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# A Note on the Existence and Uniqueness of Hölder Solutions of Nonlinear Singular Integral Equations

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The purpose of this note is to apply a generalized Kantorovich majorization principle to existence and uniqueness results for Hölder solutions of nonlinear singular integral equations. In contrast to the classical Kantorovich principle, we do not require differentiability, but only a local Lipschitz condition.

Key words: *Nonlinear singular integral equations, Hölder solutions, local Lipschitz condition, Kantorovich majorization principle, Banach-Caccioppoli principle*

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The purpose of this note is to illustrate the applicability of a generalized Kantorovich majorization principle (Theorem 1) to existence and uniqueness results for Hölder solutions of nonlinear singular integral equations (Theorem 2). This majorization principle reduces the problem of finding fixed points of abstract nonlinear operators in Banach spaces to that of finding fixed points of simple convex functions on the real line. Although this principle may be considered as a modification of the classical Banach-Caccioppoli contraction mapping principle, it is much more suitable in applications (see, for example, below). In contrast to the classical Kantorovich technique, we do not require Fréchet differentiability of the nonlinear operator involved, but only a suitable Lipschitz condition. Moreover, since this Lipschitz condition is only local, is does not lead to a strong degeneracy as a global Lipschitz condition, but covers large classes of nonlinearities arising in applications.

Let $X$ be a Banach space, and let $A : \bar{B}(x_0, R) \to X$ be a (nonlinear) operator, where $\bar{B}(x_0, R)$ denotes the closure of the ball $B(x_0, R) = \{x : x \in X, \|x - x_0\| < R\}$. Suppose that the operator $A$ satisfies a Lipschitz condition

$$\|Ax_1 - Ax_2\| \le \kappa(r)\|x_1 - x_2\| \qquad (x_1, x_2 \in \bar{B}(x_0, r); r \le R), \tag{1}$$

where $\kappa(r)$ denots the minimal Lipschitz constant for $A$ on the ball $\bar{B}(x_0, R)$, i.e.

$$\kappa(r) = \sup\left\{\frac{\|Ax_1 - Ax_2\|}{\|x_1 - x_2\|} : x_1, x_2 \in \bar{B}(x_0, r); x_1 \neq x_2\right\}.$$

Define a scalar function $a : [0, R] \to [0, \infty)$ by

$$a(r) = a_0 + \int_0^r \kappa(t)dt, \tag{2}$$

where

$$a_0 = \|x_0 - Ax_0\|. \tag{3}$$

**Theorem 1:** *Let $A : \bar{B}(x_0, R) \to X$ be an operator satisfying a Lipschitz condition (1). Suppose that the function $a$ given by (2) has a unique fixed point $r_*$ in the interval $[0, R]$, and that $a(R) \le R$. Then the operator $A$ has a fixed point $x_* \in \bar{B}(x_0, r_*)$. This fixed point may be obtained as limit of the successive approximations $x_n = A^n x_0 \in \bar{B}(x_0, r_*)$, and is unique in the ball $B(x_0, R)$.*

**Proof:** We claim that

$$\|A(x+h) - Ax\| \le a(r+\rho) - a(r) \tag{4}$$

for $\|x - x_0\| \le r$ and $\|h\| \le \rho$. In fact, for the subset $\{x, x + \frac{1}{n}h, \ldots, x + h\}$ of $[x, x+h]$ we have

$$\|A(x+h) - Ax\| \le \sum_{j=1}^{n} \left\| A\left(x + \frac{j}{n}h\right) - A\left(x + \frac{j-1}{n}h\right) \right\|$$

$$\le \sum_{j=1}^{n} \kappa\left(r + \frac{j}{n}\|h\|\right) \frac{1}{n}\|h\| \le \frac{\rho}{n} \sum_{j=1}^{n} \kappa\left(r + \frac{j}{n}\rho\right).$$

