Oscillation Criteria for Certain Second Order Difference Equations $\begin{aligned} \text{cillation Criteria for Certain Second Order} \end{aligned}$
 $\begin{aligned} \text{Ricational representation} \end{aligned} \begin{aligned} \text{Licational representation} \$

E. **THANDAPANI**

Some oscillation criteria for the second order nonlinear difference equations

$$
\Delta^2 y_{n-1} + \sum_{i=1}^m q_{in} f_i(y_n) g_i(\Delta y_{n-1}) = 0 \quad (n \in \mathbb{N})
$$

are established.

Key words: Oscillation, second order difference equations AMS **subject classification: 39A10**

1. Introduction

This paper is concerned with the oscillatory properties of solutions of the second order nonlinear difference equation

$$
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$$
\Delta^{2}y_{n-1} + \sum_{i=1}^{m} q_{in} f_{i}(y_{n}) g_{i}(\Delta y_{n-1}) = 0 \quad (n \in \mathbb{N})
$$
 (1)
re Δ is the operator defined by $\Delta y = y_{n+1} - y_{n}, \Delta^{2}y_{n-1} = \Delta(\Delta y_{n-1})$ and $\{q_{1n}\}, \dots, \{q_{mn}\}$
real sequences. The functions f_{i} and g_{i} ($i = 1, 2, ..., m$) are defined on the set R of real

where Δ is the operator defined by $\Delta y = y_{n+1} - y_n$, $\Delta^2 y_{n-1} = \Delta(\Delta y_{n-1})$ and $\{q_{1n}\}, \ldots$, $\{q_{mn}\}$ are real sequences. The functions f_i and g_j ($i = 1, 2, ..., m$) are defined on the set R of real numbers. It is interesting to study second order nonlinear difference equations because they are discrete analogues of differential equations. In addition, they do have physical applications as evidenced by [4,91.

By a *solution* of (1) we mean a real sequence $y = {y_n}_{n \geq 0}$ satisfying (1). We consider only such solutions which are nontrivial for all large *n*. A solution of (1) is said to be *non-oscillato*ry if it is essentially of constant sign. Otherwise it is called *oscillatory.* The purpose of this note is to establish some new oscillation criteria (sufficient conditions) for oscillation of all solutions of (1). For some results of this type we refer to the recent papers $[2,3,5 - 8]$.

Now for the difference equation (1) each result we shall prove requires some of the following conditions, for all $i = 1, 2, ..., m$:

 (C_1) $q_{in} \ge 0$ for all $n \in \mathbb{N}$ and, for every $N \in \mathbb{N}$, $q_{in} \ge 0$ for some $n \ge N$

- (C_2) *uf_i*(*u*) > 0 and $g_i(u)$ > 0 for *u* \neq 0
- (C_3) lim $\inf_{|u| \to \infty} (|f_i(u)|/|u|) \ge d > 0$
- *(C₄)* $f_i(u)$ is non-decreasing and $f_i(-uv) \ge f_i(uv) \ge k_1 f_i(u) f_i(v)$ for $u, v > 0$
- (C_5) $g_i(u)$ is even and non-increasing for $u > 0$
- (C₆) $g_i(u)$ is non-increasing and $g_i(-uv) \ge g_i(uv) \ge k_2 g_i(u)g_i(v)$ for $u, v > 0$
- (C_7) $f_i(u)g_i(u)/u^{\alpha} \ge d_i > 0$ for $u \ne 0$, where $\alpha \in (1, \infty)$.

In the sequel we need the following two lemmas both due to Hooker and Patula [2].

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the sequel we need the following two lemmas both due to Hooker and Patula [2].
Lemma 1: If
$$
y_N \ge 0
$$
, $\Delta^2 y_{n-1} \le 0$ and $\Delta y_n \ge 0$ for $n \ge N$, then $y_n \ge n \Delta y_{n-1}/2$ for all $n \ge 2N$.

