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Distributional Controls in Processes with Hammerstein Type Integral Equations

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The paper deals with problems of optimal control in which the control in general appears nonlinear and in distributional sense, that means as limits of regular distributional sequences. For this a generalization of necessary conditions of optimality is provided (which is also sufficient in the linear case).

Key words: *Distribution, optimal control, integral equations, necessary conditions of optimality, sufficient conditions of optimality*

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1. Introduction

Problems of optimal control were often studied for controls being piecewise continuous [2, 5], bounded measurable [6] or measurable functions [1]. However, numerous applications of optimal control in geometry, mathematical physics and engineering require an extension of these investigations to controls u in the shape of distributional vector-valued functions in a basic space $\mathcal{D}$ of r-vector-valued functions on $\mathbb{E}^1$. Referring to the theory of distributions by Gelfand and Schilow [3] and to the article [4], where a distributional version of optimal processes subject to ordinary differential equations is considered, now in the paper lying before us this conception is transmitted on Hammerstein type integral equations with bounded kernel in infinite-dimensional spaces.

2. Some preliminary notes

In the following we shall denote by $\mathbb{E}^n$ the real n-dimensional Euclidean space, $\mathbb{E}^n_+$ its positive octant, and $[a, b]$ an interval in $\mathbb{E}^1$. We denote by $\mathcal{D}(\mathbb{E}^r)$ the basic space of all infinitely differentiable finite r-vector-valued functions on $\mathbb{E}^1$. Each function $\psi \in \mathcal{D}(\mathbb{E}^r)$ vanishes outside of a bounded interval (depending on ψ). We denote by $\mathcal{D}'(\mathbb{E}^r)$ the whole set of linear continuous functionals on $\mathcal{D}(\mathbb{E}^r)$ (i.e. distributions). A distribution $\chi \in \mathcal{D}'(\mathbb{E}^r)$ is called *zero on a neighbourhood V of $s_0 \in \mathbb{E}^1$* if for every $\psi \in \mathcal{D}(\mathbb{E}^r)$ with $\psi(s) = 0$ outside of V the condition $(\chi,\psi) = 0$ holds. A point s_0 is called an *essential point of χ* if no neighbourhood of s_0 exists on which χ is zero. We denote by $\mathcal{D}'_n(\mathbb{E}^r)$ the n-fold product $\mathcal{D}'(\mathbb{E}^r) \times \ldots \times \mathcal{D}'(\mathbb{E}^r)$ and each element $\chi \in \mathcal{D}'_n(\mathbb{E}^r)$ is said to be an *n-dimensional distribution* or simply *n-distribution*. If $\chi = (\chi^1, \ldots, \chi^n)$ belongs to $\mathcal{D}'_n(\mathbb{E}^r)$, then (χ,ψ) means the n-dimensional vector $((\chi^1,\psi), \ldots, (\chi^n,\psi))$ where $\psi \in \mathcal{D}(\mathbb{E}^r)$. Remark that $\mathcal{D}'_1(\mathbb{E}^r) = \mathcal{D}'(\mathbb{E}^r)$. An n-dimensional distribution $\chi \in \mathcal{D}'_n(\mathbb{E}^r)$ is called *zero on a neighbourhood V of $s_0 \in \mathbb{E}^1$* if each distribution $\chi^i, \ldots, \chi^n$ is zero on a neighbourhood V of $s_0 \in \mathbb{E}^1$. The definition of *essential points* of $\chi \in \mathcal{D}'_n(\mathbb{E}^r)$ is standard. The set of all essential points of χ^i is denoted by the support of χ^i, briefly $\operatorname{supp}\chi^i$. For $\chi \in \mathcal{D}'_n(\mathbb{E}^r)$ we define $\operatorname{supp}\chi$ by $\bigcup_{i=1}^n \operatorname{supp}\chi^i$. For $\chi \in \mathcal{D}'_n(\mathbb{E}^r)$

