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Spectral and Polar Decomposition in AW*-Algebras

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We show the possibility and the uniqueness of spectral and polar decomposition of (normal) elements of arbitrary AW*-algebras.

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The spectral decomposition of normal linear (bounded) operators and the polar decomposition of arbitrary linear (bounded) operators on Hilbert spaces have been interesting and technically useful results in operator theory [3, 9, 13, 20]. The development of the concept of von Neumann algebras on Hilbert spaces has shown that both these decompositions are always possible *inside* the appropriate von Neumann algebra [14]. New light on these assertions was shed identifying the class of von Neumann algebras among all C*-algebras by the class of C*-algebras which possess a pre-dual Banach space, the W*-algebras. The possibility of C*-theoretical representationless descriptions of spectral and polar decomposition of elements of von Neumann algebras (and may be of more general C*-algebras) inside them has been opened up. Steps to get results in this direction were made by several authors. The W*-case was investigated by S. Sakai in 1958-60, [18, 19]. Later on J. D. M. Wright has considered spectral decomposition of normal elements of embeddable AW*-algebras, i.e., of AW*-algebras possessing a faithful von Neumann type representation on a self-dual Hilbert **A**-module over a commutative AW*-algebra **A** (on a so called Kaplansky–Hilbert module), [23, 24]. But, unfortunately, not all AW*-algebras are embeddable. In 1970 J. Dyer [5] and O. Takenouchi [21] gave (*-isomorphic) examples of type III, non-W*, AW*-factors, (see also K. Saitô [15]). Polar decomposition inside AW*-algebras was considered by I. Kaplansky [12] in 1968 and by S. K. Berberian [3] in 1972. They have shown the possibility of polar decomposition in several types of AW*-algebras, but they did not get a complete answer. In the present paper the partial result of I. Kaplansky is used that AW*-algebras without direct commutative summands and with a decomposition property for its elements like described at Corollary 5 below allow polar decomposition inside them, [3, §21: Exerc. 1] and [12, Th. 65]. For a detailled overview on these results we refer to [3].

The aim of the present paper is to show that both these decompositions are possible inside arbitrary AW*-algebras without additional assumptions to their structures.

Recall that an AW*-algebra is a C*-algebra for which the following two conditions are satisfied (cf. I. Kaplansky [10]):

(a) In the partially ordered set of projections every set of pairwise orthogonal projections has a least upper bound.

(b) Every maximal commutative $*$-subalgebra is generated by its projections, i.e., it is equal to the smallest closed $*$-subalgebra containing its projections.

An AW*-algebra **A** is called to be *monotone complete* if every increasingly directed, norm-bounded net of self-adjoint elements of **A** possesses a least upper bound in **A**. An AW*-algebra **A** is called to be *normal* if the supremum of every increasingly directed net of projections of **A** being calculated with respect to the set of all projections of **A** is it's supremum with respect to the set of all self-adjoint elements of **A** at once, cf. [25, 16]. For the most powerful results on these problems see [4] and [17].

To formulate the two theorems the following definitions are useful.

Definition 1 (J. D. M. Wright [24, p.264]): A measure m on a compact Hausdorff space X being valued in the self-adjoint part of a monotone complete AW^*-algebra is called to be *quasi-regular* if and only if

$$m(K) = \inf\{m(U) : U\text{−open sets in } X,\ K \subseteq U\}$$

for every closed set $K \subseteq X$. We remark that this condition is equivalent to the condition:

$$m(U) = \sup\{m(K) : K\text{−closed sets in } X, K \subseteq U\} \text{for every open set } U \subseteq X.$$

Further, if $m(E) = \inf\{m(U) : U\text{−open sets in } X, E \subseteq U\}$ for every Borel set $E \subseteq X$, then the measure m is called to be *regular.*

Definition 2 (M. Hamana [8, p.260] (cf. [1], [22])): A net $\{a_\alpha : \alpha \in I\}$ of elements of **A** *converges* to an element $a \in$**A** *in order* if and only if there are bounded nets $\{a_\alpha^{(k)} : \alpha \in I\}$ and $\{b_\alpha^{(k)} : \alpha \in I\}$ of self-adjoint elements of **A** and self-adjoint elements $a^{(k)} \in$**A**, $k = 1, 2, 3, 4$, such that