Lemma 2: Assume conditions (C_1) and (C_2) and let y be a non-oscillatory solution of (1) *such that* $y_n > 0$ for all $n \ge N$, for some $N \ge 0$. Then $y_{n+1} > y_n$ and $0 < \Delta y_{n+1} \le \Delta y_n$ for all $n \ge 0$. N (a similar statement holds if y_n is essentially negative).

2. Main results

In this section, we establish sufficient conditions for the oscillation of all solutions of equation (I). We begin with the following

Theorem 1: Let conditions $(C_1) - (C_3)$ and (C_5) hold. If there exists an index j such that

$$
\limsup_{n \to \infty} n \sum_{j=n}^{\infty} q_{j1} g_j(l) > d^{-1}
$$
 (2)

where *d* is as in (C_3) , then equation (1) is oscillatory.

Proof: Without loss of generality in (C_3) , we may assume $\liminf_{|u|\to\infty} (|f_i(u)|/|u|) > d$ for some index *j.* Let ybe a non-oscillatory solution of equation (1) and assume without loss of generality that $y_n > 0$ for all $n \ge N$, for some $N > 0$. By Lemma 2, y_n is increasing and Δy_n is positive and non-increasing for $n \ge N$. Summing equation (1) from *n* to ∞ ($n \ge N$), we have **an 1:** Let conditions $(C_1) - (C_3)$ and (C_5) hold. If there exists an index j such that
 $n \sum_{i=1}^{\infty} q_{ij}g_j(i) > d^{-1}$ (2)
 $a \sin(C_3)$, then equation (1) is oscillatory.

Without loss of generality in (C_3) , we may assum *nhen equation* (1) *is oscillatory.*
 nother the example in inf $|u| \rightarrow \infty$ ($|f_j(u)|/|u|$) > *d* for example a non-oscillatory solution of equation (1) and assume without loss of

for all $n \ge N$, for some $N > 0$. By Lemm

$$
\Delta y_{n-1} \ge \sum_{l=1}^{\infty} q_{jl} f_j(y_l) g(\Delta y_{l-1}). \tag{3}
$$

Using Lemma 1, we obtain

tive and non-increasing for
$$
n \ge N
$$
. Summing equation (1) from n to ∞ ($n \ge N$), we have
\n
$$
\Delta y_{n-1} \ge \sum_{l=1}^{\infty} q_{jl} f_j(y_l) g(\Delta y_{l-1}).
$$
\n(3)
\nng Lemma 1, we obtain
\n
$$
y_n \ge \frac{1}{2} \inf_{l \ge n} \frac{f_j(y_l)}{y_l} n \sum_{l=1}^{\infty} q_{jl} g_j(\Delta y_{l-1}) y_l \text{ for } n \ge 2N.
$$
\n(4)
\nwe y_n is positive and increasing, Δy_n is positive and decreasing for $n \ge 2N$ and using (C_s)
\ninequality (4) yields
\n
$$
l \ge \frac{1}{2} \inf_{l \ge n} \frac{f_j(y_l)}{y_l} n \sum_{l \ge n}^{\infty} q_{jl} g_j(l).
$$
\n(5)
\nfollows since Δy_n is positive and decreasing for $n \ge 2N$ and using condition (C_s) we can
\n l instead of Δy_{l-1} in $g_j(\Delta y_{l-1})$. From (5), we see that
\n
$$
\frac{1}{d} \ge \frac{1}{2} \inf_{n \to \infty} \sup_{l \ge n} n \sum_{l \ge n}^{\infty} q_{jl} g_j(l).
$$
\n(6)
\n n (6) and (2) we get a contradiction. This completes the proof of the theorem \blacksquare

Since y_n is positive and increasing, Δy_n is positive and decreasing for $n \geq 2N$ and using (C_s) the inequality (4) yields

$$
y_n \ge \frac{1}{2} \lim_{l \to n} \frac{y_l}{y_l} = \frac{1}{l} \frac{q_{jl}}{s_j} \left(\frac{y_j}{s_j} \right) - \frac{1}{2} \left(\frac{y_l}{s_j} \right) \quad \text{for } n \ge 2N.
$$
\n
$$
\text{Let } y_n \text{ is positive and increasing, } \Delta y_n \text{ is positive and decreasing for } n \ge 2N \text{ and using } (C_s)
$$
\n
$$
I \ge \frac{1}{2} \inf_{l \ge n} \frac{f_j(y_l)}{y_l} n \sum_{l \ge n}^{\infty} q_{jl} g_j(l).
$$
\n
$$
(5)
$$

