

Spectral Estimates for Compact Hyperbolic Space Forms and the Selberg Zeta Function for p -Spectra

Part I

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We prove an asymptotic estimation for the length spectrum with certain weights for a compact hyperbolic space form. Thereby the Selberg trace formula and a Landau difference method is used.

Key words: Length spectrum, hyperbolic space form, Selberg trace formula

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1. Introduction

Let \mathcal{G} be a properly discontinuous group of orientation preserving isometries of the n -dimensional hyperbolic space \mathbf{H}_n of constant curvature -1 without fixed points (with the exception of the identity map id) with compact fundamental domain. We consider the related Killing-Hopf space form $V = \mathbf{H}_n/\mathcal{G}$. Let Ω be the set of non-trivial free homotopy classes of V . In every class $\omega \in \Omega$ there lies exactly one closed geodesic line. We denote by $l(\omega)$ and $\nu(\omega)$ its length and multiplicity, respectively. The parallel displacement along a closed geodesic line induces an isometry of the tangent space in every point of that geodesic line with the eigenvalues $\beta_1(\omega), \dots, \beta_{n-1}(\omega), 1$ with $|\beta_i(\omega)| = 1$ ($i = 1, \dots, n-1$). Let $e_p(\omega)$ be the p^{th} elementary symmetric function of the $\beta_i(\omega)$ ($i = 1, \dots, n-1$), and put $e_0(\omega) = 1$. Further on, we introduce the weight

$$\sigma(\omega) = \frac{e^{N l(\omega)}}{\nu(\omega)} \prod_{j=1}^{n-1} \frac{1}{(e^{l(\omega)} - \beta_j(\omega))}$$

with $N = (n-1)/2$. Let S_p denote the p -spectrum of the Laplace operator $\Delta = d\delta + \delta d$. Thereby we have used the differential operator d and the codifferential operator $\delta = (-1)^{pn+n-1} * d*$ for differential p -forms, where $*$ denotes the Hodge dualization. Let $d_p^c(\mu)$ and $d_p^s(\mu)$ denote the dimension of the eigenspaces of closed ($d\alpha = 0$) and coclosed ($\delta\alpha = 0$) eigenforms α of Δ with eigenvalues μ , respectively.

Our results are based on the Selberg trace formula as a duality statement between the p -eigenvalue spectrum and the geometric spectrum of V (expressed by $l(\omega), \nu(\omega), \sigma(\omega)$ and $e_p(\omega)$). We will recall that formula in Section 2. For $a > 0$ we define $E(t, a) = \int_1^t e^{as}/s ds$. In Section 5 we will give estimations for the length spectrum with weights e_p . The main result will be

Theorem A: *We can estimate the sum*

$$P_p(T) = \sum_{\substack{\omega \in \Omega \\ \cosh l(\omega) \leq T}} e_p(\omega) \quad \text{for } T \rightarrow \infty$$

by

$$P_p(T) = \begin{cases} O(T^{N+\frac{(n-1)^2}{2n}} / \ln T) & \text{for } 1 \leq p \leq n-2 \\ E(t, n-1) + \sum_{\substack{\mu \in S_p \\ 0 < \mu < N^2(2n-1)/n^2}} d_p^\delta(\mu) E(t, N + \sqrt{N^2 - \mu}) \\ + O(T^{N+\frac{(n-1)^2}{2n}} / \ln T) & \text{for } p = 0, n-1 \end{cases}$$

with $N = \frac{n-1}{2}$, $\cosh t = T$, $T > 1$. We get the same estimation if we replace $P_p(T)$ by

$$P_p^\#(T) = \sum_{\substack{\omega \in \Omega \\ \cosh t(\omega) \leq T, \nu(\omega) = 1}} e_p(\omega).$$

GÜNTHER [12] has treated the case $p = 0$ based on a Poisson formula (cf. the literature quoted there). The introduction of $P_p(T)$ as an appropriate generalization of $P_0(T)$ is motivated by the Selberg trace formula (cf. Theorem 1). Indeed, if we exchange the 0-eigenvalue spectrum by the p -eigenvalue spectrum on one side of the Selberg trace formula, on the other side there appear additional weights $e_p(\omega)$. It seems natural that we get the same error estimate for all p (as a generalization of [12]). Of course, the results for $p = 0$ and $p = n-1$ are similar. But it is interesting that for $1 \leq p \leq n-1$ the weights $e_p(\omega)$ (which are in contrast to the case $p = 0$ no longer necessarily positive) induce a growth limitation of $P_p(T)$ for $T \rightarrow \infty$ in the sense that there is no leading term of the asymptotic development larger than the error term. In order to prove Theorem A, we use a Landau difference method and a solution of an Euler-Poisson-Darboux equation as a special function which we can use in the Selberg trace formula. We will see that in some cases these functions are better adapted to the geometric situation than functions which are usually taken in trace formulas when a Selberg zeta function is considered. In Section 6 (Part II) we will introduce a Selberg zeta function in a natural way with respect to our version of the Selberg trace formula. This zeta function is well known for the case $n = 2, p = 0$. GANGOLLI [9] treats zeta functions of Selberg's type for compact space forms of symmetric spaces of rank one from the viewpoint of representation theory. To see differences to our treatment one should compare the zeros and poles of the analytic continuation of the zeta function to the whole complex plane. The Selberg trace formula bears a striking resemblance to the explicite formulas of prime number theory. The Selberg zeta function is analogous in many ways to the classical Riemann zeta function. This enables us to study the asymptotic behaviour of the p -spectrum using techniques of analytic number theory. As a consequence of the well-known Weyl type asymptotic formula (cf. [2, 28] and Section 4) we have

$$\mathcal{N}_p(T) = \sum_{\substack{\mu \in S_p \\ \mu < T^2 + (p-N)^2}} d_p^\delta(\mu) \sim n_p T^n \quad \text{for } T \rightarrow \infty \quad (1)$$

with

$$n_p = \frac{\binom{n-1}{p} \text{vol } V}{(4\pi)^{n/2} \Gamma(\frac{n+2}{2})} \quad \text{and} \quad N = \frac{n-1}{2}. \quad (2)$$

We will prove (in Part II) the

Theorem B: The error term $\mathcal{R}_p(T)$ defined by $\mathcal{N}_p(T) = n_p T^n + \mathcal{R}_p(T)$ ($T > 1$) satisfies $|\mathcal{R}_p(T)| = O(T^{n-1} / \ln T)$.

