A Counterexample for L^2 -Estimates for Parabolic Differential Equations

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We show that the Dini (1) continuity of the coefficients of a linear parabolic differential oper**ator in non-divergence form is in some sense the weakest condition such that the solutions of the corresponding initial value problem satisfy an L i -estimate; here a function is called Dini** (α) continuous for a positive number α if the modulus of continuity ω of the function satisfies $\int_{0}^{1} e^{i\pi/2}$ $\sqrt{\alpha}$ π *d* π < ∞ . In particular, we improve a counterexample of II in which shows that an L^1 - estimate cannot hold in general if only Dini (α) continuity with $\alpha \leq 1/4$ is assumed.

Key words: *Initial value problems for second-order, parabolic equations, a priori estimates, diffusion processes*

AMS **subject classification: 35K15, 35845, 60J60**

We consider the initial value problem

al value problem
\n
$$
Lu = 0
$$
 in $(0,1) \times \mathbb{R}^n$, $u(0,\cdot) = f$

where the coefficients $a_{ij} = a_{ji}$ of the differential operator

Lu - Ltu

are assumed to be continuous and bounded. Moreover, we restrict ourselves to the uniformly parabolic case $(a_{ij}(t, x))_{i,j} \geq A \cdot \mathbb{I}$ for some positive constant A. *Lu* (t, x) , $i, j \geq A \cdot 1$ for some positive containt the coefficients of *L* are continued by \mathbb{R}^n for every $s > 0$. Then the in $Lu = 0$ for $t > s$, $u(s, \cdot) = f$

Suppose temporarily that the coefficients of *L* are continuous on $[0, \infty) \times \mathbb{R}^n$ and Hölder continuous on $[s, \infty) \times \mathbb{R}^n$ for every $s > 0$. Then the initial value problem

$$
Lu = 0 \quad \text{for } t > s, \quad u(s, \cdot) = f
$$

is uniquely solvable for every $s > 0$. The solution *u* can be written by means of the transition probability measures $P_L(t, x; s, \cdot)$ of the diffusion process with decreasing time parameter generated by the parabolic operator $L_t + \frac{\partial}{\partial t}$ as follows (cf. [5: Chapter 3]):

$$
u(t,x) = \int_{\mathbb{R}^n} f(y) P_L(t,x;s,dy).
$$

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We mention that the diffusion process generated by $L_t + \frac{\partial}{\partial t}$ is uniquely defined if the coefficients of the uniformly parabolic operator *L* are merely supposed to be continuous (see [5: Chapter 7]). The transition maps $P_1^{t_1,t_2}: C(\mathbb{R}^n) \to C(\mathbb{R}^n)$ are defined by diffusion process g

prmly parabolic op

he transition maps
 $P_L^{t_1,t_2}f(x) =$

pd $x \in \mathbb{R}^{n}$. Let $A^{(n)}$

$$
P_L^{t_1,t_2} f(x) \equiv \int_{\mathbb{R}^n} f(y) P_L(t_1,x;t_2,dy)
$$

for every $f \in C(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Let $\beta_t^{(n)}$ be the kernel of the $n-$ dimensional Brownian $P_L^{t_1,t_2} f(x) \equiv \int_{\mathbb{R}^n} f$
for every $f \in C(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Let $\beta_i^{(n)}$ be the motion at time t , i.e. $\beta_i^{(n)}(x) \equiv \frac{1}{(2\pi i)^{\frac{n}{2}}} \exp(-\frac{|x|^2}{2t})$). very $f \in C(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Let $\beta_i^{(n)}$ be the kernel of the *n*-dimensional Brownian

ion at time t , i.e. $\beta_i^{(n)}(x) \equiv \frac{1}{(2\pi i)^{\frac{n}{2}}} \exp(-\frac{|x|^2}{2t})$.

Lemma: Let $t_1, t_2 \in [0,1]$ with $t_1 > t_2$ and a fu

. . . such that $\beta_{t_1-t_2}^{(n)} * f(x)$ is positive for every $x \in \mathbb{R}^n$. Suppose that $f(x) \geq \beta_{\epsilon}^{(n)}(x)$ for some positive number ϵ and every x from the complement of an appropriate compact set $K \subset \mathbb{R}^n$. Then there exists a positive number δ such that, for every differential generator $L = \frac{\partial}{\partial t} - \frac{1}{2} \sum_{i,j=1}^n a_{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j}$ with matrix norm $||(a_{ij}(t, x)) - 1|| < \delta$ for every t and x, *LR*^{*n*} for every $f \in C(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Let $\beta_i^{(n)}$ be the kernel of the *n*-dimensional Brownian motion at time *t*, i.e. $\beta_i^{(n)}(x) \equiv \frac{1}{(2\pi i)^{\frac{n}{2}}} \exp(-\frac{|x|^2}{2t})$.

