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Error Bounds for the Truncation of Countable Linear Differential Systems Arising from Birth-Death Processes

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An infinite system of linear differential equations $x'_n(t)=\lambda_{n-1}x_{n-1}(t)-(\lambda_n+\mu_n)x_n(t)+\mu_{n+1}x_{n+1}(t)+f_n(t)$, $x_n(0)=c_n$ $(n=0,1,2,\ldots;\ t\geq 0;\ x_{-1}=0)$ is considered. The constant coefficients λ_n, μ_n are merely assumed to be non-negative. Explicit error bounds are derived for the approximation of a solution by the solutions of the finite truncated systems. They crucially depend on the ratios of the coefficients λ_n, μ_n.

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1. Introduction . Simple birth-death processes with enumerably many states can be described by a countable system of differential equations [4]

$$x'(t) = Ax(t) \quad , \; x(0) = c \;\; (t \geq 0) \; , \tag{1.1}$$

where x and c are infinite column vectors and A is a constant infinite matrix of the form

$$A = \begin{bmatrix} -\lambda_0 & \mu_1 & & & 0 \\ \lambda_0 & -\lambda_1-\mu_1 & \mu_2 & & \\ & \lambda_1 & -\lambda_2-\mu_2 & \mu_3 & \\ 0 & & \ddots & \ddots & \ddots \end{bmatrix} \tag{1.2}$$

with $\lambda_i \geq 0$ ($i \geq 0$), $\mu_i \geq 0$ ($i \geq 1$). It is known [7] that this system possesses for $c \geq 0$, $c \in l_1$ a non-negative solution which can be obtained approximately by truncation.

For numerical purposes it is of interest to know error bounds. Results in the case $\lambda_n = \lambda_n(t) \leq \lambda(t)$, $\mu_n = \mu_n(t) \leq \mu(t)$ are given in [10] , whereas the case $A = (a_{ij})$ is any bounded operator on l_1 satisfying $\sup_{1 \leq j \leq n} \sum_{i=n+1}^{\infty} |a_{ij}| \to 0$ or $\sup_{n+1 \leq j \leq \infty} \sum_{i=1}^{n} |a_{ij}| \to 0$ $(n \to \infty)$ is considered in [3]. In particular, if A is given by (1.2), then the latter conditions reduce to $\lambda_n \to 0$ or $\mu_n \to 0$, respectively.

In the present paper error bounds for inhomogeneous systems without any restrictions on the growth of the non-negative coefficients λ_n, μ_n are derived. They crucially depend on the ratios of the coefficients. In particular, if $\lambda_n/\mu_n \to g$ $(0 \le g \le \infty)$ as $n \to \infty$, then the bounds are the smaller, the closer g is to 0 or to ∞. Remark that similar results for more general systems (A is a quadrodiagonal matrix with non-negative column sums) are established in [8,9]. However, both the assumptions and the assertions are more complicated than in the present paper.

2. Preliminaries . We consider a countable system of linear differential equations of the form

$$x'(t) = Ax(t) + f(t)\ , \quad x(0) = c \quad (t \ge 0)\ , \tag{2.1}$$

where $x = (x_0, x_1, \ldots)^T$, $f = (f_0, f_1, \ldots)^T$, $c = (c_0, c_1, \ldots)^T$ are infinite column vectors and A is the matrix defined by (1.2). The following conditions are assumed to be satisfied :

$$f_n \in C[0,\infty) \qquad (n = 0, 1, 2, \ldots)\ , \tag{2.2}$$

$$c\ , \ \Big\{ \int_0^t |f_n(\tau)| d\tau \Big\}_{n \ge 0} \in l_1 \quad \text{for all } t \ge 0\ , \tag{2.3}$$

$$\lambda_n \ge 0 \quad (n \ge 0)\ , \quad \mu_n > 0 \quad (n \ge 1)\ . \tag{2.4}$$

Remark that we may assume $\mu_n > 0$ instead of $\mu_n \ge 0$ since in case $\mu_n = 0$ for finitely or infinitely many integers $n \ge 1$ the initial-value problem (2.1) can be reduced either to a finite system and a countable system satisfying (2.2) - (2.4) or to enumerably many finite systems, respectively. For convenience, we define

$$\mu_0 = 0\ , \quad \alpha_n = \lambda_n + \mu_n \quad (n \ge 0)\ . \tag{2.5}$$

Let N be any non-negative integer. Truncating all coordinates of the vectors x, f, c with indices greater than N as well as the corresponding columns and rows of the matrix A we obtain the finite initial-value problem

$$\frac{d}{dt} x^N(t) = A^N x^N(t) + f^N(t)\ , \quad x^N(0) = c^N \quad (t \ge 0)\ , \tag{2.6}$$

where $x^N = (x_0^N, x_1^N, \ldots, x_N^N)^T$, $f^N = (f_0, f_1, \ldots, f_N)^T$, $c^N = (c_0, c_1, \ldots, c_N)^T$ and

$$A^N = \begin{bmatrix} -\alpha_0 & \mu_1 & & 0 \\ \lambda_0 & -\alpha_1 & \ddots & \\ & \ddots & \ddots & \mu_N \\ 0 & & \lambda_{N-1} & -\alpha_N \end{bmatrix} .$$

