Zeitschrift für Analysis
und ihre Anwendungen
Vol. 11(1992)2, 221-227

On the Dirichlet and the Riemann-Hilbert Problem on Möbius Strips

K. H. SCHÜFFLER

In this paper we give 1) a description of harmonic functions on Möbius strips as a manifold modelled on a subspace of the (Sobolev) space of holomorphic functions on ring domains, 2) a functional-analytic theory for the Laplacian, e.g., the Dirichlet boundary value problem and 3) the Fredholm theory for the Riemann-Hilbert operator.

Key words: *Dirichlet problem, Riemann-Hilbert harmonic functions and holomorphic functions on Möbius strips*

AMS subject classification: 30 E 25, 30 F 15, 31 B 20, 35 J 15, 56 E 12, 58 E 12, 58 G 10

§0 Introduction

In this paper we study Riemann surfaces $\mathbb{M}$ of topological type of the Möbius strip and discuss the class of harmonic functions defined on $\mathbb{M}$. Although $\mathbb{M}$ is non-orientable we obtain a function-theoretic description of this class which in turn provides us with a functional-analytic approach to the classical boundary value operators including Laplace equation and the Riemann-Hilbert problem.

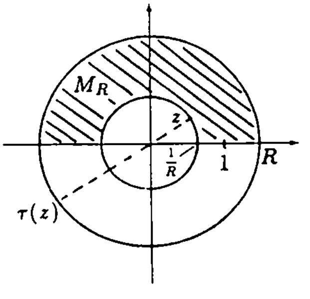
Our ideas are essentially based on the following model:

If $\mathbb{M}$ is a Möbius strip, then it can be identified in a natural way with the half-annulus

$$M_R = \left\{ z \;\middle|\; \frac{1}{R} < |z| < R \,;\; Im(z) \geq 0 \,;\; z = -\frac{1}{\bar{z}} \text{ for } Im(z) = 0 \right\}$$

for a suitable conformal parameter $R > 1$. Consequently $\mathbb{M}$ is just the annulus

$$\mathcal{R} = \left\{ z \;\middle|\; \frac{1}{R} < |z| < R \right\}$$



where points z are identified via the involution

$$\tau : \mathcal{R} \longrightarrow \mathcal{R} \,,\; \tau(z) = -\frac{1}{\bar{z}}$$

which is an anti-conformal transformation of the oriented coverings of $M(\mathcal{R})$ into themselves. Moreover, the functions on $\mathbb{M}$ are just the functions x on $\mathcal{R}$ being invariant under the action of τ, that is

$$\text{(I)} \qquad x(z) = x \circ \tau(z)\,, \; z \in \mathcal{R}$$

and it is easy to check that the harmonic functions x on $\mathcal{R}$ satisfying (I) are in 1–1 correspondence to holomorphic maps f on $\mathcal{R}$ with the additional property

$$f(z) = \overline{f \circ \tau(z)} \text{ or } f(z) = \overline{-f \circ \tau(z)}\,.$$

§1 Harmonic Functions

Since any harmonic map x defined on M_R admits a harmonic extension to the oriented covering $\mathcal{R}$ being invariant under the involution, it is easy to check that x can be written as

$$\text{(II)} \qquad x(z) = Re\, f(z)$$

for a holomorphic map f on $\mathcal{R}$. (Observe that $g = \partial_z x$ is holomorphic in $\mathcal{R}$ and the contour integral $\oint_{S^1} g$ vanishes; the symbol ∂_z stands for the Wirtinger derivative $(\partial_u - \mathbf{i}\partial_v)$, $(z = u + \mathbf{i}v)$. Hence we can define f by $f = \int^z g$, $z \in \mathcal{R}$.)

We now introduce some additional notations:
Let $m \in \mathbb{N}$, $m \geq 2$ *(Sobolev Index)* be given and consider a Möbius strip $\mathbb{M} = M_R$. We abbreviate $\mathbb{E} = \mathbb{R}^n$ or $\mathbb{E} = \mathbb{C}^n$, $D_R = \{z \big| |z| < R\}$, $D = D_1$ the unit disk in $\mathbb{C}$, as before $\mathcal{R} = \left\{z \middle| \frac{1}{R} < |z| < R\right\}$ and we define the following subspaces:

- $H^m(\mathbb{M}, \mathbb{E}) = \{x \in H^{m,2}(\mathcal{R}, \mathbb{E}) | x = x \circ \tau\}$

(this space is – as the kernel of the continous linear map $x \longmapsto x \circ \tau$ – a subspace of the Hilbert space $H^m(\mathcal{R}, \mathbb{E})$)