especially the inclusion $\operatorname{supp}\chi \subset [a,b]$ means $\operatorname{supp}\chi^i \subset [a,b]$ for all $i = 1,\dots,n$. If $\chi_k: \mathcal{D}(\mathbb{E}^r) \to \mathbb{E}^n$ $(k \in \mathbb{N})$ is a sequence of n-distributions in $\mathcal{D}_n'(\mathbb{E}^r)$, then $\chi_k \to \chi$ for $k \to \infty$ means $\chi_k^i \to \chi^i$ for $k \to \infty$ in the space $\mathcal{D}'(\mathbb{E}^r)$ for each $i = 1,\dots,n$. Let $f: \mathbb{E}^r \to \mathbb{E}^m$ be a mapping such that for each $u \in L_1^{loc}(\mathbb{E}^1, \mathbb{E}^r) \subset \mathcal{D}'(\mathbb{E}^r)$ the function $s \to f(u(s))$ is local integrable. For each $u \in \mathcal{D}'(\mathbb{E}^r)$ we denote by $f(u)$ a distribution in $\mathcal{D}'(\mathbb{E}^m)$ which is defined in the following manner: if $\{u_k\}$ is a sequence in $L_1^{loc}(\mathbb{E}^1, \mathbb{E}^r)$ which converges to u in the space $\mathcal{D}'(\mathbb{E}^r)$, then

$$\text{(i)}\quad (f(u), \psi) = \lim_{k\to\infty}(f(u_k), \psi) = \lim_{k\to\infty} \int_{-\infty}^{+\infty} f(u_k(s))\psi(s)\,ds =: \int_{-\infty}^{+\infty} f(u(s))\psi(s)\,ds \quad \forall\, \psi \in \mathcal{D}(\mathbb{E}^m)$$

under the assumption on f that this limit does not depend on the special sequence $\{u_k\}$.

We shall denote by $B^{n,m}$ the space of all $(n \times m)$-matrix functions $(a_{ij}(\cdot))$. If $f: \mathbb{E}^1 \times \mathbb{E}^r \to B^{n,m}$ satisfies the condition

for all $u \in L_1^{loc}(\mathbb{E}^1, \mathbb{E}^r)$ the function $s \to f(s, u(s))$ belongs to $L_1^{loc}(\mathbb{E}^1, B^{n,m})$,

then for each $u \in \mathcal{D}'(\mathbb{E}^r)$ the distribution $f(u)$ of $\mathcal{D}_n(\mathbb{E}^m)$ is defined as above in (i). We have only in this case to replace the integrands of (i) by $f(s, u_k(s))$ and $f(s, u(s))$, respectively.

Let be $\chi \in \mathcal{D}_n'(\mathbb{E}^m)$ and $K \in C_\infty(\mathbb{E}^1, B^{m,l})$, then we understand by $K\chi$ the n-distribution in $\mathcal{D}_n'(\mathbb{E}^l)$ which is defined by

$$\text{(ii)}\quad (K\chi, \psi) = \int_{-\infty}^{+\infty} (K\chi)(s)\,\psi(s)\,ds = (\chi, K\psi) = \int_{-\infty}^{+\infty} \chi(s)\,K(s)\,\psi(s)\,ds \quad \text{for all } \psi \in \mathcal{D}(\mathbb{E}^l).$$

Especially, if $\operatorname{supp}\chi \subset [0,1]$, then $\int_0^1 (K\chi)(s)\,ds$ expresses the value of the linear continuous operator (ii) for such $\chi \in \mathcal{D}(\mathbb{E}^l)$ which have the property $\psi^i(s) = 1$ on $[0,1]$ for $i = 1,\dots,l$.

3. Statement of control problem

Let be $I = [0,1] \subset \mathbb{E}^1$. The functions

$$f_0: I \times \mathbb{E}^n \to \mathbb{E}^1,\quad f_1: \mathbb{E}^1 \times \mathbb{E}^r \to \mathbb{E}^1 \quad\text{and}\quad g_0: I \times \mathbb{E}^n \to \mathbb{E}^n,\quad g_1: \mathbb{E}^1 \times \mathbb{E}^r \to B^{n,n}$$

are satisfying the following conditions:

a) For each $s \in I$ the functions $x \to f_0(s,x)$ and $x \to g_0(s,x)$ are continuously differentiable, i.e. they belong to the spaces $C^1(\mathbb{E}^n, \mathbb{E}^1)$ and $C^1(\mathbb{E}^n, \mathbb{E}^n)$, respectively.

b) The mappings $s \to f_0(s,\cdot)$ and $s \to g_0(s,\cdot)$ belong to the spaces $L_1(I, C^1(\mathbb{E}^n, \mathbb{E}^1))$ and $L_2(I, C^1(\mathbb{E}^n, \mathbb{E}^n))$, respectively.

c) For each $s \in I$ the functions $u \to f_1(s,u)$ and $u \to g_1(s,u)$ are continuous on $\mathbb{E}^r$.

