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A Counterexample for L^1-Estimates for Parabolic Differential Equations

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We show that the Dini (1) continuity of the coefficients of a linear parabolic differential operator in non-divergence form is in some sense the weakest condition such that the solutions of the corresponding initial value problem satisfy an L^1-estimate; here a function is called Dini (α) continuous for a positive number α if the modulus of continuity ω of the function satisfies $\int_{0+} \omega^{1/\alpha}(\tau)/\tau \, d\tau < \infty$. In particular, we improve a counterexample of Il'in which shows that an L^1-estimate cannot hold in general if only Dini (α) continuity with $\alpha < 1/4$ is assumed.

Key words: *Initial value problems for second-order, parabolic equations, a priori estimates, diffusion processes*

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We consider the initial value problem

$$Lu = 0 \quad \text{in} \quad (0,1) \times \mathbb{R}^n, \qquad u(0,\cdot) = f$$

where the coefficients $a_{ij} = a_{ji}$ of the differential operator

$$Lu = \frac{\partial u}{\partial t} - L_t u = \frac{\partial u}{\partial t} - \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial^2 u}{\partial x_i \partial x_j}$$

are assumed to be continuous and bounded. Moreover, we restrict ourselves to the uniformly parabolic case $(a_{ij}(t,x))_{i,j} \geq A \cdot \mathbb{1}$ for some positive constant A.

Suppose temporarily that the coefficients of L are continuous on $[0,\infty) \times \mathbb{R}^n$ and Hölder continuous on $[s,\infty) \times \mathbb{R}^n$ for every $s > 0$. Then the initial value problem

$$Lu = 0 \quad \text{for } t > s, \quad u(s,\cdot) = f$$

is uniquely solvable for every $s > 0$. The solution u can be written by means of the transition probability measures $P_L(t,x;s,\cdot)$ of the diffusion process with decreasing time parameter generated by the parabolic operator $L_t + \frac{\partial}{\partial t}$ as follows (cf. [5: Chapter 3]):

$$u(t,x) = \int_{\mathbb{R}^n} f(y) P_L(t,x;s,dy).$$

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We mention that the diffusion process generated by $L_t + \frac{\partial}{\partial t}$ is uniquely defined if the coefficients of the uniformly parabolic operator L are merely supposed to be continuous (see [5: Chapter 7]). The transition maps $P_L^{t_1,t_2} : C(\mathbb{R}^n) \to C(\mathbb{R}^n)$ are defined by

$$P_L^{t_1,t_2} f\ (x) \quad \equiv \quad \int_{\mathbb{R}^n} f(y)\ P_L(t_1, x; t_2, dy)$$

for every $f \in C(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Let $\beta_t^{(n)}$ be the kernel of the n-dimensional Brownian motion at time t, i.e. $\beta_t^{(n)}(x) \equiv \frac{1}{(2\pi t)^{\frac{n}{2}}} \exp(-\frac{|x|^2}{2t})$.

Lemma: *Let $t_1, t_2 \in [0,1]$ with $t_1 > t_2$ and a function $f \in C_b^2(\mathbb{R}^n)$ be given such that $\beta_{t_1-t_2}^{(n)} * f(x)$ is positive for every $x \in \mathbb{R}^n$. Suppose that $f(x) \geq \beta_\epsilon^{(n)}(x)$ for some positive number ϵ and every x from the complement of an appropriate compact set $K \subset \mathbb{R}^n$. Then there exists a positive number δ such that, for every differential generator $L = \frac{\partial}{\partial t} - \frac{1}{2}\sum_{i,j=1}^n a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j}$ with matrix norm $\|(a_{ij}(t,x)) - \mathbb{1}\| < \delta$ for every t and x, $P_L^{t_1,t_2} f(x)$ is positive for every $x \in \mathbb{R}^n$.*

Proof: Let δ_0 be a positive number with $\delta_0 < \frac{1}{3}$ and

$$(t_1 - t_2)(1 - 2\delta_0) + \frac{\epsilon}{2} \ > \ (t_1 - t_2)(1 + 3\delta_0). \tag{1}$$

For every positive number r we denote the closed ball with radius r and center 0 in $\mathbb{R}^n$ by B_r. Let 1_{B_r} be the characteristic function of B_r.