Lemma: Let $t_1, t_2 \in [0,1]$ with $P_L^{t_1,t_2} f(x)$ is positive for every $x \in \mathbb{R}^n$.

Proof: Let δ_0 be a positive number with $\delta_0 < \frac{1}{3}$ and

$$
(t_1-t_2)(1-2\delta_0)+\frac{\epsilon}{2} > (t_1-t_2)(1+3\delta_0).
$$
 (1)

For every positive number r we denote the closed ball with radius r and center 0 in \mathbb{R}^n by B_r . Let l_B , be the characteristic function of B_r .

We claim that $||(a_{ij}(t, x)) - 1|| < \delta_0$ for every t, *x* implies that $P_L^{t_1, t_2} f$ is positive outside B_{r_0} for some positive constant r_0 . The proof of the claim is based on the lower and upper bounds for transition probabilities given in [4]. It follows from [4: Theorem 21 that for appropriate positive constants C_1 , C_2 , C_3 the following holds: mumber r we denote the close

the characteristic function of I
 $|(a_{ij}(t, x)) - 1|| < \delta_0$ for every t,

we constant r_0 . The proof of th

on probabilities given in [4].

re constants C_1, C_2, C_3 the follow
 $P_L^{t_1,t_2} \beta$

be a positive number with
$$
\delta_0 < \frac{1}{3}
$$
 and $(t_1 - t_2)(1 - 2\delta_0) + \frac{\epsilon}{2} > (t_1 - t_2)(1 + 3\delta_0).$ (1) $\text{number } r$ we denote the closed ball with radius r and center 0 in \mathbb{R}^n , the characteristic function of B_r . $|(a_{ij}(t, x)) - 1|| < \delta_0$ for every t, x implies that $P_L^{t_1, t_2} f$ is positive outside, we constant r_0 . The proof of the claim is based on the lower and upper non probabilities given in [4]. It follows from [4: Theorem 2] that for r_0 constants C_1, C_2, C_3 the following holds:\n\n
$$
P_L^{t_1, t_2} \beta_{\epsilon}^{(n)} \geq C_1 P_L^{t_1, t_2} (1_{B_1} * \beta_{\epsilon/2}^{(n)})
$$
\n
$$
\geq C_2 \beta_{(t_1 - t_2)(1 - 2\delta_0)}^{(n)} * (1_{B_1} * \beta_{\epsilon/2}^{(n)})
$$
\n
$$
\geq C_3 \beta_{(t_1 - t_2)(1 - 2\delta_0) + \frac{\epsilon}{2}}^{(n)}.
$$
\n\na way that $K \subset B_{r_1}$. Then\n
$$
f \geq \beta_{\epsilon}^{(n)} - C_4 1_{B_{r_1}}
$$
\n\npositive constant C_4 . By [4: Theorem 3], we have for positive constants\n
$$
P_L^{t_1, t_2} 1_{B_{r_1}} \leq C_5 \beta_{(t_1 - t_2)(1 + 2\delta_0)}^{(n)} * 1_{B_{r_1}} \leq C_6 \beta_{(t_1 - t_2)(1 + 2\delta_0)}^{(n)}.
$$
\n\n(a) α is an now be completed as follows. By (2)–(4),

Choose r_1 in such a way that $K \subset B_r$. Then

$$
f \ge \beta_{\epsilon}^{(n)} - C_4 \mathbf{1}_{B_r} \tag{3}
$$

for an appropriate positive constant C_4 . By $[4:$ Theorem 3], we have for positive constants *C5, C6* $f \geq \beta_{\epsilon}^{(n)} - C_4 1_{B_{r_1}}$
 i e positive constant C_4 . By [4: Theorem 3], we
 $P_L^{t_1,t_2} 1_{B_{r_1}} \leq C_5 \beta_{(t_1-t_2)(1+2\delta_0)}^{(n)} * 1_{B_{r_1}} \leq C_6 \beta_{(t_1-t_2)}^{(n)}$ $C. A$ *f* $\geq \beta_i^{(n)} - C_4 1_{B_{r_1}}$
 t C_4 . By [4: Theorem 3], w

⁽ⁿ⁾

<sup>(t_{1-t2})(1+2 δ_0) * $1_{B_{r_1}} \leq C_6 \beta_6^{(n)}$

e completed as follows. By

(n)

