## Error Bounds for the Truncation of Countable Linear Differential Systems Arising from Birth-Death Processes

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An infinite system of linear differential equations  $x'_n(t) = \lambda_{n-1} x_{n-1}(t) - (\lambda_{n} + \mu_n) x_n(t)$  $*\mu_{n+1}x_{n+1}(t) * f_n(t), x_n(0) = c_n$  (n=0,1,2,...; t>0; x<sub>-1</sub>=0) is considered. The constant coefficients  $\lambda_n$ ,  $\mu_n$  are merely assumed to be non-negative. Explicit error bounds are derived for the approximation of a solution by the solutions of the finite truncated systems. They crucially depend on the ratios of the coefficients  $\lambda_n, \mu_n$ .

Key words: Birth-death processes, linear differential equations AMS subject classification: 34A30, 34A45, 60J80

1. Introduction . Simple birth-death processes with enumerably many states can be described by a countable system of differential equations [4]

$$
x'(t) = Ax(t) , x(0) = c (t \ge 0) . \qquad (1.1)
$$

where x and c are infinite column vectors and  $\bm{A}$  is a constant infinite matrix of the form

$$
A = \begin{bmatrix} -\lambda_0 & \mu_1 & & & 0 \\ \lambda_0 & -\lambda_1 - \mu_1 & \mu_2 & & \\ \lambda_1 & -\lambda_2 - \mu_2 & \mu_3 & & \\ 0 & & & & \end{bmatrix}
$$
 (1.2)

with  $\lambda_i \geq 0$  ( $i \geq 0$ ),  $\mu_i \geq 0$  ( $i \geq 1$ ). It is known [7] that this system possesses for  $c \ge 0$ ,  $c \in I_1$  a non-negative solution which can be obtained approximately by truncation.

For numerical purposes it is of interest to know error bounds. Results in the case  $\lambda_n = \lambda_n(t) \le \lambda(t)$ ,  $\mu_n = \mu_n(t) \le \mu(t)$  are given in [10], whereas the case  $A = (a_{ij})$  is any bounded operator on  $I_1$  satisfying sup  $\sum_{1 \leq i \leq n}^{\infty} |a_{ij}| \rightarrow 0$  or sup  $\sum_{n+1}^{n} |a_{ij}| \to 0$   $(n \to \infty)$  is considered in [3]. In particular, if A is given by (1.2), then the latter conditions reduce to  $\lambda_n \to 0$  or  $\mu_n \to 0$ , respectively.  $28*$ 

In the present paper error bounds for Inhomogeneous systems without any restrictions on the growth of the non-negative coefficients  $\lambda_n$ ,  $\mu_n$  are derived. They crucially depend on the ratios of the coefficients. In particular, if  $\lambda_n / \mu_n \rightarrow g$  ( $0 \le g \le \infty$ ) as  $n \rightarrow \infty$ , then the bounds are the smaller, the closer g is to 0 or to  $\infty$ . Remark that similar results for more general systems  $(A)$  is **a quadrodlagonal matrix** with non-negative column sums) are established in (8,9]. However, both the assumptions and the assertions are more complicated than in the present paper. x (*x*) as  $n \to \infty$ , then the bounds are the smaller, the closer g hark that similar results for more general systems (*A* is ix with non-negative column sums) are established in [8,9].<br>
x with non-negative column sums) a

2. **Prelimlnailes .** We consider a countable system of linear differential equations of the form

$$
x'(t) = Ax(t) + f(t), x(0) = c \t(t \ge 0), \t(2.1)
$$

where  $x = \{x_0, x_1, ... \}^T$ ,  $f = \{f_0, f_1, ... \}^T$ ,  $c = \{c_0, c_1, ... \}^T$  are infinite column vectors and *A* is the matrix defined by (1.2). The following conditions are assumed to be satisfied *C(O,w)*  $\left\{ \begin{aligned}\n &\text{for all } t \in \mathbb{R} \text{ is } t \in \mathbb{R} \text{ and } t \in \mathbb{R} \text{ is } t \in \mathbb{R} \text{ is } t \in \mathbb{R} \text{ and } t \in \mathbb{R} \text{ is } t \in \mathbb{R} \text{ and } t \in \mathbb{R} \text{ is } t \in \mathbb{R} \text{ and } t \in \mathbb{R} \text$ *x'*(*t*) =  $Ax(t) + f(t)$ ,  $x(0) = c$  ( $t \ge 0$ ), (2.1)<br> *f* = { $f_0, f_1,...$ }*T*,  $c = (c_0, c_1,...)$ *T* are infinite column vectors<br>
lefined by (1.2). The following conditions are assumed to<br>  $f_n \in C[0,\infty)$  ( $n = 0,1, 2,...$ ), (2.2)<br>  $c$ , or a countable system of linear differential equations<br>  $A x(t) + f(t)$ ,  $x(0) = c$   $(t \ge 0)$ , (2.1)<br>  $(f_0, f_1,...)$ <sup>T</sup>,  $c = (c_0, c_1,...)$ <sup>T</sup> are infinite column vectors<br>
d by (1.2). The following conditions are assumed to<br>  $T(0,\infty)$ 

$$
f_n \in C[0,\infty) \qquad (n=0,1,2,...) , \qquad (2.2)
$$

$$
c, \ \{\int\limits_0^t |f_n(\tau)|d\tau\}_{n\geq 0} \in I_1 \ \text{for all } t\geq 0 \ , \tag{2.3}
$$

$$
\lambda_n \geq 0 \quad (n \geq 0) \quad , \quad \mu_n > 0 \quad (n \geq 1) \quad . \tag{2.4}
$$

Remark that we may assume  $\mu_n > 0$  instead of  $\mu_n \geq 0$  since in case  $\mu_n = 0$  for finitely or infinitely many integers  $n \geq 1$  the initial-value problem (2.1) can be reduced either to a finite system and a countable system satisfying (2.2) - (2.4) or to enumerabiy many finite systems, respectively. For convenience, we define d by (1.2). The<br>
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( $n \ge 1$ ). (2.3)<br>
( $n \ge 1$ ). (2.4)<br>
of  $\mu_n \ge 0$  since in case  $\mu_n \approx 0$  for<br>
initial-value problem (2.1) can be<br>
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ely. For convenience, we define<br>
( finitely or infinitely many integers  $n \ge 1$  the initial-value problem (2.1)<br>reduced either to a finite system and a countable system satisfying (2.2) -<br>to enumerably many finite systems, respectively. For convenience, we

$$
\mu_0 = 0 \quad , \quad \alpha_n = \lambda_n + \mu_n \quad (n \ge 0) \quad . \tag{2.5}
$$

Let *N* be any non-negative integer. Truncating all coordinates of the vectors x, *f, c* with indices greater than *N* as well as the corresponding columns and rows of the matrix *A* we obtain the finite Initial-value problem regative integer. Truncating all coordinates of the vectors<br>
reater than N as well as the corresponding columns and<br>
we obtain the finite initial-value problem<br>  $\frac{d}{dt}x^N(t) = A^Nx^N(t) + f^N(t)$ ,  $x^N(0) = c^N$  ( $t \ge 0$ ), (2.6)

$$
\frac{d}{dt}x^N(t) = A^N x^N(t) + f^N(t) \quad , \quad x^N(0) = c^N \quad (t \ge 0) \quad , \tag{2.6}
$$



It is well known that the solution of **the initial-value** problem (2.6) can be represented in the form

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\nwell known that the solution of the initial-value problem (2.6) can be

\nd in the form

\n
$$
x_n^N(t) = \sum_{m=0}^N \left\{ \int_0^t q_{nm}^N(r) f_m(t-r) dr + c_m q_{nm}^N(t) \right\} \qquad (n = 0, 1, ..., N; \ t \ge 0) \quad (2.7)
$$

where  $(q_{0m}^N, q_{1m}^N, ..., q_{Nm}^N)^T$  for every fixed *m*  $(0 \le m \le N)$  denotes the solution  $x_n^N(t) = \sum_{m=0}^N \left\{ \int_0^t q_{nm}^N(\mathbf{r}) f_m(\mathbf{t}-\mathbf{r}) d\mathbf{r} + c_m q_{nm}^N(\mathbf{t}) \right\}$  ( $n = 0, 1, ..., N; t \ge 0$ ) (2.7)<br>where  $(q_{0m}^N, q_{1m}^N, ..., q_{Nm}^N)^T$  for every fixed  $m (0 \le m \le N)$  denotes the solution<br>of the homogeneous system (1. Is shown in [7] that Error Bounds for the Truncation of Diff. Systems 409<br>
known that the solution of the initial-value problem (2.6) can be<br>
the form<br>  $\sum_{m=0}^{N} \left\{ \int_{q_{nm}^{N}}^{N} (r) f_{m}(t-r) dr + c_{m} q_{nm}^{N}(t) \right\}$  ( $n = 0, 1, ..., N; t \ge 0$ ) (2.7)<br>  $\int$ (1.1) with  $q_{nm}^N(0) = \delta_{nm}$  (Kronecker symbol). It<br>  $r_1^1(t) \le 1$  (0  $\le m, n \le N$ ;  $t \ge 0$ ). (2.8)<br>
ence of the limits<br>  $r_n^1(t)$  ( $m, n \ge 0$ ;  $t \ge 0$ ), (2.9)<br>
we the homogeneous system (1.1) with the initial<br>
ell as the exi

$$
0 \le q_{nm}^N(t) \le q_{nm}^{N+1}(t) \le 1 \qquad (0 \le m, n \le N; t \ge 0) \; . \tag{2.8}
$$