The last term may be considered as upper Riemann-Darboux sum of the function $\kappa = \kappa(r)$ with respect to the partition $\{r, r + \frac{1}{n}\rho, \ldots, r + \rho\}$ of $[r, r+\rho]$, and hence tends to

$$\int_r^{r+\rho} \kappa(t)dt = a(r+\rho) - a(r)$$

as $n \to \infty$; this proves (4). Now, if we define $r_0 = 0$ and $r_n = a^n(r_0)$, by induction it follows that

$$\|x_m - x_n\| \le r_m - r_n \quad (n \le m). \tag{5}$$

Since the sequence $(r_n)_n$ is monotonically increasing, it converges to $r_* = \sup r_n$. By (5), we conclude that $(x_n)_n$ converges to $x_*$; the estimates $\|x_n - x_0\| \le r_*$ and $\|x_* - x_0\| \le r_*$ also follow from (5).

To prove the uniqueness assertion, suppose that $x^*$ is an arbitrary fixed point of $A$ in $B(x_0, R)$. Consider the successive approximations

$$\rho_0 = \|x^* - x_0\|, \quad \rho_n = a^n(\rho_0), \quad \xi_0 = x^*, \quad \xi_n = A^n \xi_0.$$

Applying (4) to $x = x_0$ and $h = x^* - x_0$ yields

$$\|\xi_1 - x_1\| = \|Ax^* - Ax_0\| \leq a(\|x^* - x_0\|) - a(0) = \rho_1 - r_1.$$

By induction, we get $\|\xi_n - x_n\| \leq \rho_n - r_n$. Since both $(r_n)_n$ and $(\rho_n)_n$ converge to $r_*$, we conclude that $\xi_n \to x_*$ as well. But $\xi_n \equiv x^*$ for all $n$, and hence $x^* = x_*$ ∎

We make some remarks on Theorem 1. The usefulness of this theorem consists in reducing the (hard) problem of finding fixed points of a nonlinear operator in a Banach space to the (simple) problem of finding fixed points of a scalar function. Moreover, in the "generic case" $r_* < R$ we get much more information on $x_*$ than just existence: the smaller we may choose the fixed point $r_*$ of $a$, the better we may "localize" the fixed point $x_*$ of $A$, and the larger we may choose the invariant interval $[0, R]$, the better we may "exclude" other fixed points of $A$. The case $r_* = R$, of course, is worse: we may guarantee then only uniqueness in $B(0, R)$ and existence in $\bar{B}(0, R)$.

Since the function $\kappa$ in (1) is increasing, the function $a$ in (2) is convex. Consequently, existence and uniqueness of fixed points of $a$ essentially depend on the size of the "initial value" $a_0$ in (3). To illustrate this, we have sketched in Fig.1 and Fig.2 three possible configurations in case $a$ is a strictly convex function. For small $a_0$ (Fig.1) we have a unique fixed point $r_*$ in the interval $[0, R]$. As $a_0$ increases (Fig.2), a second fixed point $r^*$ may appear, and we apply Theorem 1 by choosing $R$ between $r_*$ and $r^*$. If $a_0$ is such that the diagonal is tangential to the graph of $a$ at some point $r_*$ (Fig.2), we choose $R = r_*$ and have existence in $\bar{B}(x_0, r_*)$. Of course, in the classical Banach-Caccioppoli fixed point principle we simply have $\kappa(r) \equiv \kappa < 1$. In this case we have existence and uniqueness in $\bar{B}(x_0, R)$, where $R \geq r_* = (1-\kappa)^{-1}a_0$ may be chosen arbitrarily large (Fig.3). If $a$ is not strictly convex, the graph of $a$ may contain some