Let $K: I \times \mathbb{E}^1 \to B^{n,n}$ be a measurable bounded function such that for each $t \in I$ the function $K(t,\cdot)$ belongs to $C^\infty(\mathbb{E}^1, B^{n,n})$ and the mapping $t \to K(t,\cdot)$ is continuous on I into $L_1(I, B^{n,n})$. Finally, b is an element of $C(I, \mathbb{E}^n)$.

Under these arrangements we now formulate the following class of distributional problems:

$$F(x,u) = \int_0^1 \big(f_0(s,x(s)) + f_1(s,u(s))\big)\,ds \longrightarrow \inf \tag{1}$$

subject to state functions $x \in C(I, \mathbb{E}^n)$ and distributional controls $u \in \mathcal{D}'(\mathbb{E}^r)$ with $\operatorname{supp} f_1(u)$ and $\operatorname{supp} g_1(u)$ as subsets of I, such that the following constraint, the state Hammerstein type integral equation

$$x(t) = \int_0^1 K(t,s)(g_0(s,x(s)) + g_1(s,u(s)))\,ds + b(t). \tag{2}$$

holds. Furthermore, we demand that for each sequence of admissible processes (x_j, u_j) of (1) - (2) the following limit relations hold:

(A) $f_0(\cdot, x_j(\cdot)) \rightharpoonup f_0(\cdot, x(\cdot))$ in $L_2(I, \mathbb{E}^1)$ (i.e., weakly)

$$\lim_{j\to\infty} \int_0^1 f_{0x}(s, x_j(s))y(s)\,ds = \int_0^1 f_{0x}(s, x(s))y(s)\,ds \quad \forall\, y \in L_2(I, \mathbb{E}^n) \text{ when } x_j \to x \text{ in } L_2(I, \mathbb{E}^n).$$

Moreover, we require that for each process (x_0, u_0) of (1) - (2) a sequence of admissible regular processes $(\hat{x}_j, \hat{u}_j)$ exists such that $\hat{x}_j \in C(I, \mathbb{E}^n)$, $\hat{u}_j \in L_\infty^{loc}(\mathbb{E}^1, \mathbb{E}^r)$ and

(B) $\hat{x}_j \to x_0$ in $L_2(I, \mathbb{E}^n)$, $u_j \to u$ in $\mathcal{D}'(\mathbb{E}^r)$.

Finally, we shall denote by $\mathcal{D}_0'(f_1, g_1)$ or briefly $\mathcal{D}_0'$ the set $\{\chi \in \mathcal{D}_0'(\mathbb{E}^r) |\ \mathrm{supp} f_1(\chi), \mathrm{supp} g_1(\chi) \subset I\}$ and we suppose that for each $u \in L_\infty(I, \mathbb{E}^r)$ there exists a distribution $\bar{u}$ in $\mathcal{D}_0'$ such that, for each $\psi \in \mathcal{D}(\mathbb{E}^1)$ and $\varphi \in \mathcal{D}(\mathbb{E}^n)$,

(C) $(f_1(\bar{u}), \psi) = \int_0^1 f_1(s, u(s))\,\psi(s)\,ds$ and $(g_1(\bar{u}), \varphi) = \int_0^1 g_1(s, u(s))\,\varphi(s)\,ds$

hold. Besides of problem (1) - (2) we study further the sequence of corresponding substitutional problems of the form

$$F_j(x,u) = F(x,u) + \alpha \|x - x_0\|^2_{L_2} \longrightarrow \inf \quad \text{for } \alpha = \text{const} > 0 \tag{3}$$

where $x \in C(I, \mathbb{E}^n)$, $u \in L_\infty(I, \mathbb{E}^r)$ satisfy (2) and $|u(s)| \le M_j$ for all $s \in I$, $M_j \to \infty$ as $j \to \infty$. Let be $W = I \times \mathbb{E}^n \times \mathbb{E}^r \times L_2(I, \mathbb{E}^n) \times \mathbb{E}^1$ and let the scalar function H: $W \to \mathbb{E}^1$ be the Hamiltonian of (1) - (2) be of the form

$$H(s,x,u,\Phi,\lambda) = \lambda\big(f_0(s,x) + f_1(s,u)\big) + \Phi(s)\big(g_0(s,x) + g_1(s,u)\big). \tag{4}$$