It is well known that the solution of the initial-value problem (2.6) can be represented in the form

$$x_n^N(t) = \sum_{m=0}^{N} \left\{ \int_0^t q_{nm}^N(\tau) f_m(t-\tau)\,d\tau + c_m q_{nm}^N(t) \right\} \qquad (n = 0,1,\dots,N;\ t \ge 0) \qquad (2.7)$$

where $(q_{0m}^N, q_{1m}^N, \dots, q_{Nm}^N)^T$ for every fixed m $(0 \le m \le N)$ denotes the solution of the homogeneous system (1.1) with $q_{nm}^N(0) = \delta_{nm}$ (Kronecker symbol). It is shown in [7] that

$$0 \le q_{nm}^N(t) \le q_{nm}^{N+1}(t) \le 1 \qquad (0 \le m, n \le N;\ t \ge 0)\,. \qquad (2.8)$$

Hence, there follows the existence of the limits

$$q_{nm}(t) = \lim_{N\to\infty} q_{nm}^N(t) \qquad (m, n \ge 0;\ t \ge 0)\,, \qquad (2.9)$$

which for every fixed m solve the homogeneous system (1.1) with the initial conditions $q_{nm}(0) = \delta_{nm}$, as well as the existence of the Laplace transforms

$$Q_{nm}^N(s) = \int_0^\infty e^{-st} q_{nm}^N(t)\,dt \qquad (s > 0)\,. \qquad (2.10)$$

Moreover, in [8] there are proved the inequalities

$$0 \le \frac{d}{dt}[e^{\alpha_n t} q_{nm}^N(t)] \le \frac{d}{dt}[e^{\alpha_n t} q_{nm}^{N+1}(t)] \qquad (0 \le m, n \le N;\ t \ge 0). \qquad (2.11)$$

Finally, we notice

$$(sE - A^N)^{-1} = \begin{bmatrix} Q_{00}^N & \cdots & Q_{0N}^N \\ \vdots & & \vdots \\ Q_{N0}^N & \cdots & Q_{NN}^N \end{bmatrix}, \qquad (2.12)$$

where E denotes the unit matrix of order $N+1$. From (2.7)-(2.9) there follows

$$|x_n^{N+p}(t) - x_n^N(t)| \le \sum_{m=0}^{N} \left\{ \int_0^t [q_{nm}(\tau) - q_{nm}^N(\tau)]\,|f_m(t-\tau)|\,d\tau + |c_m|[q_{nm}(t) - q_{nm}^N(t)] \right\} \qquad (2.13)$$
$$+ \sum_{m=N+1}^{N+p} \left\{ \int_0^t |f_m(\tau)|\,d\tau + |c_m| \right\} \qquad (0 \le m, n \le N;\ t \ge 0)$$

for every integer $p \ge 1$. Hence, in order to prove the existence of $x_n(t) = \lim_{N\to\infty} x_n^N(t)$ and to obtain error bounds for $|x_n(t) - x_n^N(t)|$ it suffices to know upper bounds for the non-negative differences $q_{nm}(t) - q_{nm}^N(t)$. In the following we will show how these can be derived by means of the Laplace transformation. Thus, we first deal with certain principal minors of the matrix $sE - A^N$.

3. Some properties of principal minors of the matrix $sE - A^N$. Let

$$\Delta_i^j(s) = \begin{cases} 0 & \text{for } j < i-1 \\ 1 & \text{for } j = i-1 \\ \begin{vmatrix} s+\alpha_i & -\mu_{i+1} & & & 0 \\ -\lambda_i & s+\alpha_{i+1} & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -\mu_j \\ 0 & & & -\lambda_{j-1} & s+\alpha_j \end{vmatrix} & \text{for } j \geq i \end{cases} \qquad (i = 0, 1, 2, \ldots). \tag{3.1}$$

Every determinant Δ_i^j is a polynomial in s of degree $j-i+1$ with the leading coefficient 1. The other coefficients being sums of principal minors of the matrix $-A^N$ are non-negative, since the column sums of $-A^N$ are non-negative [1,6]. (Remark that $sE-A^N$ for $s>0$ is a non-singular M-matrix [1].) Therefore, $\Delta_i^j(s) \geq s^{j-i+1} > 0$ for $s > 0$.

Next, the recurrence formula

$$\Delta_i^j = (s+\alpha_j)\Delta_i^{j-1} - \lambda_{j-1}\mu_j\Delta_i^{j-2} \qquad (0 \leq i \leq j) \tag{3.2}$$

can be verified immediately by Laplace expansion. Using (3.2) and replacing in usual way an empty sum by 0 and an empty product by 1 the relations

$$\Delta_i^k\Delta_j^{k+1} - \Delta_j^k\Delta_i^{k+1} = \Delta_i^{j-2}\prod_{\nu=j}^{k+1}(\lambda_{\nu-1}\mu_\nu) \qquad (0 \leq i \leq j \leq k+1) \tag{3.3}$$