- $H^m_h(\mathbb{M}, \mathbb{E}) = \{x \in H^m(\mathbb{M}, \mathbb{E}) \mid \Delta x = 0\}$
- $A^m(D_R, \mathbb{C}) = \{f \in H^{m,2}(\mathcal{R}, \mathbb{C}) \mid f$ holomorphic in $D_R\}$; $A^m(\mathcal{R}, \mathbb{C})$ analogously
- $A^m_0(D_R, \mathbb{C}) = \{f \in A^m(D_R, \mathbb{C}) \mid f(0) = 0\}$
- $A^m_\pm = \{f \in A^m(\mathcal{R}, \mathbb{C}) \mid f = \pm\overline{f \circ \tau}\}$
- $A^m_{-*} = \{f \in A^m_- \mid Re \oint_{S^1} \frac{f(z)}{z} = 0\}$.

Applying the representation formula (II) to the Laurent series at zero one easily verifies the following relations:

1. $x \in H_h^m(I\!M, I\!R) \iff f \in A_+^m$
2. $x \in H_h^m(I\!M, I\!R) \iff h(z) := z\partial_z(x(z)) = (u+iv)(\partial_u - i\partial_v)x(u+iv)$ is in the space A_{-*}^{m-1}
3. $f \in A_\pm^m \iff zf'(z) \in A_\mp^{m-1}$.

   The definitions of the spaces immediatly imply

4. $A_\pm^m$ are real sub-Hilbert spaces of $A^m(\mathcal{R}, \mathbb{C})$;
   the codimension of A_{-*}^m in A_-^m is equal to 1;
   $A_+^m \cdot A_+^m = A_+^m$, $A_-^m \cdot A_+^m = A_-^m$ and $A_-^m \cdot A_-^m = A_+^m$.

Theorem 1(description of the harmonic functions): *The following statements are true:*

i) Basic relation: The function x is harmonic in $I\!M$ if and only if

$$x(z) = Re\left\{g(z) + \overline{g \circ \tau(z)}\right\} = Re\left\{g(z) + \overline{g\left(-\frac{1}{\bar z}\right)}\right\}$$

for a holomorphic function g on the disk D_R.

ii) Manifold structure: The transformation

$$\Phi : A_0^m(D, \mathbb{C}) \times I\!R \longrightarrow H_h^m(I\!M, I\!R)\ ,\ \Phi(g,\alpha)(z) = Re\left\{g\left(\frac{1}{R}z\right) + \overline{g \circ \tau(Rz)} + 2\alpha\right\}$$

is a linear homeomorphism.

iii) Variation of the conformal type: The extended transformation

$$\bigcup_{R>1} H_h^m(M_R, I\!R) \xrightarrow{\Phi} A_0^m(D, \mathbb{C}) \times I\!R \times (1,\infty)\ ,\ \Phi(g,\alpha,R) = Re\left\{g\left(\frac{1}{R}z\right) + \overline{g \circ \tau(Rz)} + 2\alpha\right\}$$

is a global chart for the fibre bundle of all harmonic functions of Sobolv class $H^{m,2}$ defined on Möbius strips.

Proof: *The basic relation is a consequence of (II) and Relation 1. comparing coefficients; ii) and iii) are simple consequneces of the basic relation.* ■

Let us remark that the transformations

$$\Phi_\pm : A_0^m(D, \mathbb{C}) \times I\!R \longrightarrow A_\pm^m\ ;\ \Phi_\pm(g;\alpha)(z) = g\left(\frac{1}{R}z\right) + \overline{g \circ \tau(Rz)} + 2\alpha\ \ (\text{ resp. } + 2i\alpha)$$

are linear homeomorphisms giving explicit descriptions of the spaces $A_\pm^m$ also of the fibre bundles – that are the union spaces over all conformal structures $\mathcal{R}$, $R > 1$.