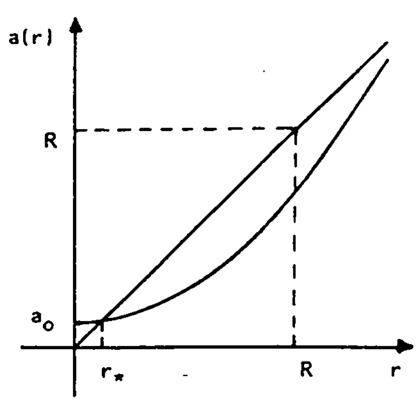


Fig. 1

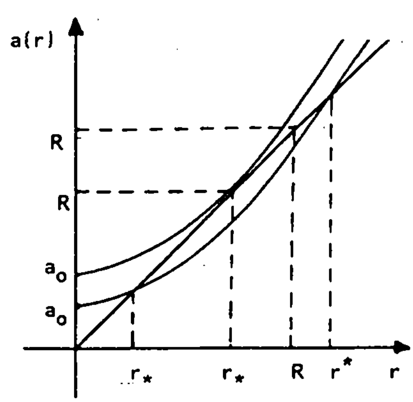


Fig. 2

segment on the diagonal starting from some fixed point $r_*$ (Fig.4). In this case we again choose $R = r_*$ and have existence in $\bar{B}(x_0, r_*)$.

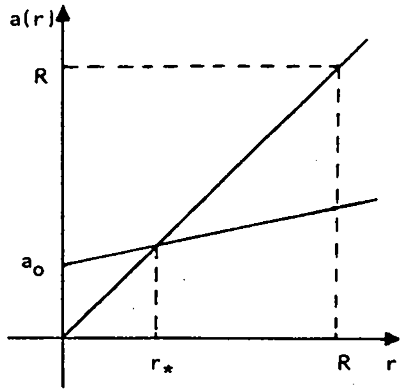


Fig. 3

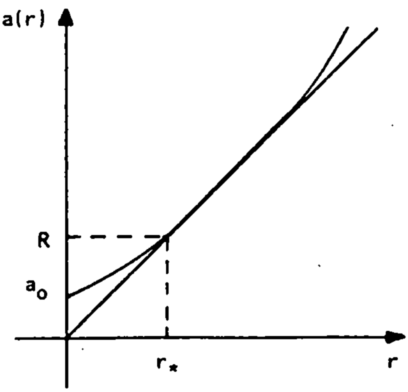


Fig. 4

For Fréchet - differentiable operators $A$, Theorem 1 goes essentially back to L.V. Kantorovich [4]. The idea of majorizing the operator $A$ by scalar functions is also due to Kantorovich (as a matter of fact, estimates of type (4) are often called "Kantorovich majorants" in the literature). The paper [14] seems to contain the first systematic application of Kantorovich majorants to nonlinear integral operators.

It is worthwhile pointing out, however, that our Lipschitz condition (1) is much weaker than Fréchet differentiability of $A$. To recall a classical example, consider the nonlinear superposition operator

$$Fx(t) = f(t, x(t)) \tag{6}$$

generated by some Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ in the Lebesgue space $L_p = L_p(\Omega)$ $(1 \leq p < \infty)$. The Lipschitz condition (1) for the operator $F$ in $L_p$ is then equivalent to a Lipschitz condition

$$|f(t,u) - f(t,v)| \leq \kappa(r)|u - v| \qquad (|u|, |v| \leq r)$$

for the generating function $f$ [1]. On the other hand, the operator $F$ is Fréchet differentiable in $L_p$ only if the generating function $f$ is linear with respect to the second variable [5].

Now we are going to apply Theorem 1 to the nonlinear singular integral equation

$$x(t) = \int_a^b \frac{k(t,\tau)f(\tau, x(\tau))}{t - \tau} d\tau \qquad (a \leq t \leq b) \tag{7}$$

in the Hölder space $C^\alpha = C^\alpha[a,b]$, equipped with the usual norm $\|x\|_{C^\alpha} = \|x\|_C + [x]_\alpha$, where

$$[x]_\alpha = \sup_{s\neq t} \frac{|x(s)-x(t)|}{|s-t|^\alpha}.$$

For simplicity, we put $x_0 = 0$ in what follows. The right-hand side of (7) defines a nonlinear operator $A$ which may be viewed as composition $A = SF$ of the superposition operator (6) and the linear singular operator