HEJHAL [14] has given this estimation in the case $n = 2, p = 0$. Weaker results for more general spaces were proved by GANGOLLI [8] and IVRII [17] for $n \geq 2, p = 0$. HEJHAL [14] remarked (for $n = 2$) that it seems hard to improve the estimation of Theorem B. The analogy between the Selberg and the Riemann zeta function is strongly apparent in our proofs. If one were able to improve the $T^{n-1}/\ln T$ - term in Theorem B, there would presumably be a corresponding improvement in the estimation $\arg \zeta(1/2 + iT) = O(\ln T / \ln \ln T)$ for the Riemann zeta function, assuming the Riemann hypothesis is valid. But no such improvement is known.

2. The Selberg trace formula

We begin by recalling the Selberg trace formula for the p -spectrum of compact real hyperbolic space forms. Selberg was first led to the trace formula around 1950 - 51. The idea of taking the trace seemed quite natural, since it looked like it would be too difficult to get hold of the individual eigenfunctions of the Laplace operator (or other related operators). Trace formulas bear a very striking resemblance to the so-called explicite formulas of prime number theory. Selberg has stated that his discovery was motivated by the classical theory of automorphic forms, cf. [14].

For further information about the historical development and about modern topics, we refer to [4, 9, 11, 13, 14, 18, 21, 24, 25, 27]. The general approach to trace formulas by means of analytic number theory begins by the development of the relevant trace formula as a duality statement (in our case between the p -eigenvalue spectrum of the Laplace operator and the geometric spectrum). Then one has to polarize the trace formula so as to define a natural zeta function with good properties under analytic continuation. With this background one is able to exploit the analogy with analytic number theory when formulating the deeper results.

In [5] the trace formula for the p -spectrum for odd $n > 3$ was given in the context of representation theory without details of the proof. In [21] we have used geodesic double differential forms, mean value operators for differential p -forms and Euler-Poisson-Darboux equations in order to prove the trace formula which we will state in Theorem 1. The used solution of a special Euler-Poisson-Darboux equation seems to be very natural for the treatment of hyperbolic space forms. We will see later on that in some cases these functions are better adapted to the geometric situation than the functions which are usually taken in the trace formulas when the Selberg zeta function is introduced. In other cases it is more appropriate to use functions which are natural with respect to the tools of analytic number theory. By the telescoping theorem of McKean and Singer (cf. [2]) we have $d_p^\delta(\mu) = d_{p+1}^\delta(\mu)$ for $\mu \in S_p \setminus \{0\}$ and $p = 0, 1, \dots, n-1$. The dimension of the space of harmonic p -eigenforms is the p^{th} Betty number B_p of the space form V . A short calculation gives the following equations for the weight $\sigma(\omega)$:

$$\sigma(\omega) = \frac{1}{\nu(\omega)e^{Nl(\omega)}} \prod_{j=1}^{n-1} (1 - \beta_j(\omega)e^{-l(\omega)})^{-1}, \quad (3)$$

$$\sigma(\omega) = \frac{1}{\nu(\omega)2^N} \prod_{j=1}^{n-1} (\cosh l(\omega) - \operatorname{Re} \beta_j(\omega))^{-1/2}. \quad (4)$$

Now we can state the Selberg trace formula as a duality between the eigenvalue spectrum and the geometric spectrum.

Theorem 1 (Selberg trace formula): *Let $h(r)$ be an analytic function in the strip $| \operatorname{Im} r | < N + \delta$ with $N = (n-1)/2$, $0 < \delta < 1/2$, which is even, $h(r) = h(-r)$, and satisfies*

$|h(r)| \leq A(1+|r|)^{-n-\delta}$. By the help of the Fourier transform $g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r)e^{-ir u} dr$ of $h(r)$ we can state the trace formula

$$\sum_{\mu \in S_p} d_p^*(\mu) h(r_p(\mu)) = \text{vol } V(S_p^n, g) + \sum_{\omega \in \Omega} l(\omega) \sigma(\omega) e_p(\omega) g(l(\omega)) \quad (5)$$

for $p = 0, \dots, n-1$ with $r_p(\mu) = \sqrt{\mu - (p-N)^2}$, where

$$(S_p^n, g) = \frac{2^{\binom{n-1}{p}}}{(4\pi)^{n/2} \Gamma(n/2)} \begin{cases} \int_0^{\infty} \left(\prod_{\substack{u=0 \\ u \neq |p-N|}}^N (r^2 + u^2) \right) h(r) dr & \text{for } n \text{ odd} \\ \int_0^{\infty} \left(\prod_{\substack{u=1/2 \\ u \neq |p-N|}}^N (r^2 + u^2) \right) h(r) r \tanh(\pi r) dr & \text{for } n \text{ even.} \end{cases} \quad (6)$$

Thereby we have used $N = \frac{n-1}{2}$ and

$$d_p^*(\mu) = \begin{cases} d_p^S(\mu) & \text{for } \mu > 0 \\ (-1)^p (B_0 - B_1 + \dots + (-1)^p B_p) + K_p & \text{for } \mu = 0 \end{cases}$$

with

$$K_p = \begin{cases} (-1)^{p+1-n/2} \pi^{-(n+1)/2} \Gamma(\frac{n+1}{2}) \text{vol } V & \text{for } p \geq n/2 \text{ (n even)} \\ 0 & \text{for the other cases.} \end{cases}$$

Further on, $\text{vol } V$ denotes the volume of the space form V .