§2 The Laplacian and Dirichlet's problem

The functions x defined on the boundary $\partial\mathbb{M}$ of a Möbius strip $\mathbb{M}$ can be described in two possible ways: First we may consider the class

$$H^{m-\frac{1}{2}}(\partial\mathbb{M},\mathbb{E}) = \left\{x \in H^{m-\frac{1}{2}}(\partial\mathcal{R},\mathbb{E}) \;\middle|\; x = x\circ\tau\right\}$$

consisting of all τ-invariant x defined on the boundary $\partial\mathcal{R} = \partial D_R \cup \partial D_{\frac{1}{R}}$. On the other hand a simple argument shows that one can apply the natural isomorphism

$$H^{m-\frac{1}{2}}(\partial\mathbb{M},\mathbb{E}) \cong H^{m-\frac{1}{2}}(\partial D_R,\mathbb{E}) \cong H^{m-\frac{1}{2}}(S^1,\mathbb{E})\,.$$

Theorem 2(Dirichlet's problem for harmonic functions): *The operator*

$$L : H_h^m(\mathbb{M},\mathbb{R}) \longrightarrow H^{m-\frac{1}{2}}(\partial\mathbb{M},\mathbb{R})\ ,\ L(x) = x_{|_{\partial\mathbb{M}}}$$

is a linear homeomorphism.

Proof: *A complete proof of this Theorem is given in [2], for the reader's convenience we sketch the main ideas: Since L is a linear homeomorphism as a map*

$$L : H_h^m(\mathcal{R},\mathbb{R}) \longrightarrow H^{m-\frac{1}{2}}(\partial\mathcal{R},\mathbb{R})$$

of larger spaces, this is clearly true for the restriction of L to the spaces $H_h^m(\mathbb{M},\mathbb{R})$ and $H^{m-\frac{1}{2}}(\partial\mathbb{M},\mathbb{R})$, respectively, provided τ-invariant boundary values admit a τ-invariant harmonic extension. In order to prove this last mentioned fact we consider $\gamma \in H^{m-\frac{1}{2}}(\partial\mathbb{M},\mathbb{R})$ and $x \in H_h^m(\mathcal{R},\mathbb{R})$ such that $L(x) = \gamma$. Observing that $x\circ\tau$ is also harmonic with $L(x\circ\tau) = \gamma$ the injectivity of L implies $x\circ\tau = x$. ■Before studying the Laplacian we observe that the operators $\partial_z, \partial_{\bar z}$ destroy the property of being τ-invariant, for example we have for τ-invariant x

$$x_z = f'(z) + \frac{1}{z^2}\overline{f'(\tau(z))}\ ,\ \Delta x(\tau z) \neq \Delta x(z)\ ,$$

whereas zx_z (and $\bar z x_{\bar z}$) are contained in the classes $A_{\pm}^m$. This motivates the study of the operator $|z|^2\Delta x(z) = (\bar z\partial_{\bar z})(z\partial_z)(x)(z)$ which has the property

Lemma 1: *The inclusion $x \in H^{m+2}(\mathbb{M},\mathbb{R})$ implies that of $|z|^2\Delta x(z) \in H^m(\mathbb{M},\mathbb{R})$.*

Proof: *The τ-invariance, $x(z) = x\circ\tau(z)$, and the fact that $\tau(z)$ is anti-conformal imply*

$$\Delta x(z) = \Delta(x\circ\tau)(z) = |\tau_{\bar z}|^2\Delta x(\tau z) = \left|\frac{1}{\bar z^4}\right|\Delta x(\tau z)\ ,$$

hence

$$|z|^2\Delta x(z) = \left|\frac{1}{z}\right|^2\Delta x(\tau z) = |\tau z|^2\Delta x(\tau z)$$

and the proof is complete. ■

Theorem 3(The Laplacian): *The operator*

$$T : H^{m+2}(\mathbb{M},\mathbb{R}) \longrightarrow H^{m}(\mathbb{M},\mathbb{R}) \times H^{m+\frac{3}{2}}(\partial\mathbb{M},\mathbb{R}) \ , T(x) = \left(|z|^2\Delta x(z) \ , \ x_{|_{\partial M}}\right)$$

is a linear homeomorphism.

Proof: *By the foregoing lemma T is well-defined. Observing that*

$$T : H^{m+2}(\mathcal{R},\mathbb{R}) \longrightarrow H^{m}(\mathcal{R},\mathbb{R}) \times H^{m+\frac{3}{2}}(\partial\mathcal{R},\mathbb{R})$$

is a linear homeomorphism inducing the operator under consideration by restriction to subspaces, injectivity and continuity of T are obvious. It remains to show that $T = (T^1, T^2)$ is onto. Recalling Theorem 2 we see that T^2 (even the restriction of T^2 to the kernel of T^1) is onto, hence we discuss the operator $T^1_{|_{\text{kernel } T^2}}$ and choose a function $y \in H^m(\mathbb{M},\mathbb{R})$, $y = y \circ \tau$. Let $x \in H^{m+2}(\mathcal{R},\mathbb{R})$ denote the unique solution of $T(x) = (y,0)$. Since

$$\begin{aligned} T(x\circ\tau) &= (|z|^2\Delta(x\circ\tau)(z),0) = (|z|^2|\tau_{\bar z}|^2\Delta x(\tau z),0) \\ &= \left(\frac{1}{|z|^2}\Delta x(\tau z),0\right) = (|\tau z|^2\Delta x(\tau z),0) \\ &= (y(\tau z),0) = (y(z),0) = T(x) \ , \end{aligned}$$