Theorem 1: *If the process (x_0, u_0) is optimal for* (1) - (2) *and each corresponding substitutional problem* (3) *is solvable, then there exist a function $\Phi \in L_2(I, \mathbb{E}^n)$ and a vector $\lambda \le 0$, not vanishing simultaneously, such that the following equations are fulfilled:*

$$\Phi(t) = \int_0^1 K^*(s,t)\big(f^*_{0x}(s,x_0(s))\lambda + g^*_{0x}(s,x_0(s))\,\Phi(s)\big)\,ds \quad \text{for a.e. } t \in I \tag{5}$$

and

$$\sup_{v \in L_\infty^{loc} \cap \mathcal{D}_0'} \int_0^1 H(s, x_0(s), v(s), \Phi, \lambda)\,ds = \int_0^1 H(s, x_0(s), u_0(s), \Phi, \lambda)\,ds. \tag{6}$$

Here and further $*$ denotes the transposition of the corresponding matrices or vectors.

Proof: Using the basic theorem, which has already been proved for more general situations in Banach spaces [1], in our case we can find for each $j \in \mathbb{N}$ an optimal solution (x_j, u_j) of the substitutional problem (3), corresponding not simultaneously vanishing elements $\Phi_j \in L_2(I, \mathbb{E}^n)$ and numbers $\lambda_j \ge 0$ such that

$$\Phi_j(t) = \int_0^1 K^*(s,t)\Big(f^*_{0x}(s,x_j(s))\lambda_j + 2(x_j(s) - x_0(s))^*\alpha\lambda_j + g^*_{0x}(s,x_j(s))\Phi_j(s)\Big)\,ds \tag{7}$$

holds as well as

$$\sup_{v \le M_j} H_j(t, x_j(t), v, \Phi_j, \lambda_j) = H_j(t, x_j(t), u_j(t), \Phi_j, \lambda_j) \text{ for a.e. } t \in I, \tag{8}$$

where $H_j\colon W \to \mathbb{E}^1$ is defined by

$$H_j(s, x, u, \Phi, \lambda) = H(s, x, u, \Phi, \lambda) + \lambda\alpha\|x - x_0(s)\|^2. \tag{9}$$

Now we introduce

$$\gamma_j = \left(\|\Phi_j\|^2_{L_2} + |\lambda_j|^2\right)^{1/2}, \ \gamma_j > 0 \text{ for all } j \in \mathbb{N},$$

and divide each equation by γ_j. Thus, using the abbreviations

$$\overline{\Phi}_j = \Phi_j/\gamma_j \quad \text{and } \overline{\lambda}_j = \lambda_j/\gamma_j$$

we obtain from(7) a modification of this equation in which Φ_j and λ_j are replaced by $\overline{\Phi}_j$ and $\overline{\lambda}_j$, respectively. We denote these modified equations by $(\overline{7})$ and $(\overline{8})$, respectively. By our construction

$$\|\overline{\Phi}_j\|^2_{L_2} + |\overline{\lambda}_j|^2 = 1 \tag{10}$$

and hence, by using well-known compactness theorems in Hilbert spaces we can find a subsequence $\{j'\}$ of $\{j\}$ such that $(\overline{\Phi}_{j'}, \overline{u}_{j'})$ converges to (Φ, λ) in the following sense:

$$\overline{\Phi}_{j'} \rightharpoonup \Phi \text{ in } L_2(I, \mathbb{E}^n) \text{ (i.e., weakly), and } \overline{\lambda}_{j'} \to \lambda \text{ in } \mathbb{E}^1. \tag{11}$$

In consequence of (B) and (C) and the optimality property of $(x_{j'}, u_{j'})$ with respect to (3), we get

$$F(x_0, u_0) \le F(x_{j'}, u_{j'}) + \alpha\|x_{j'} - x_0\|^2 \le F(\hat{x}_{j'}, \hat{u}_{j'}) + \alpha\|\hat{x}_{j'} - x_0\|^2.$$

Since $\hat{x}_{j'} \to x_0$ in $L_2(I, \mathbb{E}^n)$ and $F(\hat{x}_{j'}, \hat{u}_{j'}) \to F(x_0, u_0)$ this leads to

$$x_{j'} \to x_0 \text{ in } L_2(I, \mathbb{E}^n) \text{ and } F(x_{j'}, u_{j'}) \to F(x_0, u_0). \tag{12}$$

Further, we shall consider that $x_{j'}(t) \to x_0(t)$ holds almost everywhere in I.