Hence, there follows the existence of the limits  

$$
q_{nm}(t) = \lim_{N \to \infty} q_{nm}^N(t) \qquad (m, n \ge 0; t \ge 0) ,
$$
 (2.9)

which for every fixed *m* solve the homogeneous system (1.1) with the initial conditions  $q_{nm}(0) = \delta_{nm}$ , as well as the existence of the Laplace transforms bllows the existence of the lip<br>  $q_{nm}(t) = \lim_{N \to \infty} q_{nm}^N(t)$  (m, if  $q_{nm}$ ) (m, if  $q_0 = \delta_{nm}$ , as well as the exist<br>  $Q_{nm}^N(s) = \int_0^\infty e^{-st} q_{nm}^N(t) dt$ 1 that<br>  $0 \le q_{nm}^N(t) \le q_{nm}^{N+1}(t) \le 1$  ( $0 \le m, n \le N; t \ge 0$ ). (2.8)<br>
collows the existence of the limits<br>  $q_{nm}(t) = \lim_{N \to \infty} q_{nm}^N(t)$  ( $m, n \ge 0; t \ge 0$ ), (2.9)<br>
ry fixed *m* solve the homogeneous system (1.1) with the initia

$$
Q_{nm}^N(s) = \int_0^\infty e^{-st} q_{nm}^N(t) dt \qquad (s > 0).
$$
 (2.10)

**Moreover, In [8] there are proved the Inequalities** 

$$
0 \leq \frac{d}{dt} \left[ e^{\alpha_n t} q_{nm}^N(t) \right] \leq \frac{d}{dt} \left[ e^{\alpha_n t} q_{nm}^{N+1}(t) \right] \qquad (0 \leq m, n \leq N; t \geq 0). \tag{2.11}
$$

**Finally, we notice**

$$
\mathbf{a} \int_{0}^{\infty} e^{-st} q_{nm}^{N}(t) dt \qquad (s>0).
$$
\n(2.10)\n  
\n
$$
\mathbf{e} \text{ are proved the inequalities}
$$
\n
$$
\mathbf{e}^{\alpha} \left[ e^{\alpha} q_{nm}^{N}(t) \right] \leq \frac{d}{dt} \left[ e^{\alpha} q_{nm}^{N+1}(t) \right] \qquad (0 \leq m, n \leq N; t \geq 0).
$$
\n(2.11)\n
$$
\left[ \begin{array}{c} Q_{00}^{N} & \cdots & Q_{0N}^{N} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{N0}^{N} & \cdots & Q_{NN}^{N} \end{array} \right].
$$
\n(2.12)\n  
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where *E* denotes the unit matrix of order *N+1.* From (2.7)- (2.9) there follows

$$
(sE-A^N)^{-1} = \begin{bmatrix} Q_{00}^N \cdots Q_{0N}^N \\ \vdots \\ Q_{N0}^N \cdots Q_{NN}^N \end{bmatrix},
$$
(2.12)  
there *E* denotes the unit matrix of order *N*+1. From (2.7)-(2.9) there follows  

$$
|x_n^{N+P}(t) - x_n^N(t)| \le \sum_{m=0}^N \left\{ \int_{0}^t [q_{nm}(t) - q_{nm}^N(t)] |f_m(t-t)| dt + |c_m| [q_{nm}(t) - q_{nm}^N(t)] \right\} \quad (2.13)
$$

$$
+ \sum_{m=N+1}^{N+P} \left\{ \int_{0}^t [f_m(t)] |dt + |c_m| \right\} \qquad (0 \le m, n \le N; \ t \ge 0)
$$
or every integer  $p \ge 1$ . Hence, in order to prove the existence of *x*. (t) =  $\lim_{m \to N} x^N(t)$ 

for every integer  $p \ge 1$ . Hence, in order to prove the existence of  $x_n(t) = \lim_{N \to \infty} x_n^N(t)$ and to obtain error bounds for  $|x_n(t)-x_n^N(t)|$  it suffices to know upper bounds for the non-negative differences  $q_{nm}(t) - q_{nm}^{N}(t)$ . In the following we will show bow these can be derived by means of the Laplace transformation. Thus, we first deal with certain principal minors of the matrix  $sE - A^N$ .

3. Some properties of principal minors of the matrix  $sE - A^N$ . Let

$$
\Delta_{i}^{j}(s) = \begin{cases}\n0 & \text{for } j < i-1 \\
1 & \text{for } j < i-1 \\
1 & \text{for } j = i-1 \\
-\lambda_{i} & s + \alpha_{i+1} \\
0 & -\lambda_{j-1} & s + \alpha_{j}\n\end{cases}
$$
\n(1 = 0, 1, 2,...).\n(3.1)

Every determinant  $\Delta_i^f$  is a polynomial in *s* of degree  $f-i+1$  with the leading coefficient 1. The other coefficients being sums of principal minors of the matrix  $-A^N$  are non-negative, since the column sums of  $-A^N$  are non-negative [1,6]. (Remark that  $sE-A^N$  for  $s>0$  is a non-singular M-matrix  $[1]$ .) Therefore,  $\Delta_i^j(s) \geq s^{j-i+1} > 0$  for  $s > 0$ . *A* for so is a non-singular minors of the matrix<br> *A* are non-negative [1,6].<br> *E*-*AN* for s > 0 is a non-singular M-matrix [1].) Therefore,<br>
(for s > 0.<br>
urrence formula<br>  $A_i^j = (s + \alpha_j)A_i^{j-1} - \lambda_{j-1}\mu_jA_i^{j-2}$  (0 ≤ *i* gular M-matrix [1].) Therefore,<br>
(0  $\le$  j  $\le$  j) (3.2)<br>
on. Using (3.2) and replacing in<br>
duct by 1 the relations<br>  $\mu_v$ ) (0  $\le$  j  $\le$  j  $\le$  k+1) (3.3)<br>
(0  $\le$  j  $\le$  k+1) (3.4)<br>
In particular, we have for j=0

Next, the recurrence formula

$$
\Delta_{i}^{j} = (s + \alpha_{j}) \Delta_{i}^{j-1} - \lambda_{j-1} \mu_{j} \Delta_{i}^{j-2} \qquad (0 \leq i \leq j)
$$
 (3.2)

can be verified immediately by Laplace expansion. Using (3.2) and replacing in *i*<sub>1</sub>)  $\Delta f^{-1} - \lambda_{j-1} \mu_j \Delta f^{-2}$  (0 ≤ *i* ≤ *j*) (3.2)<br>
ly by Laplace expansion. Using (3.2) and replacing in<br>
by 0 and an empty product by 1 the relations<br>  $\Delta f \Delta f^{k+1} = \Delta f^{-2} \prod_{v=j}^{k+1} (\lambda_{v-1} \mu_v)$  (0 ≤ *i* ≤ *j* ≤ k+1

usual way an empty sum by 0 and an empty product by 1 the relations  
\n
$$
\Delta_f^k \Delta_f^{k+1} - \Delta_f^k \Delta_f^{k+1} = \Delta_f^{j-2} \prod_{v=j}^{k+1} (\lambda_{v-1} \mu_v) \qquad (0 \le i \le j \le k+1)
$$
\nand  
\n
$$
\Delta_f^k(0) = \sum_{v=1}^{k+1} \left( \prod_{o=i}^{v-1} \mu_o \prod_{o=v}^k \lambda_o \right) \qquad (0 \le j \le k+1)
$$
\n(3.4)

and

$$
\Delta_{i}^{j} = (s + \alpha_{j}) \Delta_{i}^{j-1} - \lambda_{j-1} \mu_{j} \Delta_{i}^{j-2} \qquad (0 \leq i \leq j) \qquad (3.2)
$$
  
mmmediately by Laplace expansion. Using (3.2) and replacing in  
pty sum by 0 and an empty product by 1 the relations  

$$
\Delta_{i}^{k} \Delta_{j}^{k+1} - \Delta_{j}^{k} \Delta_{i}^{k+1} = \Delta_{j}^{j-2} \prod_{v=1}^{k+1} (\lambda_{v-1} \mu_{v}) \qquad (0 \leq i \leq j \leq k+1) \qquad (3.3)
$$
  

$$
\Delta_{j}^{k}(0) = \sum_{v=j}^{k+1} \left( \prod_{\rho=1}^{v-1} \mu_{\rho} \prod_{\rho=v}^{k} \lambda_{\rho} \right) \qquad (0 \leq j \leq k+1) \qquad (3.4)
$$
  
ty induction (i, j fixed;  $k \geq j-1$ ). In particular, we have for  $j=0$   
), (2.5)  

$$
\Delta_{0}^{k}(0) = \prod_{\rho=0}^{k} \lambda_{\rho} \geq 0 \qquad (k \geq -1) \qquad (3.5)
$$
  

$$
\Delta_{1}^{k}(0) = \sum_{v=1}^{k+1} \left( \prod_{\rho=1}^{v-1} \mu_{\rho} \prod_{\rho=v}^{k} \lambda_{\rho} \right) \geq \prod_{\rho=1}^{k} \mu_{\rho} \geq 0 \qquad (k \geq 0).
$$
  
