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General Sufficient Conditions for the Convexity of a Function

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Sufficient conditions for a given function to be convex on a given segment, in terms of upper subderivative, are proved.

Key words: *Convex functions, upper subderivatives*, monotonicity of subdifferentials
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1. Introduction

Recently, the problem of characterizing an interesting class of convex functions has arisen (see [3,9]). In the case of $\mathbb{R}^2$, this class consists of, roughly speaking, functions for which the limit of directional derivatives $\lim_{n\to\infty} f'(x+\lambda_n, y+\mu_n);(p,q))$ exists, where $(x,y),(p,q)\in\mathbb{R}^2$ are given and $\{(\lambda_n,\mu_n)\}\subseteq\mathbb{R}^2$ is an arbitrary sequence of points of a given set such that $(\lambda_n,\mu_n)\to 0$. The importance of that one can see, for example, in [9], where one find an algorithm for calculating a subgradient for a function from this class. We must agree upon that taking out "well" behaving convex functions leads to investigations on "bad" one, which may prove difficult (see [11-13]). Another motivation for seeking new conditions for the convexity can be found in [8: Theorem 3.2], where, loosely speaking, we should ensure that a given function on a product set, say on $\mathbb{R}^2$, is upper semicontinuous and convex with respect to the second variable.

Herein, we provide general sufficient conditions for a function f to be convex on a given segment $[a,b]$ of a Banach cpace X (see Theorems 3.1 and 3.2). The basis virtue of them is that to obtain the convexity it is enough that, in case of upper semicontinuity, the inequality

$$\limsup_{\substack{x\to a+s(b-a)\\ x\in \operatorname{dom} f\cap \operatorname{dom}\partial f,\, x^*\in\partial f(x)\\ f(x)\to f(a+s(b-a))}} \langle x^*, b-a\rangle \le \liminf_{\substack{y\to a+t(b-a)\\ y\in \operatorname{dom} f\cap \operatorname{dom}\partial f,\, y^*\in\partial f(y)\\ f(y)\to f(a+t(b-a))}} \langle y^*, b-a\rangle$$

holds for any $0\le s<t\le 1$ (see Theorems 3.1 and 3.2 and Lemma 3.3). The conditions encompass the lower and upper semicontinuity case on a Banach space. When f is Lipschitzian it yields the monotonicity of the subdifferential, see [1,2,4 - 7].

Recently, there has been obtained a new result characterizing the convexity of lower semicontinuous functions on the whole space (see [4,5]), i.e., *a lower semicontinuous function is convex if and only if the subdifferential is monotone.* The result has been obtained for finite-dimensional spaces (see [4]) or reflexive Banach spaces (see [5]). In the case when X is one-dimensional we compare it with the above condition.

2. Basic facts on upper subderivatives

The apparatus of the paper is taken from Nonsmooth Analysis (see [1, 2, 7]). Below we shall summarize those basic facts about generalized derivatives which are used in the sequel.

Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function on a Banach space X. The upper subderivative of f at $x \in X$, $f(x) \in \mathbb{R}$, with respect to $v \in X$ is defined by

$$f^{\uparrow}(x;v) = \sup_{\varepsilon > 0} \limsup_{\substack{(y, f(y)) \to (x, f(x)) \\ t \downarrow 0}} \inf_{\|u - v\| < \varepsilon} \frac{f(y + tu) - f(y)}{t},$$

and the subdifferential $\partial f(x)$ by

$$\partial f(x) = \left\{ x^* \in X^* \mid \langle x^*, v\rangle \le f^{\uparrow}(x;v) \text{ for all } v \in X \right\},$$

where $\langle x^*, v\rangle$ denotes the value of the linear continuous functional x^* at v. Let us recall the mean value theorem for these two notions.