Hence from (A), $(\overline{7})$ and (10) we get with $\beta = \sup_{t,s \in I} |K(t,s)|$

(I) $$\lim_{j' \to \infty} \int_0^1 K^*(s,t) f^*_{0x}(s, x_{j'}(s)) \overline{\lambda}_{j'}\, ds = \int_0^1 K^*(s,t) f^*_{0x}(s, x_0(s)) \lambda\, ds$$

(II) $$\left\| \int_0^1 K^*(s,t) 2(x_{j'}(s) - x_0(s)) \alpha \overline{\lambda}_{j'}\, ds \right\| \le 2\beta|\alpha| \|x_{j'} - x_0\|_{L_2} \to 0 \text{ when } j' \to \infty$$

(III) $$\left\| \int_0^1 K^*(s,t) g^*_{0x}(s, x_{j'}(s)) \overline{\Phi}_{j'}(s)\, ds - \int_0^1 K^*(s,t) g^*_{0x}(s, x_0(s)) \Phi(s)\, ds \right\|$$

$$\le \left\| \int_0^1 K^*(s,t)\left(g^*_{0x}(s, x_{j'}(s)) - g^*_{0x}(s, x_0(s))\right) \Phi_{j'}(s)\, ds \right\| + \left\| \int_0^1 K^*(s,t) g^*_{0x}(s, x_0(s)) \left(\overline{\Phi}_{j'}(s) - \Phi(s)\right) ds \right\|$$

$$\le \beta \|\overline{\Phi}_{j'}\|_{L_2} \|g^*_{0x}(\cdot, x_j(\cdot)) - g^*_{0x}(\cdot, x_0(\cdot))\|_{L_2} + \left\| \int_0^1 K^*(s,t) g^*_{0x}(s, x_0(s)) \left(\overline{\Phi}_{j'}(s) - \Phi(s)\right) ds \right\| \xrightarrow[j' \to \infty]{} 0.$$

In fact $g^*_{0x}(s, x_{j'}(s)) - g^*_{0x}(s, x_0(s)) \to 0$ for a.e. $s \in I$ and its absolute value is restricted to $2|g_0(s, \cdot)|_{C^1}$ and $\|\overline{\Phi}_{j'}\| \le 1$ for all $j' \in \{j'\}$. Hence (I) - (III), we conclude $\overline{\Phi}_{j'}(t) \to \Phi(t)$ for a.e. $t \in I$, and with

$$\|\overline{\Phi}_{j'}\| \le \beta\left(\int_0^1 |f_0(s,\cdot)|_{C^1}\, ds + 2|\alpha|\, \|x_{j'} - x_0\|_{L_2} + \left(\int_0^1 |g_0(s,\cdot)|^2_{C^1}\, ds\right)^{1/2}\right) \le \text{const} < \infty \text{ for a.e. } t \in I$$

and for all $j' \in \{j'\}$ we have $\overline{\Phi}_{j'} \to \Phi$ in $L_2(I, \mathbb{E}^n)$. From (10) we get $\|\Phi\|^2_{L_2} + |\lambda|^2 = 1$.

Finally, we shall prove the validity of (6). From $(\bar{8})$ we obtain for each $v \in L_\infty(I, \mathbb{E}^r)$ with the property $\bar{v} \in \mathcal{D}'_0$ (see (C)) and $|v(t)| \le M_{j'}$ the inequality

$$\int_0^1 H_j(s, x_{j'}(s), v(s), \overline{\Phi}_{j'}, \overline{\lambda}_{j'})\, ds \le \int_0^1 H_j(s, x_{j'}(s), u_j(s), \overline{\Phi}_{j'}, \overline{\lambda}_{j'})\, ds \tag{13}$$

or from (4) and (9) in more particular form

$$\int_0^1 \Big(\overline{\lambda}_{j'}(f_0(s,x_{j'}(s)) + f_1(s,v(s))) + \overline{\Phi}_j(s)(g_0(s,x_{j'}(s)) + g_1(s,v(s))) + \overline{\lambda}_{j'}(x_j(s) - x_0(s))^2\Big)\, ds$$

$$\le \int_0^1 \Big(\overline{\lambda}_{j'}(f_0(s,x_{j'}(s)) + f_1(s,u_j(s))) + \overline{\Phi}_j(s)(g_0(s,x_{j'}(s)) + g_1(s,u_j(s))) + \overline{\lambda}_{j'}(x_j(s) - x_0(s))^2\Big)\, ds.$$