(3.6)  
ng (3.2), (3.5) and (3.6) the inequality

can be proved by induction  $(i, j$  fixed;  $k \geq j-1$ ). In particular, we have for  $j=0$ and  $j = 1$  by  $(2.4)$ ,  $(2.5)$ 

$$
\Delta_0^k(0) = \prod_{\rho=0}^k \lambda_\rho \ge 0 \qquad (k \ge -1)
$$
 (3.5)

and

$$
\Delta_{j}^{k}(0) = \sum_{v= j}^{\infty} \left( \prod_{\rho=1}^{\infty} \mu_{\rho} \prod_{\rho=v}^{\infty} \lambda_{\rho} \right) \qquad (0 \leq j \leq k+1)
$$
\n(3.4)

\n(3.4)

\n(3.5)

\n(2.5)

\n
$$
\Delta_{0}^{k}(0) = \prod_{\rho=0}^{k} \lambda_{\rho} \geq 0 \qquad (k \geq -1)
$$
\n
$$
\Delta_{1}^{k}(0) = \sum_{v=1}^{k+1} \left( \prod_{\rho=1}^{v-1} \mu_{\rho} \prod_{\rho=v}^{k} \lambda_{\rho} \right) \geq \prod_{\rho=1}^{k} \mu_{\rho} > 0 \qquad (k \geq 0).
$$
\n(3.6)

\n(3.2), (3.5) and (3.6) the inequality

\n
$$
\frac{k+1}{0} \cdot \left( \frac{\Delta_{0}^{k}(0) + \Delta_{1}^{k}(0)}{\Delta_{0}^{k}(0)} \right) \left[ \frac{\Delta_{0}^{k+1}(0) + \Delta_{1}^{k+1}(0)}{\Delta_{0}^{k}(0)} \right] \qquad (k \geq 0, s > 0)
$$
\n(3.7)

Furthermore, using (3.2), (3.5) and (3.6) the inequality  
\n
$$
\Delta_0^k(s)/\Delta_0^{k+1}(s) \langle [A_0^k(0)+s\Delta_1^k(0)]/[\Delta_0^{k+1}(0)+s\Delta_1^{k+1}(0)] \quad (k \ge 0, s > 0)
$$
\n(3.7)  
\ncan be proved by induction. As an immediate consequence we obtain the estimate  
\n
$$
\Delta_0^k(s)/\Delta_0^N(s) \langle [A_0^k(0)+s\Delta_1^k(0)]/[\Delta_0^N(0)+s\Delta_1^N(0)] \quad (0 \le k \le N, s > 0).
$$
\n(3.8)

can be proved by induction. As an Immediate consequence we obtain the estimate

$$
\Delta_{0}^{k}(s)/\Delta_{0}^{N}(s) \cdot [\Delta_{0}^{k}(0) + s\Delta_{1}^{k}(0)]/[\Delta_{0}^{N}(0) + s\Delta_{1}^{N}(0)] \qquad (0 \leq k \leq N, s > 0). \quad (3.8)
$$

4. Estimates of the differences  $Q_{nm}^{N+p}(s) - Q_{nm}^{N}(s)$ . The entries  $Q_{nm}^{N}$  of the matrix  $(sE-A^N)^{-1}$  have the representations

$$
Q_{nm}^{N}(s) = \begin{cases} \frac{\Delta_0^{m-1}(s)\Delta_{n+1}^{N}(s)}{\Delta_0^{N}(s)} \prod_{\rho=m}^{n-1} \lambda_{\rho} & \text{for } 0 \leq m \leq n \leq N, \\ \frac{\Delta_0^{m-1}(s)\Delta_{m+1}^{N}(s)}{\Delta_0^{N}(s)} \prod_{\rho=n+1}^{m} \mu_{\rho} & \text{for } 0 \leq n \leq m \leq N, \end{cases}
$$
(4.1)

which can be obtained easily from (2.12) by Laplace expansion. From (4.1) there follows by means of (3.3)

$$
Q_{mm}^{N+1}(s) - Q_{nm}^{N}(s) = \frac{\Delta_0^{n-1}(s)\Delta_0^{m-1}(s)}{\Delta_0^{N}(s)\Delta_0^{N+1}(s)\cdots\rho = m^{\rho}} \prod_{\rho=n+1}^{N+1} \mu_\rho \qquad (0 \leq m, n \leq N; s > 0). \quad (4.2)
$$

Hence, by (2.8) and (3.8)

$$
0 \leq Q_{nm}^{N+1}(s) - Q_{nm}^{N}(s) \leq \frac{\left[A_0^{n-1}(0) + sA_1^{n-1}(0)\right]\left[\Delta_0^{n-1}(0) + sA_1^{n-1}(0)\right]}{\left[A_0^N(0) + sA_1^N(0)\right]\left[A_0^{N+1}(0) + sA_1^{N+1}(0)\right]} \xrightarrow{\rho = m} \rho_{\rho = n+1}^{N+1} \mu_{\rho} \quad (4.3)
$$
\n
$$
(0 \leq m, n \leq N; s > 0).
$$

Next, from (3.5) and (3.6)

$$
S \prod_{\rho=1}^{N+1} \mu_{\rho} = A_0^{N+1}(0) + sA_1^{N+1}(0) - \lambda_{N+1} \left[ A_0^N(0) + sA_1^N(0) \right] \qquad (N \ge 0, s > 0).
$$
 (4.4)

Then the inequality (4.3) becomes

$$
0 \leq Q_{nm}^{N+1}(s) - Q_{nm}^{N}(s) \leq \frac{\delta_n(s)}{s} \Big[ A_0^{m-1}(0) + s A_1^{m-1}(0) \Big] \times \qquad (4.5)
$$
\n
$$
\left[ \frac{\prod_{\rho=m}^{N} \lambda_{\rho}}{A_0^{N}(0) + s A_1^{N}(0)} - \frac{\prod_{\rho=m}^{N+1} \lambda_{\rho}}{A_0^{N-1}(0) + s A_1^{N+1}(0)} \right],
$$

where  $\delta_n(s)$  is defined by

$$
\delta_n(s) = \left[ \Delta_0^{n-1}(0) + s \Delta_1^{n-1}(0) \right] / \prod_{\rho=1}^n \mu_\rho \quad (n \ge 0).
$$
 (4.6)

Therefore,

$$
0 \le Q_{nm}^{N+p}(s) - Q_{nm}^{N}(s) \le \frac{\delta_n(s)}{s} \left[ \Delta_0^{m-1}(0) + s \Delta_1^{m-1}(0) \right] \prod_{\rho=m}^{N+p} \lambda_{\rho} \times \qquad (4.7)
$$
\n
$$
\left[ \frac{1}{\Delta_0^{N}(0) + s \Delta_1^{N}(0)} - \frac{\prod_{\rho=N+1}^{N+p} \lambda_{\rho}}{\Delta_0^{N+p}(0) + s \Delta_1^{N+p}(0)} \right] \qquad (0 \le m, n \le N \le N+p; s>0) .
$$

In what follows it will be convenient to establish the numbers

it will be convenient to establish the numbers  
\n
$$
D_i^j = \Delta_i^j(0) / \prod_{\rho \in i} \lambda_\rho \qquad (1 \le i \le j)
$$
\n(4.8)

whenever  $\lambda_{\rho} > 0$  for each  $\rho = 1$ ,  $i+1, ..., j$ . According to (3.1), let  $D_i^{j-1} = 1$  and  $D_i^j = 0$  for  $j < i-1$ . From (3.4) we obtain the representation it will be convenient to establish the numbers<br>  $D_l^j = \Delta_l^j(0) / \prod_{\rho=1}^j \lambda_\rho$  (1 ≤ j ≤ j) (4.8)<br>
each  $\rho = i, i+1, ..., j$ . According to (3.1), let  $D_l^{j-1} = 1$  and<br>
om (3.4) we obtain the representation<br>  $D_l^j = \sum_{\nu=1}^{j+1} \sum_{$ *D*<sup>*j* =  $\Delta f(0)/\prod_{p=1}^{\infty} \lambda_p$  (1 ≤ *i* s) (4.8)<br>
each  $\rho = 1$ , *i* + 1, ..., *j*. According to (3.1), let  $D_f^{f-1} = 1$  and<br>
om (3.4) we obtain the representation<br>  $D_f^f = \sum_{\nu=1}^{f+1} \sum_{\rho=1}^{\nu-1} (\mu_\rho / \lambda_\rho)$  (4.9)<br>
rs</sup>

$$
D_l^j = \sum_{\nu=1}^{j+1} \prod_{\rho=1}^{\nu-1} (\mu_\rho / \lambda_\rho)
$$
 (4.9)

holding for all integers  $j \in I$ , too.

