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Error Bounds for Projection-Type Iterative Methods in Solving Linear Operator Equations

V. ISERNHAGEN

Error bounds using angles between fixed point sets of orthoprojectors are presented for generalized PSH- and SPA-methods.

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Let X and Y be Hilbert spaces. Some of the well-known iterative methods for the solution of linear operator equations

$$Ax = b \qquad (A \in L(X,Y);\ b \in Y) \tag{1}$$

can be written in the form

$$x_{n+1} = T_n x_n + D_n b, \quad T_n := I - D_n A \in L(X,X),\ D_n \in L(Y,X). \tag{2}$$

Often it is favourable also to consider the iteration of the rests

$$r_{n+1} = S_n r_n, \quad S_n := I - AD_n \in L(Y,Y),\ r_n := b - Ax_n. \tag{3}$$

If we choose the operators D_n in such a way that T_n are orthogonal projections, then, according to (2), we obtain the class of *generalized PSH-methods* (*projection methods*), which was investigated by D. Schott in [4]. The elementary variant of this class for finite-dimensional linear spaces, which is obtained by choosing $D_n = (E_n A)^+ E_n$ with row selection matrices E_n, was studied, for instance, by W. Peters [3] and G. Maess [2].

Otherwise, if the operators S_n are orthogonal projectors, then, according to (2), we obtain the class of *generalized SPA-methods* (*rest projection methods, column approximation methods*). They were also investigated by D. Schott in [4]. An elementary variant with $D_n = H_n(AH_n)^+$ and column selection matrices H_n can be found in [2, 3].

In this paper we derive error bounds for these general methods using angles between fixed point sets of orthoprojectors. More general classes of iterative methods, where T_n or S_n are so-called *relaxations* of orthogonal projectors, were presented by D. Schott in [5, 6].

Definition (see, e.g., [1, 7]): Let L_1 and L_2 be two closed subspaces of a Hilbert space H with the intersection $L = L_1 \cap L_2$. The acute angle α between L_1 and L_2 ($\alpha = \sphericalangle(L_1, L_2)$) is given by $\cos\alpha = \sup(u,v)$, where $u \in L_1 \cap L^\perp$ and $v \in L_2 \cap L^\perp$ are unit vectors, $L^\perp$ is the orthogonal complement of L and $(\cdot,\cdot)$ denotes the inner product in H.

First we formulate a theorem for the generalized PSH-methods. For that we denote the orthoprojector onto a linear subspace M by P_M. The proving technique is similar as in [1], where a special result is given.

Theorem 1: *Let the following conditions be fulfilled:*

(i) *The equation $Ax = b$ has a generalized solution x^* with respect to (D_n), i.e. $D_n A x^* = D_n b$ holds for all n.*

(ii) *The operators T_n are orthoprojectors.*

Then for the generalized PSH-method (2) *the error bound*

$$\|(x_{n+1} - P_{\mathfrak{R}} x_0) - x^*\|^2 \le \left(1 - \prod_{i=0}^{n-1} \sin^2\alpha_i\right) \|(x_0 - P_{\mathfrak{R}} x_0) - x^*\|^2$$

holds for all $x_0 \in X$, where $\mathfrak{R} = \bigcap_i \mathfrak{R}(T_i)$ and $\alpha_i = \sphericalangle(\mathfrak{R}(T_i), \bigcap_{j=i+1}^{n} \mathfrak{R}(T_j))$.

Proof: There is no loss of generality in assuming that $x^* = 0$, because the statement is independent of linear translations. Considering the fact that $P_{\mathfrak{R}} x_0$ is a fixed point for all T_n and setting $v = x_0 - P_{\mathfrak{R}} x_0$, the inequality to be proved is

$$\|T_n T_{n-1} \cdots T_0 v\|^2 \le \left(1 - \prod_{i=0}^{n-1} \sin^2\alpha_i\right) \|v\|^2 \quad \text{for all} \quad v \in \mathfrak{R}^\perp. \tag{4}$$

This will be shown by induction.

Of course $\|T_n v\|^2 \le \|v\|^2$ is true for all $v \in \mathfrak{R}(T_n)^\perp$. Now we assume

$$\|T_n \dots T_{i+1} v\|^2 \le \left(1 - \prod_{j=i+1}^{n-1} \sin^2\alpha_j\right) \|v\|^2 \quad \text{for all} \quad v \in (\mathfrak{R}(T_n) \cap \dots \cap \mathfrak{R}(T_{i+1}))^\perp. \tag{5}$$

Let $v \in (\mathfrak{R}(T_n) \cap \dots \cap \mathfrak{R}(T_i))^\perp$. Then we can write $v = w + u$ with $w \in \mathfrak{R}(T_i)$ and $u \in \mathfrak{R}(T_i)^\perp = \mathfrak{N}(T_i)$. Hence, because of $T_i u = 0$, the equation yields $T_n \dots T_{i+1} T_i v = T_n \dots T_{i+1} w$. If we decompose w in the form

$$w = w' + w'', \quad \text{where } w' \in \mathfrak{R}(T_n) \cap \dots \cap \mathfrak{R}(T_{i+1}) \text{ and } w'' \in (\mathfrak{R}(T_n) \cap \dots \cap \mathfrak{R}(T_{i+1}))^\perp,$$

then in view of $T_n \dots T_{i+1} w' = w'$ we obtain $T_n \dots T_{i+1} w = w' + T_n \dots T_{i+1} w''$. Since $(T_n \dots T_{i+1} w'', w') = (w'', w') = 0$, it follows that $T_n \dots T_{i+1} w'' \in (\mathfrak{R}(T_n) \cap \dots \cap \mathfrak{R}(T_{i+1}))^\perp$. Therefore we have

$$\|w\|^2 = \|w'\|^2 + \|w''\|^2 \quad \text{and} \quad \|T_n \dots T_{i+1} w\|^2 \le \|w'\|^2 + \|T_n \dots T_{i+1} w''\|^2.$$

