math. J. **33** (1966), 413 - 416.

[15] GUDDER, S.: *Expectation and transition probability.* Intern. J. Theor. Phys. **20** (1981), 383 - 395.

[16] GUDDER, S. and J.-P. MARCHAND: *Noncommutative probability on vN-algebras.* J. Math. Phys. **13** (1972), 799 - 805.

[17] GUDDER, S., MARCHAND, J.-P. and W. WYSS: *Bures distance and relative entropy.* J. Math. Phys. **20** (1979), 1963 - 1966.

[18] JOST, R.: *Quantenmechanik I.* Zürich: Verlag Vereins Math. und Phys. ETH Zürich 1969.

[19] SAKAI, S.: *A Radon-Nikodym theorem in W^*-algebras.* Bull. Amer. Math. Soc. **71** (1965), 149 - 151.

[20] SAKAI, S.: *C^*-Algebras and W^*-Algebras.* Berlin-Heidelberg-New-York: Springer -Verlag 1971.

[21] SEILER, R.: *Die Berry-Phase.* Phys. Bl. **47** (1991), 914 - 916.

[22] SIMON, B.: *Holonomy, the quantum adiabatic theorem and Berry's phase.* Phys. Rev. Lett. **51** (1983), 2167 - 2170.

[23] TAKESAKI, M.: *Theory of Operator Algebras I.* Berlin-Heidelberg-New-York: Springer - Verlag 1979.

[24] UHLMANN, A.: *The "transition probability" in the state space of a *-algebra.* Rep. Math. Phys. **9** (1976), 273 - 279.

[25] UHLMANN, A.: *The transition probability for states of *-algebras.* Ann. Phys. **42** (1985), 524 - 532.

[26] UHLMANN, A.: *Parallel transport and quantum holonomy along density operators.*

Rep. Math. Phys. **24** (1986), 229 - 240.

[27] UHLMANN, A.: *A gauge field governing parallel transport along mixed states.* Lett. Math. Phys. **21** (1991), 229 - 236.

[28] UHLMANN, A.: *Parallel transport and holonomy along density operators.* In: Differential Geometric Methods in Theoretical Physics (eds.: D. D. Doebner and J. D. Henning). Singapore : World Sci. Publ. 1987, pp. 246 - 254.

[29] UHLMANN, A.: *Parallel transport of phases.* In: Differential Geometry, Group Representation, and Quantization. Lecture Notes in Physics: Vol. 379 (eds.: J. D. Henning et al). Berlin-Heidelberg-New-York: Springer-Verlag 1991, pp. 55 - 72.

[30] UHLMANN, A.: *On Berry phases along mixtures of states.* Ann. Phys. (Leipzig) **7/46** (1989), 63 - 69.

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