3. Euler-Poisson-Darboux equations

We will use the uniquely determined solution $z(t, \lambda, \mu, n)$ of the Euler-Poisson-Darboux equation

$$\left(\frac{d^2}{dt^2} + \lambda \coth t \frac{d}{dt} + \mu + \frac{\lambda^2 - (n-1)^2}{4} \right) z(t, \lambda, \mu, n) = 0 \quad (7)$$

for $\lambda \geq 0$ with the initial conditions

$$z(0, \lambda, \mu, n) = 1, \quad \frac{d}{dt} z(t, \lambda, \mu, n)|_{t=0} = 0. \quad (8)$$

We need this function in our treatment of mean value operators for differential forms in the hyperbolic space \mathbf{H}_n (cf. [21]). We can express $z(t, \lambda, \mu, n)$ in terms of the Gauß hypergeometric function F :

$$z(t, \lambda, \mu, n) = \left(\frac{\cosh t + 1}{2} \right)^{\frac{1-\lambda}{2}} F \left(\frac{1}{2} - \chi(\mu), \frac{1}{2} + \chi(\mu), \frac{\lambda+1}{2}, \frac{1 - \cosh t}{2} \right) \quad (9)$$

with

$$N = \frac{n-1}{2}, \quad \chi(\mu) = \sqrt{N^2 - \mu} \text{ for } \mu \leq N^2 \text{ and } \chi(\mu) = i\sqrt{\mu - N^2} \text{ for } \mu > N^2.$$

For our purpose, it is more convenient to use

$$Z(T, \lambda, \mu, n) = \sinh^{2\lambda-1} t \, z(t, 2\lambda, \mu, n) \quad (10)$$

with

$$T = \cosh t \quad (11)$$

instead of $z(t, \lambda, \mu, n)$. In our former paper [21] we have expressed the results in terms of $z(t, \lambda, \mu, n)$, but it is quite obvious how to transform the basic equations, which we will recall here. The Euler-Poisson-Darboux equation (7) can be written as

$$\left((T^2 - 1) \frac{d^2}{dT^2} + (3 - 2\lambda) T \frac{d}{dT} + (\mu + (\lambda - 1)^2 - N^2) \right) Z(T, \lambda, \mu, n) = 0. \quad (12)$$

It valids $Z(1, \lambda, \mu, n) = 0$ for $\lambda \geq 1$. The recursion formula

$$\frac{d}{dT} Z(T, \lambda + 1, \mu, n) = (2\lambda + 1) Z(T, \lambda, \mu, n) \quad (13)$$

plays a key role when we use a Landau difference method in order to prove results about the length spectrum. Notice that

$$Z(T, \lambda_2, \mu, n) = \frac{2^{\lambda_2 - \lambda_1}}{B(\lambda_1 + \frac{1}{2}, \lambda_2 - \lambda_1)} \int_1^T (T - S)^{\lambda_2 - \lambda_1 - 1} Z(S, \lambda_1, \mu, n) dS \quad (14)$$

for $\lambda_2 \geq \lambda_1 + 1$, where B denotes the Beta function. If we use $Z(T, \lambda, \mu, n)$ in connection with the Selberg trace formula, it is useful to write the special case $\lambda_1 = 0$ of the last equation as a Fourier transformation:

$$Z(T, \lambda, \mu, n) = \frac{2^{\lambda - 1}}{B(1/2, \lambda)} \int_{-\infty}^{\infty} \{T - \cosh \tau\}_+^{\lambda - 1} e^{i\tau} d\tau \quad (15)$$

with

$$\{s\}_+^\lambda = \begin{cases} s^\lambda & \text{for } s \geq 0 \\ 0 & \text{for } s < 0 \end{cases}$$

and

$$\nu = \begin{cases} \sqrt{\mu - N^2} & \text{for } \mu \geq N^2 \\ \sqrt{N^2 - \mu} i & \text{for } \mu < N^2, \end{cases} \quad N = \frac{n - 1}{2}.$$

If we replace $h(r_p(\mu))$ and $g(l(\omega))$ in Theorem 1 by $B(1/2, \lambda) Z(T, \lambda, \mu + N^2 - (p - N)^2, n)$ and $\{2(T - \cosh l(\omega))\}_+^{\lambda - 1}$, respectively, we get the formula we originally deduced in [21] in order to prove the Selberg trace formula. We use $\lambda > n$ in order to ensure absolute convergence of the left-hand side of (18) as a consequence of

$$|Z(T, \lambda, \mu, n)| \leq c T^{\lambda - 1} (\mu - N^2)^{-\lambda/2} \quad \text{for } \mu > N^2 + \epsilon_0, \epsilon_0 > 0. \quad (16)$$

Note that the constant c is independent of T and μ . Further on, we will need the equation

$$Z(T, 1, \mu, n) = \begin{cases} \sinh(\sqrt{N^2 - \mu} t) / \sqrt{N^2 - \mu} & \text{for } \mu < N^2 \\ t & \text{for } \mu = N^2 \\ \sin(\sqrt{\mu - N^2} t) / \sqrt{\mu - N^2} & \text{for } \mu > N^2 \end{cases} \quad (17)$$

with $T = \cosh t$. In the next theorem we will state the mentioned version of the Selberg trace formula.

Theorem 2: For $\lambda > n, T \geq 1$ we have

$$\begin{aligned} & \sum_{\mu \in S_p} d_p^*(\mu) Z(T, \lambda, \mu + N^2 - (p - N)^2, n) \\ &= Q(T, \lambda, p) + \frac{\Gamma(\lambda + 1/2) 2^{\lambda - 1}}{\Gamma(\lambda) \sqrt{\pi}} \sum_{\substack{\omega \in \Omega \\ \cosh l(\omega) \leq T}} l(\omega) \sigma(\omega) e_p(\omega) [T - \cosh l(\omega)]^{\lambda - 1} \end{aligned} \quad (18)$$

with

$$Q(T, \lambda, p) = \frac{\binom{n-1}{p} \text{vol } V}{\pi^{n/2} \Gamma(n/2)} \sum_{u=0}^{N-|N-p|} \frac{\Gamma(\frac{n}{2}-u) \Gamma(\lambda+\frac{1}{2})}{2^{2u} \Gamma(\lambda-N+u)} (-1)^u \times \left(\prod_{v=0}^{u-1} ((p-N)^2 - (N-v)^2) \right) (2(T-1))^{\lambda-1-N+u}. \quad (19)$$

The sum on the right-hand side of the equation (18) is finite in contrast to the general situation in Theorem 1.