Notice that, if  $\lambda_{\alpha} > 0$  for every integer  $\rho \geq i$  and any fixed  $i \geq 1$ , the sequence  $(D_j^j)_{j \geq 1}$  is monotonically increasing. For these *I* we define

$$
D_j = \lim_{j \to \infty} D_j^j \tag{4.10}
$$

Notice that, if  $\lambda_{\rho} > 0$  for  $(D_f^I)_{J \ge 1}$  is monotonically inci<br>  $D_I = \lim_{J \to \infty}$ <br>
Obviously, we have  $1 \le D_I \le \infty$ <br>
Therefore,  $D_L \le \infty$  for some *i* Obviously, we have  $1 \times D_i \le \infty$ . Remark that  $D_i^j = D_i^{k-1} + (D_i^k - D_i^{k-1}) D_{k+1}^j$  ( $i \le k \le j$ ).  $D_i = \lim_{j \to \infty} D_i^j$ .<br>Obviously, we have  $1 \leftarrow D_i \leftarrow \infty$ . Remark that  $D_i^j = D_i^{k-1} + (D_i^k - 1)$ .<br>Therefore,  $D_k \leftarrow \infty$  for some  $k \geq i$  implies  $D_k \leftarrow \infty$  for all  $k \geq i$ .<br>Now we distinguish two cases.

Now we distinguish two cases.

*Case 1*: Assume that  $\lambda_{\alpha} > 0$  for all  $\rho \ge 0$ . Then (4.7) can be rewritten as

holding for all integers 
$$
j < 1
$$
, too. Notice that, if  $\lambda_{\rho} > 0$  for every integer  $\rho \geq 1$  and any fixed  $i \geq 1$ , the sequence  $|D_j^I|_{j \geq 1}$  is monotonically increasing. For these  $I$  we define  $D_I = \lim_{J \to \infty} D_I^I$ . (4.10) Obviously, we have  $1 < D_I \leq \infty$ . Remark that  $D_I^J = D_I^{k-1} + (D_I^k - D_I^{k-1}) D_{k+1}^I$   $(1 \leq k \leq f)$ . Therefore,  $D_k < \infty$  for some  $k \geq 1$  implies  $D_k < \infty$  for all  $k \geq 1$ . Now we distinguish two cases. Case 1: Assume that  $\lambda_{\rho} > 0$  for all  $\rho \geq 0$ . Then (4.7) can be rewritten as  $0 \leq Q_{nm}^{N+p}(s) - Q_{nm}^{N}(s) \leq \delta_n(s)(\lambda_0 + sD_I^{m-1}) \frac{D_I^{N+p} - D_I^N}{(\lambda_0 + sD_I^{N+p})(\lambda_0 + sD_I^N)}$  (4.11)  $(0 \leq m, n \leq N < N + p; s > 0)$ . Case 2: Assume that finitely or infinitely many  $\lambda_{\rho}$  vanish. Let  $\rho = \min(\rho \geq 0; \lambda_{\rho} = 0)$ ,  $\bar{\rho} = \sup\{\rho \geq 0 : \lambda_{\rho} = 0\}$ . Evidently,  $0 \leq \rho \leq \bar{\rho} \leq \infty$ . Then (4.7) for  $N < \rho$  implies  $0 \leq Q_{nm}^{N+p}(s) - Q_{nm}^{N}(s) \leq \frac{1}{s} \delta_n(s) \frac{\lambda_0 + sD_I^{m-1}}{\lambda_0 + sD_I^N}$   $(0 \leq m, n \leq N < N + p; s > 0)$  (4.12) For

*Case 2:* Assume that finitely or infinitely many  $\lambda_{\alpha}$  vanish. Let  $\rho = \min\{\rho \ge 0: \lambda_{\alpha} = 0\}$ ,

$$
(0 \le m, n \le N \le N + p ; s > 0).
$$
  
\n2: Assume that finitely or infinitely many  $\lambda_{\rho}$  vanish. Let  $\rho = \min\{\rho \ge 0 : \lambda_{\rho} = 0\}$ ,  
\n $p\{\rho \ge 0 : \lambda_{\rho} = 0\}$ . Evidently,  $0 \le \rho \le \bar{\rho} \le \infty$ . Then (4.7) for  $N \le \rho$  implies  
\n $0 \le Q_{nm}^{N+p}(s) - Q_{nm}^{N}(s) \le \frac{1}{s} \delta_n(s) \frac{\lambda_0 + sD_1^{m-1}}{\lambda_0 + sD_1^{N}}$   $(0 \le m, n \le N \le N + p; s > 0)$  (4.12)

For  $N \ge \rho$  we immediately obtain by (4.7)

$$
Q_{nm}^{N+p}(s) = Q_{nm}^{N}(s) \qquad (0 \leq m \leq \rho_N, \ 0 \leq n \leq N \leq N+p; \ s>0), \qquad (4.13)
$$

where  $\rho_N = \max\{ \rho \le N: \ \lambda_\rho = 0 \}$ ; moreover, if  $\rho_N \le N$  (i.e.  $\lambda_N > 0$ ), then (4.7) yields

$$
Q_{nm}^{N+p}(s) = Q_{nm}^{N}(s) \quad (0 \leq m \leq \rho_N, 0 \leq n \leq N \leq N+p; s>0), \tag{4.13}
$$
  
\n
$$
P_N = \max \{ \rho \leq N: \lambda_{\rho} = 0 \}; \text{ moreover, if } \rho_N \leq N \text{ (i.e. } \lambda_N > 0), \text{ then (4.7) yields}
$$
  
\n
$$
0 \leq Q_{nm}^{N+p}(s) - Q_{nm}^{N}(s) \leq \frac{1}{s} \delta_n(s) D_{\rho_N+1}^{m-1} \left[ 1/D_{\rho_N}^N + 1 - \frac{P_{N+p}}{\rho} (\lambda_{\rho} / \mu_{\rho}) / D_{\rho_{N+p}+1}^{N+p} \right] \tag{4.14}
$$
  
\n
$$
(\rho \leq \rho_N \leq m \leq N; 0 \leq n \leq N \leq N+p; s>0),
$$

*PN*<br>
because  $\Delta_0^m(0) = 0$  for  $\rho_N \le m$  and  $\Delta_1^m(0) = D_{\rho_N+1}^m \prod_{\rho=1}^{\rho_N} \mu_{\rho} \prod_{\rho= \rho_N \sim 1}^m \lambda_{\rho}$  for  $\rho_N \le m \le N$ 

*by (3.5), (3.6)* and (4.9). The estimate (4.14) can be reduced to

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\n
$$
0 \leq Q_{nm}^{N+p}(s) - Q_{nm}^{N}(s) \leq \frac{1}{s} \delta_n(s) D_{\rho_N+1}^{m-1} / D_{\rho_N+1}^{N}
$$
\n(4.15)

\n
$$
(\rho \leq \rho_N \cdot m \leq N; 0 \leq n \leq N \cdot (N+p; s > 0)
$$
\nwith the upper bounds in (4.14) and (4.15) are equal when  $N \leq 5$  and  $n$  is

Remark that the upper bounds in  $(4.14)$  and  $(4.15)$  are equal, when  $N \triangleleft p$  and  $p$  is sufficiently large. If  $\bar{p} \le \infty$  (i.e., only a finite number of the coefficients  $\lambda_p$  vanishes), then for every  $N > \bar{\rho}$  we have  $\rho_N = \rho_{N+p} = \bar{\rho}$  and therefore (4.14) reduces to Error Bounds for the Tr<br>
0  $\leq Q_{nm}^{N+p}(s) - Q_{nm}^{N}(s) \leq \frac{1}{s} \delta_n(s) D_{p_N+1}^{m-1} / D_{p_N+1}^{N}$ <br>  $(\rho \leq \rho_N \cdot m \leq N; 0 \leq$ <br>
c that the upper bounds in (4.14) and (4.15) as<br>
ently large. If  $\bar{\rho} \leq \infty$  (i.e., only a finite numb *I A i*  $\frac{1}{P}$  *f*  $\frac{1}{P}$  *f m* **f** *f i f i f i f i f i f i f i f* ( $\rho \le \rho_N$ ,  $m \le N$ ;  $0 \le n \le N$ ,  $N+p$ ;  $s > 0$ )<br>
at the upper bounds in (4.14) and (4.15) are equal, when  $N \le \bar{\rho}$  and  $\rho$  is<br>
<sup>*'*</sup> large. If  $\bar{\rho} \le \infty$  (i.e., only a finite number of the coefficients  $\lambda_{\rho}$  va-<br>
en

$$
0 \leq Q_{nm}^{N+p}(s) - Q_{nm}^{N}(s) \leq \frac{1}{s} \delta_n(s) D_{\bar{\rho}+1}^{m-1} [1/D_{\bar{\rho}+1}^{N} - 1/D_{\bar{\rho}+1}^{N+p}]
$$
(4.16)  
 $(\bar{\rho} \langle m \leq N, 0 \leq n \leq N \langle N+p; s \rangle 0).$ 