(t_{1-t2})(1-2 δ_0)+ $\frac{1}{2} - C_4 C_6 \beta_{(n)}^{(n)}$

he above claim is </sup> positive con
 $h^{1, l_2} 1_{B_{r_1}} \leq 0$

aim can no
 $P_L^{t_1, t_2} f \geq 0$ *C₃,* $\beta_{(i_1-i_2)(1+2\delta_0)}^{(n)}$ *A* 1 $B_{r_1} \leq C_6 \beta_{(i_1-i_2)(1+2\delta_0)}^{(n)}$
 C₃ $\beta_{(i_1-i_2)(1-2\delta_0)+\frac{5}{2}}^{(n)}$ *<i>C₄C₆* $\beta_{(i_1-i_2)(1+3\delta_0)}^{(n)}$
 C₃ $\beta_{(i_1-i_2)(1-2\delta_0)+\frac{5}{2}}^{(n)}$ *<i>C₄C₆* $\beta_{(i_1-i_2)(1+3\$

$$
D_L^{t_1,t_2}1_{B_{r_1}} \leq C_5 \beta_{(t_1-t_2)(1+2\delta_0)}^{(n)} * 1_{B_{r_1}} \leq C_6 \beta_{(t_1-t_2)(1+2\delta_0)}^{(n)}.
$$
 (4)

The proof of the claim can now be completed as follows. By $(2)-(4)$,

$$
P_L^{t_1,t_2}f \geq C_3\beta_{(t_1-t_2)(1-2\delta_0)+\frac{\epsilon}{2}}^{(n)}-C_4C_6\beta_{(t_1-t_2)(1+3\delta_0)}^{(n)}.
$$

The existence of an r_0 such that the above claim is correct follows from (1).

(b) α *Counterexample for L*¹-Estimates 403
 (iii) α *f* the number $C_7 \equiv \inf_{\|x\| \le r_0} {\beta_{t_1-t_2}^{(n)} * f(x)}$ is $\beta_{t_1-t_2}^{(n)} * f(x)$ is $\beta_{t_1-t_2}^{(n)} * f(x)$ is $\beta_{t_1-t_2}^{(n)} * f(x)$ positive. By [4: Theorem 11, On account of the assumptions on f the number $C_7 \equiv \inf_{\|x\| \le r_0} {\beta_{t_1-t_2}^{(n)}} * f(x)$ is

A Counterexample for
$$
L^1
$$
-Estimates 403\nOn account of the assumptions on f the number $C_7 \equiv \inf_{\|x\| \le r_0} \{\beta_{i_1-i_2}^{(n)} * f(x)\}$ is positive. By [4: Theorem 1],\n
$$
(P_L^{t_1,t_2}f - \beta_{i_1-i_2}^{(n)} * f)(x) = \int_{t_2}^{t_1} P_L^{t_1,t} \left(\frac{1}{2} \sum_{i,j=1}^n a_{ij}(t,x) \frac{\partial^2}{\partial x_i \partial x_j} - \frac{\Delta}{2}\right) (\beta_{i-t_2}^{(n)} * f)(x) dt.
$$
\nSince $f \in C_b^2(\mathbb{R}^n)$, we can choose a positive number δ_1 such that $||(a_{ij}(t,x)) - 1|| < \delta_1$ for

since $f \in U_b(\mathbb{R})$, we can choose a positive number o_1 such that $||(a_{ij}(t, x)) - 1|| < o_1$ for every t, x implies $||P_L^{t_1, t_2} f - \beta_{t_1 - t_2}^{(n)} * f||_{\infty} < C_7$. Set $\delta \equiv \min{\{\delta_0; \delta_1\}}$. The assertion of the Lemma follows immediately \blacksquare positive. By
 $(P_L^{t_1,t_2}f$.

Since $f \in C$

every t, x in

Lemma foll

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by wint of the assumptions on f the number $C_7 \equiv \text{ir}$
 $\left[\begin{array}{l} 4 \colon \text{Theorem 1} \end{array} \right]$,
 $-\beta_{i_1-i_2}^{(n)} * f(x) = \int_{t_2}^{t_1} P_L^{t_1,t} \left(\frac{1}{2} \sum_{i,j=1}^n a_{ij}(t,x) \frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial^2}{\partial x_i \partial x_j} \right)$
 $\left[\begin{array}{l} \beta_k^2(\mathbb{R}^n) \$

For every λ with $0 < \lambda \leq 1$ we denote by f_{λ} the function from $C_{b}^{2}(\mathbb{R}^{n})$ which is given

$$
f_{\lambda}(x) \equiv f\left(\frac{x}{\sqrt{\lambda}}\right)
$$
 for every $x \in \mathbb{R}^n$;

here *f* stands for a function which satisfies the conditions of the Lemma. Using the scaling transformation $(t, x) \mapsto (\lambda t, \sqrt{\lambda} x)$ and taking into account that $||(a_{ij}(t, x)) - 1|| < \delta$ for every t, x implies a similar condition for the transformed coefficients, we obtain the following corollary.