(i) $0 \leq a_\alpha^{(k)} - a^{(k)} \leq b_\alpha^{(k)}$, $k = 1, 2, 3, 4, \alpha \in I$,

(ii) $\{b_\alpha^{(k)} : \alpha \in I\}$ is decreasingly directed and has greatest lower bound zero,

(iii) $\sum_{k=1}^4 (\mathrm{i})^k a_\alpha^{(k)} = a_\alpha$ for every $\alpha \in I$, $\sum_{k=1}^4 (\mathrm{i})^k a^{(k)} = a$ (where $\mathrm{i} = \sqrt{-1}$).

We denote this type of convergence by $\mathrm{LIM}\{a_\alpha : \alpha \in I\} = a$.

By [6, p.260] the order limit of $\{a_\alpha : \alpha \in I\}$ does not depend on the special choice of the nets $\{a_\alpha^{(k)} : \alpha \in I\}, \{b_\alpha^{(k)} : \alpha \in I\}$ and of the elements $a^{(k)}$, $k = 1, 2, 3, 4$. If **A** is a commutative AW^*-algebra, then the notion of order convergence defined above is equivalent to the order convergence in **A** which was defined by H.Widom [22] earlier. Note that (cf. [8, Lemma 1.2]) if $\mathrm{LIM}\{a_\alpha : \alpha \in I\} = a$, $\mathrm{LIM}\{b_\beta : \beta \in J\} = b$, then

(i) $\mathrm{LIM}\{a_\alpha + b_\beta : \alpha \in I, \beta \in J\} = a + b$,

(ii) $\mathrm{LIM}\{x a_\alpha y : \alpha \in I\} = xay$ for every $x, y \in A$,

(iii) $\mathrm{LIM}\{a_\alpha b_\beta : \alpha \in I, \beta \in J\} = ab$,

(iv) $a_\alpha \leq b_\alpha$ for every $\alpha \in I = J$ implies $a \leq b$,

(v) $\|a\|_A \leq \limsup\{\|a_\alpha\|_A : \alpha \in I\}$.

Furthermore, we need the following lemma describing the key idea of the present paper and being of interest on its own.

Lemma 3 : *Let* **A** *be an* AW^**-algebra and* **B** $\subseteq$ **A** *be a commutative* C^**-subalgebra. Then the monotone closures* $\hat{\mathbf{B}}(\mathbf{D})$, $\hat{\mathbf{B}}(\mathbf{D}')$ *of* **B** *inside arbitrary two maximal commutative* C^**-subalgebras* **D**, **D**′ *of* **A** *which contain* **B**, *respectively, are* $*$*-isomorphic commutative* AW^**-algebras. Moreover, all monotone closures* $\hat{\mathbf{B}}(\mathbf{D})$ *of* **B** *of this type coincide as* C^**-subalgebras of* **A** *if* **A** *is normal.*

Proof: Let $\mathbf{D}$ be a maximal commutative C*-subalgebra of $\mathbf{A}$ containing $\mathbf{B}$. By definition $\mathbf{D}$ is generated by its projections. Let $p \in \hat{\mathbf{B}}(\mathbf{D})$ be a projection. Suppose $p \notin \mathbf{B}$. Then p is the supremum of the set $\mathcal{P} = \{x \in \mathbf{B}_h^+ \subseteq \mathbf{D} : x \leq p\}$ by [7, Lemma 1.7]. In particular, $(1_A - p)$ is the maximal annihilator projection of $\mathcal{P}$ inside $\mathbf{D}$. But, $\mathcal{P}^2 = \mathcal{P}$ and, hence, the supremum of $\mathcal{P}$ being calculated inside $\mathbf{D}'$ is a projection p' again, and $(1_A - p')$ is the maximal annihilator projection of $\mathcal{P}$ in $\mathbf{D}'$. Changing the positions of $\mathbf{D}$ and $\mathbf{D}'$ one finds a one-to-one correspondence between the projections of $\hat{\mathbf{B}}(\mathbf{D})$ and $\hat{\mathbf{B}}(\mathbf{D}')$.