**S. Reaults . Using** the **above estimates we** are **in position to derive explicit** error bounds for the approximation of the limits  $q_{nm}^-$  defined by (2.9). For conve**nience, we** Introduce **the abbreviations**  *i* (*time the coefficients*  $\lambda_p$  va-<br> *the*  $\rho_N = \rho_{N+p} = \bar{\rho}$  and therefore (4.14) reduces to<br> *s*)  $D_{\bar{\rho}+1}^{m-1} [1/D_{\bar{\rho}+1}^N - 1/D_{\bar{\rho}+1}^{N+p}]$  (4.16)<br>
( $\bar{\rho} < m \le N$ ,  $0 \le n \le N \le N+p$ ;  $s > 0$ ).<br> *tes* we are in posit  $Q_{nm}^{N+p}(s) - Q_{nm}^{N}(s) \leq \frac{1}{s} \delta_n(s) D_{\beta+1}^{m-1} [1/D_{\beta+1}^{N} - 1/D_{\beta+1}^{N+p}]$  (4.16)<br>
( $\bar{\rho} \langle m \leq N, 0 \leq n \leq N \langle N+p; s \rangle 0$ ).<br> **alts** . Using the above estimates we are in position to derive explicit er-<br>
unds for the approx

$$
K_n(s,t) = \frac{1}{s} \delta_n(s)(s + \alpha_n) e^{st}, \quad k_n(t) = \inf_{s > 0} K_n(s,t) \quad (n \ge 0, \ t \ge 0) \; . \tag{5.1}
$$

**Theorem** 1: Assume (2.4). Then for every fixed integer  $N \ge 0$  and  $m, n = 0, 1, ..., N$ 

$$
0 \le q_{nm}(t) - q_{nm}^N(t) \le \min\left\{1 - q_{nm}^N(t), \ A_{nm}^N(t)\right\} \qquad (t \ge 0) \tag{5.2}
$$

*where either*

$$
0 \leq q_{nm}(t) - q_{nm}^{N}(t) \leq \min\left\{1 - q_{nm}^{N}(t), A_{nm}^{N}(t)\right\} \qquad (t \geq 0) , \qquad (5.2)
$$
\n
$$
e^{i\theta t}
$$
\n
$$
A_{nm}^{N} = \begin{cases} \frac{1}{\lambda_0} \alpha_n \delta_n(0) (D_1 - D_1^{N}) & \text{in case } 1 \text{ if } D_1 < \infty \\ \frac{1}{\lambda_0} \left\{K_n(s, t) \frac{\lambda_0 + s D_1^{m-1}}{\lambda_0 + s D_1^{N}}\right\} & \text{in case } 1 \text{ if } D_1 = \infty \\ \text{and in case } 2 \text{ for } N < \rho, \end{cases} (5.3)
$$
\n
$$
case 2 \text{ for } N \geq \rho,
$$

*or, in case 2 for*  $N \ge \rho$ ,

e either  
\n
$$
A_{nm}^N = \begin{cases}\n\frac{1}{\lambda_0} \alpha_n \delta_n(0) (D_1 - D_1^N) & \text{in case 1 if } D_1 \leq \infty, \\
\inf_{\alpha > 0} \left\{ K_n(s, t) \frac{\lambda_0 + sD_1^{m-1}}{\lambda_0 + sD_1^N} \right\} & \text{in case 1 if } D_1 = \infty\n\end{cases}
$$
\n(5.3)  
\n
$$
t \text{ case 2 for } N \geq \rho,
$$
\n
$$
t \text{ case 2 for } N \geq \rho,
$$
\n
$$
A_{nm}^N = \begin{cases}\n0 & \text{for } 0 \leq m \leq \rho_N \ (s \bar{\rho} \leq \infty), \\
k_n(t) D_{\rho_N+1}^{m-1} / D_{\rho_N+1}^N & \text{for } \rho_N \leq m \leq N \leq \bar{\rho} \ (s \infty), \\
k_n(t) D_{\rho+1}^{m-1} / D_{\rho+1}^N & \text{for } \bar{\rho} = \rho_N \leq m \leq N \ \text{if } D_{\bar{\rho}+1} = \infty,\n\end{cases}
$$
\n(5.4)  
\n
$$
k_n(t) D_{\rho+1}^{m-1} [1/D_{\rho+1}^N - 1/D_{\bar{\rho}+1}] \text{ for } \bar{\rho} = \rho_N \leq m \leq N \ \text{if } D_{\bar{\rho}+1} \leq \infty.
$$
\ne, for fixed  $m, n \geq 0$ ,  $q_{nm}^N(t)$  converges uniformly in t on every bounded interval.  
\n
$$
\frac{\alpha_n \delta_n(0)}{\lambda_0} = \begin{cases}\n1 & \text{for } n = 0, \\
(1 + \lambda_n / \mu_n) \prod_{\rho=1}^{m-1} (\lambda_\rho / \mu_\rho) \kappa (1 + \lambda_n / \mu_n) [D_1^{m-1} - D_1^{m-2}) \ \text{for } n \geq 1.\n\end{cases}
$$
\n(5.5)

*Hence, for fixed m,n*  $\geq 0$ ,  $q_{nm}^N(t)$  *converges uniformly in t on every bounded interval.* 

**R.mrk., 1. It is easy to show** that In **case 1** 

**marks:** 1. It is easy to show that in case 1  
\n
$$
\frac{\alpha_n \hat{\zeta}_n(0)}{\lambda_0} = \begin{cases}\n1 & \text{for } n = 0, \\
(1 + \lambda_n / \mu_n) \prod_{\rho=1}^n (\lambda_\rho / \mu_\rho) \kappa (1 + \lambda_n / \mu_n) (D_1^{n-1} - D_1^{n-2}) & \text{for } n \ge 1.\n\end{cases}
$$
\n(5.5)

**2.** In case 2 we have for  $n > p$ 

E. WAGNER

\n**case 2 we have for** 
$$
n \cdot \rho
$$

\n
$$
k_n = (1 + \lambda_n / \mu_n) \sum_{v = \rho_{n-1} + 1}^n \left( \prod_{\rho = v}^{n-1} \lambda_\rho / \mu_\rho \right) = (1 + \lambda_n / \mu_n) \sum_{v = \rho_{n-1} + 1}^n \left( D_v^{n-1} - D_v^{n-2} \right)^{-1}
$$
\n(6.6)

\n**See**  $A_0^{n-1}(0) = 0$  implies  $k_n(t) = K_n(+0, t)$ . **3.** In case 2 there follows directly

\n(1.1) with  $c = \delta$ , that  $a^N$  (t) = 0 for 0 is  $m \leq a$  (n \leq N, t \geq 0). Thus,

**because**  $A_0^{n-1}(0) = 0$  implies  $k_n(t) = K_n(*0,t)$ . **3.** In case 2 there follows directly from (1.1) with  $c_n = \delta_{nm}$  that  $q_{nm}^N(t) = 0$  for  $0 \le m \le p \le n \le N$ ,  $t \ge 0$ . Thus,  $\int_{0}^{1} \frac{1}{\lambda_{p}} \lambda_{p} \mu_{p} = (1 + \lambda_{n} \lambda_{n}) \sum_{v=0}^{n} (D_{v}^{n-1} - D_{v}^{n-2})^{-1}$  (5.6)<br>  $\int_{0}^{1} (t) = K_{n}(\cdot 0, t)$ . S. In case 2 there follows directly<br>  $\int_{0}^{N} f(t) dt = 0$  for  $0 \le m \le p \le n \le N$ ,  $t \ge 0$ . Thus,<br>
for  $0 \le m \le p \le n$ ,

$$
q_{nm}(t) = 0 \quad \text{for} \quad 0 \leq m \leq \rho \leq n, \quad t \geq 0 \tag{5.7}
$$

**4.** The infima in  $(5.3)$  can be approached by setting  $a = 1/t$   $(t > 0)$ , because  $e^{st}/s$  has a minimum at  $s = 1/t$ .

**Proof of Theorem** 1: Obviously, by  $(2.8)$  0  $\leq q_{nm}(t) - q_{nm}^N(t) \leq 1 - q_{nm}^N(t)$ . From (2.10) there follows

1) with 
$$
c_n = \delta_{nm}
$$
 that  $q_{nm}^N(t) = 0$  for  $0 \leq m \leq p < n \leq N$ ,  $t \geq 0$ . Thus,

\n $q_{nm}(t) = 0$  for  $0 \leq m \leq p < n$ ,  $t \geq 0$ . (5.7)

\ninflma in (5.3) can be approached by setting  $s = 1/t$  (t) 0), because

\nas a minimum at  $s = 1/t$ .