Corollary: *Under the assumptions of the Lemma there exists a positive number* ^S such that, for every L with $||(a_{ij}(t, x)) - 1|| < \delta$ for every t and x, $P_L^{\lambda t_1, \lambda t_2} f_\lambda(x)$ is positive *for every* $x \in \mathbb{R}^n$.

We will restrict ourselves to the consideration of one-dimensional processes in the remainder of this paper. Therefore we will write simply β_t for the one-dimensional heat kernel.

Our main result is the following theorem.

Theorem: Let $\omega : [0, \infty) \to \mathbb{R}$ *be an increasing continuous function with* $\omega(0) = 0$. *Suppose that* $\lambda \mapsto \frac{\omega(\lambda)}{\lambda}$ *is decreasing on* $(0, \infty)$ *and* $\int_0^1 \frac{\omega(\lambda)}{\lambda} d\lambda = +\infty$. Let δ_0 be a positive *number.* The measural interpose of the Lemma there exists a posity L with $||(a_{ij}(t, x)) - 1|| < \delta$ for every t and x , $P_L^{\lambda t_1, \lambda t_2} f_\lambda$

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aper. Therefore we will write simply be an increasing continuous
 \Rightarrow **P** *be* an increasing continuous

ecreasing on $(0, \infty)$ and $\int_0^1 \frac{\omega(\lambda)}{\lambda} d\lambda = +$
 \therefore innous function $a : \mathbb{R}^1 \times [0,1] \rightarrow \mathbb{R}$ v
 \therefore $a(t_2, x_2) | \leq \omega(|(t_1 - t_2, x_1 - x_2)|)$
 $\$

Then there exists a continuous function $a: \mathbb{R}^1 \times [0,1] \to \mathbb{R}$ *with* $||a-1||_{\infty} < \delta_0$ and

$$
|a(t_1,x_1)-a(t_2,x_2)| \leq \omega(|(t_1-t_2,x_1-x_2)|) \tag{5}
$$

for every $t_1, t_2 \in [0, 1]$ and $x_1, x_2 \in \mathbb{R}$ such that for the corresponding initial value problem

$$
\frac{\partial u}{\partial t} = \frac{1}{2} a(t,x) \frac{\partial^2 u}{\partial x^2}, \qquad u(0,x) = f(x)
$$

for any given $c > 0$ *there is a function f such that* $||u(1, \cdot)||_1 > c \cdot ||$

$$
||u(1,\cdot)||_1 > c \cdot ||f(\cdot)||_1.
$$

 $||u(1, \cdot)||_1 > c \cdot ||f(\cdot)||_1.$
Proof: Let *b* be a function from $C_0^{\infty}(\mathbb{R}^1)$ with $0 \le b \le 1$, $|b'| \le 2$, and $b(\lambda) = 1$ for
rv λ with $|\lambda| > 2$, and $b(\lambda) = 0$ for every λ with $|\lambda| < 1$. We set every λ with $|\lambda| \geq 2$, and $b(\lambda)=0$ for every λ with $|\lambda| \leq 1$. We set

$$
||u(1, \cdot)||_1 > c \cdot ||f(\cdot)||_1.
$$

et b be a function from $C_0^{\infty}(\mathbb{R}^1)$ with $0 \le b \le 1$, $|b'| \le 2$, and $|\lambda| \ge 2$, and $b(\lambda) = 0$ for every λ with $|\lambda| \le 1$. We set

$$
a(t, x) \equiv 1 + C_1 \cdot b\left(\frac{x^2}{t}\right) \omega(t) \quad \text{for every } (t, x) \in (0, 1] \times \mathbb{R}
$$

and $a(0, x) \equiv 1$ for every $x \in \mathbb{R}$. The constant C_1 will be chosen later in such a way that $0 < C_1 < \min\{\frac{1}{20}; \frac{1}{\omega(1)}; \frac{6}{\omega(1)}\}$. Obviously, we have $||a - 1||_{\infty} < \delta_0$.