Theorem 2.1 [10]: *Let X be a real Banach space, $a, b \in X$, $a \neq b$ and $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and finite at a and b. Then, for every $x \in [a, b]$ such that $x \neq b$ and the inequality*

$$f(x) + \frac{f(b) - f(a)}{\|b - a\|}\|x - b\| \le f(y) + \frac{f(b) - f(a)}{\|b - a\|}\|y - b\| \tag{2.1}$$

holds for all $y \in [a, b]$, *there exist sequences $\{x_k\} \subseteq X$ and $\{x_k^*\} \subseteq X^*$ such that*

$$\lim_{k \to \infty} x_k = x \quad \text{and} \quad \limsup_{k \to \infty} f(x_k) \le f(a) + \frac{f(b) - f(a)}{\|b - a\|}\|x - a\|$$

$$x_k^* \in \partial f(x_k) \ \forall k \quad \text{and} \quad \liminf_{k \to \infty} \langle x_k^*, b - a\rangle \ge f(b) - f(a).$$

Throughout the paper for any $a, b \in X$ we denote by $[a, b]$ the set $\{a + t(b - a) \mid 0 \le t \le 1\}$. We write $x <_{[a,b]} y$ if there exist $0 \le s < t \le 1$ such that $x = a + s(b - a)$ and $y = a + t(b - a)$. Further we denote by $\operatorname{dom}\partial f$ the set $\{x \in X \mid \partial f(x) \neq \emptyset\}$.

3. Convexity on a segment

In this section we provide sufficient conditions for lower and upper semicontinuous functions to be convex on a given segment $[a, b]$. Before we do it, let us refer to known facts on convex functions.

From the classical differential calculus we know that the convexity of a real function is related to the monotonicity of its derivative. When we use a more sophisticated tool, for example, the subdifferential calculus, we still have to do with monotonicity (see, e.g., [7: Proposition 7A]). This strongly suggest that the monotonicity of a derivative is essential for the convexity. However, it is worth mentioning that the subdifferential can be empty on a given segment (see [10: Example 4.1]), so we can not follow directly the methods of subdifferential calculus and some refinements are needed. Let us also notice that the funcion $f\colon \mathbb{R}^2 \to \mathbb{R}$, where $f(x,y) = -|y|^{1/2} - x^2$ is not convex on the line $L = \{(x,0) \mid x \in \mathbb{R}\}$ but $f^{\uparrow}$ is equal to $-\infty$ on it. So we have the monotonicity, but $f_{|L}$ is not convex (examples where f' does not exist can be obtained from this one replacing $-x^2$ by a proper non-convex function $\varphi: \mathbb{R} \to \mathbb{R}$ which does not possess the right derivative).

Theorem 3.1: *Let X be a Banaxh space and $a, b \in X$ with $a \neq b$.* Assume that $f: X \to \mathbb{R} \cup \{+\infty\}$ *is lower semicontinuous. If the implication*

$$x <_{[a,b]} y \text{ for all } x, y \in [a,b] \cap \operatorname{dom} f \Longrightarrow \limsup_{\substack{(u,f(u))\to(x,f(x))\\ u\in\operatorname{dom}\partial f,\, u^*\in\partial f(u)}} \langle u^*, b-a\rangle \le \liminf_{\substack{(v,f(v))\to(y,f(y))\\ v\in\operatorname{dom}\partial f,\, v^*\in\partial f(v)}} \langle v^*, b-a\rangle \tag{3.1}$$

holds, then $f_{|[a,b]}$ is convex.

Proof: Let us introduce an auxiliary function g on $[0,1]$ by $g(s) = f(a + s(b-a))$. If $f_{|[a,b]}$ is not convex, then there exist $0 \le s_1 < s_2 < s_3 \le 1$ such that

$$g(s_2) > \frac{s_3 - s_2}{s_3 - s_1} g(s_1) + \frac{s_2 - s_1}{s_3 - s_1} g(s_3). \tag{3.2}$$

This implies that $g(s_1), g(s_3) \in \mathbb{R}$ and $(g(s_2) - g(s_1))/(s_2 - s_1) > (g(s_3) - g(s_2))/(s_3 - s_2)$. Let $x_i = a + s_i(b-a)$ for $i = 1,2,3$. We get

$$\frac{f(x_2) - f(x_1)}{\|x_2 - x_1\|} > \frac{f(x_3) - f(x_2)}{\|x_3 - x_2\|} \tag{3.3}$$