Moreover, the product projection p_1p_2 of two projections $p_1, p_2 \in \hat{\mathbf{B}}(\mathbf{D})$ corresponds to the supremum of the intersection set of the two appropriate sets $\mathcal{P}_1$ and $\mathcal{P}_2$ of elements of $\mathbf{B}$, and hence, to the product projection $p_1'p_2'$ of the corresponding two projections $p_1', p_2' \in \hat{\mathbf{B}}(\mathbf{D}')$. That is, the found one-to-one correspondence between the sets of projections of $\hat{\mathbf{B}}(\mathbf{D})$ and of $\hat{\mathbf{B}}(\mathbf{D}')$ preserves the lattice properties of these nets. Since $\hat{\mathbf{B}}(\mathbf{D})$ and $\hat{\mathbf{B}}(\mathbf{D}')$ are commutative AW*-algebras (i.e. both they are linearly spanned by their projection lattices as linear spaces and as Banach lattices), this one-to-one correspondence extends to a $*$-isomorphism of $\hat{\mathbf{B}}(\mathbf{D})$ and $\hat{\mathbf{B}}(\mathbf{D}')$.

Now, fix such a set $\mathcal{P} \subseteq \mathbf{B}$. By [10] there exists a global maximal annihilator projection $(1_A - q)$ of $\mathcal{P}$ in $\mathbf{A}$. The problem arrising in this situation can be formulated as follows: does q commute with $\mathbf{B}$, i.e. is q an element of $\mathbf{D}$ and, hence, of every maximal commutative C*-subalgebra $\mathbf{D}'$ of $\mathbf{A}$ containing $\mathbf{B}$? Obviously, $(1_A - q)$ is the supremum of the set of all those annihilator projections $\{(1_A - p)\}$ which we have constructed above, but only in the sense of a supremum in the net of all projections of $\mathbf{A}$ since monotone completeness or normality of $\mathbf{A}$ are not supposed, in general. So we have to assume that $\mathbf{A}$ is normal, cf. [25, 16], to be sure in our subsequent conclusions. Then there follows that $(1_A - q)$has to be the supremum of the setof all those projections $\{(1_A - p)\}$ in the self-adjoint part of $\mathbf{A}$. Hence, q commutes with $\mathbf{B}$ since each of the projections p do. This means that q belongs to every maximal commutative C*-subalgebra $\mathbf{D}$ of $\mathbf{A}$ containing $\mathbf{B}$ because of their maximality, and $q = p$ for every $p \in \mathbf{D}$ since $q \leq p$, and p was the supremum of $\mathcal{P}$ inside $\mathbf{D}_h^+$.

Since $\mathcal{P}$ was fixed arbitrarily one concludes that $\hat{\mathbf{B}}(\mathbf{D})$ does not depend on the choice of $\mathbf{D}$ inside normal AW*-algebras $\mathbf{A}$. ∎

Theorem 4 (cf. [24, Th.3.1 and Th.3.2]): *Let $\mathbf{A}$ be a normal AW*-algebra and $a \in \mathbf{A}$ be a normal element. Let $\mathbf{B} \subseteq \mathbf{A}$ be that commutative C*-subalgebra in $\mathbf{A}$ being generated by the elements $\{1_A, a, a^*\}$, and denote by $\hat{\mathbf{B}}$ the smallest commutative AW*-algebra inside $\mathbf{A}$ containing $\mathbf{B}$ and being monotone complete inside every maximal commutative C*-subalgebra of $\mathbf{A}$. Then there exists a unique quasi-regular $\hat{\mathbf{B}}$-valued measure m on the spectrum $\sigma(a) \subset \mathbf{C}$ of $a \in \mathbf{A}$, the values of which are projections in $\hat{\mathbf{B}}$ and for which the integral*

$$\int_{\sigma(a)} \lambda \, dm_\lambda \quad = \quad a$$

exists in the sense of order convergence in $\hat{\mathbf{B}} \subseteq \mathbf{A}$.