4. Basic estimations for the p -eigenvalue spectra

Since our version of the Selberg trace formula only involves that part of the p -eigenvalue spectrum which corresponds to the coclosed eigenforms of the Laplace operator, we also use this partition of the p -spectrum if we exploit basic asymptotic formulas. By the Telescopage theorem of McKean and Singer (cf. [2]), it is not difficult to check that it is sufficient to use the coclosed part of the p -spectrum. We define

$$N_p(T) = \sum_{\mu \in S_p, \mu < T} d_p(\mu) \quad (20)$$

and start with the well-known Weyl asymptotic formula

$$N_p(T) \sim \frac{1}{(4\pi)^{n/2} \Gamma(\frac{n+2}{2})} \binom{n}{p} \text{vol } V T^{n/2} \text{ for } T \rightarrow \infty. \quad (21)$$

Using (21), we immediately see (1). By the Selberg trace formula we also get the well-known formula for the Euler-Poincaré characteristic $\chi(V)$ of the orientable space form V in the case of even n :

$$\chi(V) = (-1)^{n/2} \pi^{-(n+1)/2} \Gamma(\frac{n+1}{2}) \text{vol } V. \quad (22)$$

Therefore

$$N_p(T) \sim (-1)^{n/2} \binom{n-1}{p} \frac{1}{\Gamma(n+1)} \chi(V) T^n. \quad (23)$$

The equation $N_p(T) = O(T^{n/2})$ implies in the usual way by partial integration

$$\sum_{\mu \in S_p, 0 < \mu \leq T} d_p(\mu) \mu^{-s} = \begin{cases} O(T^{n/2-s}) & \text{for } s < n/2 \\ O(\ln T) & \text{for } s = n/2 \end{cases}, \quad (24)$$

$$\sum_{\mu \in S_p, T < \mu} d_p(\mu) \mu^{-s} = O(T^{n/2-s}) \text{ for } s > n/2. \quad (25)$$

5. Estimates for the length spectrum

5.1. Difference operators for spectral functions

In order to get results about the length spectrum (for $p = 0, n = 2$) HEJHAL [14] has used the fact that the Selberg zeta function is analogous to the classical Riemann zeta function (cf. [26])

in many ways. Thereby methods of analytic number theory are applied, and the train of thought is quite long. We will reach the desired result more direct if we start with the Selberg trace formula in the version of Theorem 2. We use a Landau difference method which already was used in the case $p = 0$ by GÜNTHER [12]. We begin with the definitions (cf. [12] for $p = 0$) of the spectral functions

$$P_p(T) = \sum_{\substack{\omega \in \Gamma \\ \cosh l(\omega) \leq T}} e_p(\omega), \quad P_p^\#(T) = \sum_{\substack{\omega \in \Gamma \\ \cosh l(\omega) \leq T, \nu(\omega)=1}} e_p(\omega) \tag{26}$$

and

$$G_{p,m}(T) = \sum_{\substack{\omega \in \Gamma \\ \cosh l(\omega) \leq T}} e_p(\omega) l(\omega) \sigma(\omega) \frac{1}{\Gamma(m+1)} [T - \cosh l(\omega)]^m \tag{27}$$

for $p = 0, 1, \dots, n - 1$ with $m = 0, 1, 2, \dots$. We remark that the sums are finite for each fixed T . For a function F from \mathbb{R} into an arbitrary vector space we use the difference operator

$$(\nabla_m^a F)(T) = \sum_{\nu=0}^m (-1)^{m-\nu} \binom{m}{\nu} F(T + \nu S) \quad \text{with } S = T^a, a \in (0, 1). \tag{28}$$

We obtain

$$G_{p,m}(T) = \int_1^T \int_1^{T_m} \dots \int_1^{T_2} G_{p,0}(T_1) dT_1 \dots dT_{m-1} dT_m. \tag{29}$$

It will be useful to consider more generally

$$F_m(T) = \int_1^T \int_1^{T_m} \dots \int_1^{T_2} F(T_1) dT_1 \dots dT_{m-1} dT_m$$

for an integrable function F from \mathbb{R} into an arbitrary vector space.

Lemma 3: *It holds*

$$(\nabla_m^a F_m)(T) - S^m F(T) = \int_T^{T+S} \int_{T_m}^{T_m+S} \dots \int_{T_2}^{T_2+S} [F(T_1) - F(T)] dT_1 \dots dT_{m-1} dT_m \tag{30}$$

$$(\nabla_m^a G_{p,m})(T) - S^m G_{p,0}(T) = \int_T^{T+S} \int_{T_m}^{T_m+S} \dots \int_{T_2}^{T_2+S} [G_{p,0}(T_1) - G_{p,0}(T)] dT_1 \dots dT_{m-1} dT_m. \tag{31}$$

Proof: The first assertion follows from

$$(\nabla_m^a F_m)(T) = \int_T^{T+S} \int_{T_m}^{T_m+S} \dots \int_{T_2}^{T_2+S} F(T_1) dT_1 \dots dT_{m-1} dT_m. \tag{32}$$

By using $F(T) = G_{p,0}(T)$, we get the second one ■

We will use Lemma 3 in a situation, in which we have some 'a priori' knowledge about $F(T_1) - F(T)$ for $T_1 \in [T, T + mS]$ as well as information about $(\nabla_m^a F_m)(T)$ in order to get a result about $F(T)$ (cf. [12]). The following lemma supplies us with such information.

Lemma 4: For $T_1 > T$ we can estimate

$$|G_{p,0}(T_1) - G_{p,0}(T)| \leq \sum_{\substack{\omega \in \mathfrak{N} \\ T < \cosh l(\omega) \leq T_1}} l(\omega) \sigma(\omega) \binom{n-1}{p} \leq \binom{n-1}{p} (G_{0,0}(T_1) - G_{0,0}(T)), \quad (33)$$

Proof: Using the definition of $G_{p,0}(T)$, we find that

$$G_{p,0}(T_1) - G_{p,0}(T) = \sum_{\substack{\omega \in \mathfrak{N} \\ T < \cosh l(\omega) \leq T_1}} e_p(\omega) l(\omega) \sigma(\omega).$$

The lemma follows immediately ■

5.2. Application of the trace formula

We will use the Selberg trace formula in order to get an estimation of the spectral function $G_{p,0}(T)$. More precisely, we can state the following proposition as the main result of this subsection.