and

$$\Delta_j^k(0) = \sum_{\nu=j}^{k+1}\left(\prod_{\rho=j}^{\nu-1}\mu_\rho \prod_{\rho=\nu}^{k}\lambda_\rho\right) \qquad (0 \leq j \leq k+1) \tag{3.4}$$

can be proved by induction (i, j fixed; $k \geq j-1$). In particular, we have for $j=0$ and $j=1$ by (2.4), (2.5)

$$\Delta_0^k(0) = \prod_{\rho=0}^{k}\lambda_\rho \geq 0 \qquad (k \geq -1) \tag{3.5}$$

and

$$\Delta_1^k(0) = \sum_{\nu=1}^{k+1}\left(\prod_{\rho=1}^{\nu-1}\mu_\rho \prod_{\rho=\nu}^{k}\lambda_\rho\right) \geq \prod_{\rho=1}^{k}\mu_\rho > 0 \qquad (k \geq 0). \tag{3.6}$$

Furthermore, using (3.2), (3.5) and (3.6) the inequality

$$\Delta_0^k(s)/\Delta_0^{k+1}(s) < [\Delta_0^k(0) + s\Delta_1^k(0)]/[\Delta_0^{k+1}(0) + s\Delta_1^{k+1}(0)] \quad (k \geq 0,\ s > 0) \tag{3.7}$$

can be proved by induction. As an immediate consequence we obtain the estimate

$$\Delta_0^k(s)/\Delta_0^N(s) < [\Delta_0^k(0) + s\Delta_1^k(0)]/[\Delta_0^N(0) + s\Delta_1^N(0)] \qquad (0 \leq k < N,\ s > 0). \tag{3.8}$$

4. Estimates of the differences $Q_{nm}^{N+p}(s) - Q_{nm}^{N}(s)$. The entries Q_{nm}^{N} of the matrix $(sE-A^N)^{-1}$ have the representations

$$Q_{nm}^{N}(s) = \begin{cases} \dfrac{\Delta_0^{m-1}(s)\Delta_{n+1}^{N}(s)}{\Delta_0^{N}(s)} \prod\limits_{\rho=m}^{n-1} \lambda_\rho & \text{for} \quad 0 \le m \le n \le N, \\ \dfrac{\Delta_0^{n-1}(s)\Delta_{m+1}^{N}(s)}{\Delta_0^{N}(s)} \prod\limits_{\rho=n+1}^{m} \mu_\rho & \text{for} \quad 0 \le n \le m \le N, \end{cases} \tag{4.1}$$

which can be obtained easily from (2.12) by Laplace expansion. From (4.1) there follows by means of (3.3)

$$Q_{nm}^{N+1}(s) - Q_{nm}^{N}(s) = \frac{\Delta_0^{n-1}(s)\Delta_0^{m-1}(s)}{\Delta_0^{N}(s)\Delta_0^{N+1}(s)} \prod_{\rho=m}^{N} \lambda_\rho \prod_{\rho=n+1}^{N+1} \mu_\rho \quad (0 \le m, n \le N;\ s > 0). \tag{4.2}$$

Hence, by (2.8) and (3.8)

$$0 \le Q_{nm}^{N+1}(s) - Q_{nm}^{N}(s) \le \frac{[\Delta_0^{n-1}(0)+s\Delta_1^{n-1}(0)][\Delta_0^{m-1}(0)+s\Delta_1^{m-1}(0)]}{[\Delta_0^{N}(0)+s\Delta_1^{N}(0)][\Delta_0^{N+1}(0)+s\Delta_1^{N+1}(0)]} \prod_{\rho=m}^{N} \lambda_\rho \prod_{\rho=n+1}^{N+1} \mu_\rho \tag{4.3}$$
$$(0 \le m, n \le N;\ s > 0).$$

Next, from (3.5) and (3.6)

$$s \prod_{\rho=1}^{N+1} \mu_\rho = \Delta_0^{N+1}(0) + s\Delta_1^{N+1}(0) - \lambda_{N+1}[\Delta_0^{N}(0) + s\Delta_1^{N}(0)] \quad (N \ge 0,\ s > 0). \tag{4.4}$$

Then the inequality (4.3) becomes

$$0 \le Q_{nm}^{N+1}(s) - Q_{nm}^{N}(s) \le \frac{\delta_n(s)}{s}[\Delta_0^{m-1}(0) + s\Delta_1^{m-1}(0)] \times \left[\frac{\prod\limits_{\rho=m}^{N} \lambda_\rho}{\Delta_0^{N}(0) + s\Delta_1^{N}(0)} - \frac{\prod\limits_{\rho=m}^{N+1} \lambda_\rho}{\Delta_0^{N+1}(0) + s\Delta_1^{N+1}(0)} \right], \tag{4.5}$$

where $\delta_n(s)$ is defined by

$$\delta_n(s) = [\Delta_0^{n-1}(0) + s\Delta_1^{n-1}(0)] / \prod_{\rho=1}^{n} \mu_\rho \quad (n \ge 0). \tag{4.6}$$