\nof Theorem 1: Obviously, by (2.8)  $0 \leq q_{nm}(t) - q_{nm}^N(t) \leq 1 - q_{nm}^N(t)$ .

\nObserve that the following conditions are given by:

\n $Q_{nm}^{N+p}(s) - Q_{nm}^N(s) = \int_0^\infty e^{-s} \left[ q_{nm}^{N+p}(t) - q_{nm}^N(t) \right] dr$  (s) 0.

\nNow, properties of the Laplace transformation, we obtain

Using known properties of the Laplace transformation we obtain

$$
t_{\ell,s}
$$
 has a minimum at  $s = 1/t$ .  
\nProof of Theorem 1: Obviously, by (2.8)  $0 \le q_{nm}(t) - q_{nm}^{N}(t) \le 1 - q_{nm}^{N}(t)$ .  
\nom (2.10) there follows  
\n $Q_{nm}^{N+p}(s) - Q_{nm}^{N}(s) = \int_{0}^{\infty} e^{-st} [q_{nm}^{N+p}(r) - q_{nm}^{N}(r)] dr$  (s) 0). (5.8)  
\nsing known properties of the Laplace transformation we obtain  
\n $s[Q_{nm}^{N+p}(s-\alpha_n) - Q_{nm}^{N}(s-\alpha_n)] = \int_{0}^{\infty} e^{-st} \frac{d}{dt} \{e^{\alpha_n t} [q_{nm}^{N+p}(r) - q_{nm}^{N}(r)]\} d\tau$  (s)  $\alpha_n$ ). (5.9)  
\nnce the integrand on the right-hand side is non-negative (see (2.11)),  
\n $s[Q_{nm}^{N+p}(s-\alpha_n) - Q_{nm}^{N}(s-\alpha_n)] \ge e^{(\alpha_n - s)t} [q_{nm}^{N+p}(t) - q_{nm}^{N}(t)]$  (s)  $\alpha_n, t \ge 0$ ). (5.10)  
\nence  
\n $q_{nm}^{N+p}(t) - q_{nm}^{N}(t) \le (s+\alpha_n) e^{st} [Q_{nm}^{N+p}(s) - Q_{nm}^{N}(s)]$  (s) 0,  $t \ge 0$ ). (5.11)  
\nfollows from (5.8) by using (4.13) and the estimates (4.11), (4.12), (4.15) and

Since the integrand on the right-hand side is non-negative (see  $(2.11)$ ),

$$
s\big[Q_{nm}^{N+p}(s-\alpha_n) - Q_{nm}^N(s-\alpha_n)\big] \geq e^{(\alpha_n - s)t} \big[q_{nm}^{N+p}(t) - q_{nm}^N(t)\big] \quad (s \geq \alpha_n, t \geq 0) \,.
$$
 (5.10)

**Hence**

$$
q_{nm}^{N\ast p}(t) - q_{nm}^N(t) \le (s+\alpha_n) e^{st} \left[Q_{nm}^{N\ast p}(s) - Q_{nm}^N(s)\right] \qquad (s>0, t>0). \tag{5.11}
$$

**It follows from (5.8)** by using (4.13) and the estimates (4.11), (4.12), (4.15) and (4.16) that, for fixed  $m, n \ge 0$ ,  $q_{nm}^N(t) \rightharpoonup q_{nm}(t)$  as  $N \rightharpoonup \infty$ , uniformly in t on every bounded interval. Furthermore, letting  $p \rightarrow \infty$  and moreover, if  $D_1 \leftarrow \infty$  in case 1,  $s \rightarrow 0$ , we obtain the upper bounds  $A_{nm}^N$ .

Now, it is easy to prove that the solutions  $x^N = (x_0^N, x_1^N, ..., x_N^N)^T$  of the finite **initial-value** problems (2.6) converge componentwise to a solution of the infinite initial-value problem (2.1) as well as to determine explicit error bounds, provided the conditions (2.2) - (2.4) are satisfied. We use the abbreviations initial-value problems (2.6) converge componentwise<br>
ie initial-value problem (2.1) as well as to determine<br>
ied the conditions (2.2) - (2.4) are satisfied. We use t<br>  $R_N = R_N(t) = \sum_{m=0}^{N} \left\{ \int_{0}^{t} [f_m(\tau)|d\tau + |c_m| \right\}$ ,  $R$ moreover, if  $D_1 \, \cdot \, \infty$  in<br>  $x_0^N, x_1^N, \ldots, x_N^N$  of the<br>  $\pm \infty$  to a solution of the<br>
explicit error bounds,<br>
the abbreviations<br>  $\lim_{t \to \infty} R_N(t)$ . (5.12)<br>
than N.

$$
R_N = R_N(t) = \sum_{m=0}^{N} \left\{ \int_{0}^{t} f_m(\tau) \, d\tau + |c_m| \right\} , \quad R = R(t) = \lim_{N \to \infty} R_N(t) . \tag{5.12}
$$

Let *M* denote *any* fixed non-negative integer not greater than *N.*

*Theorem* **21** *Assume (2.2)-(2.4). Then the Initial-value problem (2.1) has a*  **Theorem 2:** Assume<br>
solution  $x = \{x_0, x_1, ...$ <br>  $x_n = x_0$ *)T where*  **Error Bounds for the Truncation of Diff.** Systems 415<br> **ssume (2.2) - (2.4). Then the initial-value problem (2.1) has a**<br>  $x_1, ..., Y$ , where<br>  $x_n = x_n(t) = \lim_{N \to \infty} x_n^N(t)$  (n= 0,1,2,...; t > 0), (5.13)<br>
i every bounded interva

$$
x_n = x_n(t) = \lim_{N \to \infty} x_n^N(t) \qquad (n = 0, 1, 2, \dots; t \ge 0), \tag{5.13}
$$

*uniformly In t on every bounded interval, with the following error bounds:* 

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\nTheorem 2: Assume (2.2) - (2.4). Then the initial-value problem (2.1) has a

\nition 
$$
x = \{x_0, x_1, \ldots\}^T
$$
, where

\n
$$
x_n = x_n(t) = \lim_{N \to \infty} x_n^N(t) \quad (n = 0, 1, 2, \ldots; t \ge 0),
$$
\nSimilarly in t on every bounded interval, with the following error bounds:

\n
$$
|x_n(t) - x_n^N(t)|
$$
\n
$$
\begin{cases}\n\min\{1, \frac{1}{\lambda_0} \alpha_n \delta_n(0) (D_1 - D_1^N)\} R_N + R - R_N & \text{in case 1 if } D_1 < \infty, \\
m \min\{1, A_{nM}^N\} R_M + R - R_M & \text{in case 2 for } N < \rho,\n\end{cases}
$$
\n
$$
R - R_N
$$
\n
$$
\begin{cases}\nR - R_{\rho_N} & \text{in case 2 for } \rho \le N \le \bar{\rho} \le \infty, -(5.14) \\
\min\{1, A_{nM}^N\} (R_M - R_{\rho}) + R - R_M & \text{in case 2 for } \rho \le N \le \bar{\rho} \le \infty, -(5.14) \\
\min\{1, A_{nM}^N\} (R_M - R_{\rho}) + R - R_M & \text{in case 2 for } \bar{\rho} \le N \le \bar{\rho} \le \infty, -(5.14) \\
\min\{1, k_n[1 - D_{\rho+1}^N / D_{\rho+1}]\} (R_N - R_{\rho}) + R - R_N & \text{in case 2 for } \bar{\rho} \le N \text{ if } D_{\rho+1} < \infty \\
0 \le n \le N, t \ge 0).\n\end{cases}
$$

**Proof:** Using the error bounds given in (5.3), (5.4) and noting that  $A_{nm}^N(\tau) \leq A_{nm}^N(t)$  $\begin{cases}\n\min\{1, A_{nM}^N\} \left(R_M - R_{\tilde{\rho}}\right) + R - R_M & \text{in case 2 for } \tilde{\rho} \cdot M \leq N \text{ if } D_{\tilde{\rho}+1} = \infty, \\
\min\{1, k_n\{1 - D_{\tilde{\rho}+1}^N / D_{\tilde{\rho}+1}\}\} \left(R_N - R_{\tilde{\rho}}\right) + R - R_N & \text{in case 2 for } \tilde{\rho} \cdot N \text{ if } D_{\tilde{\rho}+1} < \infty \\
& (0 \leq n \leq N, t \geq 0).\n\end{cases}$ <br> **Proo Proof:** Using the error bounds given in (5.3), (5.4) and noting that  $A_{nm}^N(\tau) \le A_{nm}^N(t)$ <br>for  $0 \le \tau \le t$ ,  $m \le M$  and that  $q_{nm}^N(t) \le 1$ , the uniform convergence  $x_n^N \to x_n$  as<br> $N \to \infty$  and the error bounds (5.14) can be for  $0 \le n \le N$ ,  $t \ge 0$ ).<br> **Froof:** Using the error bounds given in (5.3), (5.4) and noting that  $A_{nm}^N(\tau) \le A_{nm}^N(t)$ <br>
for  $0 \le \tau \le t$ ,  $m \le M$  and that  $q_{nm}^N(t) \le 1$ , the uniform convergence  $x_n^N \rightarrow x_n$  as<br>  $N \rightarrow \infty$  and th equation of the system (2.6) we see that the limits  $x_n$  solve the infinite initialvalue problem (2.1).  $\frac{d}{dt} x_n^N(t) \rightarrow \frac{d}{dt} x_n(t)$  as  $N \rightarrow$ <br>ry bounded interval. Thereform<br>if (2.6) we see that the limits x<br>les. 1. If  $c_m \ge 0$ ,  $f_m \ge 0$  ( $m \ge 0$ ,<br> $c_n^N(t) \le x_n^{N+1}(t)$  ( $0 \le n \le N$ ,  $t \ge 0$ ). T<br>5.14) are valid for  $x_n(t) - x_n^N(t)$ <br>