From previous discussions we can conclude $g_0(\cdot, x_{j'}(\cdot)) \to g_0(\cdot, x_0(\cdot))$ in L_2 such that

$$\int_0^1 \overline{\Phi}_j(s) g_0(s, x_{j'}(s))\, ds \to \int_0^1 \Phi(s) g_0(s, x(s))\, ds$$

is obvious. After that it is easy to prove

$$\lim_{j' \to \infty} \int_0^1 H_j(s, x_{j'}(s), v(s), \overline{\Phi}_{j'}, \overline{\lambda}_{j'})\, ds = \int_0^1 H(s, x_0(s), v(s), \Phi, \lambda)\, ds. \tag{14}$$

We have also

$$\begin{aligned} \lim_{j' \to \infty} \int_0^1 \overline{\lambda}_{j'}(f_0(s,x_{j'}(s)) + f_1(s,u_j(s)))\, ds &= \lim_{j' \to \infty} (\overline{\lambda}_j F(x_{j'}, u_{j'})) \\ = \lambda F(x_0, u_0) &= \int_0^1 \lambda\big(f_0(s,x_0(s)) + f_1(s,u_0(s))\big) ds. \end{aligned} \tag{15}$$

From (5), $(\bar{7})$ and (2), after changing the order of integration we obtain

$$\int_0^1 \overline{\Phi}_j(s) g_1(s, u_j(s))\, ds$$

$$= \int_0^1 \left(\int_0^1 K^*(t,s)\big[f^*_{0x}(t, x_{j'}(t))\bar{\lambda}_{j'} + 2\alpha(x_{j'}(t) - x_0(t))\bar{\lambda}_{j'} + g^*_{0x}(t, x_{j'}(t))\overline{\Phi}_j(t)\big] dt \cdot g_1(s, u_j(s)\right) ds$$

$$= \int_0^1 \left(\bar{\lambda}_{j'} f_{0x}(t, x_{j'}(t)) \int_0^1 K(t,s) g_1(s, u_j(s))\, ds\right) dt$$

$$+ \int_0^1 \left(2\alpha(x_{j'}(t) - x_0(t)) \int_0^1 K(t,s) g_1(s, u_j(s))\, ds\right) dt$$

$$+ \int_0^1 \left(\overline{\Phi}_j(t) g_{0x}(t, x_{j'}(t)) \int_0^1 K(t,s) g_1(s, u_j(s))\, ds\right) dt$$

$$= \int_0^1 \left(\bar{\lambda}_{j'} f_{0x}(t, x_{j'}(t)) \left[x_{j'}(t) - b(t) - \int_0^1 K(t,s) g_0(s, x_{j'}(s))\, ds\right]\right) dt$$

$$+\int_0^1\Big(2\alpha(x_{j'}(t)-x_0(t))\Big[x_{j'}(t)-b(t)-\int_0^1 K(t,s)g_0(s,x_{j'}(s))\,ds\Big]\Big)dt$$

$$+\int_0^1\Big(\overline{\Phi}_{j'}(t)g_{0x}(t,x_{j'}(t))\Big[x_{j'}(t)-b(t)-\int_0^1 K(t,s)g_0(s,x_{j'}(s))\,ds\Big]\Big)dt$$

$$\to\int_0^1\Big(\lambda f_{0x}(t,x_0(t))\Big[x_0(t)-b(t)-\int_0^1 K(t,s)g_0(s,x_0(s))\,ds\Big]\Big)dt$$

$$+\int_0^1\Big(\Phi(t)g_{0x}(t,x_0(t))\Big[x_0(t)-b(t)-\int_0^1 K(t,s)g_0(s,x_0(s))\,ds\Big]\Big)dt$$

$$=\int_0^1\Big(\lambda f_{0x}(t,x_0(t))\int_0^1 K(t,s)g_1(s,u_0(s))\,ds\Big)dt+\int_0^1\Big(\Phi(t)g_{0x}(t,x_0(t))\int_0^1 K(t,s)g_1(s,u_0(s))\,ds\Big)dt$$

$$=\int_0^1\Big(\int_0^1 K^*(t,s)\Big[f_{0x}^*(t,x_0(t))\lambda+g_{0x}^*(t,x_0(t))\Phi(t)\Big]dt\cdot g_1(s,u_0(s))\Big)ds=\int_0^1\Phi(s)g_1(s,u_0(s))\,ds.$$