$$Sx(t) = \int_a^b \frac{k(t,\tau)x(\tau)}{t-\tau} d\tau. \tag{8}$$

The operator (8) has been studied extensively in the classical Hölder space $C^\alpha (0 < \alpha < 1)$, as well as in several generalized Hölder spaces. Sufficient conditions for the boundedness of $S$ in the space $C^\alpha[0,\pi]$, for instance, have been obtained first by I.I. Privalov [10]; they build on the classical Zygmund inequality [15]

$$\omega(Sx;t) \leq C\left\{\int_0^t \frac{\omega(x;\tau)}{\tau} d\tau + t\int_t^\pi \frac{\omega(x;\tau)}{\tau^2} d\tau\right\},$$

where $\omega(y;\cdot)$ denotes the modulus of continuity of the function $y$. We point out, however, that for guaranteeing the boundedness of the operator (8) in the space $C^\alpha$ one usually has to impose quite restrictive conditions on the kernel function $k = k(t,\tau)$. For the reader's convenience we recall that a typical set of such conditions is as follows:

(a) $k(a,a) = k(b,b) = 0,$

(b) $\omega(k(\cdot,\tau);\delta)\log\frac{b-a}{\delta} = O(\delta^\alpha),$

(c) $\int_0^\delta \frac{\omega(k(t,\cdot);\sigma)}{\sigma} d\sigma = O(\delta^\alpha),$

(d) $\int_\delta^{\frac{b-a}{2}} \frac{\omega(k(t,\cdot);\sigma)}{\sigma^2} d\sigma = O(\delta^{\alpha-1}).$

The sufficiency of the conditions $(a) - (d)$ for the boundedness of the operator $S$, as well as upper estimates for its norm $\|S\|$ in $C^\alpha$, may be found in the monograph [3].

In order to apply Theorem 1 to the operator $A = SF$ we have to find an explicit formula, or at least a good upper estimate, for the Lipschitz constant $\kappa(r)$ in (1). Since for the linear part (8) we simply have $\kappa(r) \equiv \|S\|$, it is the nonlinear part (6) whose Lipschitz constant

$$\kappa(r) = \sup\left\{\frac{\|Fx-Fy\|_{C^\alpha}}{\|x-y\|_{C^\alpha}} : \|x\|_{C^\alpha}, \|y\|_{C^\alpha} \leq r\right\} \tag{9}$$

requires a more careful analysis. Before carrying out this analysis, we emphasize the dependence of (9) on $r > 0$. In fact, in [6,7] it was shown that the superposition operator (6) may satisfy a global Lipschitz condition in the space $C^\alpha$ (i.e. $\kappa(r) \equiv$ const in (9)) only if the corresponding function $f$ is affine in the second variable (i.e. $f(t,u) = a(t) + b(t)u$ with fixed coefficients $a, b \in C^\alpha$). This degeneracy phenomenon caused the prejudice that it does not make sense to apply the Banach - Caccioppoli principle (or its generalizations) to problems involving superposition operators in Hölder spaces, and may be the reason why in many papers on the equation (7) one tries to impose (unnatural) compactness conditions to employ the Schauder principle, rather than the Banach - Caccioppoli principle. Similarly, led by the same prejudice, we applied in a previous paper [2] Sadovskij's fixed point principle for condensing operators [11] to the equation (7). Apparently, E.P. Sobolevskij [12, 13] was the first who emphasized the need of studying *local* Lipschitz conditions in this case (i.e. $\kappa(r)$ depends actually on $r$ in (9)). In fact, in case $f = f(u)$ it is shown in [13] that the function (9) is finite on $(0,\infty)$ if and only if the derivative $g(u) = f'(u)$ exists and satisfies $|g(u) - g(v)| \leq \kappa(r)|u - v|$ for $|u|, |v| \leq r$.