Therefore,

$$0 \le Q_{nm}^{N+p}(s) - Q_{nm}^{N}(s) \le \frac{\delta_n(s)}{s}[\Delta_0^{m-1}(0) + s\Delta_1^{m-1}(0)] \prod_{\rho=m}^{N} \lambda_\rho \times \left[\frac{1}{\Delta_0^{N}(0) + s\Delta_1^{N}(0)} - \frac{\prod\limits_{\rho=N+1}^{N+p} \lambda_\rho}{\Delta_0^{N+p}(0) + s\Delta_1^{N+p}(0)} \right] \tag{4.7}$$
$$(0 \le m, n \le N < N+p;\ s > 0).$$

In what follows it will be convenient to establish the numbers

$$D_i^j = \Delta_i^j(0)/\prod_{\rho=i}^{j}\lambda_\rho \qquad (1 \le i \le j) \tag{4.8}$$

whenever $\lambda_\rho > 0$ for each $\rho = i, i+1, \ldots, j$. According to (3.1), let $D_i^{i-1} = 1$ and $D_i^j = 0$ for $j < i-1$. From (3.4) we obtain the representation

$$D_i^j = \sum_{\nu=i}^{j+1}\prod_{\rho=i}^{\nu-1}(\mu_\rho/\lambda_\rho) \tag{4.9}$$

holding for all integers $j < i$, too.

Notice that, if $\lambda_\rho > 0$ for every integer $\rho \ge i$ and any fixed $i \ge 1$, the sequence $\{D_i^j\}_{j\ge i}$ is monotonically increasing. For these i we define

$$D_i = \lim_{j\to\infty} D_i^j . \tag{4.10}$$

Obviously, we have $1 < D_i \le \infty$. Remark that $D_i^j = D_i^{k-1} + (D_i^k - D_i^{k-1})\, D_{k+1}^j \quad (i \le k \le j)$. Therefore, $D_k < \infty$ for some $k \ge i$ implies $D_k < \infty$ for all $k \ge i$.

Now we distinguish two cases.

Case 1: Assume that $\lambda_\rho > 0$ for all $\rho \ge 0$. Then (4.7) can be rewritten as

$$0 \le Q_{nm}^{N+p}(s) - Q_{nm}^{N}(s) \le \delta_n(s)\big(\lambda_0 + sD_1^{m-1}\big)\,\frac{D_1^{N+p} - D_1^{N}}{\big(\lambda_0 + sD_1^{N+p}\big)\big(\lambda_0 + sD_1^{N}\big)} \tag{4.11}$$

$$(0 \le m, n \le N < N+p\,;\; s > 0).$$

Case 2: Assume that finitely or infinitely many λ_ρ vanish. Let $\underline{\rho} = \min\{\rho \ge 0 : \lambda_\rho = 0\}$, $\bar\rho = \sup\{\rho \ge 0 : \lambda_\rho = 0\}$. Evidently, $0 \le \underline{\rho} \le \bar\rho \le \infty$. Then (4.7) for $N < \underline{\rho}$ implies

$$0 \le Q_{nm}^{N+p}(s) - Q_{nm}^{N}(s) \le \frac{1}{s}\,\delta_n(s)\,\frac{\lambda_0 + sD_1^{m-1}}{\lambda_0 + sD_1^{N}} \qquad (0 \le m, n \le N < N+p;\; s > 0) \tag{4.12}$$

For $N \ge \underline{\rho}$ we immediately obtain by (4.7)

$$Q_{nm}^{N+p}(s) = Q_{nm}^{N}(s) \qquad (0 \le m \le \rho_N,\; 0 \le n \le N < N+p\,;\; s > 0), \tag{4.13}$$

where $\rho_N = \max\{\rho \le N : \lambda_\rho = 0\}$; moreover, if $\rho_N < N$ (i.e. $\lambda_N > 0$), then (4.7) yields

$$0 \le Q_{nm}^{N+p}(s) - Q_{nm}^{N}(s) \le \frac{1}{s}\,\delta_n(s) D_{\rho_N+1}^{m-1}\Big[1/D_{\rho_N+1}^{N} - \prod_{\rho=\rho_N+1}^{\rho_{N+p}}(\lambda_\rho/\mu_\rho)/\,D_{\rho_{N+p}+1}^{N+p}\Big] \tag{4.14}$$

$$(\underline{\rho} \le \rho_N < m \le N;\; 0 \le n \le N < N+p\,;\; s > 0),$$

because $\Delta_0^m(0) = 0$ for $\rho_N \le m$ and $\Delta_1^m(0) = D_{\rho_N+1}^{m}\prod_{\rho=1}^{\rho_N}\mu_\rho\prod_{\rho=\rho_N+1}^{m}\lambda_\rho$ for $\rho_N \le m \le N$ by (3.5), (3.6) and (4.9). The estimate (4.14) can be reduced to

$$0 \le Q_{nm}^{N+p}(s) - Q_{nm}^{N}(s) \le \frac{1}{s}\delta_n(s) D_{\rho_N+1}^{m-1} / D_{\rho_N+1}^{N} \tag{4.15}$$

$$(\underline{\rho} \le \rho_N < m \le N;\ 0 \le n \le N < N+p;\ s > 0)$$

Remark that the upper bounds in (4.14) and (4.15) are equal, when $N < \bar{\rho}$ and p is sufficiently large. If $\bar{\rho} < \infty$ (i.e., only a finite number of the coefficients λ_ρ vanishes), then for every $N > \bar{\rho}$ we have $\rho_N = \rho_{N+p} = \bar{\rho}$ and therefore (4.14) reduces to

