Zeitschrift für Analysis
und ihre Anwendungen
Vol. 11 (1992) 3, 425 - 429

Oscillation Criteria for Certain Second Order Difference Equations

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Some oscillation criteria for the second order nonlinear difference equations

$$\Delta^2 y_{n-1} + \sum_{i=1}^{m} q_{in} f_i(y_n) g_i(\Delta y_{n-1}) = 0 \quad (n \in \mathbb{N})$$

are established.

Key words: *Oscillation, second order difference equations*
AMS subject classification: 39 A 10

1. Introduction

This paper is concerned with the oscillatory properties of solutions of the second order nonlinear difference equation

$$\Delta^2 y_{n-1} + \sum_{i=1}^{m} q_{in} f_i(y_n) g_i(\Delta y_{n-1}) = 0 \quad (n \in \mathbb{N}) \tag{1}$$

where Δ is the operator defined by $\Delta y = y_{n+1} - y_n$, $\Delta^2 y_{n-1} = \Delta(\Delta y_{n-1})$ and $\{q_{1n}\}, \ldots, \{q_{mn}\}$ are real sequences. The functions f_i and g_i $(i = 1, 2, \ldots, m)$ are defined on the set $\mathbb{R}$ of real numbers. It is interesting to study second order nonlinear difference equations because they are discrete analogues of differential equations. In addition, they do have physical applications as evidenced by [4, 9].

By a *solution* of (1) we mean a real sequence $y = \{y_n\}_{n \geq 0}$ satisfying (1). We consider only such solutions which are nontrivial for all large n. A solution of (1) is said to be *non-oscillatory* if it is essentially of constant sign. Otherwise it is called *oscillatory*. The purpose of this note is to establish some new oscillation criteria (sufficient conditions) for oscillation of all solutions of (1). For some results of this type we refer to the recent papers [2,3,5 - 8].

Now for the difference equation (1) each result we shall prove requires some of the following conditions, for all $i = 1, 2, \ldots, m$:

(C_1) $q_{in} \geq 0$ for all $n \in \mathbb{N}$ and, for every $N \in \mathbb{N}$, $q_{in} > 0$ for some $n > N$

(C_2) $u f_i(u) > 0$ and $g_i(u) > 0$ for $u \neq 0$

(C_3) $\liminf_{|u| \to \infty} (|f_i(u)|/|u|) \geq d > 0$

(C_4) $f_i(u)$ is non-decreasing and $f_i(-uv) \geq f_i(uv) \geq k_1 f_i(u) f_i(v)$ for $u, v > 0$

(C_5) $g_i(u)$ is even and non-increasing for $u > 0$

(C_6) $g_i(u)$ is non-increasing and $g_i(-uv) \geq g_i(uv) \geq k_2 g_i(u) g_i(v)$ for $u, v > 0$

(C_7) $f_i(u) g_i(u)/u^{\alpha} \geq d_1 > 0$ for $u \neq 0$, where $\alpha \in (1, \infty)$.

In the sequel we need the following two lemmas both due to Hooker and Patula [2].

Lemma 1: *If* $y_N \geq 0$, $\Delta^2 y_{n-1} \leq 0$ *and* $\Delta y_n > 0$ *for* $n \geq N$, *then* $y_n \geq n \Delta y_{n-1}/2$ *for all* $n \geq 2N$.

Lemma 2: *Assume conditions* (C_1) *and* (C_2) *and let* y *be a non-oscillatory solution of* (1) *such that* $y_n > 0$ *for all* $n \geq N$, *for some* $N \geq 0$. *Then* $y_{n+1} > y_n$ *and* $0 < \Delta y_{n+1} \leq \Delta y_n$ *for all* $n \geq N$ *(a similar statement holds if* y_n *is essentially negative).*

2. Main results

In this section, we establish sufficient conditions for the oscillation of all solutions of equation (1). We begin with the following

Theorem 1: *Let conditions* (C_1) - (C_3) *and* (C_5) *hold. If there exists an index* j *such that*

$$\limsup_{n\to\infty} n \sum_{l=n}^{\infty} q_{jl}\, g_j(l) > d^{-1} \tag{2}$$

where d *is as in* (C_3), *then equation* (1) *is oscillatory.*

Proof: Without loss of generality in (C_3), we may assume $\liminf_{|u|\to\infty} (|f_j(u)|/|u|) > d$ for some index j. Let y be a non-oscillatory solution of equation (1) and assume without loss of generality that $y_n > 0$ for all $n \geq N$, for some $N > 0$. By Lemma 2, y_n is increasing and Δy_n is positive and non-increasing for $n \geq N$. Summing equation (1) from n to ∞ $(n \geq N)$, we have

$$\Delta y_{n-1} \geq \sum_{l=1}^{\infty} q_{jl} f_j(y_l) g(\Delta y_{l-1}). \tag{3}$$

Using Lemma 1, we obtain

$$y_n \geq \frac{1}{2} \inf_{l\geq n} \frac{f_j(y_l)}{y_l}\, n \sum_{l=1}^{\infty} q_{jl}\, g_j(\Delta y_{l-1}) y_l \quad \text{for } n \geq 2N. \tag{4}$$