$x\circ\tau$ is a solution, and since T is 1-1 on $H^{m+2}(\mathcal{R},\mathbb{R})$ we get $x = x\circ\tau$. ■

The preceeding argument gives a sharper version of the lemma, more precisely we have

$$x \text{ is } \tau\text{-invariant} \iff |z|^2\Delta x \text{ is } \tau\text{-invariant} \ .$$

§3 The Riemann-Hilbert Problem

Many problems coming from applications can be reduced to the study of operators of the form

$$\begin{aligned} A^m(\mathcal{G},\mathbb{C}) \ni w &\longmapsto Re\{\bar\lambda w\}_{|_{\partial\mathcal{G}}} \in H^{m-\frac{1}{2}}(\partial\mathcal{G},\mathbb{R}) \\ H^{m+1}_h(\mathcal{G},\mathbb{R}) \ni x &\longmapsto Re\{\bar\lambda x_z\}_{|_{\partial\mathcal{G}}} \end{aligned}$$

or of similar type; here $\mathcal{G}$ denotes a domain in the complex plane or more general an (oriented) Riemann surface with boundary and λ is a given complex function on $\mathcal{G}$ without zeros. The above mentioned operators are Fredholm with index depending on the topological type of $\mathcal{G}$ and on the so-called *geometrical index* which measures the increasement of the argument of λ. It is well-known that the above mentioned Riemann-Hilbert problems are closely connected to the uniqueness and stability Theorems for the Plateau-Douglas problem. This is also true for non-orientable surfaces of the type of the Möbius strip, for details compare [2] and the papers quoted there, and from the facts described in §1 it becomes plausible that now the spaces $A^m(\mathcal{G},\mathbb{C})$

of the holomorphic functions has to be replaced by the spaces $A^m_\mp$. We therefore study the operators

$$\begin{aligned} RH : A^m_\mp &\longrightarrow H^{m-\frac{1}{2}}(\partial M, \mathbb{R}) \ , \ RH(w) = Re\{\bar{\lambda} w\}_{|_{\partial M}} \\ RH : H^{m+1}_h(M, \mathbb{R}) &\longrightarrow H^{m-\frac{1}{2}}(\partial M, \mathbb{R}) \ , \ RH(x) = Re\{\bar{\lambda} x_z\}_{|_{\partial M}} \end{aligned}$$

and – more general – the inhomogeneous boundary value operator

$$S : H^{m+2}(M, \mathbb{R}) \longrightarrow H^m(M, \mathbb{R}) \times H^{m+\frac{1}{2}}(\partial M, \mathbb{R}) \ , \ S(x) = (T(x), RH(x)) \, .$$

Theorem 4(The Riemann-Hilbert operator for harmonic functions): *The operator*

$$RH : A^m_\mp \longrightarrow H^{m-\frac{1}{2}}(\partial M, \mathbb{R}) \ , \ RH(w) = Re\{\bar{\lambda} w\}_{|_{\partial M}}$$

is Fredholm with index given by index $RH = 2\kappa(\lambda)$. *Here* $\lambda \in H^m(\partial M, \mathbb{C})$ *is a given function without zeros and* $\kappa(\lambda)$ *denotes the increasement of the argument of* λ *along* ∂M.

Corollary: *The index of the Fredholm operator*

$$RH : H^{m+1}_h(M, \mathbb{R}) \longrightarrow H^{m-\frac{1}{2}}(\partial M, \mathbb{R}) \ , \ RH(x) = Re\{\bar{\lambda} x_z\}_{|_{\partial M}}$$

is given by $2\kappa(\lambda) - 1$.