$$0 \le Q_{nm}^{N+p}(s) - Q_{nm}^{N}(s) \le \frac{1}{s}\delta_n(s) D_{\bar{\rho}+1}^{m-1}[1/D_{\bar{\rho}+1}^{N} - 1/D_{\bar{\rho}+1}^{N+p}] \tag{4.16}$$

$$(\bar{\rho} < m \le N,\ 0 \le n \le N < N+p;\ s > 0).$$

5. Results . Using the above estimates we are in position to derive explicit error bounds for the approximation of the limits q_{nm} defined by (2.9). For convenience, we introduce the abbreviations

$$K_n(s,t) = \frac{1}{s}\delta_n(s)(s+\alpha_n)e^{st},\quad k_n(t) = \inf_{s>0} K_n(s,t) \quad (n \ge 0,\ t \ge 0). \tag{5.1}$$

Theorem 1: *Assume* (2.4). *Then for every fixed integer* $N \ge 0$ *and* $m, n = 0,1,\dots,N$

$$0 \le q_{nm}(t) - q_{nm}^{N}(t) \le \min\{1 - q_{nm}^{N}(t),\ A_{nm}^{N}(t)\} \qquad (t \ge 0), \tag{5.2}$$

where either

$$A_{nm}^{N} = \begin{cases} \frac{1}{\lambda_0}\alpha_n\delta_n(0)(D_1 - D_1^N) & \text{in case 1 if } D_1 < \infty, \\ \inf_{s>0}\left\{K_n(s,t)\,\frac{\lambda_0 + sD_1^{m-1}}{\lambda_0 + sD_1^N}\right\} & \text{in case 1 if } D_1 = \infty \text{ and in case 2 for } N < \underline{\rho}, \end{cases} \tag{5.3}$$

or, in case 2 for $N \ge \underline{\rho}$,

$$A_{nm}^{N} = \begin{cases} 0 & \text{for } 0 \le m \le \rho_N\ (\le \bar{\rho} \le \infty), \\ k_n(t) D_{\rho_N+1}^{m-1} / D_{\rho_N+1}^{N} & \text{for } \rho_N < m \le N < \bar{\rho}\ (\le \infty), \\ k_n(t) D_{\bar{\rho}+1}^{m-1} / D_{\bar{\rho}+1}^{N} & \text{for } \bar{\rho} = \rho_N < m \le N \text{ if } D_{\bar{\rho}+1} = \infty, \\ k_n(t) D_{\bar{\rho}+1}^{m-1}[1/D_{\bar{\rho}+1}^{N} - 1/D_{\bar{\rho}+1}] & \text{for } \bar{\rho} = \rho_N < m \le N \text{ if } D_{\bar{\rho}+1} < \infty. \end{cases} \tag{5.4}$$

Hence, for fixed $m, n \ge 0$, $q_{nm}^{N}(t)$ *converges uniformly in t on every bounded interval.*

Remarks: 1. It is easy to show that in case 1

$$\frac{\alpha_n\delta_n(0)}{\lambda_0} = \begin{cases} 1 & \text{for } n = 0, \\ (1+\lambda_n/\mu_n)\prod_{\rho=1}^{n-1}(\lambda_\rho/\mu_\rho) = (1+\lambda_n/\mu_n)(D_1^{n-1} - D_1^{n-2}) & \text{for } n \ge 1. \end{cases} \tag{5.5}$$

2. In case 2 we have for $n > \rho$

$$k_n = (1+\lambda_n/\mu_n) \sum_{\nu=\rho_{n-1}+1}^{n} \Big(\prod_{\rho=\nu}^{n-1} \lambda_\rho/\mu_\rho\Big) = (1+\lambda_n/\mu_n) \sum_{\nu=\rho_{n-1}+1}^{n} (D_\nu^{n-1} - D_\nu^{n-2})^{-1} \tag{5.6}$$

because $\Delta_0^{n-1}(0) = 0$ implies $k_n(t) = K_n(+0,t)$. **3.** In case 2 there follows directly from (1.1) with $c_n = \delta_{nm}$ that $q_{nm}^N(t) = 0$ for $0 \le m \le \rho < n \le N$, $t \ge 0$. Thus,

$$q_{nm}(t) = 0 \quad \text{for} \quad 0 \le m \le \rho < n, \quad t \ge 0 . \tag{5.7}$$

4. The infima in (5.3) can be approached by setting $s = 1/t$ $(t > 0)$, because e^{st}/s has a minimum at $s = 1/t$.

Proof of Theorem 1: Obviously, by (2.8) $0 \le q_{nm}(t) - q_{nm}^N(t) \le 1 - q_{nm}^N(t)$. From (2.10) there follows

$$Q_{nm}^{N+p}(s) - Q_{nm}^N(s) = \int_0^\infty e^{-s\tau} [q_{nm}^{N+p}(\tau) - q_{nm}^N(\tau)]\, d\tau \quad (s > 0). \tag{5.8}$$