This follows since Δy_n is positive and decreasing for $n \ge 2N$ and using condition (C_s) we can take *l* instead of Δy_{I-1} in $g_i(\Delta y_{I-1})$. From (5), we see that

$$
\frac{1}{d} \ge \frac{1}{2} \inf_{n \to \infty} \sup_{I \subset n} \sum_{j=1}^{\infty} q_{jI} g_j(I). \tag{6}
$$

From (6) and (2) we get a contradiction. This completes the proof of the theorem **U**

Remark: Theorem lisa discrete analogue of [l:Theorem *11.*

 $\frac{1}{d} \ge \frac{1}{2}$ inf sup $n \sum_{i=n}^{\infty} q_{j1} g_j(i)$. (6)

n (6) and (2) we get a contradiction. This completes the proof of the theorem **I**
 Remark: Theorem 1 is a discrete analogue of [1:Theorem 1].
 Theorem 2: *Assum* $=$ ∞ for some index j, then all solutions of equation (1) are oscillatory.

Proof: Suppose y is a non-oscillatory solution of equation (1) and assume without loss of generality that $y_n > 0$ for all $n \ge N$, for some $N > 0$. By Lemma 2, y_n is increasing and Δy_n is positive and non-increasing for $n \geq N$. Using Lemma 1 and (C_6) , we obtain from equation (1) Oscillation Criteria for Difference Equations 427

uppose *y* is a non-oscillatory solution of equation (1) and assume without loss of

at $y_n > 0$ for all $n \ge N$, for some $N > 0$. By Lemma 2, y_n is increasing and Δy_n Consultation Criteria for Difference Equations 427

²*Y*₂. *2y_n* is a non-oscillatory solution of equation (1) and assume without loss of
 ind non-increasing for n ≥ <i>N, for some *N* > 0. By Lemma 2, y_n is in **Proof:** Suppose *y* is a non-
rrality that $y_n > 0$ for all *n*
iive and non-increasing for
 $\Delta^2 y_{n-1} + k_2 q_{jn} g_j(2/n) g_j(y)$
iply both sides of (7) by *n* j
 $\sum_{n=2N}^{k-1} n y_n^{-\alpha} \Delta^2 y_{n-1} + \sum_{n=2N}^{k-1} n \Delta^2 y_{n-1}$
in apply

$$
\Delta^2 y_{n-1} + k_2 q_{in} g_i(2/n) g_i(y_n) f_i(y_n) \le 0 \quad \text{for } n \ge 2N. \tag{7}
$$

Multiply both sides of (7) by $n y_n^{-\alpha}$ and use (C₇) to obtain

$$
\Delta^2 y_{n-1} + k_2 q_{jn} g_j(2/n) g_j(y_n) f_j(y_n) \le 0 \quad \text{for } n \ge 2N.
$$
\n(7)\n
$$
\text{inly both sides of (7) by } n y_n^{-\alpha} \text{ and use } (C_7) \text{ to obtain}
$$
\n
$$
\sum_{n=2N}^{k-1} n y_n^{-\alpha} \Delta^2 y_{n-1} + \sum_{n=2N}^{k-1} d_i k_2 n q_{jn} g_j(2/n) \le 0 \quad \text{for } k \ge 2N.
$$
\n(8)