If $\mathbf{A}$ is not normal, then for every maximal commutative C-subalgebra $\mathbf{D}$ of $\mathbf{A}$ containing $\mathbf{B}$ there exists a unique spectral decomposition of $a \in \mathbf{A}$ inside the monotone closure $\hat{\mathbf{B}}(\mathbf{D})$ of $\mathbf{B}$ with respect to $\mathbf{D}$. But it is unique only in the sense of the $*$-isomorphy of $\hat{\mathbf{B}}(\mathbf{D})$ and $\hat{\mathbf{B}}(\mathbf{D}')$ for every two different maximal commutative C*-subalgebras $\mathbf{D}, \mathbf{D}'$ of $\mathbf{A}$ containing $\mathbf{B}$.*

If $\mathbf{A}$ is a W^*-algebra, then m is regular and the integral exists in the sense of norm convergence.

Proof: By the Gelfand–Naimark representation theorem the commutative C*-subalgebra $\mathbf{B} \subseteq \mathbf{A}$ being generated by the elements $\{1_A, a, a^*\}$ is $*$-isomorphic to the commutative C*-algebra $C(\sigma(a))$ of all complex-valued continuous functions on the spectrum $\sigma(a) \subset \mathbf{C}$ of $a \in \mathbf{A}$. Denote this $*$-isomorphism by ϕ, $\phi : C(\sigma(a)) \longrightarrow \mathbf{B}$. The isomorphism ϕ is isometric and preserves order relations between self-adjoint elements and, hence, positivity of self-adjoint elements. Therefore, ϕ is a positive mapping.

Selecting an arbitrary maximal abelian C*-subalgebra $\mathbf{D}$ of $\mathbf{A}$ containing $\mathbf{B}$ one can complete $\mathbf{B}$ to $\hat{\mathbf{B}}(\mathbf{D})$ with respect to the order convergence in $\mathbf{D}$. Note that $\hat{\mathbf{B}}(\mathbf{D}) \subseteq \mathbf{A}$ does not depend on the choice of $\mathbf{D}$ by the previous lemma if $\mathbf{A}$ is normal.

Now, by [24] , [23, Th.4.1] there exists a unique positive quasi-regular $\hat{\mathbf{B}}(\mathbf{D})$-valued measure m with the property that

$$\int_{\sigma(a)} f(\lambda)\, dm_\lambda \quad = \quad \phi(f)$$

for every $f \in C(\sigma(a))$. Since $\phi^{-1}(a)(\lambda) = \lambda$ for every $\lambda \in \sigma(a) \subset \mathbf{C}$ by the definition of ϕ one gets

$$\int_{\sigma(a)} \lambda\, dm_\lambda \quad = \quad a.$$

Moreover, since the extension $\hat{\phi}$ of ϕ to the set of all bounded Borel functions on $\sigma(a)$ fulfils $\hat{\phi}(\chi_E)^2 = \hat{\phi}(\chi_E^2) = \hat{\phi}(\chi_E)$ for the characteristic function χ_E of every Borel set $E \in \sigma(a)$ the measure m is projection-valued, cf. [24]. One finishes refering to Lemma 3 ∎

The following corollary is essential to get the polar decomposition theorem.

Corollary 5 : *Let $\mathbf{A}$ be an AW^*-algebra and $x \in \mathbf{A}$ be different from zero. Then there exists a projection $p \in \mathbf{A}_h^+$, $p \neq 0$, and an element $a \in \mathbf{A}_h^+$ such that a, p and $(xx^*)^{1/2}$ commute pairwise, and $a(xx^*)^{1/2} = (axx^*a)^{1/2} = p$.*

Proof: Consider the commutative C*-subalgebra $\mathbf{B}$ of $\mathbf{A}$ being generated by the elements $\{1_A, xx^*\}$. By the spectral theorem there exists a unique positive quasi-regular measure m on the Borel sets of $\sigma((xx^*)^{1/2}) \subset \mathbf{R}^+$ being projection-valued in the monotone closure $\hat{\mathbf{B}}(\mathbf{D}) \subseteq \mathbf{A}$ of $\mathbf{B}$ with respect to an arbitrarily chosen, but fixed, maximal commutative C*-subalgebra $\mathbf{D}$ of $\mathbf{A}$ containing $\mathbf{B}$, and satisfying the equality