Using known properties of the Laplace transformation we obtain

$$s[Q_{nm}^{N+p}(s-\alpha_n) - Q_{nm}^N(s-\alpha_n)] = \int_0^\infty e^{-s\tau} \frac{d}{d\tau} \Big\{ e^{\alpha_n \tau} [q_{nm}^{N+p}(\tau) - q_{nm}^N(\tau)] \Big\} d\tau \quad (s > \alpha_n). \tag{5.9}$$

Since the integrand on the right-hand side is non-negative (see (2.11)),

$$s[Q_{nm}^{N+p}(s-\alpha_n) - Q_{nm}^N(s-\alpha_n)] \ge e^{(\alpha_n - s)t} [q_{nm}^{N+p}(t) - q_{nm}^N(t)] \quad (s > \alpha_n,\ t \ge 0). \tag{5.10}$$

Hence

$$q_{nm}^{N+p}(t) - q_{nm}^N(t) \le (s+\alpha_n)\, e^{st} [Q_{nm}^{N+p}(s) - Q_{nm}^N(s)] \quad (s > 0,\ t \ge 0). \tag{5.11}$$

It follows from (5.8) by using (4.13) and the estimates (4.11), (4.12), (4.15) and (4.16) that, for fixed $m, n \ge 0$, $q_{nm}^N(t) \to q_{nm}(t)$ as $N \to \infty$, uniformly in t on every bounded interval. Furthermore, letting $p \to \infty$ and moreover, if $D_1 < \infty$ in case 1, $s \to +0$, we obtain the upper bounds A_{nm}^N.

Now, it is easy to prove that the solutions $x^N = (x_0^N, x_1^N, \dots, x_N^N)^T$ of the finite initial-value problems (2.6) converge componentwise to a solution of the infinite initial-value problem (2.1) as well as to determine explicit error bounds, provided the conditions (2.2) - (2.4) are satisfied. We use the abbreviations

$$R_N = R_N(t) = \sum_{m=0}^{N} \Big\{ \int_0^t |f_m(\tau)|\, d\tau + |c_m| \Big\} , \quad R = R(t) = \lim_{N\to\infty} R_N(t) . \tag{5.12}$$

Let M denote any fixed non-negative integer not greater than N.

Theorem 2: *Assume* (2.2) - (2.4). *Then the initial-value problem* (2.1) *has a solution* $x = \{x_0, x_1, \dots\}^T$, *where*

$$x_n = x_n(t) = \lim_{N\to\infty} x_n^N(t) \qquad (n = 0,1,2,\dots;\ t \geq 0), \tag{5.13}$$

uniformly in t on every bounded interval, with the following error bounds :

$$|x_n(t) - x_n^N(t)|$$

$$\leq \begin{cases} \min\{1, \frac{1}{\lambda_0}\alpha_n\delta_n(0)(D_1 - D_1^N)\}R_N + R - R_N & \text{in case 1 if } D_1 < \infty, \\ \min\{1, A_{nM}^N\}R_M + R - R_M & \text{in case 1 if } D_1 = \infty, \\ R & \text{in case 2 for } N < \underline{\rho}, \\ R - R_{\rho_N} & \text{in case 2 for } \underline{\rho} \leq N \leq \bar{\rho} \leq \infty, \\ \min\{1, A_{nM}^N\}(R_M - R_{\bar{\rho}}) + R - R_M & \text{in case 2 for } \bar{\rho} < M \leq N \text{ if } D_{\bar{\rho}+1} = \infty, \\ \min\{1, k_n(1 - D_{\bar{\rho}+1}^N/D_{\bar{\rho}+1})\}(R_N - R_{\bar{\rho}}) + R - R_N & \text{in case 2 for } \bar{\rho} < N \text{ if } D_{\bar{\rho}+1} < \infty \end{cases} \tag{5.14}$$

$$(0 \leq n \leq N,\ t \geq 0).$$

Proof: Using the error bounds given in (5.3), (5.4) and noting that $A_{nm}^N(\tau) \leq A_{nM}^N(t)$ for $0 \leq \tau \leq t$, $m \leq M$ and that $q_{nm}^N(t) \leq 1$, the uniform convergence $x_n^N \to x_n$ as $N \to \infty$ and the error bounds (5.14) can be obtained from (2.13). It immediately follows by (2.6) that $\frac{d}{dt}x_n^N(t) \to \frac{d}{dt}x_n(t)$ as $N \to \infty$ for every fixed $n \geq 0$, uniformly in t on every bounded interval. Therefore, letting $N \to \infty$ in each equation of the system (2.6) we see that the limits x_n solve the infinite initial-value problem (2.1).

6. Remarks and examples. 1. If $c_m \geq 0$, $f_m \geq 0$ $(m \geq 0,\ t \geq 0)$, then from (2.7) and (2.8) there follows $0 \leq x_n^N(t) \leq x_n^{N+1}(t)$ $(0 \leq n \leq N,\ t \geq 0)$. Therefore, $x_n^N(t) \leq x_n(t)$ and the upper bounds in (5.14) are valid for $x_n(t) - x_n^N(t)$. Furthermore, by (2.6)

$$\frac{d}{dt}\Big(\sum_{\nu=0}^N x_\nu^N\Big) = -\lambda_N x_N^N + \sum_{\nu=0}^N f_\nu \leq \sum_{\nu=0}^N f_\nu .$$

Integrating from 0 to t, we obtain

$$\sum_{\nu=0}^N x_\nu^N(t) \leq \sum_{\nu=0}^N \{c_\nu + \int_0^t f_\nu(\tau)d\tau\}.$$

From this there follows $x \in l_1$ and $\|x\| \leq R$.