Now let us consider the case when $f(x_2) \in \mathbb{R}$, choose $x \in [x_1, x_2)$ and $y \in (x_2, x_3]$ such that

$$f(x) + \frac{f(x_2) - f(x_1)}{\|x_2 - x_1\|}\|x - x_2\| \le f(z) + \frac{f(x_2) - f(x_1)}{\|x_2 - x_1\|}\|z - x_2\| \text{ for every } z \in [x_1, x_2]$$

$$f(y) + \frac{f(x_2) - f(x_3)}{\|x_2 - x_3\|}\|y - x_2\| \le f(z) + \frac{f(x_2) - f(x_3)}{\|x_2 - x_3\|}\|z - x_2\| \text{ for every } z \in [x_3, x_2].$$

In particular, we have

$$f(x) + \frac{f(x_2) - f(x_1)}{\|x_2 - x_1\|}\|x - x_2\| \le f(x_2) \quad\text{and}\quad f(y) + \frac{f(x_2) - f(x_3)}{\|x_2 - x_3\|}\|y - x_2\| \le f(x_2),$$

thus the auxiliary functions p and q, where

$$p(z) = \begin{cases} f(x) + \dfrac{f(x_2) - f(x_1)}{\|x_2 - x_1\|}\|x - x_2\|, & z = x_2 \\ f(z), & z \neq x_2 \end{cases} \quad\text{and}\quad q(z) = \begin{cases} f(y) + \dfrac{f(x_2) - f(x_3)}{\|x_2 - x_3\|}\|y - x_2\|, & z = x_2 \\ f(z), & z \neq x_2 \end{cases}$$

are lower semicontinuous and $\partial p(z) = \partial f(z) = \partial q(z)$ for $z \neq x_2$. Theorem 2.1 applied for the functions p and q ensures the existence of sequences $\{u_k\} \subseteq \operatorname{dom} p$, $\{v_k\} \subseteq \operatorname{dom} q$, $\{u_k^*\}$ and $\{v_k^*\}$ such that

$$\lim_{k\to\infty} (u_k, p(u_k)) = (x, f(x)) \tag{3.4}$$

$$\lim_{k\to\infty} (v_k, q(v_k)) = (y, f(y)) \tag{3.5}$$

$$u_k^* \in \partial p(u_k) \text{ and } v_k^* \in \partial q(v_k) \text{ for all } k \tag{3.6}$$

$$\limsup_{k\to\infty}\left\langle u_k^*, \frac{x_2 - x}{\|x_2 - x\|}\right\rangle \geq \frac{p(x_2) - p(x)}{\|x_2 - x\|} = \frac{f(x_2) - f(x_1)}{\|x_2 - x_1\|}$$

$$\limsup_{k\to\infty}\left\langle v_k^*, \frac{x_2 - y}{\|x_2 - y\|}\right\rangle \geq \frac{q(x_2) - q(y)}{\|x_2 - y\|} = \frac{f(x_2) - f(x_3)}{\|x_2 - x_3\|}.$$

The last two inequalities, by (3.2), imply $\limsup_{k\to\infty}\langle u_k^*, b - a\rangle > \liminf_{k\to\infty}\langle v_k^*, b - a\rangle$, which, by (3.4) - (3.6), contradicts (3.1). When $f(x_2) = +\infty$ we can run a proof as before, replacing f by $\tilde{f}$ given as $\tilde{f}(z) = f(z)$ for $z \neq x_2$ and $\tilde{f}(z) = \alpha$ for $z = x_2$, where α is such that (3.2) still holds ∎

In the view of the above proof, it might seem that instead of f we can consider the function g, $g(t) = f(ta + (1 - t)b)$ on $[0,1]$. In this case, we may admit in (3.1) only those pairs $(t, g(t))$ for which $t \in (0,1)$. However, in some particular situation, this restriction would lead to a false assertion. For example, let us consider the function $f\colon \mathbb{R} \to \mathbb{R}$, where $f(t) = 1$ for $t < 0$ and $f(t) = -t^{1/2}$ for $t \geq 0$. This function is lower semicontinuous but is not convex on the segment $[-1,0]$, condition (3.1) is violated for $x = -1/2$ and $y = 0$. The function g is equal to 0 for $t = 0$ and to 1 for $0 < t \leq 1$. Since $\partial g(t) = \{0\}$ for $t \in (0,1)$, so (3.1) is fulfilled. It is worth noticing that if we reduce our considerations to the one-dimensional case, then (3.1) is equivalent to the monotonicity of the multifunction $t \to \partial f(t)$ (if wee assume that (3.1) holds for every $a, b \in \mathbb{R}$). In fact, if the multifunction is monotone (i.e., for all $t_1, t_2 \in \mathbb{R}$, the inclusions $t_1^* \in \partial f(t_1)$ and $t_2^* \in \partial f(t_2)$ imply the inequality $(t_1^* - t_2^*)(t_1 - t_2) \geq 0$), then