Due to the induction assumption (5) we find

$$\|T_n \dots T_{i+1} w''\|^2 \le \left(1 - \prod_{j=i+1}^{n-1} \sin^2\alpha_j\right) \|w''\|^2.$$

Combining this with the latter formulas we get

$$\begin{aligned} \|T_n \dots T_{i+1} w\|^2 &\le \|w'\|^2 + \left(1 - \prod_{j=i+1}^{n-1} \sin^2\alpha_j\right) \|w''\|^2 \\ &\le \left(1 - \prod_{j=i+1}^{n-1} \sin^2\alpha_j\right) \|w\|^2 + \left(\prod_{j=i+1}^{n-1} \sin^2\alpha_j\right) \|w'\|^2 \end{aligned}$$

On one hand we have $w \in \mathfrak{R}(T_i)$ and on the other hand $w \in (\mathfrak{R}(T_n) \cap \dots \cap \mathfrak{R}(T_i))^\perp$, which can be seen from

$$x \in \mathfrak{N}(T_n) \cap \ldots \cap \mathfrak{N}(T_i) \quad \text{and} \quad (w, x) = (v, x) - (u, x) = 0.$$

Analogously we find

$$w' \in \mathfrak{N}(T_n) \cap \ldots \cap \mathfrak{N}(T_{i+1}) \quad \text{and} \quad w' \in (\mathfrak{N}(T_n) \cap \ldots \cap \mathfrak{N}(T_i))^{\perp}.$$

Definition 1 results in the relation $(w,w')\|w\|^{-1}\|w'\|^{-1} \le \cos\alpha_i$. Taking $(w,w') = \|w'\|^2$ and $\|w\| \le \|v\|$ into account, we get the formula

$$\|T_n \cdots T_{i+1} T_i v\|^2 \le \left(1 - \prod_{j=i}^{n-1} \sin^2\alpha_j\right)\|v\|^2.$$

For $i = 0$ the required inequality arises ∎

If we choose the sequence (D_n) cyclically with the cycle length L, then we obtain the cyclically summarized stationary iterative method

$$x^{(n+1)} = Tx^{(n)} + Db, \quad T = T_{L-1} \cdots T_1 T_0, \quad D = \sum_{i=0}^{L-1} T_{L-1} \cdots T_{i+1} D_i.$$

Now it is easy to prove the cyclewise error bound

$$\|(x^{(n)} - P_{\mathfrak{N}} x^{(0)}) - x^*\|^2 \le \left(1 - \sum_{i=0}^{L-1} \sin^2\alpha_i\right)^n \|(x^{(0)} - P_{\mathfrak{N}} x^{(0)}) - x^*\|^2.$$

Here $\mathfrak{N}$ denotes $\bigcap_{i=0}^{L-1}\mathfrak{N}(T_i)$.

The error bound (6) has been proven by Smith, Solmon and Wagner in [7] for the elementary version of this class of methods, the so-called *Kaczmarz's method.* In their paper the authors mentioned above investigated interesting applications of this method to the field of image reconstruction from its projections (computerized tomography). Hamaker and Solmon [1] used the error bound to improve the rate of convergence of the Kaczmarz's procedure in the field of computerized tomography.

It is obvious that the error bound of Theorem 1 can also be used for considerations concerning convergence acceleration of the generalized methods. But it seems to be complicate to formulate a general heuristics, when the factor containing the angle quantity in the error estimate become small.

By analogy to Theorem 1 we can give an error bound for generalized SPA-methods. Therefore the proof can be omitted here.

Theorem 2: *Let the following conditions be fulfilled:*

(i) *There exists a rest vector r^* with $S_n r^* = r^*$ for all n.*

(ii) *The operators S_n are orthoprojectors.*

Then for the generalized SPA*-method* (3) *the error bound*

$$\|(r_{n+1} - P_{\mathfrak{N}} r_0) - r^*\|^2 \le \left(1 - \prod_{i=0}^{n-1} \sin^2\alpha_i\right)\|(r_0 - P_{\mathfrak{N}} r_0) - r^*\|^2$$

holds for all $r_0 = b - Ax_0$ with arbitrary $x_0 \in X$, where

$$\mathfrak{R} = \bigcap_i \mathfrak{R}(S_i) \quad \text{and} \quad \alpha_i = \sphericalangle(\mathfrak{R}(S_i), \bigcap_{j=i+1}^{n} \mathfrak{R}(S_j)).$$

We remark that obviously the condition $S_n r^* = r^*$ means $AD_n r^* = A(D_n b - D_n A x^*) = 0$. Thus the condition (i) of Theorem 2 is fulfilled if this holds for the condition (i) of Theorem 1.

Moreover it is possible again to derive a corresponding estimate for the cyclical method (see (6)). Such an error bound for generalized SPA-methods presented in this paper couldn't be found in literature so far. Again the error bounds can also be used as a starting point for considerations concerning convergence acceleration.

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Volker Isernhagen
Lärchenstraße 4
D (*Ost*) - 2600 Güstrow