Taking this into account we get from (15)

$$\int_0^1 H_{j'}(s,x_{j'}(s),u_{j'}(s),\overline{\Phi}_{j'},\overline{\lambda}_{j'})\,ds\to\int_0^1 H(s,x_0(s),u_0(s),\Phi,\lambda)\,ds.$$

Hence (14) and from the last conclusion we have

$$\int_0^1 H(s,x_0(s),v(s),\Phi,\lambda)\,ds\le\int_0^1 H(s,x_0(s),u_0(s),\Phi,\lambda)\,ds\quad\text{for all } v\in L_\infty\cap\mathcal{D}_0'. \tag{16}$$

On the other hand, according to (B) there exists a sequence of regular admissible distributional controls $\hat{u}_j$ such that $\hat{u}_j\to u_0$ in $\mathcal{D}'(\mathbb{E}^r)$. Hence for each $\psi\in\mathcal{D}(\mathbb{E}^1)$ and $\varphi\in\mathcal{D}(\mathbb{E}^n)$ we get

$$(f_1(\hat{u}_j),\psi)=\int_0^1 f_1(s,\hat{u}_j(s))\psi(s)\,ds\to(f_1(u_0),\psi)=\int_0^1 f_1(s,u_0(s))\psi(s)\,ds$$

$$(g_1(\hat{u}_j),\varphi)=\int_0^1 g_1(s,\hat{u}_j(s))\varphi(s)\,ds\to(g_1(u_0),\varphi)=\int_0^1 g_1(s,u_0(s))\varphi(s)\,ds$$

and therefore

$$\lim_{j\to\infty}\int_0^1 H(s,x_0(s),\hat{u}_j(s),\Phi,\lambda)\,ds=\int_0^1 H(s,x_0(s),u_0(s),\Phi,\lambda)\,ds. \tag{17}$$

The conditions (15) and (17) together imply the proposition (6) ∎

4. Sufficient optimality conditions

We consider now the following control problem:

$$F(x,u)=\int_0^1\big(f_0(s)x(s)+f_1(s)u(s)\big)\,ds\longrightarrow\inf \tag{1'}$$

$$x(t)=\int_0^1 K(t,s)\big(g_0(s)x(s)+g_1(s)u(s)\big)\,ds+b(t), \tag{2'}$$

where

$b, x \in C(I, \mathbb{E}^n)$, $u \in \mathcal{D}'(\mathbb{E}^r)$ with $\operatorname{supp} u \subset I = [0,1]$

$f_0 \in L_2(I, \mathbb{E}^n)$, $g_0 \in L_2(I, \mathbb{E}^n)$ and $f_1 \in C^\infty(\mathbb{E}^1, \mathbb{E}^r)$, $g_1 \in C^\infty(\mathbb{E}^1, B^{r,n})$

and the function $K: I \times \mathbb{E}^1 \to B^{n,n}$ satisfies the same conditions as in Section 3. We shall assume again that condition (B) holds. However, the other conditions (a,b,c, A,C) from Section 3 are automatically fulfilled here. The corresponding substitutional problems according to Section 3 we denote by (3)'. The Hamiltonian in our case has now the form

$$H(s,x,u,\Phi,\lambda) = \lambda(f_0(s)x + f_1(s)u) + \Phi(s)(g_0(s)x + g_1(s)u). \tag{4'}$$

Theorem 2 (Sufficient Optimality Condition): *Let $(x_0(\cdot), u_0(\cdot))$ be an admissible process satisfying the condition (2)' and let each corresponding substitutional problem (3)' be solvable. Let there exist $\Phi \in L_2(I, \mathbb{E}^n)$ and $\lambda < 0$ in $\mathbb{E}^1$ such that*

$$\Phi(t) = \int_0^1 K^*(s,t)(f_1^*(s) + g_0^*(s)\Phi(s))\,ds \quad \text{for a.e. } t \in I \tag{5'}$$

and

$$\sup_{v \in L_\infty \cap \mathcal{D}_0'} \int_0^1 H(s, x_0(s), v(s), \Phi, \lambda)\,ds = \int_0^1 H(s, x_0(s), u_0(s), \Phi, \lambda)\,ds. \tag{6'}$$

Then $(x_0(\cdot), u_0(\cdot))$ is an optimal solution of the problem (1)' - (2)'.