We claim that $\|(a_{ij}(t,x)) - \mathbb{1}\| < \delta_0$ for every t, x implies that $P_L^{t_1,t_2} f$ is positive outside B_{r_0} for some positive constant r_0. The proof of the claim is based on the lower and upper bounds for transition probabilities given in [4]. It follows from [4: Theorem 2] that for appropriate positive constants C_1, C_2, C_3 the following holds:

$$\begin{aligned} P_L^{t_1,t_2}\beta_\epsilon^{(n)} &\geq C_1 P_L^{t_1,t_2}(1_{B_1} * \beta_{\epsilon/2}^{(n)}) \\ &\geq C_2 \beta_{(t_1-t_2)(1-2\delta_0)}^{(n)} * (1_{B_1} * \beta_{\epsilon/2}^{(n)}) \\ &\geq C_3\ \beta_{(t_1-t_2)(1-2\delta_0)+\frac{\epsilon}{2}}^{(n)} . \end{aligned} \tag{2}$$

Choose r_1 in such a way that $K \subset B_{r_1}$. Then

$$f \geq \beta_\epsilon^{(n)} - C_4 1_{B_{r_1}} \tag{3}$$

for an appropriate positive constant C_4. By [4: Theorem 3], we have for positive constants C_5, C_6

$$P_L^{t_1,t_2} 1_{B_{r_1}} \ \leq \ C_5\ \beta_{(t_1-t_2)(1+2\delta_0)}^{(n)} * 1_{B_{r_1}} \ \leq \ C_6\ \beta_{(t_1-t_2)(1+2\delta_0)}^{(n)} . \tag{4}$$

The proof of the claim can now be completed as follows. By (2)–(4),

$$P_L^{t_1,t_2} f \ \geq \ C_3 \beta_{(t_1-t_2)(1-2\delta_0)+\frac{\epsilon}{2}}^{(n)} \ - \ C_4 C_6 \beta_{(t_1-t_2)(1+3\delta_0)}^{(n)}.$$

The existence of an r_0 such that the above claim is correct follows from (1).

On account of the assumptions on f the number $C_7 \equiv \inf_{||x||\le r_0}\{\beta^{(n)}_{t_1-t_2} * f(x)\}$ is positive. By [4: Theorem 1],

$$(P_L^{t_1,t_2} f - \beta^{(n)}_{t_1-t_2} * f)(x) = \int_{t_2}^{t_1} P_L^{t_1,t} \left(\frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} - \frac{\Delta}{2}\right)(\beta^{(n)}_{t-t_2} * f)(x)\, dt.$$

Since $f \in C_b^2(\mathbb{R}^n)$, we can choose a positive number δ_1 such that $||(a_{ij}(t,x)) - \mathbb{1}|| < \delta_1$ for every t, x implies $||P_L^{t_1,t_2} f - \beta^{(n)}_{t_1-t_2} * f||_\infty < C_7$. Set $\delta \equiv \min\{\delta_0; \delta_1\}$. The assertion of the Lemma follows immediately ∎

For every λ with $0 < \lambda \le 1$ we denote by f_λ the function from $C_b^2(\mathbb{R}^n)$ which is given by

$$f_\lambda(x) \equiv f\left(\frac{x}{\sqrt{\lambda}}\right) \qquad \text{for every } x \in \mathbb{R}^n;$$

here f stands for a function which satisfies the conditions of the Lemma. Using the scaling transformation $(t,x) \mapsto (\lambda t, \sqrt{\lambda}\, x)$ and taking into account that $||(a_{ij}(t,x)) - \mathbb{1}|| < \delta$ for every t, x implies a similar condition for the transformed coefficients, we obtain the following corollary.