$$\limsup_{\substack{(t,f(t))\to(x,f(x))\\ t\in\operatorname{dom}\partial(f),\ t^*\in\partial(f)(t)}} \langle t^*, b - a\rangle \leq \limsup_{\substack{t\to x\\ t^*\in\partial(f)(t)}} \langle t^*, b - a\rangle$$

$$\leq \liminf_{\substack{t\to y\\ t^*\in\partial(f)(t)}} \langle t^*, b - a\rangle \leq \liminf_{\substack{(t,f(t))\to(y,f(y))\\ t\in\operatorname{dom}\partial(f),\ t^*\in\partial(f)(t)}} \langle t^*, b - a\rangle.$$

So the monotonicity forces that (3.1) holds, and vice verse, if (3.1) holds, then

$$\sup_{t_1^*\in\partial f(t_1)} \langle t_1^*, t_1 - t_2\rangle \leq \inf_{t_2^*\in\partial f(t_2)} \langle t_2^*, t_1 - t_2\rangle.$$

Thus, in the one-dimensional case, Theorem 3.1 is equivalent to Poliquin's result ([5]; for an extension see [4]) which states that a lower semicontinuous function f on $\mathbb{R}^n$ is convex if and only if the multifunction $\partial f(\cdot)$ is monotone.

The next theorem deals with upper semicontinuous functions. Unexpectedly, the monotonicity alone is not sufficient for its convexity. We give a proper example after that theorem.

Theorem 3.2: *Let X be a Banach space and $a, b \in X$ with $a \neq b$. Assume that $f\colon X \to \mathbb{R} \cup \{-\infty\}$ is an upper semicontinuous function such that the following implication holds:*

For every $x_1 <_{[a,b]} x_2 <_{[a,b]} x_3$ there is an $\tilde{x}_2 \in [x_1, x_3]\setminus\{x_2\}$ such that

$$\frac{f(x_2) - f(x_1)}{\|x_2 - x_1\|} > \frac{f(x_3) - f(x_2)}{\|x_3 - x_2\|} \Longrightarrow \left|\begin{array}{l} \dfrac{f(\tilde{x}_2) - f(x_1)}{\|\tilde{x}_2 - x_1\|} > \dfrac{f(x_3) - f(x_2)}{\|x_3 - x_2\|} \text{ and } x_1 <_{[a,b]} \tilde{x}_2 <_{[a,b]} x_2 \\ \textbf{or} \\ \dfrac{f(x_2) - f(x_1)}{\|x_2 - x_1\|} > \dfrac{f(x_3) - f(\tilde{x}_2)}{\|x_3 - \tilde{x}_2\|} \text{ and } x_2 <_{[a,b]} \tilde{x}_2 <_{[a,b]} x_3. \end{array}\right. \tag{3.7}$$

If the implication

$$x <_{[a,b]} y \quad \forall\, x, y \in [a,b] \cap \operatorname{dom}(-f)$$
$$\Longrightarrow \limsup_{\substack{(u,f(u))\to(x,f(x))\\ u\in\operatorname{dom}\partial(-f),\, u^*\in\partial(-f)(u)}} \langle -u^*, b-a\rangle \le \liminf_{\substack{(v,f(v))\to(y,f(y))\\ v\in\operatorname{dom}\partial(-f),\, v^*\in\partial(-f)(v)}} \langle -v^*, b-a\rangle \tag{3.8}$$

holds, then $f_{|[a,b]}$ is convex.