Upon applying the result

$$
\sum_{n=2N}^{k-1} n y_{n}^{-\alpha} \Delta^{2} y_{n-1} + \sum_{n=2N}^{k-1} d_{1} k_{2} n q_{jn} g_{j}(2/n) \le 0 \quad \text{for } k \ge 2N.
$$
\n
$$
\sum_{n=2N}^{n-1} u_{i} \Delta v_{i} = u_{n} v_{n} - u_{p} v_{p} - \sum_{i=p}^{n-1} v_{i+1} \Delta u_{i}
$$
\n
$$
\sum_{i=p}^{n-1} u_{i} \Delta v_{i} = u_{n} v_{n} - u_{p} v_{p} - \sum_{i=p}^{n-1} v_{i+1} \Delta u_{i}
$$
\n8) with $u_{n} = n y_{n}^{-\alpha}$ and $v_{n} = \Delta y_{n-1}$, this yields for any $k \ge 2N$
\n
$$
ky_{k}^{-\alpha} \Delta y_{k-1} - Ny_{2}^{-\alpha} \Delta y_{2N-1} - \sum_{n=2N}^{k-1} \Delta y_{n} \Delta (ny_{n}^{-\alpha}) + \sum_{n=2N}^{k-1} d_{1} k_{2} n q_{jn} g_{j}(2/n) \le 0.
$$
\n
$$
\text{view of Lemma 2 and the hypothesis, (9) implies}
$$
\n
$$
\sum_{n=2N}^{k-1} \Delta y_{n} \Delta (ny_{n}^{-\alpha}) \to +\infty \text{ as } k \to \infty.
$$
\n
$$
\text{shall show that (10) is impossible. For, from (10) we have}
$$
\n(10)

to (8) with $u_n = ny_n^{-\alpha}$ and $v_n = \Delta y_{n-1}$, this yields for any $k \ge 2N$

$$
ky_k^{-\alpha} \Delta y_{k-1} - Ny_2^- \tilde{N} \Delta y_{2N-1} - \sum_{n=2N}^{k-1} \Delta y_n \Delta (ny_n^{-\alpha}) + \sum_{n=2N}^{k-1} d_1 k_2 n q_{jn} g_j(2/n) \le 0.
$$
 (9)

In view of Lemma 2 and the hypothesis, (9) implies

$$
\sum_{n=2N}^{k-1} \Delta y_n \, \Delta (n y_n^{-\alpha}) \to +\infty \text{ as } k \to \infty. \tag{10}
$$

We shall show that (10) is impossible. For, from (10) we have

$$
\sum_{i=p} u_{i} \Delta v_{i} = u_{n}v_{n} - u_{p}v_{p} - \sum_{i=p} v_{i+1} \Delta u_{i}
$$

\n) with $u_{n} = n y_{n}^{-\alpha}$ and $v_{n} = \Delta y_{n-1}$, this yields for any $k \ge 2N$
\n $k y_{k}^{-\alpha} \Delta y_{k-1} - Ny_{2}^{-\alpha} \Delta y_{2N-1} - \sum_{n=2N}^{k-1} \Delta y_{n} \Delta (ny_{n}^{-\alpha}) + \sum_{n=2N}^{k-1} d_{i} k_{2} n q_{jn} g_{j}(2/n) \le 0.$ (9)
\new of Lemma 2 and the hypothesis, (9) implies
\n
$$
\sum_{n=2N}^{k-1} \Delta y_{n} \Delta (ny_{n}^{-\alpha}) \to +\infty \text{ as } k \to \infty.
$$
 (10)
\nshall show that (10) is impossible. For, from (10) we have
\n
$$
\sum_{n=2N}^{k-1} \Delta y_{n} \Delta (ny_{n}^{-\alpha}) = \sum_{n=2N}^{k-1} (y_{n+1}^{-\alpha} \Delta y_{n} + n \Delta y_{n} \Delta y_{n}^{-\alpha}) \le \sum_{n=2N}^{k-1} y_{n+1}^{-\alpha} \Delta y_{n}.
$$
 (11)
\ncomplete the proof it suffices to show that
\n
$$
\sum_{n=2N}^{\infty} y_{n+1}^{-\alpha} \Delta y_{n} < \infty.
$$
 (12)
\n $h(x) = v_{n} + (\Delta v_{n})(x - n) (n \le x \le n + 1, n \ge 2N).$ Then