Corollary: *Under the assumptions of the Lemma there exists a positive number δ such that, for every L with $||(a_{ij}(t,x)) - \mathbb{1}|| < \delta$ for every t and x, $P_L^{\lambda t_1, \lambda t_2} f_\lambda(x)$ is positive for every $x \in \mathbb{R}^n$.*

We will restrict ourselves to the consideration of one-dimensional processes in the remainder of this paper. Therefore we will write simply β_t for the one-dimensional heat kernel.

Our main result is the following theorem.

Theorem: *Let $\omega : [0,\infty) \to \mathbb{R}$ be an increasing continuous function with $\omega(0) = 0$. Suppose that $\lambda \mapsto \frac{\omega(\lambda)}{\lambda}$ is decreasing on $(0,\infty)$ and $\int_0^1 \frac{\omega(\lambda)}{\lambda} d\lambda = +\infty$. Let δ_0 be a positive number.*

Then there exists a continuous function $a : \mathbb{R}^1 \times [0,1] \to \mathbb{R}$ with $||a - 1||_\infty < \delta_0$ and

$$|a(t_1,x_1) - a(t_2,x_2)| \le \omega(|(t_1-t_2, x_1-x_2)|) \tag{5}$$

for every $t_1, t_2 \in [0,1]$ and $x_1, x_2 \in \mathbb{R}$ such that for the corresponding initial value problem

$$\frac{\partial u}{\partial t} = \frac{1}{2}\, a(t,x)\, \frac{\partial^2 u}{\partial x^2}, \qquad u(0,x) = f(x)$$

for any given $c > 0$ there is a function f such that

$$||u(1,\cdot)||_1 > c \cdot ||f(\cdot)||_1.$$

Proof: Let b be a function from $C_b^\infty(\mathbb{R}^1)$ with $0 \le b \le 1$, $|b'| \le 2$, and $b(\lambda) = 1$ for every λ with $|\lambda| \ge 2$, and $b(\lambda) = 0$ for every λ with $|\lambda| \le 1$. We set

$$a(t,x) \equiv 1 + C_1 \cdot b\left(\frac{x^2}{t}\right)\omega(t) \qquad \text{for every } (t,x) \in (0,1] \times \mathbb{R}$$

and $a(0,x) \equiv 1$ for every $x \in \mathbb{R}$. The constant C_1 will be chosen later in such a way that $0 < C_1 < \min\{\frac{1}{20}; \frac{1}{\omega(1)}; \frac{\delta_0}{\omega(1)}\}$. Obviously, we have $\|a-1\|_\infty < \delta_0$.

We check that Condition (5) is met. Let $t_1, t_2 \in [0,1]$ and $x_1, x_2 \in \mathbb{R}$ be given. In view of the continuity of a we may and will assume that t_1, t_2 are not equal to zero. Moreover we suppose that $t_2 \le t_1$. It is obvious from the definition of a that $|a(t_1,x_1) - a(t_2,x_2)| \le \frac{1}{2}\omega(t_1)$. Since $\lambda \mapsto \frac{\omega(\lambda)}{\lambda}$ is decreasing, we have $\frac{1}{2}\omega(t_1) \le \omega(\frac{t_1}{2})$. On account of the assumption that ω is increasing, Condition (5) is established if $|(t_1-t_2,\ x_1-x_2)| \ge \frac{1}{2}t_1$. Now assume that $|(t_1-t_2,\ x_1-x_2)| < \frac{1}{2}t_1$. In particular, we have $\frac{1}{2}t_1 < t_2$. Thus,

$$|a(t_1,x_1) - a(t_2,x_2)| \le C_1\ |\omega(t_1) - \omega(t_2)| + C_1 \left| b\left(\frac{x_1^2}{t_1}\right) - b\left(\frac{x_2^2}{t_2}\right)\right| \omega(t_2). \tag{6}$$

Since $|b'| \le 2$,

$$\left|\frac{\partial}{\partial t} b\left(\frac{x^2}{t}\right)\right| \le \frac{2x^2}{t^2} \quad \text{and} \quad \left|\frac{\partial}{\partial x} b\left(\frac{x^2}{t}\right)\right| \le \frac{4|x|}{t}.$$