Proof: Let us introduce an auxiliary function g by $g(s) = f(a + s(b-a))$ for all $s \in [0,1]$. If $f_{|[a,b]}$ is not convex, then there exist $0 \le s_1 < s_2 < s_3 \le 1$ such that

$$g(s_2) > \frac{s_3 - s_2}{s_3 - s_1} g(s_1) + \frac{s_2 - s_1}{s_3 - s_1} g(s_3).$$

This implies that $g(s_2) \in \mathbb{R}$ and $\frac{g(s_2) - g(s_1)}{s_2 - s_1} > \frac{g(s_3) - g(s_2)}{s_3 - s_2}$. Let $x_i = a + s_i(b-a)$ for $i = 1, 2, 3$. We get

$$\frac{(-f)(x_1) - (-f)(x_2)}{\|x_2 - x_1\|} > \frac{(-f)(x_2) - (-f)(x_3)}{\|x_3 - x_2\|}.$$

Now, by (3.7), there are $\tilde{x}_2, \tilde{\tilde{x}}_2 \in [x_1, x_3] \cap \operatorname{dom}(-f)$ such that

$$x_1 <_{[a,b]} \tilde{x}_2 <_{[a,b]} \tilde{\tilde{x}}_2 <_{[a,b]} x_3 \quad \text{and} \quad \frac{(-f)(x_1) - (-f)(\tilde{x}_2)}{\|x_2 - \tilde{x}_1\|} > \frac{(-f)(\tilde{\tilde{x}}_2) - (-f)(x_3)}{\|\tilde{\tilde{x}}_3 - x_2\|}.$$

Repeating he method of the proof of Theorem 3.1 we get sequences $\{u_k\}, \{u_k^*\}, \{v_k\}, \{v_k^*\}$ such that $\{u_k\}$ and $\{v_k\}$ converge to $x \in [x_1, \tilde{x}_2]$ and $y \in [\tilde{\tilde{x}}, x_3]$, respectively, and $u_k^* \in \partial(-f)(u_k)$, $v_k^* \in \partial(-f)(v_k)$ for every $k \in \mathbb{N}$, and

$$\limsup_{k\to\infty} \left\langle u_k^*, \frac{x_1 - \tilde{x}_2}{\|x_1 - \tilde{x}_2\|} \right\rangle \ge \frac{(-f)(x_1) - (-f)(\tilde{x}_2)}{\|x_1 - \tilde{x}_2\|}$$

$$\limsup_{k\to\infty} \left\langle v_k^*, \frac{x_3 - \tilde{\tilde{x}}_2}{\|x_3 - \tilde{\tilde{x}}_2\|} \right\rangle \ge \frac{(-f)(x_3) - (-f)(\tilde{\tilde{x}}_2)}{\|x_3 - \tilde{\tilde{x}}_2\|}$$

The last two inequalities are a contradiction to (3.8) ∎

Let us notice that if a function f is continuous, then (3.7) holds automatically. However, this assumption can not be dropped when the function is upper semicontinuous. Let us consider the non-convex function $f(x) = 0$ for $x \neq 0$ and $f(x) = 1$ for $x = 1$, which satisfies only (3.8). For the time being, we focus our attention on locally Lipschitzian functions. We know that in this case $-\partial f(x) = \partial(-f)(x)$ (see [2: Proposition 2.3.1]). Thus Theorems 3.1 and 3.2 yield the same condition. Moreover, (3.1) is equivalent to the monotonicity of the multifunction ∂f on $[a,b]$ (see [2: Proposition 2.1.5]). Now, we may say that Theorems 3.1 and 3.2 in the case are neither more nor less than the "classical" sufficient condition for the convexity (see [7: Proposition 7A]).

Finally observe that the relation $\partial(-f)(x) = -\partial f(x)$ may fail, even when the function is continuous. However, (3.1) and (3.8) are equivalent in this case, as it will be shown in the next lemma: But for some classes of functions conditions (3.1) and (3.8) are not equivalent. Indeed, let us consider the function $f\colon \mathbb{R} \to \mathbb{R}$, where $f(x) = -x$ for $x \in [0,1)$ and $f(x) = 0$ for $x \in [1,2]$. Conditions (3.1) and (3.7) are fulfilled, with $a = 0$ and $b = 2$, but the function f is not convex. Thus (3.8) is violated since

$$\partial(-f)(1) = (-\infty, 0] \quad \text{and} \quad \liminf_{\substack{(t,(-f)(t))\to(3/2,(-f)(3/2))\\ t^*\in\partial(-f)(t)}} \langle -t^*, 2\rangle = 0.$$