We check that Condition (5) is met. Let $t_1, t_2 \in [0,1]$ and $x_1, x_2 \in \mathbb{R}$ be given. In view of the continuity of a we may and will assume that t_1, t_2 are not equal to zero. Moreover we suppose that $t_2 \leq t_1$. It is obvious from the definition of a that $|a(t_1, x_1) - b(t_1, x_2)|$ $|a(t_2, x_2)| \leq \frac{1}{2}\omega(t_1)$. Since $\lambda \mapsto \frac{\omega(\lambda)}{\lambda}$ is decreasing, we have $\frac{1}{2}\omega(t_1) \leq \omega(\frac{t_1}{2})$. On account of the assumption that *w* is increasing, Condition (5) is established if $|(t_1-t_2, x_1-x_2)| \geq \frac{1}{2}t_1$.
Now assume that $|(t_1-t_2, x_1-x_2)| < \frac{1}{2}t_1$. In particular, we have $\frac{1}{2}t_1 < t_2$. Thus, $I(x, x) \equiv 1$ for every $x \in \mathbb{R}$. The constant C_1 will be chosen later if $\langle \min\{\frac{1}{20}; \frac{1}{\omega(1)}\}\rangle$. Obviously, we have $||a - 1||_{\infty} \langle \delta_0$.

check that Condition (5) is met. Let $t_1, t_2 \in [0, 1]$ and x_1, x_2

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 $x_2 \in \mathbb{R}$ be given. In

re not equal to zero.

(of a that $|a(t_1, x_1) -$
 $\frac{1}{2} \omega(\frac{t_1}{2})$. On account of
 $|1-t_2, x_1-x_2| \geq \frac{1}{2}t_1$.
 $\frac{1}{2}t_1 < t_2$. Thus,
 $b\left(\frac{x_2^2}{t_2}\right) \omega(t_2)$. $\begin{array}{l} \text{try } x \in \mathbb{R}. \text{ The equation } \frac{\delta_0}{\omega(1)} \}. \text{ Obvioid,} \ \text{dition (5) is} \ \text{of } a \text{ we make } t_2 \leq t_1. \ \text{since } \lambda \mapsto \frac{\omega(\lambda)}{2} \text{ is increasing,} \ -t_2, \ x_1 - x_2 \big| \ x_2 \big| \leq C_1 \ \text{and} \ \text{by } \left(\frac{x^2}{t} \right) \big| \leq C_1 \text{ constant out} \end{array}$ For the text of the set of the set of the set of the set that $t_2 \leq t_1$. It is obvious f

Since $\lambda \mapsto \frac{\omega(\lambda)}{2}$ is decreasing
 ω is increasing, Condition (5)
 $t_1 - t_2, x_1 - x_2$) $\langle t_2, t_1, t_1 \rangle$
 $(t_2, x_2) \leq C_1 |\omega$ the $\left|\frac{1}{2} \leq t_1$. It is down
 $\left|\frac{1}{2} \leq t_2 \right| \leq \frac{1}{2}$, it is decreasing, Condition
 $\left| \frac{1}{2}, x_1 - x_2 \right| \right| \leq \frac{1}{2}$
 $\left| \frac{1}{2}, x_1 - x_2 \right| \leq \frac{1}{2}$
 $\left| \frac{1}{2}, x_1 \right| \leq \frac{1}{2}$
 $\left| \frac{1}{2}, x_1 \right| \leq \frac{1$

$$
|a(t_1,x_1)-a(t_2,x_2)| \leq C_1 |\omega(t_1)-\omega(t_2)|+C_1 \left|b\left(\frac{x_1^2}{t_1}\right)-b\left(\frac{x_2^2}{t_2}\right)\right| \omega(t_2). \hspace{0.5cm} (6)
$$

Since $|b'| \leq 2$,

$$
a(t_2, x_2)| \leq C_1 |\omega(t_1) - \omega(t_2)| + C_1 |b\left(\frac{x_1}{t_1}\right) - b\left(\frac{x_2}{t_1}\right)|
$$

$$
\left|\frac{\partial}{\partial t} b\left(\frac{x^2}{t}\right)\right| \leq \frac{2x^2}{t^2} \text{ and } \left|\frac{\partial}{\partial x} b\left(\frac{x^2}{t}\right)\right| \leq \frac{4|x|}{t}.
$$

$$
\left|\nabla_{t,x} b\left(\frac{x^2}{t}\right)\right| \leq \frac{10}{t} \text{ for every } t \text{ with } 0 < t \leq 1.
$$

Since $(t, x) \mapsto b(\frac{x^2}{t})$ is constant outside the set of all t, x with $x^2 \leq 2t$, we can conclude that

$$
|\nabla_{t,x} b(\frac{x^2}{t})| \leq \frac{10}{t} \quad \text{for every } t \text{ with } 0 < t \leq 1.
$$

Hence,

is constant outside the set of all
$$
t, x
$$
 with $x^2 \le$
\n
$$
\nabla_{t,x} b(\frac{x^2}{t})| \le \frac{10}{t} \quad \text{for every } t \text{ with } 0 < t \le 1
$$
\n
$$
\left| b\left(\frac{x_1^2}{t_1}\right) - b\left(\frac{x_2^2}{t_2}\right) \right| \le \frac{10}{t_2} |(t_1 - t_2, x_1 - x_2)|.
$$
\nthat $|(t_1 - t_2, x_1 - x_2)| < t_2$ and that $\lambda \mapsto \frac{1}{t_2}$

Hence,
 $|\nabla_{t,x} b(\frac{x}{t})| \leq \frac{10}{t}$ for every t with $0 < t \leq 1$.