2. In [7] it is shown that the following conditions are sufficient in order that $\sum_{n=0}^\infty q_{nm}(t) = 1$ and that $\{q_{0m}, q_{1m}, \dots\}^T$ is the only non-negative solution of (1.1) with $c_n = \delta_{nm}$ $(m \geq 0$ fixed) :

(i) $m \le \bar{\rho}$ in case 2,

(ii) $D_m + \sum_{\rho=m}^{\infty} D_{\rho+1}/\lambda_\rho = \infty$ both in case 1 and in case 2 with $\bar{\rho} < \infty$ for $m > \bar{\rho}$.

In particular, (i) is satisfied for every $m \ge 0$ if $\bar{\rho} = \infty$ and (ii) is satisfied if $D_m = \infty$ or $\sum_{\rho=m}^{\infty} 1/\lambda_\rho = \infty$. Sufficient conditions for $\sum_{n=0}^{\infty} q_{nm}(t) < 1$ for some $t > 0$ and corresponding uniqueness have also been proved in [7].

3. Suppose that $c_m = f_m = 0$ for $m > m_0 \ge 0$, $t \ge 0$. Then $R = R_m = R_{m_0}$ for $m \ge m_0$ and in case 2, by (5.14), $x_n^N(t) = x_n(t)$ for $N \ge \rho_N \ge m_0$. In other words: If $m_0 \le \bar{\rho}$ and $\rho^* = \min\{\rho \ge m_0:\ \lambda_\rho = 0\}$, then $x_n^N = x_n$ $(0 \le n \le N)$ for every $N \ge \rho^*$. In fact, by (2.6) it is immediately seen that $x_n^N = 0$ for $\rho^* + 1 \le n \le N$. Hence, $x_n = 0$ for $n \ge \rho^* + 1$ and (2.1) can be reduced to a finite initial-value problem.

4. The upper bounds A_{nm}^N defined by (5.3) tend to zero as $N \to \infty$ of the same order as either $D_1 - D_1^N$ if $D_1 < \infty$ or $1/D_1^N$ if $D_1 = \infty$. The behaviour of D_1^N as $N \to \infty$ is determined by the ratios μ_ρ/λ_ρ for large ρ. In particular, putting $\mu_\rho/\lambda_\rho = \omega_\rho$ $(\rho \ge 1)$ one can obtain easily from (4.9) that $D_1 - D_1^N = O(1/(N+1)!)$ if $\omega_\rho = 1/\rho$, $D_1 - D_1^N = O(c^N)$ if $\omega_\rho = c < 1$, $1/D_1^N = O(1/N)$ if $\omega_\rho = 1$, $1/D_1^N = O(c^{-N})$ if $\omega_\rho = c > 1$ and $1/D_1^N = O(1/N!)$ if $\omega_\rho = \rho$. This shows that the more different the behaviour of λ_ρ, μ_ρ for large ρ is the faster the upper bounds $A_{nm}^N(t)$ for fixed n, m, t tend to zero as $N \to \infty$. Similar observations can be made in case 2.

Example 1: In the theory of kinetics of compartmentalized free-radical polymerization reactions one was led to consider the homogeneous initial-value problem (1.1) where $\lambda_n = \lambda > 0$, $\mu_n = n\mu$ $(\mu > 0,\ n \ge 0)$, $c_0 = 1$, $c_1 = c_2 = \ldots = 0$ [2] . Since $D_1^N = \sum_{\nu=0}^{N} \nu!(\mu/\lambda)^\nu \to \infty$ as $N \to \infty$ (see (4.9)), it has (cf. Remark 2) the unique non-negative solution $\{q_{00}, q_{10}, \ldots\}^T$. (Remark that q_{n0} is the concentration of loci of reaction system which contain n propagating radicals.) Using (3.5),(3.6),(4.6),(5.1) and (5.3) we obtain the upper bounds

$$A_{n0}^N = \inf_{s>0} \left\{ \left(1 + \frac{\lambda + n\mu}{s}\right) \frac{(\lambda/\mu)^n}{n!} e^{st} \frac{\lambda + sD_1^{n-1}}{\lambda + sD_1^N} \right\} = O\left(\frac{(\lambda/\mu)^N}{N!}\right) \qquad (N \to \infty) .$$

Remark that the estimates given in [8,9] fail since the assumption $\sum_{n=1}^{\infty} 1/\alpha_n < \infty$ is violated .