Since y_n is positive and increasing, Δy_n is positive and decreasing for $n \geq 2N$ and using (C_5) the inequality (4) yields

$$l \geq \frac{1}{2} \inf_{l\geq n} \frac{f_j(y_l)}{y_l}\, n \sum_{l=n}^{\infty} q_{jl}\, g_j(l). \tag{5}$$

This follows since Δy_n is positive and decreasing for $n \geq 2N$ and using condition (C_5) we can take l instead of Δy_{l-1} in $g_j(\Delta y_{l-1})$. From (5), we see that

$$\frac{1}{d} \geq \frac{1}{2} \inf_{n\to\infty} \sup n \sum_{l=n}^{\infty} q_{jl}\, g_j(l). \tag{6}$$

From (6) and (2) we get a contradiction. This completes the proof of the theorem ∎

Remark: Theorem 1 is a discrete analogue of [1:Theorem 1].

Theorem 2: *Assume conditions* $(C_1), (C_2), (C_6), (C_7)$ *are satisfied. If* $\sum_{n=2N}^{\infty} n\, q_{jn}\, g_j(2/n) = \infty$ *for some index* j, *then all solutions of equation* (1) *are oscillatory.*

Proof: Suppose y is a non-oscillatory solution of equation (1) and assume without loss of generality that $y_n > 0$ for all $n \geq N$, for some $N > 0$. By Lemma 2, y_n is increasing and Δy_n is positive and non-increasing for $n \geq N$. Using Lemma 1 and (C_6), we obtain from equation (1)

$$\Delta^2 y_{n-1} + k_2 q_{jn} g_j(2/n) g_j(y_n) f_j(y_n) \leq 0 \quad \text{for } n \geq 2N. \tag{7}$$

Multiply both sides of (7) by $n y_n^{-\alpha}$ and use (C_7) to obtain

$$\sum_{n=2N}^{k-1} n y_n^{-\alpha} \Delta^2 y_{n-1} + \sum_{n=2N}^{k-1} d_1 k_2 n q_{jn} g_j(2/n) \leq 0 \quad \text{for } k \geq 2N. \tag{8}$$

Upon applying the result

$$\sum_{i=p}^{n-1} u_i \Delta v_i = u_n v_n - u_p v_p - \sum_{i=p}^{n-1} v_{i+1} \Delta u_i$$

to (8) with $u_n = n y_n^{-\alpha}$ and $v_n = \Delta y_{n-1}$, this yields for any $k \geq 2N$

$$k y_k^{-\alpha} \Delta y_{k-1} - N y_{2N}^{-\alpha} \Delta y_{2N-1} - \sum_{n=2N}^{k-1} \Delta y_n \Delta(n y_n^{-\alpha}) + \sum_{n=2N}^{k-1} d_1 k_2 n q_{jn} g_j(2/n) \leq 0. \tag{9}$$

In view of Lemma 2 and the hypothesis, (9) implies

$$\sum_{n=2N}^{k-1} \Delta y_n \Delta(n y_n^{-\alpha}) \to +\infty \text{ as } k \to \infty. \tag{10}$$

We shall show that (10) is impossible. For, from (10) we have

$$\sum_{n=2N}^{k-1} \Delta y_n \Delta(n y_n^{-\alpha}) = \sum_{n=2N}^{k-1} (y_{n+1}^{-\alpha} \Delta y_n + n \Delta y_n \Delta y_n^{-\alpha}) \leq \sum_{n=2N}^{k-1} y_{n+1}^{-\alpha} \Delta y_n. \tag{11}$$

To complete the proof it suffices to show that

$$\sum_{n=2N}^{\infty} y_{n+1}^{-\alpha} \Delta y_n < \infty. \tag{12}$$

Let $h(x) = y_n + (\Delta y_n)(x - n)$ $(n \leq x \leq n+1,\ n \geq 2N)$. Then

$$h(n) = y_n,\ h(n+1) = y_{n+1} \quad \text{and} \quad h'(x) = \Delta y_n > 0 \quad (n < x < n+1,\ n \geq 2N).$$

Then h is continuous and increasing for $n \geq 2N$. We have thus

$$\begin{aligned} \sum_{n=2N}^{k} y_{n+1}^{-\alpha} \Delta y_n &= \sum_{n=2N}^{k} \int_n^{n+1} y_{n+1}^{-\alpha} \Delta y_n \, dx \\ &< \sum_{n=2N}^{k} \int_n^{n+1} h^{-\alpha}(x) h'(x) \, dx \leq \frac{1}{1-\alpha}\left(h^{1-\alpha}(k+1) - h^{1-\alpha}(2N)\right). \end{aligned}$$