Proof: In this case the set $\mathcal{D}_0'$ has the trivial form $\mathcal{D}_0' = \{u \in \mathcal{D}'(\mathbb{E}^r) : \operatorname{supp} u \subset I\}$. Let us assume the contrary. Then an admissible process $(x_1(\cdot), u_1(\cdot))$ exists such that $u_1 \in \mathcal{D}'(\mathbb{E}^r)$ with $\operatorname{supp} u_1 \subset I$ and

$$\int_0^1 (f_0(s)x_1(s) + f_1(s)u_1(s))\,ds < \int_0^1 (f_0(s)x_0(s) + f_1(s)u_0(s))\,ds$$

holds. From this we obtain

$$\int_0^1 f_0(s)(x_0(s) - x_1(s))\,ds + \int_0^1 f_1(s)(u_0(s) - u_1(s))\,ds > 0$$

and, because of (B), there exists an admissible process $(\hat{x}_1, \hat{u}_1)$ satisfying the conditions (2)', $u_1 \in L_\infty^{\text{loc}} \cap \mathcal{D}_0'$ and

$$\int_0^1 f_0(s)(x_0(s) - \hat{x}_1(s))\,ds + \int_0^1 f_1(s)(u_0(s) - \hat{u}_1(s))\,ds > 0. \tag{7'}$$

Condition (6)' implies

$$\int_0^1 \left[H(s, x_0(s), u_0(s), \Phi, \lambda) - H(s, x_0(s), \hat{u}_1(s), \Phi, \lambda)\right] ds \geq 0. \tag{8'}$$

By $\alpha: I \to \mathbb{E}^n$ we denote the function $\alpha(s) = f_0^*(s)\lambda + g_0^*(s)\Phi(s)$. From (4)' and (5)' we have

$$\alpha(s) = f_0^*(s)\lambda + \int_0^1 g_0^*(t)K^*(t,s)\alpha(t)\,dt \tag{9'}$$

and

$$H(s,x,u,\Phi,\lambda) = \lambda(f_0(s)x + f_1(s)u) + \int_0^1 \alpha(t)K(t,s)(g_0(s)x + g_1(s)u)\,dt. \tag{10'}$$

From (10)′ and the inequality (8)′ we receive

$$\int_0^1 \lambda f_1(s)(u_0(s) - \hat{u}_1(s))\,ds + \int_0^1\int_0^1 \alpha(t)K(t,s)g_1(s)(u_0(s) - \hat{u}_1(s))\,dt\,ds \geq 0. \tag{11'}$$

Since $(x_0(\cdot), u_0(\cdot))$ and $(\hat{x}_1(\cdot), \hat{u}_1(\cdot))$ satisfy the condition (2)′ we get

$$x_0(t) - \hat{x}_1(t) = \int_0^1 K(t,s)g_0(s)(x_0(s) - \hat{x}_1(s))\,ds + \int_0^1 K(t,s)g_1(s)(u_0(s) - \hat{u}_1(s))\,ds.$$

Now multiplying both sides of this equation by $\alpha(t)$, after integration we obtain

$$\int_0^1\int_0^1 \alpha(t)K(t,s)g_1(s)(u_0(s) - \hat{u}_1(s))\,dt\,ds$$
$$= \int_0^1 \alpha(t)(x_0(t) - \hat{x}_1(t))\,dt - \int_0^1\int_0^1 \alpha(t)K(t,s)g_0(s)(x_0(s) - \hat{x}_1(s))\,dt\,ds.$$

We substitute this expression in inequality (11)′ and conclude

$$\int_0^1 \lambda f_1(s)(u_0(s) - \hat{u}_1(s))\,ds + \int_0^1 \alpha(t)(x_0(t) - \hat{x}_1(t))\,dt - \int_0^1\int_0^1 \alpha(t)K(t,s)g_0(s)(x_0(s) - \hat{x}_1(s))\,dt\,ds \geq 0. \tag{12'}$$

From (9)′

$$\int_0^1 \alpha(t)(x_0(t) - \hat{x}_1(t))\,dt = \int_0^1 \lambda f_0(s)(x_0(s) - \hat{x}_1(s))\,ds + \int_0^1\int_0^1 \alpha(t)K(t,s)g_0(s)(x_0(s) - \hat{x}_1(s))\,dt\,ds$$

and because of (12)′ we obtain

$$\lambda\left\{\int_0^1 f_0(s)(x_0(s) - \hat{x}_1(s))\,ds + \int_0^1 f_1(s)(u_0(s) - \hat{u}_1(s))\,ds\right\} \geq 0.$$

Since $\lambda < 0$ and (7)′ the last inequality is a contradiction, and therefore Theorem 2 is proved ∎

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