Since $(t,x) \mapsto b(\frac{x^2}{t})$ is constant outside the set of all t, x with $x^2 \le 2t$, we can conclude that

$$|\nabla_{t,x}\, b(\frac{x^2}{t})| \le \frac{10}{t} \qquad \text{for every } t \text{ with } 0 < t \le 1.$$

Hence,

$$\left| b\left(\frac{x_1^2}{t_1}\right) - b\left(\frac{x_2^2}{t_2}\right)\right| \le \frac{10}{t_2}\ |(t_1-t_2,\ x_1-x_2)|.$$

Taking into account that $|(t_1-t_2,\ x_1-x_2)| < t_2$ and that $\lambda \mapsto \frac{\omega(\lambda)}{\lambda}$ is decreasing, we can estimate the second term on the right-hand side of (6) as follows:

$$C_1 \left| b\left(\frac{x_1^2}{t_1}\right) - b\left(\frac{x_2^2}{t_2}\right)\right| \omega(t_2) \le \frac{1}{2}\omega(|(t_1-t_2,\ x_1-x_2)|).$$

On the other hand,

$$0 \le \omega(t_1) - \omega(t_2) \le \omega(t_1) - \frac{t_2}{t_1}\omega(t_1) = \frac{t_1-t_2}{t_1}\omega(t_1) \le \omega(t_1-t_2).$$

This completes the proof of Condition (5).

Let $\tau \in (0, \frac{1}{2})$ be given. We aim to estimate $P_L^{1,0}\beta_\tau$ from below. Explicit calculation yields (recall that $0 < a \le 2$)

$$\frac{1}{2}a(t,x)\frac{\partial^2}{\partial x^2}\beta_\tau(x) \ge -\frac{1}{\tau}\beta_\tau(x) \qquad \text{for every } t, x.$$

The maximum principle (or [4: Corollary 1]) yields immediately

$$P_L^{\tau,0}\beta_\tau \ge \exp(-1)\beta_\tau. \tag{7}$$

In order to apply a comparison result (see [4: Theorem 1]) we set

$$\tilde{L}_t \equiv \frac{\Delta}{2} + \frac{1}{2}C_1C_2\,\frac{\omega(t)}{t} \qquad \text{for every } t \in [\tau, \frac{1}{2}],$$

the positive constant C_2 will be chosen later. The corresponding transition maps will be denoted by $\tilde{P}^{t_1,t_2}$ (the transition maps $\tilde{P}^{t_1,t_2}$ are well-defined although $\tilde{L}_t$ contains a zero order term, cf. for instance [4]). We have

$$\tilde{P}^{t,\tau}\ \beta_\tau\ =\ \exp\left(\frac{1}{2}C_1C_2\int_\tau^t \frac{\omega(\lambda)}{\lambda}\,d\lambda\right)\ \beta_t \tag{8}$$

for every $t \in [\tau,\ \frac{1}{2}]$. By [4: Theorem 1],

$$P_L^{\frac{1}{2},\tau}\ \beta_\tau\ -\ \tilde{P}^{\frac{1}{2},\tau}\ \beta_\tau\ \ =\ \ \int_\tau^{\frac{1}{2}} P_L^{\frac{1}{2},t}\ (L_t\ -\ \tilde{L}_t)\ \tilde{P}^{t,\tau}\beta_\tau\ dt. \tag{9}$$