$$\int_{\sigma((xx^*)^{1/2})} \lambda\, dm_\lambda = (xx^*)^{1/2}$$

in the sense of order convergence in $\hat{\mathbf{B}}(\mathbf{D}) \subseteq \mathbf{A}$. Now, if $(xx^*)^{1/2}$ is a projection, then set $a = 1_A$, $p = xx^*$. If $(xx^*)^{1/2}$ is invertible in $\mathbf{A}$, then set $p = 1_A$, $a = (xx^*)^{-1/2}$. Otherwise consider a number $\mu \in \sigma((xx^*)^{1/2})$, $0 < \mu < \|x\|$, and set $K = [0, \mu] \cap \sigma((xx^*)^{1/2})$. The value $m(K) \in \hat{\mathbf{B}}(\mathbf{D})$ is a projection different from zero. It commutes with every spectral projection of $(xx^*)^{1/2}$ and with $(xx^*)^{1/2}$ itself. Since m is a quasi-regular measure one has

$$\int_{\sigma((xx^*)^{1/2})\setminus K} \lambda\, d(m_\lambda(1_A - m(K))) = (1_A - m(K))(xx^*)^{1/2}.$$

Therefore, one finds $p = (1_A - m(K))$ and $a = ((1_A - m(K))(xx^*))^{-1/2}$, where the inverse is taken inside the C*-subalgebra $(1_A - m(K))\mathbf{B}(\mathbf{D}) \subseteq \mathbf{A}$. Since $\mu < \|x\|$ the projection p is different from zero. The existence of $a \in \mathbf{A}_h^+$ is guaranteed by $0 < \mu$ ∎

Now we go on to show the polar decomposition theorem for AW*-algebras using results of S. K. Berberian and I. Kaplansky. Previously we need it for commutative AW*-algebras. Note that the proof of the following lemma works equally well for all monotone complete C*-algebras.

Lemma 6 : *Let* **A** *be a commutative AW*-algebra. For every $x \in \mathbf{A}$ there exists a unique partial isometry $u \in \mathbf{A}$ such that $x = (xx^*)^{1/2}u$ and uu^* is the range projection of $(xx^*)^{1/2}$.*

Proof: Throughout the proof we use freely the order convergence inside monotone complete C*-algebras as defined at Definition 2.

First, suppose x to be self-adjoint. The sequence $\{u_n = x(1/n + |x|)^{-1} : n \in \mathbf{N}\}$ is bounded in norm by the representation theory. It consists of self-adjoint elements, and the sequences $\{|x|(1/n + |x|)^{-1}\}$ and $\{(|x| - x)(1/n + |x|)^{-1}\}$ are monotone increasing. Hence, the sequence $\{u_n\}$ is order converging inside **A**, $\mathrm{LIM}\, u_n = u$, and $u \in \mathbf{A}_h$. Furthermore, the sequence $\{u_n|x| : n \in \mathbf{N}\}$ converges to x in order, i.e., $x = u|x|$. From the equality $x^2 = |x|u^*u|x|$ one draws $u^*u \geq rp(|x|)$ (where $rp(|x|)$ denotes the range projection of $|x|$ being an element of **A**). Hence, $u^*u = rp(|x|)$ by construction and u is a partial isometry.

Now suppose $x \in \mathbf{A}$ to be arbitrarily chosen. Consider again the sequence $\{u_n\}$ of elements of **A** as defined at the beginning. One has to show the fundamentality of it with respect to the order convergence. Since **A** is monotone complete the existence of its order limit u inside **A** will be guaranteed in this case. For the self-adjoint part of the elements of $\{u_n\}$ the inequality

$$\begin{aligned}
0 &\leq [x((1/n + |x|)^{-1} - (1/m + |x|)^{-1}) + ((1/n + |x|)^{-1} - (1/m + |x|)^{-1})x^*]^2 \\
&\leq 2[x((1/n + |x|)^{-1} - (1/m + |x|)^{-1})^2 x^* + \\
&\quad + ((1/n + |x|)^{-1} - (1/m + |x|)^{-1})x^*x((1/n + |x|)^{-1} - (1/m + |x|)^{-1})]
\end{aligned}$$

is valid for every $n, m \in \mathbf{N}$. The expression of the right side converges weakly to zero as n, m go to infinity in each faithful $*$-representation of **A** on Hilbert spaces. Therefore, it is bounded in norm and converges in order to zero as n, m go to infinity because of the positivity of the expression. Since taking the square root preserves order relations between positive elements of a C*-algebra and since self-adjoint elements have polar decomposition inside **A** the order fundamentality of the sequence $\{1/2(u_n + u_n^*) : n \in \mathbf{N}\}$ turns out. The order convergence of the anti-self-adjoint part of the sequence $\{u_n\}$, $\{1/2i \cdot (u_n - u_n^*)\}$, can be shown in an analogous way. Hence, there exists $\mathrm{LIM}\, u_n = u$ inside **A**.