To complete the proof it suffices to show that

$$
\sum_{n=2N}^{\infty} y_n^{-\alpha} \Delta y_n < \infty. \tag{12}
$$

Let $h(x) = y_n + (\Delta y_n)(x - n)$ ($n \le x \le n + 1, n \ge 2N$). Then

$$
h(n) = y_n, h(n+1) = y_{n+1} \text{ and } h'(x) = \Delta y_n > 0 \quad (n < x < n+1, n \ge 2N).
$$

Then *h* is continuous and increasing for $n \geq 2N$. We have thus

$$
n=2N
$$

\n
$$
n=2N
$$
<

Since α > 1 and *h* is an increasing function, it follows that (12) holds \blacksquare

Theorem 3: Let conditions (C_1) , (C_2) and (C_1) hold and let in addition condition (C_7) hold for $\alpha \in (0,1)$. If

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\n
$$
\sum_{n=2N}^{\infty} q_{jn} f_j(n/2) = \infty \text{ for some index } j,
$$
\nall solutions of (1) are oscillatory.

then all solutions of (1) *are oscillatory.*

Proof: Suppose y is a non-oscillatory solution of (1) and assume without loss of generality that $y_n > 0$ for $n \ge N$, for some $N \ge 1$. By Lemma 2, y_n is increasing and Δy_n is positive and non-increasing for $n \ge N$. Dividing (1) by $(\Delta y_{n-1})^{\alpha}$ and applying Lemma 1 and using (C_4) and (C_1) and summing from 2N to k, we obtain **(13)**
 ∞ for some index j,

(13) are oscillatory.

is a non-oscillatory solution of (1) and assume without loss of generality

¹, for some $N \ge 1$. By Lemma 2, y_n is increasing and Δy_n is positive and
 $\ge N$.

$$
\sum_{n=2N}^{k} \frac{\Delta^2 y_{n-1}}{(\Delta y_{n-1})^{\alpha}} + \sum_{n=2N}^{k} k_1 d_1 q_{jn} f_j(n/2) \le 0 \quad \text{for } n \ge 2N.
$$
 (14)

Let $h(x) = y_n + (\Delta y_n)(x - n)$ $(n \le x \le n + 1, n \ge 2N)$ as in the proof of Theorem 2. The function *h* Let $h(x) = y_n + (\Delta y_n)(x - h)$ ($h \ge x \le h + 1$, $h \ge 2iv$) as in the proof of Theorem 2. The function *h*
is positive and continuous and increasing since $h'(x) = \Delta y_n > 0$ ($n \le x \le n + 1$, $n \ge 2N$). Let $b(x)$
= $h(x + 1) - h(x) > 0$. Then *b* for $n - 1 < x < n$, we have d summing
 $\frac{\Delta^2 y_{n-1}}{zN(\Delta y_{n-1})^2}$
 $\frac{\Delta^2 y_{n-1}}{zN(\Delta y_{n-1})^2}$
 $\frac{\Delta y_n}{zN_1} = \frac{z}{n}$
 $\frac{z}{N_{n-1}}$
 $\frac{z}{N_{n-1}}$ $= h(x + 1) - h(x) > 0$. Then *b* is continuous and non-increasing and $b(x) \le b(n - 1) = \Delta y_{n-1}$. Now $\frac{1}{(1-x)^{\alpha}} + \sum_{n=2}^{k} k_1 d_1 q_{jn} f_2$
 $(\Delta y_n)(x - n)$ ($n \le x < x$)

continuous and increasing the set of *h*(x) = $y_n + (\Delta y_{n-1})^{\alpha}$
h(x) = $y_n + (\Delta y_n)(x - n)$ (*n* s x < *n* + 1, *n* ≥ 2*N*
sitive and continuous and increasing since *h*'
 $x + 1$) - *h*(x) > 0. Then *b* is continuous and non
 $y - 1 < x < n$, we have
 $\frac{\Delta^2 y_{n-1}}{(\Delta y$ $\vert \text{continuity} \rangle > 0. \text{ T}$
 $n, \text{ we have } \frac{n}{n-1}$
 $= \int_{n-1}^{n} \frac{y_{n-1}}{(x-y)^{\alpha}}$ + $\sum_{n=2N}^{k} k_1 d_1 q$
 $(x - n)$ ($n \le x$