Proposition 5: We obtain the following estimations:

(i) For $1 \leq p \leq n-2$ we have

$$|G_{p,0}(T)| = O\left(T^{(n-1)^2/(2n)}\right).$$

(ii) For $p=0$ and $p=n-1$ we have

$$\left| G_{p,0}(T) - 2 \sum_{\substack{\mu \in S_p \\ \mu < N^2(2n-1)/n^2}} d_p^*(\mu) \sinh\left(\sqrt{N^2 - \mu} \, t / \sqrt{N^2 - \mu}\right) \right| = O\left(T^{(n-1)^2/(2n)}\right).$$

We will return to the proof of Proposition 5 at the end of this subsection. We remark that we would have much more problems if the right-hand side of equation (27) would be an infinite series. We now apply the Selberg trace formula in the version of Theorem 2 and get

Lemma 6: We have

$$G_{p,m}(T) = \frac{\pi^{1/2}}{\Gamma(m+3/2)} 2^{-m} \left(\sum_{\mu \in S_p} d_p^*(\mu) Z(T, m+1, \mu + N^2 - (p-N)^2, n) - Q(T, m+1, p) \right).$$

The series on the right-hand side is absolutely convergent for $m \geq n$ as a consequence of formulae (16) and (25).

The case $m = n$ is of special interest. We define

$$H_{p,m}(T) = \frac{\pi^{1/2}}{\Gamma(m+3/2)} 2^{-m} \sum_{\substack{\mu \in S_p \\ \mu \leq (p-N)^2}} d_p^*(\mu) Z(T, m+1, \mu + N^2 - (p-N)^2, n) \quad (34)$$

and

$$R_{p,m}(T) = G_{p,m}(T) - H_{p,m}(T). \quad (35)$$

Next we need information about $\nabla_n^\alpha G_{p,n}$ in order to prove Proposition 5 by the help of Lemma 3.

Proposition 7: *The following statements are true:*

(i) *We have*

$$H_{p,0}(T) = 2 \sum_{\substack{\mu \in S_p \\ \mu \leq (p-N)^2}} d_p^\alpha(\mu) \frac{\sinh\left(\sqrt{(p-N)^2 - \mu} t\right)}{\sqrt{(p-N)^2 - \mu}}$$

with $T = \cosh t$. The following estimations hold with $a = (n+1)/(2n)$:

$$(ii) \quad |(\nabla_n^\alpha H_{p,n})(T) - T^{na} H_{p,0}(T)| = O(T^{n-1+\frac{n+1}{2n}})$$

$$(iii) \quad |(\nabla_n^\alpha R_{p,n})(T)| = O(T^{n-1+\frac{n+1}{2n}})$$

$$(iv) \quad |(\nabla_n^\alpha G_{p,n})(T) - T^{na} H_{p,0}(T)| = O(T^{n-1+\frac{n+1}{2n}}).$$

If $\mu = (p-N)^2 \in S_p$, we understand that $\sinh\left(\sqrt{(p-N)^2 - \mu} t\right) / \sqrt{(p-N)^2 - \mu} = t$.

Proof: By virtue of (17) we get (i). The equation

$$H_{p,n}(T) = \int_1^T \int_1^{T_1} \dots \int_1^{T_{n-1}} H_{p,0}(T_1) dT_1 \dots dT_{n-1} dT_n$$

immediately follows from the recursion formula (13) for $Z(T, \lambda, \mu, n)$ and from the definition (34). By (29) and (35), we obtain

$$R_{p,n}(T) = \int_1^T \int_1^{T_1} \dots \int_1^{T_{n-1}} R_{p,0}(T_1) dT_1 \dots dT_{n-1} dT_n.$$

Using $F(T) = H_{p,0}(T)$, a look at (30) for $n = m$ shows

$$(\nabla_n^\alpha H_{p,n})(T) = S^n H_{p,0}(T) + \int_T^{T+S} \int_T^{T_1+S} \dots \int_T^{T_{n-1}+S} [H_{p,0}(T_1) - H_{p,0}(T)] dT_1 \dots dT_{n-1} dT_n. \quad (36)$$

Starting with (i), a simple application of the mean value theorem shows

$$H_{p,0}(T_1) - H_{p,0}(T) = O(T^{N-1+a}) \quad (37)$$

for $T_1 \in [T, T + nT^a]$. Using (36) it follows that

$$|(\nabla_n^\alpha H_{p,n})(T) - T^{na} H_{p,0}(T)| = O(T^{N-1+(n+1)a}). \quad (38)$$

The assertion (ii) of the Proposition follows at once if we use $a = (n+1)/(2n)$. It will turn out that this value of a is the optimal choice if we compare (38) with the estimations used for the proof of assertion (iii).

In order to get information about $\nabla_n^\alpha R_{p,n}$, we break up $R_{p,n}$ into contributions

$$R_{p,n}^I(T) = \frac{\pi^{1/2}}{\Gamma(n + \frac{3}{2})} 2^n \left(\sum_{\substack{\mu \in S_p \\ (p-N)^2 < \mu \leq T^b}} d_p^\alpha(\mu) Z(T, n+1, \mu + N^2 - (p-N)^2, n) \right), \quad (39)$$

and

$$R_{p,n}^{II}(T) = R_{p,n}(T) - R_{p,n}^I(T) \tag{40}$$

with $b \in (0, 1)$. We use (32) with $F(T) = R_{p,n}^I(T)$, $m = n$ and get

$$(\nabla_n^\alpha R_{p,n}^I)(T) = \int_T^{T+S} \int_{T_n}^{T_n+S} \dots \int_{T_2}^{T_2+S} R_{p,0}^I(T_1) dT_1 \dots dT_{n-1} dT_n.$$