**6. Remarks and examples.** 1. If  $c_m \ge 0$ ,  $f_m \ge 0$   $(m \ge 0, t \ge 0)$ , then from (2.7) and (2.8) there follows  $0 \le x_n^{N}(t) \le x_n^{N+1}(t)$  ( $0 \le n \le N$ ,  $t \ge 0$ ). Therefore,  $x_n^{N}(t) \le x_n(t)$  and the upper bounds in (5.14) are valid for  $x_n(t) - x_n^{\mathcal{N}}(t)$ . Furthermore, by (2.6) and examples. 1. If  $c_m$ <sup>2</sup><br>
follows  $0 \le x_n^{N}(t) \le x_n^{N+1}(t)$ <br>
bounds in (5.14) are valid<br>  $\frac{d}{dt} \left( \sum_{v=0}^N x_v^N \right) =$ <br>
from 0 to t, we obtain<br>  $\sum_{v=0}^N x_v^N(t) \le \sum_{v=0}^N$ <br>
there follows  $x \in I_1$  and  $\parallel$ <br>
it is shown t

$$
\frac{d}{dt}\Big(\sum_{v=0}^{N}x_{v}^{N}\Big) = -\lambda_{N}x_{N}^{N} + \sum_{v=0}^{N}f_{v} \leq \sum_{v=0}^{N}f_{v}.
$$

Integrating from 0 to t, we obtain

$$
\sum_{v=0}^N x_v^N(t) \leq \sum_{v=0}^N \Big\{ c_v + \int_0^t f_v(\tau) d\tau \Big\}.
$$

From this there follows  $x \in I_1$  and  $||x|| \le R$ .

2. In (7) it is shown that the following conditions are sufficient In order that 5. Remarks and examp<br>(2.8) there follows  $0 \le t$ <br>the upper bounds in<br>integrating from 0 to<br>integrating from 0 to<br> $\frac{8}{10}$ <br>2. In (7) it is shown<br>2. In (7) it is shown<br> $\frac{6}{10}$ <br> $q_{nm}(t) = 1$  and the<br>(1.1) with  $c_n = \delta_{nm}$  (  $i \infty$  +  $\int_{0}^{1} f_{v}(t) dt$  }.<br>  $x \parallel \le R$ .<br>
wing conditions are sufficient in order the<br>
... }<sup>T</sup> is the only non-negative solution of (2.8) there follows  $0 \le x_n^N(t) \le x_n^{N+1}$ <br>the upper bounds in (5.14) are  $v_i$ <br> $\frac{d}{dt} \left( \sum_{v=0}^N x_v^N \right)$ <br>Integrating from 0 to t, we obte<br> $\sum_{v=0}^N x_v^N(t) \le \sum_{v=0}^N x_v^N(t) \le$ <br>From this there follows  $x \cdot l_1$  an<br>2. In [7]

(i)  $m \leq \bar{p}$  in case 2,

**(i)**  $m \leq \bar{p}$  in case 2,<br> **(ii)**  $D_m \leq \bar{p}$  in case 2,<br> **In particular, (i) is satisfied for every**  $m \geq 0$  **if**  $\bar{p} = \infty$  **and (ii) is satisfied if**  $D_m$ **.<br>
<b>In particular, (i) is satisfied for every**  $m \geq 0$  **if \bar{p or**  $\sum_{p=1}^{\infty} D_{p+1} / \lambda_p = \infty$  both in case 1 and in case 2 with  $\bar{\rho} \leftarrow \infty$  for  $m \cdot \bar{\rho}$ .<br>
In particular, (i) is satisfied for every  $m \ge 0$  if  $\bar{\rho} = \infty$  and (ii) is satisfied if  $D_m = \infty$ <br>
or  $\sum_{p=m}^{\infty} \frac{1}{p}$ **responding uniqueness have also been proved In (7).** 

**3.** Suppose that  $c_m = f_m = 0$  for  $m > m_0 \ge 0$ ,  $t \ge 0$ . Then  $R = R_m = R_{m_0}$  for  $m \ge m_0$  and  $N \ge \rho_N \ge m_0$ . In other words: If  $m_0$ (ii)  $D_m + \sum_{r=1}^{\infty} D_{\rho+1}/\lambda_{\rho} = \infty$  both in case 1 and in case 2 with  $\bar{\rho} \cdot \infty$  for  $m > \bar{\rho}$ .<br>
In particular, (i) is satisfied for every  $m \ge 0$  if  $\bar{\rho} = \infty$  and (ii) is satisfied if  $D_m = \infty$ <br>
or  $\sum_{\rho=m}^{\infty} \frac$ (2.6) it is immediately seen that  $x_n^N = 0$  for  $\rho^* \rightarrow 1 \le n \le N$ . Hence,  $x_n = 0$  for  $n \ge \rho^* \rightarrow 1$ **and (2.1) can be reduced to a finite initial-value problem.** 

**4.** The upper bounds  $A_{nm}^N$  defined by (5.3) tend to zero as  $N \rightarrow \infty$  of the same order as either  $D_1 - D_1^N$  if  $D_1 \leftarrow \infty$  or  $1/D_1^N$  if  $D_1 = \infty$ . The behaviour of  $D_1^N$  as  $N \rightarrow \infty$  is determined by the ratios  $\mu_{\rho}/\lambda_{\rho}$  for large  $\rho$ . In particular, putting  $\mu_p/\lambda_p = \omega_p$  ( $\rho \ge 1$ ) one can obtain easily from (4.9) that  $D_1 - D_1^N = O(1/(N+1))$ If  $\omega_p = 1/p$ ,  $D_1 - D_1^N = O(c^N)$  if  $\omega_p = c(1, 1/D_1^N = O(1/N)$  if  $\omega_p = 1$ ,  $1/D_1^N = O(c^{-N})$ if  $\omega_o = c \times 1$  and  $1/D_1^N = O(1/N!)$  if  $\omega_o = \rho$ . This shows that the more different the behaviour of  $\lambda_{\rho}$ ,  $\mu_{\rho}$  for large  $\rho$  is the faster the upper bounds  $A_{nm}^N(t)$  for fixed  $n, m, t$  tend to zero as  $N \rightarrow \infty$ . Similar observations can be made in case 2. **c** and the set of  $\lambda_p = \omega_p$  ( $\rho \ge 1$ )<br>  $\omega_p = 1/\rho$ ,  $D_1 - 1$ <br>  $\omega_p = c \cdot 1$  and<br>
behaviour of<br>
d *n*, *m*, *t* tend<br>
ample 1. In the on reactions<br>
where  $\lambda_n = \frac{1}{\rho}$ <br>  $\frac{1}{\rho}$ <br>  $\frac{1}{\rho}$ <br>
( $\frac{1}{\rho}$  ( $\mu/\lambda$ )<br>  $\rightarrow$ <br>
(

**Example 1:** In the theory of kinetics of compartmentalized free-radical polymeri**zation reactions one was led to consider the homogeneous initial-value problem**  if  $\omega_{\rho} = c \times 1$  and  $1/D_1^N = O(1/N!)$  if  $\omega_{\rho} = \rho$ . This shows that the more different<br>the behaviour of  $\lambda_{\rho}$ ,  $\mu_{\rho}$  for large  $\rho$  is the faster the upper bounds  $A_{nm}^N(t)$  for<br>fixed  $n, m, t$  tend to zero as  $N \rightarrow \in$  $=\sum_{n=-\infty}^{N} v!(\mu/\lambda)^{v}$  +  $\infty$ as  $N \rightarrow \infty$ . Similar observations can be made in case 2.<br>
heory of kinetics of compartmentalized free-radical polymericus intervals and to consider the homogeneous initial-value problem<br>  $\mu_n = n\mu$  ( $\mu > 0$ ,  $n \ge 0$ ),  $c_0$ *V =*   $\int_{\sqrt{20}}^{\sqrt{2}} \sqrt{f(\mu/\lambda)^2} \to \infty$  as  $N \to \infty$  (see (4.9)), it has (cf. Remark 2) the unique non-<br>
negative solution  $(q_{00}, q_{10}, ...)^T$ . (Remark that  $q_{n0}$  is the concentration of loci of<br>
reaction system which contain *n* **reaction system which contain** *n* **propagating radicals.) Using (3.S),(3.6),(4.6) ,(S.1) and (5.3) we obtain the upper bounds** solution  $\{q_{00}, q_{10}\}$ <br>system which contains the upper b<br>obtain the upper b<br> $A_{n0}^N = \inf_{s>0} \left\{ \left(1 + \frac{\lambda^2}{s} \right) \right\}$ 

$$
A_{n0}^N = \inf_{s>0} \left\{ \left( 1 + \frac{\lambda + n\mu}{s} \right) \frac{(\lambda/\mu)^n}{n!} e^{st} \frac{\lambda + sD_1^{n-1}}{\lambda + sD_1^N} \right\} = O\left( \frac{(\lambda/\mu)^N}{N!} \right) \qquad (N \to \infty) .
$$