Hence,
 $\left| b\left(\frac{x_1^2}{t_1}\right) - b\left(\frac{x_2^2}{t_2}\right) \right| \leq \frac{10}{t_2} |(t_1 - t_2, x_1 - x_2)|$.

Taking into account that $|(t_1 - t_2, x_1 - x_2)| < t_2$ and that $\lambda \mapsto \frac{\omega$ can estimate the second term on the right-hand side of (6) as follows:

to account that
$$
|(t_1 - t_2, x_1 - x_2)| < t_2
$$
 and that $\lambda \mapsto \frac{\omega(\lambda)}{\lambda}$ is
ate the second term on the right-hand side of (6) as follows:

$$
C_1 \left| b\left(\frac{x_1^2}{t_1}\right) - b\left(\frac{x_2^2}{t_2}\right) \right| \omega(t_2) \leq \frac{1}{2}\omega(|(t_1 - t_2, x_1 - x_2)|).
$$

her hand,

$$
0 \leq \omega(t_1) - \omega(t_2) \leq \omega(t_1) - \frac{t_2}{t_1}\omega(t_1) = \frac{t_1 - t_2}{t_1}\omega(t_1) \leq \omega(t_1)
$$

pletes the proof of Condition (5).
 $\in (0, \frac{1}{2})$ be given. We aim to estimate $P_L^{1,0}\beta_\tau$ from below. Exp
call that $0 < a \leq 2$)

$$
\frac{1}{2}a(t, x)\frac{\partial^2}{\partial x^2}\beta_\tau(x) \geq -\frac{1}{\tau}\beta_\tau(x) \quad \text{for every } t, x.
$$

mu principle (or [4: Corollary 1]) yields immediately

On the other hand,

there hand,

\n
$$
0 \leq \omega(t_1) - \omega(t_2) \leq \omega(t_1) - \frac{t_2}{t_1} \omega(t_1) = \frac{t_1 - t_2}{t_1} \omega(t_1) \leq \omega(t_1 - t_2).
$$
\npletes the proof of Condition (5).

\n
$$
\in (0, \frac{1}{2}) \text{ be given. We aim to estimate } P_L^{1,0} \beta_\tau \text{ from below. Explicit to call that } 0 < a \leq 2
$$
\n
$$
\frac{1}{2} a(t, x) \frac{\partial^2}{\partial x^2} \beta_\tau(x) \geq -\frac{1}{\tau} \beta_\tau(x) \quad \text{for every } t, x.
$$

This completes the proof of Condition (5).

Let $\tau \in (0, \frac{1}{2})$ be given. We aim to estimate $P_L^{1,0}\beta_\tau$ from below. Explicit calculation yields (recall that $0 < a \leq 2$) $p(t_1) - \frac{t_2}{t_1} \omega(t_1) = \frac{t_1 - t_2}{t_1} \omega(t_1) \leq \omega(t_1 - t_2).$
 PL $\omega(t_1 - t_2)$.
 *PL*⁰*P*, *PL*⁰*P*, *P*(*PL*⁰*P*, *P*(*PPL*⁰*P*), *P*(*PPL*⁰*P*), *PL*^{*P*}_{*P*}(*x*) for every *t*, *x*.
 *PL*⁰*P*_{*P*} 2 exp(

$$
\frac{1}{2}a(t,x)\frac{\partial^2}{\partial x^2}\beta_\tau(x) \ \geq \ -\frac{1}{\tau}\beta_\tau(x) \qquad \text{for every } t,x
$$

The maximum principle (or [4: Corollary 1]) yields immediately

$$
P_L^{\tau,0} \beta_\tau \ge \exp(-1)\beta_\tau. \tag{7}
$$

In order to apply a comparison result (see [4: Theorem 1]) we set

$$
\frac{1}{2}a(t,x)\frac{\partial^2}{\partial x^2}\beta_\tau(x) \ge -\frac{1}{\tau}\beta_\tau(x) \quad \text{for every } t, x.
$$

\nciple (or [4: Corollary 1]) yields immediately
\n
$$
P_L^{\tau,0}\beta_\tau \ge \exp(-1)\beta_\tau.
$$