Example 2: Let $\lambda_n = 1$, $\mu_n = n$, $c_n = 1/n!$, $f_n = 0$ $(n = 0,1,2,\ldots;\ t \ge 0)$. Since $D_1^n = \sum_{\nu=0}^{n} \nu!$, we have case 1 with $D_1 = \infty$. Therefore, by (5.14) and in view of Remark 1,

$$0 \le x_n(t) - x_n^N(t) \le \min\{1, A_{nM}^N\} R_M + R - R_M ,$$

where by (5.3) (cf. Example 1 with $\lambda = \mu = 1$)

$$A_{nM}^N = \inf_{s>0} \left\{ K_n(s,t) \frac{1 + sD_1^{M-1}}{1 + sD_1^N} \right\} = \inf_{s>0} \left\{ \left(1 + \frac{n+1}{s}\right)\left(1 + s\sum_{\nu=0}^{n-1} \nu!\right) \frac{e^{st}}{n!} \frac{1 + sD_1^{M-1}}{1 + sD_1^N} \right\}$$

and by (5.12) $R_M = \sum_{m=0}^{M} 1/m! \to R = e \ (M \to \infty)$. Obviously,

(i) $A_{nM}^{N}(t) \le (n+2)\left(1+D_1^{n-1}\right)\frac{e^t}{n!}\left(1+D_1^{M-1}\right)/\left(1+D_1^{N}\right)$

for $t \ge 0$ (putting $s=1$),

(ii) $A_{nM}^{N}(t) \le \left[1+(n+1)t\right]\left(t+D_1^{n-1}\right)\frac{e}{n!t}\left(t+D_1^{M-1}\right)/\left(t+D_1^{N}\right)$

for $t > 0$ (putting $s=1/t$) and

$$R - R_M = \sum_{m=M+1}^{\infty} 1/m! < \frac{M+2}{(M+1)(M+1)!} .$$

Suppose that we wish to find an integer N such that $0 \le x_n(t) - x_n^N(t) < 10^{-3}$ for some n, t. We start from the sufficient condition

$$A_{nM}^{N}(t)e + \frac{M+2}{(M+1)(M+1)!} < 10^{-3} \qquad (0 \le n, M \le N;\ t \ge 0) ,$$

which yields $M \ge 6$. Choosing $M=6$, the inequality is satisfied for e.g. $n=0$ if $A_{06}^{N} < 2{,}8\cdot10^{-4}$ or, in view of (ii), if $D_1^N = \sum_{\nu=0}^{N} \nu! > \left[10^4 e(t+1)(t+154) - t\right]/2{,}8 - t$. Since the right-hand side is a monotone increasing function of t ($t \ge 0$) we obtain $N \ge 10$ for $0 \le t \le 1$, $N \ge 11$ for $0 \le t \le 10$ and $N \ge 12$ for $0 \le t \le 100$. Choosing $N=12$, we have $A_{n6}^{12} < 2{,}8\cdot10^{-4}$ for $n=1,2,\dots,12$ if (see (ii))

$$\left[1+(n+1)t\right]\left(t+D_1^{n-1}\right)\frac{e}{n!t}\left(t+154\right)/\left(t+D_1^{12}\right) < 2{,}8\cdot10^{-4} .$$

For $n=1$ few computations give the unessentially stronger inequality $2t^3 + 311t^2 - 53404\,t + 154 < 0$, so that $A_{16}^{12} < 2{,}8\cdot10^{-4}$ and, consequently, $x_1 - x_1^{12} < 10^{-3}$ for all t lying between both positive zeros of the polynomial on the left-hand side, at least for $0{,}003 \le t \le 103$. For $0 \le t \le 0{,}003$ we use (i) to show that $A_{16}^{12} < 2{,}8\cdot10^{-4}$. Finally, we have $x_1 - x_1^{12} < 10^{-3}$ at least for $0 \le t \le 103$. In the same way we can proceed for $n=2,3,\dots,12$. For example, $0 \le x_5 - x_5^{12} < 10^{-3}$ holds at least for $0 \le t \le 945$.

On the other side, in order to determine an integer N being as small as possible such that the only condition $0 \le x_5 - x_5^N < 10^{-3}$ for $0 \le t \le 100$ is satisfied, we start from the inequality $(1+6t)(t+34)(t+154) \le 0{,}012t\left(t + \sum_{\nu=0}^{N} \nu!\right)$ implying (ii) with $n=5$, $M=6$. If we put $t=100$ we obtain $N \ge 11$. Choosing $N=11$. In fact the inequality is satisfied at least for $0{,}02 \le t \le 208$. By using (i) it can be shown as above that $0 \le x_5 - x_5^{11} < 10^{-3}$ for $0 \le t \le 0{,}02$, too. Remark that the present example has the constant solution $x_n = 1/n!$ ($n=0,1,\dots$). By numerical computations one can obtain real upper bounds δ_n for $x_n - x_n^{12}(t)$ ($0 \le n \le 12$) holding in the intervall $0 \le t \le 10^3$. They and the corresponding relative errors $\delta_n/x_n = \delta_n n!$ are exhibited for some n in the following table.

n	1	2	4	6	8	11	12
δ_n	7,1E-7	3,6E-7	3,0E-8	1,4E-9	3,0E-10	2,0E-10	1,8E-10
δ_n/x_n	7,1E-7	7,2E-7	7,2E-7	1,1E-6	1,2E-5	8,0E-3	8,6E-2

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