A Representation Formula for Three-Dimensional Stokes Flows

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In this note there is derived a representation formula for the velocity field and the pressure of three-dimensional Stokes flows via three (scalar and unique) harmonic functions. This formula yields the complete system of interior solutions of the Stokes equations by R. N. Kaufmann when expanding the harmonic functions in terms of spherical harmonics.

1. Introduction. In this note we derive a representation formula for the velocity $\vec{v}$ and the pressure p of a three-dimensional Stokes flow via three (uniquely determined) harmonic functions. When expanding these harmonic functions in terms of spherical harmonics our representation yields the complete system of the (interior) solutions of the Stokes equations by R.N. Kaufmann [2; 3]. While the well-known relation between the Stokes equations and the biharmonic equation does not lead to a characterization of the Stokes functions, the representation here reduces the Stokes system (like [4]) to the Laplace equation. The proof of our result is elementary, and although it is similar to the proof of another representation (using partly the same identities; see [4]) there seems to be no direct connection between these two results.

2. The result. To formulate our result we need some notation. We shall use the usual *inner product* and *vector product* in $I\!\!R^3$, and the differential operators grad, div, curl, and Δ denote the *gradient, divergence, curl,* and *Laplacian* of scalar functions respectively vector fields in $I\!\!R^3$. Moreover, a scalar function Φ (or a vector field componentwise) is called *harmonic* in a domain $G \subset I\!\!R^3$ if $\Phi \in C_2(G)$ with $\Delta\Phi(\vec{x}) = 0$ for $\vec{x} \in G$; and a vector field $\vec{v}$ is called a *Stokes function* in a domain $G \subset I\!\!R^3$ if $\vec{v} \in C_2(G)$ and if there exists a (corresponding pressure-) function $p \in C_1(G)$ such that

$$\Delta\vec{v}(\vec{x}) = \operatorname{grad} p(\vec{x}), \quad \operatorname{div}\vec{v}(\vec{x}) = 0 \quad \text{for } \vec{x} \in G. \tag{1}$$

Finally, we have to assume that G is *star-shaped with respect to the origin*, i.e. $t\vec{x} \in G$ for all $t \in [0,1]$ and $\vec{x} \in G$; and it is not clear whether this assumption can be avoided (see [4] and [5]).

Theorem: *Assume that $G \subset I\!\!R^3$ is a star-shaped domain with respect to the origin. Then a vector field $\vec{v}$ on G is a Stokes function with corresponding pressure p in G, if and only if there exist (scalar) harmonic functions Ψ, Φ, and φ in G such that*

$$\vec{v}(\vec{x}) = \operatorname{grad}\varphi(\vec{x}) + \vec{x} \times \operatorname{grad}\Phi(\vec{x}) + \operatorname{curl}\{r^2\vec{x} \times \operatorname{grad}\Psi(\vec{x})\} \quad \text{where } r = |\vec{x}| \tag{2}$$

and

$$p(\vec{x}) = -6\Psi(\vec{x}) - 10h(\vec{x}) - 4\,\vec{x}\cdot grad\,h(\vec{x})\,,\quad h(\vec{x}) = \vec{x}\cdot grad\,\Psi(\vec{x})\quad for\ \vec{x}\in G\,.\quad (3)$$

These harmonic functions are uniquely determined by the Stokes function $\vec{v}$ (and its pressure p) under the normalization $p(0) = \Psi(0) = \Phi(0) = \varphi(0) = 0$, and they are given by the formulae

$$\Psi(\vec{x}) = \tfrac{1}{2}\int_0^1\{\sqrt{\tau}-1\}\,p\,(\tau\vec{x})\,d\tau\,,$$

$$\varphi(\vec{x}) = \int_0^1\vec{x}\cdot\tilde{v}\,(\tau\,\vec{x})\,d\tau\,,\qquad\qquad(4)$$

$$\Phi(\vec{x}) = -\int_0^1 w(\tau\vec{x})\,d\tau\quad for\ \vec{x}\in G\,,$$

where

$$\tilde{v}\,(\vec{x}) = \vec{v}(\vec{x}) - curl\,\{r^2\,\vec{x}\cdot grad\,\Psi(\vec{x})\}\,,$$
$$w(\vec{x}) = \vec{x}\cdot\int_0^1 curl\,\vec{v}^*(\tau\vec{x})\,d\tau\quad with\ \vec{v}^*(\vec{x}) = \tilde{v}\,(\vec{x}) - grad\,\varphi(\vec{x})\,.\qquad(5)$$

Remark: If one puts into (2) the spherical harmonics (see, e.g., [6; 7]) for φ , Φ , and Ψ resp. (while setting the other to functions equal to zero resp.), then the complete system of interior solutions of (1) by R.N. Kaufman [2; 3] is obtained. Observe that Kaufman's system was obtained by methods of representation theory of groups (see [1]), while our derivation here is elementary.