\ncomparison result (see [4: Theorem 1]) we set
\n
$$
\tilde{L}_t \equiv \frac{\Delta}{2} + \frac{1}{2}C_1C_2 \frac{\omega(t)}{t} \quad \text{for every } t \in [\tau, \frac{1}{2}],
$$

the positive constant C_2 will be chosen later. The corresponding transition maps will be denoted by \tilde{P}^{t_1,t_2} (the transition maps \tilde{P}^{t_1,t_2} are well-defined although \tilde{L}_t contains a zero order term, cf. for instance [4]). We have terexample for L^1 -Estimates 405

ssponding transition maps will be

fined although \tilde{L}_t contains a zero
 $\frac{1}{2} d\lambda$ β_t (8)

transition maps
$$
P^{1,1}
$$
 are well-defined, although L_t contains a zero

\n $\tilde{P}^{t,\tau} \beta_{\tau} = \exp\left(\frac{1}{2}C_1C_2 \int_{\tau}^{t} \frac{\omega(\lambda)}{\lambda} d\lambda\right) \beta_t$

\n[4: Theorem 1],

\n $-\tilde{P}^{\frac{1}{2},\tau} \beta_{\tau} = \int_{\tau}^{\frac{1}{2}} P_L^{\frac{1}{2},t} (L_t - \tilde{L}_t) \tilde{P}^{t,\tau} \beta_{\tau} dt.$

\n(9)

\ntion of a , we obtain that

for every $t \in [\tau, \frac{1}{2}]$. By [4: Theorem 1],

$$
\tilde{P}^{t,\tau} \beta_{\tau} = \exp\left(\frac{1}{2}C_1C_2 \int_{\tau} \frac{\omega(\lambda)}{\lambda} d\lambda\right) \beta_t
$$
\n(8)
\n
$$
\frac{1}{2}
$$
]. By [4: Theorem 1],
\n
$$
P_L^{\frac{1}{2},\tau} \beta_{\tau} - \tilde{P}^{\frac{1}{2},\tau} \beta_{\tau} = \int_{\tau}^{\frac{1}{2}} P_L^{\frac{1}{2},t} (L_t - \tilde{L}_t) \tilde{P}^{t,\tau} \beta_{\tau} dt.
$$
\n(9)
\nthe definition of *a* we obtain that

From (8) and the definition of a we obtain that

$$
\begin{array}{rcl}\n\left((L_t - \tilde{L}_t) \tilde{P}^{t,\tau} \beta_\tau \right) (x) & = & c_t \left(C_1 \omega(t) \ b\left(\frac{x^2}{t}\right) \frac{\partial^2}{\partial x^2} \beta_t(x) - C_1 C_2 \frac{\omega(t)}{t} \beta_t(x) \right) \\
& = & c_t C_1 \frac{\omega(t)}{t} \left(b\left(\frac{x^2}{t}\right) \left(\frac{x^2}{t} - 1\right) - C_2 \right) \ \beta_t(x)\n\end{array}
$$

for a positive number c_t which depends only on t and for every $t \in [\tau, \frac{1}{2}]$. Since $b(\lambda^2)(\lambda^2-1)$ is non-negative for every λ and bounded below by a positive number outside a compact set, we can choose the positive constant *C2* such that the following function is positive:

$$
\beta_1 * ((b(\cdot^2)(|\cdot|^2-1) - C_2)\beta_1).
$$

In view of the Lemma and the Corollary which we have already proved we can choose the positive number δ such that the function $P_L^{2t,t} \left((b(\frac{x^2}{t}) (\frac{x^2}{t} - 1) - C_2) \cdot \beta_t(x) \right)$ is positive In view of the Lemma and the Corollary which we have already proved we can choose the positive number δ such that the function $P_L^{2t,t} \left((b(\frac{x^2}{t}) (\frac{x^2}{t} - 1) - C_2) \cdot \beta_t(x) \right)$ is positive for every *a* with $||a - 1||_{\in$ for every $t \in [\tau, \frac{1}{2}]$. By (9), $\begin{aligned}\n &= c_t C_1 \frac{1}{t} \\
 &= c_t C_1 \frac{1}{t}\n \end{aligned}$

(mber c_t which depends only for every λ and bounded

ose the positive constant ($\beta_1 * ((b(\cdot^2))$)