The mean value theorem implies $(\nabla_n^\alpha R_{p,n}^I)(T) = S^n R_{p,0}^I(T^*)$ with $T^* \in [T, T + nS]$, $S = T^\alpha$. Of course, T^* depends on T and a . By using (16), (24) and (39), we obtain

$$|R_{p,0}^I(T^*)| \leq c^* \sum_{\substack{\mu \in S_p \\ (p-N)^2 < \mu \leq (T^*)^b}} \frac{1}{[\mu - (p - N)^2]^{1/2}} \leq c^{**} T^{bN}$$

and thereby

$$|(\nabla_n^\alpha R_{p,n}^I)(T)| = O(T^{an+bN}). \tag{41}$$

We define

$$Q^\#(T, n + 1, p) = \frac{\pi^{1/2}}{\Gamma(n + \frac{3}{2})} 2^n Q(T, n + 1, p).$$

To estimate $(\nabla_n^\alpha R_{p,n}^{II})(T)$ we exploit definition (28):

$$|(\nabla_n^\alpha R_{p,n}^{II})(T)| \leq \sum_{\nu=0}^n \binom{n}{\nu} |(R_{p,n}^{II})(T + \nu S) + Q^\#(T + \nu S, n + 1, p)| + |(\nabla_n^\alpha Q^\#(\cdot, n + 1, p))(T)|.$$

Now we are using (16):

$$|Z(T, n + 1, \mu, n)| \leq c^\# T^n [\mu - N^2]^{-(n+1)/2}.$$

By (19) we see that $Q(T, n + 1, p)$ is a polynomial of degree $n - |N - p|$ in T . Therefore the degree is at most n . Using again the mean value theorem, we get $|(\nabla_n^\alpha Q(\cdot, n + 1, p))(T)| = O(T^{an})$. By using Lemma 6, (34), (35), (39) and (40), we obtain

$$|R_{p,n}^{II}(T + \nu S) + Q^\#(T + \mu S, n + 1, p)| \leq c^{\#\#} T^n \sum_{\substack{\mu \in S_p \\ (p-N)^2 < \mu, T^b < \mu}} d_p^*(\mu) (\mu - (p - N)^2)^{-\frac{n+1}{2}}.$$

We apply (25) and get as an immediate consequence

$$|(\nabla_n^\alpha R_{p,n}^{II})(T)| = O\left(T^{n+b(\frac{n}{2} - \frac{n+1}{2})}\right) + O(T^{an}). \tag{42}$$

We compare (38), (41) and (42) and get the optimal choice $b = 2(1 - a)$, $a = (n + 1)/(2n)$. Thereby the estimation (iii) is an immediate consequence of (40) - (42). The assertion (iv) follows from (ii), (iii) and $(\nabla_n^\alpha G_{p,n})(T) = (\nabla_n^\alpha H_{p,n})(T) + (\nabla_n^\alpha R_{p,n})(T)$ ■

As we have mentioned above, Proposition 7/(iv) gives us information about the first term of the left-hand side of equation (31) of Lemma 3. With the help of the following Propositions 8 and 9 we will consider the integrand appearing in equation (31).

Proposition 8: We get the estimation $G_{0,0}(T) = H_{0,0}(T) + O\left(T^{(n-1)^2/(2n)}\right)$.

Proof: The functions

$$G_{0,0}(T) = \sum_{\substack{\omega \in \Omega \\ \cosh l(\omega) \leq T}} l(\omega) \sigma(\omega) \quad (43)$$

and

$$H_{0,0}(T) = 2 \sum_{\mu \in S_p, \mu \leq N^2} d_0^*(\mu) \frac{\sinh(\sqrt{N^2 - \mu} t)}{\sqrt{N^2 - \mu}} \quad (44)$$

are monotonically increasing. We use (32) with $m = n$, $F = R_{0,0} = G_{0,0} - H_{0,0}$ and get

$$(\nabla_n^\alpha R_{0,n})(T) = \int_T^{T+S} \int_{T_n}^{T_n+S} \dots \int_{T_2}^{T_2+S} [G_{0,0}(T_1) - H_{0,0}(T_1)] dT_1 \dots dT_{n-1} dT_n.$$

The monotony of the functions $G_{0,0}$ and $H_{0,0}$ implies

$$G_{0,0}(T) - H_{0,0}(T + nS) \leq S^{-n} (\nabla_n^\alpha R_{0,n})(T) \quad (45)$$

and

$$S^{-n} (\nabla_n^\alpha R_{0,n})(T) \leq G_{0,0}(T + nS) - H_{0,0}(T). \quad (46)$$

By (37), it follows that

$$H_{0,0}(T + nS) = H_{0,0}(T) + O(T^{N-1+a}). \quad (47)$$

Using Proposition 7/(iii), (45) and (47) we get

$$G_{0,0}(T) \leq H_{0,0}(T) + O(T^{N-1+a}) + O(T^{n-1+(n+1)/(2n)-na}).$$

We use $a = (n+1)/(2n)$ as above and get

$$G_{0,0}(T) \leq H_{0,0}(T) + O(T^{(n-1)^2/(2n)}). \quad (48)$$

If we use the transformation $T \rightarrow T^\# = T + nT^a$ and analogous considerations starting with (46) instead of (45), we get

$$G_{0,0}(T) \geq H_{0,0}(T) + O(T^{(n-1)^2/(2n)}). \quad (49)$$

The equations (48) and (49) complete the proof ■

Proposition 9: For $T_1 \in (T, T + nS)$ we get $|G_{p,0}(T_1) - G_{p,0}(T)| = O(T^{(n-1)^2/(2n)})$.