Since $\alpha > 1$ and h is an increasing function, it follows that (12) holds ∎

Theorem 3: *Let conditions* (C_1), (C_2) *and* (C_4) *hold and let in addition condition* (C_7) *hold for* $\alpha \in (0,1)$. *If*

$$\sum_{n=2N}^{\infty} q_{jn} f_j(n/2) = \infty \quad \textit{for some index } j, \tag{13}$$

then all solutions of (1) *are oscillatory.*

Proof: Suppose y is a non-oscillatory solution of (1) and assume without loss of generality that $y_n > 0$ for $n \geq N$, for some $N \geq 1$. By Lemma 2, y_n is increasing and Δy_n is positive and non-increasing for $n \geq N$. Dividing (1) by $(\Delta y_{n-1})^{\alpha}$ and applying Lemma 1 and using (C_4) and (C_1) and summing from $2N$ to k, we obtain

$$\sum_{n=2N}^{k} \frac{\Delta^2 y_{n-1}}{(\Delta y_{n-1})^{\alpha}} + \sum_{n=2N}^{k} k_1 d_1 q_{jn} f_j(n/2) \leq 0 \quad \text{for } n \geq 2N. \tag{14}$$

Let $h(x) = y_n + (\Delta y_n)(x - n)$ $(n \leq x < n+1,\ n \geq 2N)$ as in the proof of Theorem 2. The function h is positive and continuous and increasing since $h'(x) = \Delta y_n > 0$ $(n < x < n+1,\ n \geq 2N)$. Let $b(x) = h(x+1) - h(x) > 0$. Then b is continuous and non-increasing and $b(x) \leq b(n-1) = \Delta y_{n-1}$. Now for $n - 1 < x < n$, we have

$$\frac{\Delta^2 y_{n-1}}{(\Delta y_{n-1})^{\alpha}} = \int_{n-1}^{n} \frac{\Delta^2 y_{n-1}}{(\Delta y_{n-1})^{\alpha}}\, dx \geq \int_{n-1}^{n} \frac{b'(x)}{b^{\alpha}(x)}\, dx.$$

It follows that

$$\sum_{n=2N}^{k} \frac{\Delta^2 y_{n-1}}{(\Delta y_{n-1})^{\alpha}} \geq \int_{2N-1}^{k} \frac{b'(x)}{b(x)}\, dx = \frac{b^{1-\alpha}(k) - b^{1-\alpha}(2N-1)}{1-\alpha}. \tag{15}$$

But $b^{1-\alpha}(k) > 0$ for all $k > 2N$, so the sum on the left in (15) is bounded below, contradicting (13), which completes the proof ∎

Remark: Theorems 2 and 3 generalize Theorems 4.1 and 4.3, respectively, of Hooker and Patula [2].

Theorem 4: *Let conditions* (C_1), (C_2) *and* (C_6) *hold and let in addition condition* (C_7) *hold for* $\alpha = 1$. *If there exists a positive sequence* $\{h_n\}_{n \geq N}$ *such that*

$$\sum_{n=2N}^{\infty} h_n \left(\sum_{i=1}^{m} q_{in} g_i(2/n) - \left(\frac{\Delta h_n}{2h_n}\right)^2 \right) = \infty, \tag{16}$$

then all solutions of equation (1) *are oscillatory.*

Proof: Suppose y is a non-oscillatory solution of equation (1) and assume without loss of generality that $y_n > 0$ for $n \geq N$, for some $N \geq 1$. By Lemma 2, y_n is increasing and Δy_n is positive and non-increasing for $n \geq N$. Let us denote $Z_n = h_n \Delta y_{n-1}/y_n$ $(n \geq N)$. Then from equation (1) we get

$$\Delta Z_n = -\frac{h_n}{y_n} \sum_{i=1}^{m} q_{in} f_i(y_n) g_i(\Delta y_{n-1}) + \frac{\Delta h_n \Delta y_n}{y_{n+1}} - \frac{h_n \Delta y_n \Delta y_n}{y_n y_{n+1}}.$$

In view of monotonicity of y_n and Δy_n, (C_6), (C_7) and Lemma 1 we see that

$$\Delta Z_n \leq -d_1 k_2 h_n \sum_{i=1}^{m} q_{in} g_i(2/n) + Z_{n+1} \frac{\Delta h_n}{h_{n+1}} - Z_{n+1}^2 \frac{h_n}{h_{n+1}^2}$$

$$= -d_1 k_2 h_n \sum_{i=1}^{m} q_{in}\, g_i(2/n) - \frac{h_n}{h_{n+1}^2}\left(Z_{n+1} - \frac{\Delta h_n\, h_{n+1}}{2h_n}\right)^2 + \frac{(\Delta h_n)^2}{4h_n} \quad (n \geq 2N).$$

Hence

$$\Delta Z_n \leq -d_1 k_2 h_n \sum_{i=1}^{m} q_{in}\, g_i(2/n) + \frac{(\Delta h_n)^2}{4h_n} \quad (n \geq 2N).$$

Summing the above inequality from $2N$ to n, we get

$$Z_{n+1} \leq Z_{2N} - d_1 k_2 \sum_{l=2N}^{n} h_l\left(\sum_{i=1}^{m} q_{il}\, g_i(2/l) - \left(\frac{\Delta h_l}{2h_l}\right)^2\right).$$

Now by (16), it is easy to see that Z_n is essentially negative, which is a contradiction ∎

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Received 22.10.1991

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