3. Proof of the result. First, we note some identities from the vector analysis, which are, of course, partly well known and are contained in [4; Lemma 1]. But these formulae may be verified directly as well, and we assume the existence and/or continuity of the partial derivatives involved. Throughout, $\vec{v}$ denotes a vector field and Φ a scalar function on some domain $G \subset I\!R^3$. Moreover, $\vec{x}\in G$ with $r = |\vec{x}|$, and $t\in I\!R$. We require the following identities:

$$\text{curl}\,\{grad\,\Phi\} = 0\,,\quad \text{div}\,\{curl\,\vec{v}\} = 0\,,\quad \text{div}\,\{\vec{x}\times grad\,\Phi\} = 0 \qquad (6)$$

$$\vec{x}\times grad\,\{r^2\Phi\} = r^2\,\vec{x}\times grad\,\Phi \qquad (7)$$

$$t\frac{d}{dt}\{\Phi(t\,\vec{x})\} = \vec{x}\cdot grad\,\{\Phi(t\,\vec{x})\} \qquad (8)$$

$$t\frac{d}{dt}\{\vec{v}(t\,\vec{x})\} = -\vec{v}(t\,\vec{x}) + grad\,\{\vec{x}\cdot\vec{v}(t\,\vec{x})\} \qquad (9)$$

$$\frac{d}{dt}\{t\,\vec{v}(t\,\vec{x})\} = grad\,\{\vec{x}\cdot\vec{v}\}(t\,\vec{x}) - t\,\vec{x}\times curl\,\vec{v}(t\vec{x}) \qquad (10)$$

$$\text{div}\,\{grad\,\Phi\} = \Delta\Phi\,,\quad \Delta\{grad\,\Phi\} = grad\,\{\Delta\Phi\}\,,\quad \Delta\{curl\,\vec{v}\} = curl\,\{\Delta\vec{v}\} \qquad (11)$$

$$\text{div}\,\{\Phi\vec{x}\} = 3\Phi + \vec{x}\cdot grad\,\Phi \qquad (12)$$

$$\text{curl}\,\{\Phi\vec{x}\} = -\vec{x}\times grad\,\Phi \qquad (13)$$

$$\text{curl}\,\{curl\,\vec{v}\} = grad\,\{div\,\vec{v}\} - \Delta\vec{v} \qquad (14)$$

$$\text{curl}\,\{\vec{x}\times grad\,\Phi\} = \vec{x}\cdot\Delta\Phi - grad\,\{\Phi + \vec{x}\cdot grad\,\Phi\} \qquad (15)$$

$$\Delta\{\Phi\vec{x}\} = 2\operatorname{grad}\Phi + (\Delta\Phi)\vec{x} \tag{16}$$

$$\Delta\{\vec{x}\cdot\operatorname{grad}\Phi\} = \vec{x}\cdot\operatorname{grad}\{\Delta\Phi\} - 2\Delta\Phi \tag{17}$$

$$\Delta\{\vec{x}\times\operatorname{grad}\Phi\} = \vec{x}\times\operatorname{grad}\{\Delta\Phi\} \tag{18}$$

$$\Delta\{r^2\Phi\} = 6\Phi + 4\vec{x}\cdot\operatorname{grad}\Phi + r^2\Delta\Phi \tag{19}$$

$$\Delta\{\vec{x}\cdot\vec{v}\} = \vec{x}\cdot(\Delta\vec{v}) + 2\operatorname{div}\vec{v}\ . \tag{20}$$

Proof of the Theorem: Assume first that $\vec{v}$ and p are given by (2) and (3), where φ, Φ, and Ψ are harmonic in the domain G. This part of the proof does not require that G is star-shaped with respect to the origin. Then, by (6) and (11), we obtain

$$\operatorname{div}\vec{v} = \Delta\varphi + 0 + 0 = 0\ .$$

Moreover, the formulae (7), (11), (15), (17), (18), and (19) yield

$$\begin{aligned}
\Delta\vec{v} &= \operatorname{grad}\{\Delta\varphi\} + \vec{x}\times\operatorname{grad}\{\Delta\Phi\} + \operatorname{curl}[\Delta\{\vec{x}\times\operatorname{grad}(r^2\Psi)\}]\\
&= \operatorname{curl}\{\vec{x}\times\operatorname{grad}\Delta(r^2\Psi)\} = \operatorname{curl}\{\vec{x}\times\operatorname{grad}(6\Psi + 4\vec{x}\cdot\operatorname{grad}\Psi)\}\\
&= \operatorname{grad}\{-6\Psi - 10\,h - 4\,\vec{x}\cdot\operatorname{grad}h\} = \operatorname{grad}p\ .
\end{aligned}$$

Hence, $\vec{v}$ is a Stokes function with corresponding pressure p in G.