Proof: Using (37) again, Proposition 8 implies $G_{0,0}(T_1) - G_{0,0}(T) = O(T^{(n-1)^2/(2n)})$ for $T_1 \in (T, T + nS)$. The assertion follows from the application of Lemma 4 ■

Proof of Proposition 5: By (31) with $a = (n+1)/(2n)$, Proposition 7/(i),(ii),(iv) and Proposition 9 we conclude that

$$\left| G_{p,0}(T) - 2 \sum_{\mu \in S_p, \mu \leq (p-N)^2} d_p^*(\mu) \frac{\sinh(\sqrt{(p-N)^2 - \mu} t)}{\sqrt{(p-N)^2 - \mu}} \right| = O(T^{\frac{(n-1)^2}{2n}}).$$

By using

$$\frac{\sinh(\sqrt{(p-N)^2 - \mu} t)}{\sqrt{(p-N)^2 - \mu}} = O(T^{\frac{(n-1)^2}{2n}})$$

for $\mu \leq (p - N)^2$ if $1 \leq p \leq n - 2$ and for $\mu \geq N^2(2n - 1)/n^2$ if $p = 0, n - 1$, we complete the proof ■

We want to remark that we have used a similar version of this Landau difference method in [22, 23] to get results about the asymptotic behaviour of double p -form valued kernels of mean value operators for differential p -forms.

5.3. Proof of Theorem A

The aim of this subsection is the proof of the spectral estimates of Theorem A with the help of Proposition 5. Before we can reach this goal, we still have to consider another spectral function. We define

$$G_p^*(T) = \sum_{\substack{\omega \in \Omega \\ \cosh l(\omega) \leq T}} (\cosh^N l(\omega)) \sigma(\omega) e_p(\omega). \quad (50)$$

For $a > 0$ we introduce $E(t, a) = \int_1^t e^{as}/s ds$. It is well known that

$$E(t, a) \sim \frac{e^{at}}{at} \quad \text{for } t \rightarrow \infty. \quad (51)$$

The asymptotic behaviour of the spectral function G_p^* is described by

Proposition 10: *Using $T = \cosh t$, we obtain the asymptotic estimate*

$$G_p^*(T) = \begin{cases} O\left(T^{N + \frac{(n-1)^2}{2n}} / \ln T\right) & \text{for } 1 \leq p \leq n - 2 \\ 2^{-N} \sum_{\substack{\mu \in S_p \\ \mu < N^2(2n-1)/n^2}} d_p^*(\mu) E(t, N + \sqrt{N^2 - \mu}) \\ + O\left(T^{N + \frac{(n-1)^2}{2n}} / \ln T\right) & \text{for } p = 0, n - 1. \end{cases}$$

Proof: We obtain $G_p^*(T)$ as a Stieltjes integral

$$G_p^*(\cosh t) = \int_{\epsilon}^t \frac{\cosh^N s}{s} dG_{p,0}(\cosh \cdot)(s)$$

with $\epsilon = \min_{\omega \in \Omega} \{l(\omega)/2\}$. Put

$$\tilde{H}_p(t) = \begin{cases} 0 & \text{for } 1 \leq p \leq n - 2 \\ 2 \sum_{\substack{\mu \in S_p \\ \mu < N^2(2n-1)/n^2}} d_p^*(\mu) \frac{\sinh(\sqrt{N^2 - \mu} t)}{\sqrt{N^2 - \mu}} & \text{for } p = 0, n - 1 \end{cases} \quad (52)$$

and $\tilde{R}_p(t) = G_{p,0}(\cosh t) - \tilde{H}_p(t)$. By the calculus for Stieltjes integrals, we get

$$G_p^*(\cosh t) = \int_{\epsilon}^t \frac{\cosh^N s}{s} d\tilde{H}_p(s) + \int_{\epsilon}^t \frac{\cosh^N s}{s} d\tilde{R}_p(s).$$

Proposition 5 immediately implies $\tilde{R}_p(t) = O\left((\cosh t)^{(n-1)^2/(2n)}\right)$, and we get by partial integration

$$\int_{\epsilon}^t \frac{\cosh^N s}{s} d\tilde{R}_p(s) = O\left(E(t, N + \frac{(n-1)^2}{2n})\right). \quad (53)$$

For $\mu < N^2(2n-1)/n^2$ we consider

$$\begin{aligned} \int_{s=\epsilon}^t \frac{\cosh^N s}{s} d \frac{\sinh(\sqrt{N^2 - \mu} s)}{\sqrt{N^2 - \mu}} &= \int_{\epsilon}^t \frac{\cosh^N s}{s} \cosh(\sqrt{N^2 - \mu} s) ds \\ &= \int_{\epsilon}^t \frac{1}{s} \left(2^{-(N+1)} e^{(N+\sqrt{N^2 - \mu})s} + O(e^{(N+\frac{n-1}{2n})s}) \right) ds \\ &= 2^{-(N+1)} E(t, N + \sqrt{N^2 - \mu}) + O\left(E(t, N + \frac{(n-1)^2}{2n})\right). \end{aligned}$$

Therefore we obtain

$$\int_{\epsilon}^t \frac{\cosh^N s}{s} d\tilde{H}_p(s) = \begin{cases} 0 & \text{for } 1 \leq p \leq n-2 \\ 2^{-N} \sum_{\substack{\mu \in S_p \\ \mu < N^2(2n-1)/n^2}} d_p^*(\mu) E(t, N + \sqrt{N^2 - \mu}) \\ \quad + O\left(E(t, N + \frac{(n-1)^2}{2n})\right) & \text{for } p = 0, n-1. \end{cases} \quad (54)$$

The equations (53) and (54) complete the proof ■

Next we define the spectral function

$$G_p^\#(T) = \sum_{\substack{\omega \in \Omega \\ \cosh l(\omega) \leq T}} \frac{e_p(\omega)}{2^N \nu(\omega)}. \quad (55)$$

Proposition 11: We obtain the asymptotic estimate

$$G_p^\#(T) = \begin{cases} O\left(T^{N+\frac{(n-1)^2}{2n}} / \ln T\right) & \text{for } 1 \leq p \leq n-2 \\ 2^{-N} \sum_{\substack{\mu \in S_p \\ \mu < N^2(2n-1)/n^2}} d_p^*(\mu) E(t, N + \sqrt{N^2 - \mu}) \\ \quad + O\left(T^{N+\frac{(n-1)^2}{2n}} / \ln T\right) & \text{for } p = 0, n-1. \end{cases}$$

Proof: From the definitions (50) and (55), we deduce

$$G_p^*(T) - G_p^\#(T) = \sum_{\substack{\omega \in \Omega \\ \cosh l(\omega) \leq T}} \left((\cosh^N l(\omega)) \sigma(\omega) - \frac{1}{2^N \nu(\omega)} \right) e_p(\omega).$$