Building on this idea, we shall derive now an estimate for the function (9) in the nonautonomous case $f = f(t,u)$. To this end, suppose that the partial derivative $g(t,u) = \partial f(t,u)/\partial u$ exists and defines a superposition operator

$$Gz(t) = g(t, z(t)) \tag{10}$$

in the space $C^\alpha$. From the well-known formula

$$f(t,x(t)) - f(t,y(t)) = [x(t) - y(t)] \int_0^1 g[t,(1-\lambda)x(t) + \lambda y(t)]d\lambda$$

and the fact that $C^\alpha$ is a normed algebra we conclude that

$$\|Fx - Fy\|_{C^\alpha} \leq \|x - y\|_{C^\alpha} \left\| \int_0^1 g[\cdot,(1-\lambda)x(\cdot) + \lambda y(\cdot)]d\lambda \right\|_{C^\alpha}$$

and hence

$$\kappa(r) \leq \sup\{\|Gz\|_{C^\alpha} : \|z\|_{C^\alpha} \leq r\}.$$

Thus, our problem reduces to estimating the growth of the superposition operator (10) on each ball $\bar{B}(0,r) \subset C^\alpha$. But in [2] it was shown that

$$\sup\{\|Gz\|_{C^\alpha} : \|z\|_{C^\alpha} \leq r\} = \max\{\gamma_C(r), \gamma_\alpha(r)\},$$

where

$$\gamma_C(r) = \sup\{|g(t,u)| : a \leq t \leq b; |u| \leq r\}$$

and

$$\gamma_\alpha(r) = \sup\left\{ \frac{|g(t,u) - g(s,v)|}{|t-s|^\alpha} : a \leq t, s \leq b; |u|, |v| \leq r; |u - v| \leq r|t-s|^\alpha \right\}.$$

In this way, we arrive at the following

**Theorem 2:** *Suppose that the operators (6), (8) and (10) act in the space $C^\alpha = C^\alpha[a,b]$ and are bounded. Moreover, suppose that the scalar function $a : [0,\infty) \to [0,\infty)$ defined by*

$$a(r) = a_0 + \|S\| \int_0^r \max\{\gamma_C(t), \gamma_\alpha(t)\}\, dt \tag{11}$$

*($a_0 = \|SF(0)\|_{C^\alpha}$) has a unique fixed point $r_*$ in some interval $[0,R]$, and that $a(R) \le R$. Then equation (7) has a unique solution $x_* \in \bar{B}(0,r_*) \subset C^\alpha$. This solution may be obtained as limit of the successive approximations $x_n = (SF)^n(0) \in \bar{B}(0,r_*)$ and is unique in the ball $B(0,R)$.*

In some cases, the scalar function (11) may be calculated explicitly. We illustrate this by means of a very elementary example. Suppose that the nonlinearity in (7) is a quadratic polynomial $f(u) = u^2 + pu + q$, where

$$0 < p < 1, \quad (1-p)^2 > 4|q|\|S(1)\|_{C^\alpha}. \tag{12}$$

A trivial computation shows that $\gamma_C(r) = 2r + p$ and $\gamma_\alpha(r) = 2r$, hence

$$a(r) = r^2 + pr + |q|\|S(1)\|_{C^\alpha}.$$

By (12), the function $r - a(r)$ has two different positive roots

$$R_\pm = \frac{1}{2}\left\{1 - p \pm [(1-p)^2 - 4|q|\|S(1)\|_{C^\alpha}]^{1/2}\right\}.$$

Consequently, the condition $a(R) \le R$ holds for any $R \in [R_-, R_+]$, and Theorem 2 applies. Observe that $\kappa(r) \to \infty$, as $r \to \infty$, in this example, and hence the classical (global) Banach-Caccioppoli principle does not apply.

We intentionally confined ourselves to the Hölder space $C^\alpha$ when dealing with equation (7). One could expect that another good choice would be the Lebesgue space $L_p = L_p(a,b)$ $(1 < p < \infty)$, since quite effective formulas for the norm of the linear operator (8) are available in this space, too (see, e.g., [8,9]). However, according to a classical theorem of M.A. Krasnosel'skij [5], the requirement $F(L_p) \subseteq L_p$ necessarily leads to nonlinearities (6) of sublinear growth, i.e. $|f(t,u)| \le a(t) + b|u|$ for some $a \in L_p$ and $b \ge 0$, and thus even excludes the elementary quadratic example we considered above.

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