Now, suppose that $\vec{v}$ is a Stokes function with pressure p, where $p(0) = 0$. Then, by [8], $\vec{v}$, $p \in C_\infty(G)$ with $\Delta p = \operatorname{div}\{\operatorname{grad}p\} = \operatorname{div}\{\Delta\vec{v}\} = 0$, such that p is harmonic in G. Moreover, assume that Ψ, φ, and Φ are given by (4) and (5). Actually, these formulae make sense only, if G is *star-shaped* with respect to the origin.

(i) The definition of Ψ, $\Delta p = 0$, formula (8), and partial integration imply that Ψ is harmonic, and that

$$h(\vec{x}) = \vec{x}\cdot\operatorname{grad}\Psi(\vec{x}) = -\tfrac{1}{2}\int_0^1\{\tfrac{3}{2}\sqrt{\tau} - 1\}\,p(\tau\vec{x})\,d\tau\ ,$$

$$\vec{x}\cdot\operatorname{grad}h(\vec{x}) = -\tfrac{1}{4}p(\vec{x}) + \tfrac{1}{2}\int_0^1\{\tfrac{9}{4}\sqrt{\tau} - 1\}\,p(\tau\vec{x})\,d\tau\ .$$

Thus, $\Psi(0) = -\tfrac{1}{6}p(0) = 0$, and (3) holds, i.e.

$$-6\Psi(\vec{x}) - 10\,h(\vec{x}) - 4\,\vec{x}\cdot\operatorname{grad}h(\vec{x}) = p(\vec{x})\ . \tag{21}$$

(ii) Let $\tilde{\vec{v}}(\vec{x}) = \vec{v}(\vec{x}) - \vec{v}_1(\vec{x})$ with $\vec{v}_1(\vec{x}) = \operatorname{curl}\{r^2\vec{x}\times\operatorname{grad}\Psi(\vec{x})\}$. Then the first part of the proof shows that

$$\Delta\tilde{\vec{v}} = 0\quad\text{and}\quad\operatorname{div}\tilde{\vec{v}} = 0\ .$$

Hence, by (20), $\Delta\{\vec{x}\cdot\tilde{\vec{v}}\} = 0$. Now, the definiton of φ, formula (9), and partial integration imply that φ is harmonic, $\varphi(0) = 0$, and that

$$\vec{x}\cdot\operatorname{grad}\varphi(\vec{x}) = \vec{x}\cdot\tilde{\vec{v}}(\vec{x})\ . \tag{22}$$

(iii) Finally, let $\vec{v}^*(\vec{x}) = \tilde{\vec{v}}(\vec{x}) - \vec{v}_2(\vec{x})$ with $\vec{v}_2(\vec{x}) = \operatorname{grad}\varphi(\vec{x})$. Then, by (22),

$$\Delta\vec{v}^* = 0\ ,\quad\operatorname{div}\vec{v}^* = 0\ ,\quad\text{and}\quad\vec{x}\cdot\vec{v}^*(\vec{x}) = 0\ .$$

This, the definition of Φ (and w), the formulae (6), (9), (10), (11), (20), and partial integration imply that $w(0) = \Phi(0) = 0$, $\Delta w = 0$, so that Φ is harmonic, that $\operatorname{grad}w(\vec{x}) = \operatorname{curl}\vec{v}^*(\vec{x})$, and that

$$\vec{x}\times\operatorname{grad}\Phi(\vec{x}) = \vec{v}^*(\vec{x})\ . \tag{23}$$

Hence, equality (2) holds, and it remains to show the *uniqueness* of φ, Φ, and Ψ. For this, let φ, Φ, and Ψ be harmonic with $\varphi(0) = \Phi(0) = \Psi(0) = 0$, such that

$$\operatorname{grad} \varphi(\vec{x}) + \vec{x} \times \operatorname{grad} \Phi(\vec{x}) + \operatorname{curl} \{r^2 \vec{x} \times \operatorname{grad} \psi(\vec{x})\} \equiv 0 ,$$

and

$$6\Psi(\vec{x}) + 10\,h(\vec{x}) + 4\,\vec{x} \cdot \operatorname{grad} h(\vec{x}) \equiv 0 \quad \text{where} \quad h(\vec{x}) = \vec{x} \cdot \operatorname{grad} \Psi(\vec{x}) .$$