uous and inc

then *b* is contained
 $\frac{\triangle^2 y_{n-1}}{\triangle y_{n-1}} dx$
 $\ge \int_{2N-1}^{k} \frac{b'(x)}{b(x)} dx$
 $\ge 2N,50$ $h(x) = y_n + (\Delta y_n)(x - n)$
sitive and continuous and
 $x + 1 - h(x) > 0$. Then b is
 $h = 1 < x < n$, we have
 $\frac{\Delta^2 y_{n-1}}{(\Delta y_{n-1})^{\alpha}} = \int_{n-1}^{n} \frac{\Delta^2 y_{n-1}}{(\Delta y_{n-1})^{\alpha}}$
allows that
 $\sum_{n=2N}^{k} \frac{\Delta^2 y_{n-1}}{(\Delta y_{n-1})^{\alpha}} \ge \int_{2N-1}^{k}$
 $h^{$ $k_1 d_1 q_{jn} f_j(n/2)$:
 $(n \le x \le n+1, n)$

ind increasing site

is continuous an
 $\frac{n}{\alpha} dx \ge \int_{n-1}^{n} \frac{b'(x)}{b(x)} dx$
 $= \frac{b^{1-\alpha}}{b(x)}$
 $2N$, so the sum coof for $n \ge 2N$.

2*N*) as in the proof
 $\frac{1}{2} h'(x) = \Delta y_n > 0$

2*n* - *a* Δy_n

2*n* - *a* Δx .

2*n* - *a* Δx .

2*n* - *a* Δy_n

4*n* - *a* Δy_n

4*n* - *a* Δy_n

4*n* - *a* (15)

$$
\frac{\Delta^2 y_{n-1}}{(\Delta y_{n-1})^{\alpha}} = \int_{n-1}^{n} \frac{\Delta^2 y_{n-1}}{(\Delta y_{n-1})^{\alpha}} dx \ge \int_{n-1}^{n} \frac{b'(x)}{b^{\alpha}(x)} dx
$$

It follows that

$$
\sum_{n=2}^{k} \frac{\Delta^2 y_{n-1}}{(\Delta y_{n-1})^{\alpha}} \ge \int_{2N-1}^{k} \frac{b'(x)}{b(x)} dx = \frac{b^{1-\alpha}(k) - b^{1-\alpha}(2N-1)}{1-\alpha}.
$$
 (15)

But $b^{1-\alpha}(k)$ > 0 for all $k > 2N$, so the sum on the left in (15) is bounded below, contradicting (13), which completes the proof \blacksquare

Remark: Theorems 2 and 3 generalize Theorems 4.1 and 4.3, respectively, of Hooker and Patula [2].

for α = 1. *If there exists a positive sequence* $\{h_n\}_{n\ge N}$ *such that*

Theorem 4: Let conditions
$$
(C_1)
$$
, (C_2) and (C_6) hold and let in addition condition (C_7) hold
\n $t = 1$. If there exists a positive sequence $\{h_n\}_{n \ge N}$ such that
\n
$$
\sum_{n=2N}^{\infty} h_n \left(\sum_{i=1}^{m} q_{in} g_i (2/n) - \left(\frac{\Delta h_n}{2h_n} \right)^2 \right) = \infty,
$$
\n(16)

then all solutions of equation (1) *are oscillatory.*

Proof: Suppose y is a non-oscillatory solution of equation (1) and assume without loss of generality that $y_n > 0$ for $n \ge N$, for some $N \ge 1$. By Lemma 2, y_n is increasing and Δy_n is positive and non-increasing for $n \ge N$. Let us denote $Z_n = h_n \Delta y_{n-1}/y_n$ ($n \ge N$). Then from equation (1) we get lutions of equals of equal that $y_n > 0$
that $y_n > 0$
d non-increase that $y_n > 0$
l) we get
 $y_n = -\frac{h_n}{y_n} \sum_{j=1}^m q_j$ of equation
 $\frac{1}{2}$ y is a not
 $\frac{1}{2}$ of star n

increasing
 $\frac{1}{2}$
 $\frac{1}{2}$
 y y 2 1. By Lemma 2
 w a 1. By Lemma 2
 *i y*_n $\frac{1}{2}$ *i*_n $\frac{h_n \Delta y_n}{y_n}$ $\frac{h_n \Delta y_n}{y_n y_{n+1}}$