From (8) and the definition of a we obtain that

$$\begin{aligned}((L_t - \tilde{L}_t)\tilde{P}^{t,\tau}\beta_\tau)\ (x)\ &=\ c_t\left(C_1\omega(t)\ b(\frac{x^2}{t})\ \frac{\partial^2}{\partial x^2}\beta_t(x)\ \ -\ \ C_1C_2\frac{\omega(t)}{t}\beta_t(x)\right)\\ &=\ c_tC_1\frac{\omega(t)}{t}\left(b(\frac{x^2}{t})\ (\frac{x^2}{t}-1)\ -\ C_2\right)\ \beta_t(x)\end{aligned}$$

for a positive number c_t which depends only on t and for every $t \in [\tau, \frac{1}{2}]$. Since $b(\lambda^2)(\lambda^2-1)$ is non-negative for every λ and bounded below by a positive number outside a compact set, we can choose the positive constant C_2 such that the following function is positive:

$$\beta_1 * \left((b(\cdot^2)(|\cdot|^2-1)\ - C_2)\beta_1\right).$$

In view of the Lemma and the Corollary which we have already proved we can choose the positive number δ such that the function $P_L^{2t,t}\left((b(\frac{x^2}{t})\ (\frac{x^2}{t}-\ 1) - C_2)\cdot\beta_t(x)\right)$ is positive for every a with $||a-1||_\infty < \delta$. Thus, we can conclude that $P_L^{1,t}(L_t - \tilde{L}_t)\tilde{P}^{t,\tau}\ \beta_\tau$ is positive for every $t \in [\tau, \frac{1}{2}]$. By (9),

$$P_L^{1,\tau}\ \beta_\tau\ -\ P_L^{1,\frac{1}{2}}\tilde{P}^{\frac{1}{2},\tau}\ \beta_\tau\ \ =\ \ \int_\tau^{\frac{1}{2}} P_L^{1,t}\ (L_t\ -\ \tilde{L}_t)\ \tilde{P}^{t,\tau}\beta_\tau\ dt\ \ \geq\ 0. \tag{10}$$

Finally, we remark that $a(t,\cdot)\frac{\partial^2}{\partial x^2}\ \beta_t\ \ \geq\ \ \frac{\partial^2}{\partial x^2}\ \beta_t$ for every t. The maximum principle (or [4: Corollary 1]) yields immediately

$$P_L^{1,\frac{1}{2}}\beta_{1/2}\ \ \geq\ \ \beta_1. \tag{11}$$

We can now conclude from (7), (8), (10), and (11) that

$$\begin{aligned}P_L^{1,0}\beta_\tau\ \ &\geq\ \ \exp(-1)\ P_L^{1,\tau}\ \beta_\tau\\ &\geq\ \ \exp(-1)\ P_L^{1,\frac{1}{2}}\tilde{P}^{\frac{1}{2},\tau}\beta_\tau\\ &=\ \ P_L^{1,\frac{1}{2}}\ \exp\left(\frac{1}{2}C_1C_2\int_\tau^{\frac{1}{2}}\frac{\omega(\lambda)}{\lambda}\,d\lambda\ -\ 1\right)\ \beta_{\frac{1}{2}}\\ &\geq\ \ \exp\left(\frac{1}{2}C_1C_2\int_\tau^{\frac{1}{2}}\frac{\omega(\lambda)}{\lambda}\,d\lambda\ -\ 1\right)\ \beta_1.\end{aligned}$$

Since $||\beta_\tau||_1 = 1$ for every τ, we obtain in particular that

$$||P_L^{1,0}\beta_\tau||_1 \quad \geq \quad \exp\left(\frac{1}{2}C_1C_2\int_\tau^{\frac{1}{2}}\frac{\omega(\lambda)}{\lambda}\,d\lambda\ -\ 1\right)||\beta_\tau||_1.$$

In view of the assumptions on ω the proof is now complete ∎

Example: We define the function ω by $\omega(0) \equiv 0$, $\omega(\lambda) \equiv -\frac{1}{\ln\lambda}$ for every $\lambda \in (0, \frac{1}{e}]$, and $\omega(\lambda) \equiv 1$ for every $\lambda \in (\frac{1}{e}, \infty)$. It is easy to check that the assumptions of the Theorem are met. Moreover, a is Dini (α) continuous for every $\alpha \in (0,1)$ (cf. the abstract of the present paper or [2] for the definition of Dini (α) continuity). In view of the results in [2] corresponding counterexamples for L^p-estimates with $0 < p < 1$ and differential operators with Dini $(\frac{1}{2})$ continuous coefficients cannot exist.

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