Now, from the existence of $\mathrm{LIM} u_n|x| = x$ one derives the equality $x = u|x|$. The equality $x^*x = |x|u^*u|x|$ shows that $u^*u \geq rp(|x|)$ and, consequently, $u^*u = rp(|x|)$ by construction, i.e., u is a partial isometry.

To show the uniqueness of polar decomposition inside **A** suppose $x = v|x|$ for a partial isometry v with $v^*v = rp(|x|)$. Then $v|x| = u|x|$, i.e., $v = v \cdot rp(|x|) = u$ ∎

Theorem 7 : *Let* **A** *be an AW*-algebra. For every $x \in \mathbf{A}$ there exists a unique partial isometry $u \in \mathbf{A}$ such that $x = (xx^*)^{1/2}u$ and uu^* is the range projection of $(xx^*)^{1/2}$.*

Proof: By [12, Th. 65] polar decomposition is possible inside of all AW*-algebras without direct commutative summands under the supposition that every element of it has the property of Corollary 5, (see also [3, §21: Exerc. 1]). Since polar decomposition works separately in every direct summand we have only to compare Corollary 5, Lemma 6 and the result of I. Kaplansky ∎

The result of Theorem 7 is of interest also because monotone completeness was not necessary to show it, what is a little bit surprising.

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Book reviews

V. P. KHAVIN and N. K. NIKOL'SKIJ (Eds.): **Commutative Harmonic Analysis** I. General Survey - Classical Aspects (Encyclopaedia of Mathematical Sciences: Vol. 15). Berlin - Heidelberg: Springer-Verlag 1991, IX + 268 pp, 1 fig.

Commutative harmonic analysis is one of the eldest branches of analysis with a big influence on the development of the whole mathematics. Its history is starting with Euler, Bernoulli and Fourier, which gave fundamental ideas and stimulating suggestions without having a clear conception of a correct realization. This was made much time later by Dirichlet, Cantor, Riemann and Lebesgue. Only the applications of modern theories of functional analysis especially distribution theory and the theory of commutative groups leads to a new revival of methods of harmonic analysis in a much more higher level. Commutative harmonic analysis is densely connected with other parts of mathematics (spectral theory, theory of orthogonal systems, integral transforms, commutative Banach algebras, ...) and in this way this is basic knowledge of each analytic working mathematician. Nevertheless there are many intrinsic results which show the relative independence of this disciplin.

The book under review is splitting into the following three relatively independent parts:

I. Methods and structure of commutative harmonic analysis
II. Classical themes of Fourier analysis
III. Methods of the theory of singular integrals.

The authors try not only to collect worth mentioning results in theory and application of harmonic analysis but also give ingenious suggestions and useful hints for the proofs. In Part I most of the proofs are given completely. Let us have a look through the chapters of Part I - III.

Part I is divided into five chapters. In Chapter 1 there is given a special version on Fourier series in order to recall basic facts of Fourier analysis. The proofs are mostly given completely and ingeniously. The construction of harmonic analysis in $\mathbb{R}$ is entirely done from viewpoint of distribution theory. Translation invariant linear operators and convolutions play the main role in Chapter 2. Basic facts on S'- and L_2-theory of Fourier transform are presented in a convincing manner. Many very useful examples of different character (Heisenberg's uncertainty principle, central limit theorem, Jacobi's identity for the Θ-function, hypoellipticity) additionally support the understanding. Fourier analysis on topological groups is the main topic of Chapter 3. Herein one can find the well-known Peter-Weyl theorem and statements of almost periodic functions (Theorem of Bohr). The authors also give very nice applications, for instance a description of the algorithm of Schönhage and Strassen about the fast Fourier transform or the quadratic reciprocity law. The historical essay in Chapter 4 suggests the reader the general idea of commutative harmonic analysis. Some remarks on spectral analysis as well as intrinsic problems of Fourier analysis conclude the first Part.