Lemma 3.3: *Let X be a Banach space and $f\colon X\to \mathbb{R}$ a continuous function. Then*

$$\limsup_{\substack{u\to x\\ u\in \mathrm{dom}\,\partial f,\, u^*\in\partial f(u)}} \langle u^*, p\rangle = \limsup_{\substack{v\to x\\ v\in \mathrm{dom}\,\partial(-f),\, v^*\in\partial(-f)(v)}} \langle -v^*, p\rangle$$

and

$$\liminf_{\substack{u\to x\\ u\in \mathrm{dom}\,\partial f,\, u^*\in\partial f(u)}} \langle u^*, p\rangle = \liminf_{\substack{v\to x\\ v\in \mathrm{dom}\,\partial(-f),\, v^*\in\partial(-f)(v)}} \langle -v^*, p\rangle$$

for every $p \in X$.

Proof: Let us consider the non-trivial case when $p \neq 0$. Let α be equal to the left-hand side of the first equality and β to the right-hand one. For any $\varepsilon > 0$ we can find sequences $\{x_n\}, \{t_n\} \subseteq X$ such that

$$x_n \to x,\ t_n \downarrow t \quad \text{and} \quad \frac{(-f)(x_n) - (-f)(x_n + t_n p)}{t_n} \geq \alpha - \varepsilon \text{ for all } n.$$

By Theorem 2.1 we can find sequences $\{v_n\} \subseteq \mathrm{dom}\,\partial(-f)$ and $\{v_n^*\} \subseteq X^*$ such that

$$v_n \to x \quad \text{and} \quad v_n^* \in \partial(-f)(v_n),\ \limsup_{n\to\infty} \langle v_n^*, -p\rangle \geq \alpha - \varepsilon.$$

Thus $\beta \geq \alpha$, similarly $\alpha \geq \beta$. The second equality can be obtained from the first one when we take $-p$ instead of p ∎

Below we present a result which can be helpful to prove the convexity of a continuous function (see, for example, [11]).

Corollary 3.4: *Let $a, b \in \mathbb{R}$, $a \neq b$, and $f\colon [a,b] \to \mathbb{R}$ a continuous function such that the limit*

$$f'(x;1) = \lim_{t\downarrow 0} \frac{f(x+t) - f(x)}{t}$$

exists for every $x \in [a,b]$. If the function $f'(\cdot;1)$ is non-decreasing, then (3.1) *holds on every segment $[c,d] \subset [a,b]$ and f is convex.*

Proof: Assume that for some $c < d$ (3.1) is violated. By Lemma 3.3 for some $x, y \in [c,d]$ with $x <_{[c,d]} y$ we get

$$\alpha \equiv \limsup_{\substack{u\to x\\ u\in \mathrm{dom}\,\partial f,\, u^*\in\partial f(u)}} \langle u^*, d-c\rangle > \liminf_{\substack{v\to y\\ v\in \mathrm{dom}\,\partial(-f),\, v^*\in\partial(-f)(v)}} \langle -v^*, d-c\rangle \equiv \beta.$$

We infer the existence of sequences $\{x_n\}, \{y_n\}, \{t_n\}, \{s_n\}$ such that $x_n \to x$, $y_n \to y$, $t_n \downarrow 0$, $s_n \downarrow 0$ and numbers $M \in (\alpha, \beta)$, $\varepsilon > 0$ such that, for all n,

$$\frac{f(x_n + t_n(d-c)) - f(x_n)}{t_n} \geq M + \varepsilon \quad \text{and} \quad \frac{(-f)(y_n + s_n(d-c)) - (-f)(y_n)}{s_n} \geq -M + \varepsilon.$$

The continuity of f forces that there exist $\tilde{x}_n \in [x_n, x_n + t_n(d-c))$ and $\tilde{y}_n \in [y_n, y_n + s_n(d-c))$

such that

$$f'(\tilde{x}_n; d-c) \geq M+\varepsilon \quad \text{and} \quad (-f)'(\tilde{y}_n; d-c) \geq -M+\varepsilon$$

Thus

$$f'(\tilde{x}_n; 1) \geq \frac{M+\varepsilon}{d-c} \geq \frac{M-\varepsilon}{d-c} \geq f'(\tilde{y}_n; 1) \text{ for all } n.$$

This is a contradiction ∎

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