By using (3) we can easily check that, for a sufficiently large $c^\#$, we have

$$\left| (\cosh^N l(\omega)) \sigma(\omega) - \frac{1}{2^N \nu(\omega)} \right| \leq c^\# \cosh^{N-1} l(\omega) \sigma(\omega). \quad (56)$$

By Proposition 10 and (51), it follows that $G_p^*(T) = O(T^{n-1}/\ln T)$. In the same way we can prove

$$\sum_{\substack{\omega \in \Omega \\ \cosh l(\omega) \leq T}} (\cosh^{N-1} l(\omega)) \sigma(\omega) e_p(\omega) = \begin{cases} O(T^{n-2}/\ln T) & \text{for } n > 2 \\ O(T^{1/2}) & \text{for } n = 2. \end{cases}$$

By using (56), we deduce that

$$|G_p^*(T) - G_p^\#(T)| = \begin{cases} O(T^{n-2}/\ln T) & \text{for } n > 2 \\ O(T^{1/2}) & \text{for } n = 2 \end{cases}$$

and thereby $|G_p^*(T) - G_p^\#(T)| = O(T^{N+(n-1)^2/(2n)}/\ln T)$. Now the assertion is a consequence of Proposition 10 ■

Proof of Theorem A: We define the spectral function

$$P_{p,j}^\#(T) = \sum_{\substack{\omega \in \Omega \\ \cosh l(\omega) \leq T, v(\omega)=1}} e_p(\omega^j)$$

for $j \in \mathbb{N}$. In view of the definitions, we have $P_{p,1}^\#(T) = P_p^\#(T)$. One immediately checks that

$$2^N G_p^\#(T) = P_p^\#(T) + \sum_{j=2}^\infty \frac{1}{j} P_{p,j}^\#(\cosh \frac{t}{j}), \tag{57}$$

$$P_p(T) = \sum_{j=1}^\infty P_{p,j}^\#(\cosh \frac{t}{j}) \tag{58}$$

with $\cosh t = T$. Now we can use a standard procedure (cf. [12]). We have

$$\begin{aligned} |2^N G_p^\#(T) - P_p^\#(T)| &= \left| \sum_{j=2}^\infty \frac{1}{j} P_{p,j}^\#(\cosh \frac{t}{j}) \right| \leq \binom{n-1}{p} \sum_{j=2}^\infty \frac{1}{j} P_0^\#(\cosh \frac{t}{j}) \\ &\leq \binom{n-1}{p} (2^N G_0^\#(T) - P_0^\#(T)). \end{aligned} \tag{59}$$

We observe that $P_0^\#(\cosh \frac{t}{j}) = 0$ for $j > \frac{t}{l_0}$ with $l_0 = \min_{\omega \in \Omega} \{l(\omega)\}$. The definition $e_0(\omega) = 1$ implies $P_{0,j}^\#(T) = P_0^\#(T)$ for all j . Thus we get

$$\begin{aligned} 0 \leq 2^N G_0^\#(T) - P_0^\#(T) &= \sum_{2 \leq j \leq t/l_0} \frac{1}{j} P_0^\#(\cosh \frac{t}{j}) \\ &\leq \sum_{2 \leq j \leq t/l_0} 2^N \frac{1}{j} G_0^\#(\cosh \frac{t}{j}) \leq 2^N \frac{t}{l_0} G_0^\#(\cosh \frac{t}{2}). \end{aligned} \tag{60}$$

Thereby we have used the monotony of $G_0^\#$. Proposition 11 implies $G_0^\#(T) = O(T^{2N}/\ln T)$. Therefore we get $t G_0^\#(\cosh t/2) = O(T^N)$. We apply (60) to obtain $P_0^\#(T) = 2^N G_0^\#(T) + O(T^N)$. Using (59), it follows $|2^N G_p^\#(T) - P_p^\#(T)| = O(T^N)$. Similarly we get $P_p(T) = P_p^\#(T) + O(T^N)$ if we use (58) instead of (57). Thus we get $|2^N G_p^\#(T) - P_p(T)| = O(T^N)$. With the help of these equations we see that the asymptotic spectral estimations of P_p and $P_p^\#$ are given as an immediate consequence of Proposition 11. We remark that $d_p^*(0) = 1$ for $p = 0$ and $p = n - 1$ ■

Our proof has essentially used the asymptotic spectral estimations of G_p^* and $G_p^\#$ which are also interesting by themselves.

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Book review

ALAN F. BEARDON: *Iteration of Rational Functions*. Complex Analytical Dynamical Systems (Graduate Texts in Mathematics: Vol. 132). Berlin - Heidelberg - New York: Springer-Verlag 1991, XVI + 280 pp., 35 Fig.

The iteration of rational maps became popular after Mandelbrot's work in 1978 which introduced the famous set which now is called after him. He iterated the polynomial $p(z) = z^2 + c$ and was interested in the behaviour of this iteration for different complex parameters c instead of using only real ones. The underlying theory starts with G. Julia and P. Fatou around 1920 and was developed much further after using computer graphics to see the beauty of the considered subsets of the complex plane.

The aim of the book is to develop this theory clearly, the author gives a lot of examples illustrating the different kinds of behaviour. So Chapter 1 fully consists of examples. Chapter 2 recalls elementary facts about rational maps, Chapters 3-5 introduce the Julia and Fatou sets and prove their main properties. Chapters 6-9 consider periodic points, forward invariant components, non-wandering domains, and critical points. Chapter 10 introduces the Hausdorff dimension and shows that for rational maps of degree ≥ 2 the Hausdorff dimension of the Julia set is non-zero. The concluding Chapter 11 again fully consists of examples illustrating the advanced features developed in the preceding chapters.

The book contains several illustrations showing the fractal structure of the considered sets. Most of these pictures appeared elsewhere before. One should not expect the book to be a guide for producing strange fractals of extraordinary beauty, this is not the aim of the book. It shows the relevant mathematical ideas and gives examples where they apply. For mathematical simplicity these examples are sometimes far from producing strange graphics.

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