(C_A),(C₇) and Le *i* is a non-oscillatory solution of equation of the interval of the $n \ge N$, for some $N \ge 1$. By Lemm
reasing for $n \ge N$. Let us denote $Z_n =$
 $q_{in} f_i(y_n) g_i(\Delta y_{n-1}) + \frac{\Delta h_n \Delta y_n}{y_{n+1}} - \frac{h_n \Delta y_n}{y_n y_n}$
icity of y_n and Δy

$$
\Delta Z_n = -\frac{h_n}{y_n} \sum_{i=1}^m q_{in} f_i(y_n) g_i(\Delta y_{n-1}) + \frac{\Delta h_n \Delta y_n}{y_{n+1}} - \frac{h_n \Delta y_n \Delta y_n}{y_n y_{n+1}}.
$$

In view of monotonicity of y_n and Δy_n , (C_6) , (C_7) and Lemma 1 we see that

$$
\Delta Z_n = -\frac{h_n}{y_n} \sum_{j=1}^m q_{in} f_i(y_n) g_i(\Delta y_{n-1}) + \frac{\Delta h_n \Delta y_n}{y_{n+1}} - \frac{h_n \Delta y_n}{y_n y_n}
$$

iew of monotonicity of y_n and Δy_n , (C_6) , (C_7) and

$$
\Delta Z_n \le -d_1 k_2 h_n \sum_{i=1}^m q_{in} g_i(2/n) + Z_{n+1} \frac{\Delta h_n}{h_{n+1}} - Z_{n+1}^2 \frac{h_n}{h_{n+1}^2}
$$

Oscillation Criteria for Difference Equation
\n
$$
= -d_1 k_2 h_n \sum_{i=1}^m q_{in} g_i(2/n) - \frac{h_n}{h_{n+1}^2} \left(Z_{n+1} - \frac{\Delta h_n h_{n+1}}{2 h_n}\right)^2 + \frac{(\Delta h_n)^2}{4 h_n} \quad (n \ge 2N).
$$
\n
$$
\le -d_1 k_2 h_n \sum_{i=1}^m q_{in} g_i(2/n) + \frac{(\Delta h_n)^2}{4 h_n} \quad (n \ge 2N).
$$
\nthe above inequality from 2N to n, we get

Hence

$$
\Delta Z_n \le -d_1 k_2 h_n \sum_{i=1}^m q_{in} g_i (2/n) + \frac{(\Delta h_n)^2}{4 h_n} \quad (n \ge 2N).
$$

ming the above inequality from 2N to *n*, we get

$$
Z_{n+1} \le Z_{2N} - d_1 k_2 \sum_{i=1}^n h_i \left(\sum_{i=1}^m q_{ii} g_i (2/l) - \left(\frac{\Delta h_i}{2 h_i} \right)^2 \right).
$$

Summing the above inequality from *2N* to *n,* we get

$$
\Delta Z_n \leq -a_1 K_2 n_n \sum_{i=1}^{\infty} q_{in} g_i (2/n) + \frac{a_1}{4 h_n} \quad (n \geq 2N)
$$

ming the above inequality from 2N to *n*, we get

$$
Z_{n+1} \leq Z_{2N} - d_1 k_2 \sum_{l=2N}^{n} h_l \Big(\sum_{i=1}^{m} q_{il} g_i (2/l) - \Big(\frac{\Delta h_l}{2 h_l} \Big)^2 \Big).
$$

Now by (16), it is easy to see that Z_n is essentially negative, which is a contradiction \blacksquare

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