First, fix $\vec{x} \in G$, and put $y(t) = \Psi(t\,\vec{x})$. Then $y \in C_\infty[0,1]$, $y(0) = 0$, and $t\,y'(t) = h(t\,\vec{x})$, $t^2\,y''(t) = t\,\vec{x}\operatorname{grad} h(t\,\vec{x}) - h(t\,\vec{x})$ by (8). Hence,

$$6\,y(t) + 14\,t\,y'(t) + 4\,t^2\,y''(t) = 0 , \tag{24}$$

and this Euler equation has the unique solution $y(t) \equiv 0$ (which is in $C_\infty[0,1]$). This implies $\Psi \equiv 0$, which yields

$$\operatorname{grad} \varphi(\vec{x}) + \vec{x} \times \operatorname{grad} \Phi(\vec{x}) = 0 .$$

Similarly, let $y(t) = \varphi(t\,\vec{x})$. Then $y(0) = 0$, and $\vec{x} \cdot \operatorname{grad} \varphi(\vec{x}) = 0$ yields by (8)

$$t\,y'(t) = 0 , \tag{25}$$

and this shows $\varphi \equiv 0$, so that $\vec{x} \times \operatorname{grad} \Phi(\vec{x}) = 0$. Hence, by (13), $\operatorname{curl}\{\Phi\vec{x}\} = 0$, and the formulae (12), (14), and (16) show

$$0 = \operatorname{curl}\operatorname{curl}\{\Phi\vec{x}\} = \operatorname{grad}\{\operatorname{div}(\Phi\vec{x})\} - \Delta\{\Phi\vec{x}\} = \operatorname{grad}\{5\Phi + \vec{x}\cdot\operatorname{grad}\Phi\} .$$

Putting $y(t) = \Phi(t\,\vec{x})$ similarly as above, we obtain from (8) $t\,y'(t) = t\,\vec{x}\operatorname{grad}\Phi(t\,\vec{x})$ $=: g(t\,\vec{x})$, $t\,(t\,y'(t))' = t\,\vec{x}\operatorname{grad} g(t\,\vec{x})$, and therefore

$$t^2\,y''(t) + 6\,t\,y'(t) = 0 . \tag{26}$$

Since $y(0) = \Phi(0) = 0$, it follows $\Phi \equiv 0$, and this completes the proof. ∎

Remark: The explicitly solvable Euler equations (24), (25), and (26) but with corresponding inhomogenity were actually used to derive the formulae (4) and (5), which express Ψ, φ, and Φ in terms of $\vec{v}$ and p such that (21), (22), and (23) hold.

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Book reviews

B. ANGER and C. PORTENIER: **Radon Integrals.** An Abstract Approach to Integration and Riesz Representation through Function Cones (Progress in Mathematics: Vol. 103). Boston - Basel - Berlin: Birkhäuser Verlag 1992; IV + 332 pp.

Radon measures are the central objects in topological measure theory. In the framework of locally compact spaces, there are two equivalent canonical definitions. As a set function, a Radon measure is an inner compact regular Borel measure, finite on compact sets. As a functional, it is simply a positive linear form, defined on the vector lattice of continuous real-valued functions with compact support.

In the last decades, in particular because of the developments of abstract probability theory and mathematical physics attention has been focussed on measures on general topological spaces, which are no longer locally compact, e.g. spaces of continuous functions or Schwartz distributions. It is the main goal of the authors to introduce a new concept of a Radon integral, embedded in a general functional-analytic theory of integration in an abstract Riemann spirit. The authors present a unified approach to both the integration-theoretical and set-theoretical aspects of measure theory, based on two concepts: that of a regular linear functional on a function cone, and that of an upper functional as an abstract version of an upper integral, possibly without convergence properties. The general concept covers also such different aspects as Radon measures in the sense of Choquet, contents on lattices of sets, Loomis' abstract Riemann integration for positive linear forms on vector lattices, as well as the Daniell and Bourbaki integration theory.

The monograph should be addressed to mathematicians doing research in measure and integration theory, functional analysis, stochastics and mathematical physics. Furthermore, graduate students working on advanced topics in the field will benefit from it.

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