(emma and the Corollary v
 δ such that the function
 $||a - 1||_{\infty} < \$ er c_t which depevery λ and λ
the positive c
 $\beta_1 *$
ma and the Cosuch that the
 $-1||_{\infty} < \delta$. T.
By (9),
 $- P_L^{1, \frac{1}{2}} \tilde{P}_{2, \alpha}^{\frac{1}{2}, \tau}$ for
that $a(t, \cdot) \frac{\partial^2}{\partial x}$
) yields imme *ary* which we have already proved we can choose the
 dion $P_L^{2t,t} \left((b(\frac{x^2}{t}) (\frac{x^2}{t} - 1) - C_2) \cdot \beta_t(x) \right)$ is positive

we can conclude that $P_L^{1,t}(L_t - \tilde{L}_t) \tilde{P}^{t,\tau} \beta_{\tau}$ is positive
 $= \int_{\tau}^{\frac{1}{2}} P_L^{1,t} (L_t$ for every a with $||a-1||_{\infty} < \delta$. Thus, we can conclude that $P_L^{1,t}(L_t - \tilde{L}_t) \tilde{P}^{t,\tau}$ β_{τ} is positive

$$
P_L^{1,\tau} \beta_\tau - P_L^{1,\frac{1}{2}} \tilde{P}_\tau^{\frac{1}{2},\tau} \beta_\tau = \int_\tau^{\frac{1}{2}} P_L^{1,t} \left(L_t - \tilde{L}_t \right) \tilde{P}^{t,\tau} \beta_\tau \, dt \geq 0. \qquad (10)
$$

Finally, we remark that $a(t, \cdot) \frac{\partial^2}{\partial x^2} \beta_t$ $\frac{\partial^2}{\partial x^2}$ β_t for every t. The maximum principle (or [4: Corollary 1]) yields immediately

$$
P_L^{1,\frac{1}{2}}\beta_{1/2} \geq \beta_1. \tag{11}
$$

We can now conclude from (7), (8), (10), and (11) that

$$
\beta_{\tau} - P_{L}^{1.7} P^{\frac{1}{2},\tau} \beta_{\tau} = \int_{\tau} P_{L}^{1.1} (L_{t} - \bar{L}_{t}) \bar{P}^{t,\tau} \beta_{\tau} dt \ge
$$

\n
$$
\begin{array}{lll}\n\text{mark that } a(t, \cdot) \frac{\partial^{2}}{\partial x^{2}} \beta_{t} \geq \frac{\partial^{2}}{\partial x^{2}} \beta_{t} \text{ for every } t. \text{ The maxi} \\
\text{by 1]}) yields immediately\n\end{array}
$$
\n
$$
P_{L}^{1, \frac{1}{2}} \beta_{1/2} \geq \beta_{1}.
$$
\n
$$
\text{nclude from (7), (8), (10), and (11) that}
$$
\n
$$
P_{L}^{1,0} \beta_{\tau} \geq \exp(-1) P_{L}^{1, \tau} \beta_{\tau}
$$
\n
$$
\geq \exp(-1) P_{L}^{1, \frac{1}{2}} \bar{P}^{\frac{1}{2}, \tau} \beta_{\tau}
$$
\n
$$
= P_{L}^{1, \frac{1}{2}} \exp\left(\frac{1}{2} C_{1} C_{2} \int_{\tau}^{\frac{1}{2}} \frac{\omega(\lambda)}{\lambda} d\lambda - 1\right) \beta_{\frac{1}{2}}
$$
\n
$$
\geq \exp\left(\frac{1}{2} C_{1} C_{2} \int_{\tau}^{\frac{1}{2}} \frac{\omega(\lambda)}{\lambda} d\lambda - 1\right) \beta_{1}.
$$

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Since $||\beta_{\tau}||_1 = 1$ for every τ , we obtain in particular that

ER

\n1 for every
$$
\tau
$$
, we obtain in particular that

\n
$$
||P_L^{1,0}\beta_\tau||_1 \geq \exp\left(\frac{1}{2}C_1C_2\int_\tau^{\frac{1}{2}}\frac{\omega(\lambda)}{\lambda} d\lambda - 1\right)||\beta_\tau||_1.
$$
\nassumptions on ω the proof is now complete

In view of the assumptions on ω the proof is now complete \blacksquare

Example: We define the function ω by $\omega(0) \equiv 0$, $\omega(\lambda) \equiv -\frac{1}{\ln \lambda}$ for every $\lambda \in (0, \frac{1}{\epsilon}]$. and $\omega(\lambda) \equiv 1$ for every $\lambda \in (\frac{1}{\epsilon}, \infty)$. It is easy to check that the assumptions of the Theorem are met. Moreover, a is Dini (α) continuous for every $\alpha \in (0,1)$ (cf. the abstract of the present paper or [2] for the definition of Dini (α) continuity). In view of the results in [2] corresponding counterexamples for L^p -estimates with $0 < p < 1$ and differential operators with Dini $(\frac{1}{2})$ continuous coefficients cannot exist.

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