Proof (of the Theorem): *First we identify the spaces* $H^{m-\frac{1}{2}}(\partial M, \mathbb{R})$ *and* $H^{m-\frac{1}{2}}(\partial D_R, \mathbb{R})$ *via the map (which gives a linear homeomorphism)*

$$\Psi : \partial D_R \longrightarrow \partial M \ , \ \Psi(z) = \begin{cases} z & \text{for} \quad Im(z) \geq 0 \\ -\frac{1}{z} & \text{for} \quad Im(z) \leq 0 \end{cases} .$$

We write w *and* λ *instead of* $w \circ \Psi$ *and* $\lambda \circ \Psi$, *respectively. According to Theorem 1 we can use the linear homeomorphisms* Φ_+ *and* Φ_- *as global charts for the spaces* A^m_+ *and* A^m_-, *respectively, so that we have to calculate the index of the composition* $RH \circ \Phi_\pm$ *which is to be calculated to*

$$RH \circ \Phi_\pm : A^m_0(D, \mathbb{C}) \times \mathbb{R} \longrightarrow H^{m-\frac{1}{2}}(\partial D_R, \mathbb{R})$$

$$RH \circ \Phi_\pm(g, \alpha) = Re\left\{\bar{\lambda}\left(g\left(\frac{1}{R}z\right) + 2\alpha\right)\right\}_{|_{\partial D_R}} \pm Re\left\{\overline{\lambda (g \circ \tau)(Rz)}\right\}_{|_{\partial D_R}}$$

that is , $RH \circ \Phi_\pm$ *is splitted in the sum of two operators,* $RH \circ \Phi_\pm = T_1 + T_2$. *Now the functions* $z \longmapsto \overline{g \circ \tau(Rz)}$ *are holomorphic outside the disc* $D_{\frac{1}{R}}$. *Quoting well-known theorems of Montel respectively Sobolev's embedding theorems we see that the second term in the above equation,* T_2, *is a compact operator. Hence the Fredholm property as well as the desired index formula will follow if we can prove the corresponding facts for the operator* T_1. *According to [1] the index of the map*

$$A^m(D, \mathbb{C}) \ni g \longmapsto Re\left\{\bar{\lambda} g\left(\frac{1}{R}z\right)\right\}_{|_{\partial D_R}}$$

is $2\kappa(\lambda) + 1$ *(the case of the unit disk) and the space* $A^m_0(D, \mathbb{C}) \times \mathbb{R}$ *has codimension 1 in the space* $A^m(D, \mathbb{C}) \cong A^m_0(D, \mathbb{C}) \times \mathbb{C}$. *This proves the theorem.*

The corollary follows from the fact that in the composition

$$\begin{array}{ccccc} H_h^{m+1}(\mathbb{M},\mathbb{R}) & \xrightarrow{\Lambda} & A_-^m & \xrightarrow{\Psi} & H^{m-\frac{1}{2}}(\partial D_R,\mathbb{R}) \\ x & \longmapsto & zx_z & \longmapsto & Re\{\bar{\lambda}zx_z\} \end{array}$$

(since $\Lambda(H_h^{m+1}(\mathbb{M},\mathbb{R})) = A_{-*}^m$*) the first operator* Λ *is of co-rank 1.*

Theorem 5(The *inhomogeneous* Riemann-Hilbert operator): *The operator*

$$S : H^{m+2}(\mathbb{M},\mathbb{R}) \longrightarrow H^m(\mathbb{M},\mathbb{R}) \times H^{m+\frac{1}{2}}(\partial D_R,\mathbb{R}) \ , \ S(x) = \left(|z|^2\Delta x(z) \ , \ Re\{\bar{\lambda}zx_z\}_{|_{\partial D_R}}\right)$$

is of Fredholm type with index given by $index\, S = 2\kappa(\lambda) - 1$.

Proof: *If we write* $S = (S^1, S^2)$*, then the foregoing arguments show that*

$$index\left(S^2_{|_{\text{kernel } S^1}}\right) = 2\kappa(\lambda) - 1 \ .$$

On the other hand the map S^1 *is onto by Theorem 4, so that the general index relation [3, Hilfssatz 1.4] immediately implies the desired formula for the index of* S. ∎

References

[1] GAKHOV, F.D.: *Boundary value problems.* Pergamon Press: Oxford–London 1966.

[2] SCHÜFFLER, KH.: *Minimalflächen auf Möbiusbändern.* ZAA 9 (1990), 503 - 517.

[3] SCHÜFFLER, KH. AND TOMI, F.: *Ein Indexsatz für Flächen konstanter mittlerer Krümmung.* Math.Z. 182 (1983), 245 - 257.

[4] SCHÜFFLER, KH.: *On holomorphic functions on Riemann surfaces and the Riemann-Hilbert problem.* Analysis 9 (1989), 283 - 296.

Recieved 23.12.1991

Prof.Dr. Karlheinz Schüffler
Mathematisches Institut der
Heinrich-Heine Universität Düsseldorf
Universitätsstraße 1
D-4000 Düsseldorf 1