**Remark that the estimates given in (8,9) fail since the assumption**  $\sum_{i=1}^{\infty} 1/\alpha_n < \infty$ **Is violated.** 

**Example 2**. Let  $\lambda_n=1$ ,  $\mu_n=n$ ,  $c_n=1/n!$ ,  $f_n=0$  (n=0,1,2,...; t o). Since  $D_1^n=\sum_{i=0}^n y_i$ , we have case 1 with  $D_1 = \infty$ . Therefore, by (5.14) and in view of Remark 1,  $\lambda * sD_i^{\infty}$  <br>
i fail since the assumption  $\sum_{n=1}^{\infty}$ <br>  $\frac{1}{n^2} = 0$  ( $n = 0, 1, 2, ...$ ;  $t \ge 0$ ). Since  $k$ <br>  $\frac{1}{t}$  (5.14) and in view of Remark<br>  $k = R_M$ .<br>  $\frac{1}{t} = 1$ <br>  $\frac{1}{t}$   $\left(1 + \frac{n+1}{s}\right)\left(1 + \frac{n-1}{s} \frac{1}{\sqrt$ 

$$
0 \le x_n(t) - x_n^N(t) \le \min\{1, A_{nM}^N\} R_M + R - R_M.
$$

where by  $(5.3)$  (cf. Example 1 with  $\lambda \approx \mu = 1$ )

$$
x_{n}(t) - x_{n}^{N}(t) \le \min\{1, A_{nM}^{N}\} R_{M} + R - R_{M}
$$
  
by (5.3) (cf. Example 1 with  $\lambda \propto \mu = 1$ )  

$$
A_{nM}^{N} = \inf_{s>0} \left\{ K_{n}(s,t) \frac{1 + sD_{1}^{M-1}}{1 + sD_{1}^{N}} \right\} = \inf_{s>0} \left\{ \left(1 + \frac{n+1}{s} \right) \left(1 + \frac{n-1}{s} \right) + \frac{s}{n!} \frac{1 + sD_{1}^{M-1}}{1 + sD_{1}^{N}} \right\}
$$

and by (5.12)  $R_M = \sum_{m=0}^{M}$ <br>(i)  $A_{M}^{N}(t) \leq (n+2)^{2}$  $\sum_{m=0}^{M} 1/m! \rightarrow R = e \ (M \rightarrow \infty)$ . Obviously,

(i) 
$$
A_{nM}^N(t) \le (n+2)(1+D_1^{n+1}) \frac{e^t}{n!} (1+D_1^{M-1})/(1+D_1^N)
$$

for  $t \geq 0$  (putting  $s = 1$ )

**,**<br>【 】

$$
A_{nM}(t) \le (n+2)(1+D_1^-) \frac{1}{n!} (1+D_1^-) / (1+D_1^-)
$$
  
for  $t \ge 0$  (putting  $s = 1$ ),  
(ii)  $A_{nM}^N(t) \le [1 + (n+1)t] (t + D_1^{n-1}) \frac{e}{n!} (t + D_1^{M-1}) / (t + D_1^N)$ 

for  $t>0$  (putting  $s=1/t$ ) and

$$
R - R_M = \sum_{m=M+1}^{\infty} 1/m! \leftarrow \frac{M+2}{(M+1)(M+1)!}
$$

*R*<br> *R*  $A_{nM}^{N}(t) \leq (n+2)(1+D_{1}^{n-1})\frac{e^{t}}{n!}(1+D_{1}^{M-1})/(1+D_{1}^{N})$ <br> *r*  $t \geq 0$  (putting  $s = 1$ ).<br> *R*  $A_{nM}^{N}(t) \leq [1+(n+1)t][t+D_{1}^{n-1}]\frac{e}{n!t}(t+D_{1}^{M-1})/(t+D_{1}^{N})$ <br> *r*  $t \geq 0$  (putting  $s = 1/t$ ) and<br> *R*  $-R_{M}$ **for some** *n.* **L. We start from the sufficient condition**   $A_{nM}^N(t) \leq (n+2)$ <br>
(putting  $s=1$ )<br>  $A_{nM}^N(t) \leq [1+(n+1)k]$ <br>
putting  $s=1/t$ )<br>  $R - R_M = \sum_{m=M+1}^{\infty} 1$ <br>  $R - R_M = \sum_{m=M+1}^{\infty} 1$  $(1 + D_1^{T-1}) \frac{e^t}{n!}$ <br>  $+1) t \left[ \left( t + D_1^{T-1} \right) \right]$ <br>
and<br>  $\sqrt{m!} \leftarrow \frac{M+2}{(M+1)!}$ <br>
to find an<br>
from the aufi $\frac{M+2}{(M+1)!}$  < 10<br>
baing  $M = 6$ .

$$
A_{IM}^N(t) = + \frac{M+2}{(M+1)(M+1)!} \cdot 10^{-3} \qquad (0 \le n, M \le N; t \ge 0)
$$

which yields  $\dot{M} \ge 6$ . Choosing  $\dot{M} = 6$ , the inequality is satisfied for e.g.  $n=0$  if (ii)  $A_{nM}^N(t) \leq [1+(n+1)t] (t+l)$ <br>
for  $t>0$  (putting  $s=1/t$ ) and<br>  $R - R_M = \sum_{m=M+1}^{\infty} 1/m! \leq \overline{M}$ <br>
Suppose that we wish to fin<br>
for some *n*, *t*. We start from the<br>  $A_{nM}^N(t) = \frac{M \cdot 2}{(M+1)(M+1)!}$ <br>
which yields  $M \geq 6$ **i**  $0^{-3}$  **b**  $0 \le n, M \le N$ ;  $t \ge 0$ <br> **if**  $D_1^N = \sum_{\nu=0}^N v!$  >  $\left[10^4 e(t+1)(t+154) - t\right] / 2,8$ Since the right-hand side is a monotone increasing function of  $t$   $(t \ge 0)$  we obtain  $N \ge 10$  for  $0 \le t \le 1$ ,  $N \ge 11$  for  $0 \le t \le 10$  and  $N \ge 12$  for  $0 \le t \le 100$ . Choosing  $N = 12$ .  $N \ge 10$  for  $0 \le t$ <br>we have  $A_{R6}^{12}$ or, in view of (ii), if  $D_1^N = \sum_{v=0}^{12} v!$ <br>  $+$  hand side is a monotone increasing f<br>  $\le 1$ ,  $N \ge 11$  for  $0 \le t \le 10$  and  $N \ge 12$  for<br>  $\le 2.8 \cdot 10^{-4}$  for  $n = 1, 2, ..., 12$  if (see (ii))<br>  $\therefore N = \frac{n-11}{2}$ ,  $\le 1.6 \le t \le$ 

 $[1+(n+1)t](t+D_1^{n-1}) \frac{e}{n!t}(t+154)/(t+D_1^{12})$   $\in$  2,8.10<sup>-4</sup>.

For  $n=1$  few computations give the unessentially stronger inequality  $2t^3 \div 311$ we have  $A_{B}^{12} \times 2.8 \cdot 10^{-4}$  for  $n=1, 2, ..., 12$  if (see (ii))<br>  $\left[1 + (n+1)t\right](t + D_1^{n-1}) \frac{e}{n!t}(t + 154)/(t + D_1^{12}) \times 2.8 \cdot 10^{-4}$ .<br>
For  $n=1$  few computations give the unessentially stronger inequality  $2t^3 + 311t^2$ <br>  $-$ **16 lying between both positive zeros of the polynomial on the left-hand side, at least for 0.003**  $\le t \le 103$ . For  $0 \le t \le 0.003$  we use (i) to show that  $A_{16}^{12} \cdot 2.8 \cdot 10^{-4}$ . **FILM 4.53404**  $t \cdot 154 \cdot 0$ , so that  $A_{16}^{12} \cdot 2.8 \cdot 10^{-4}$  and, consequently,  $x_1 - x_1^{12} \cdot 10^{-3}$  for all lying between both positive zeros of the polynomial on the left-hand side at least for  $0.003 \le t \le 103$ . For ceed for  $n=2,3,...,12$ . For example,  $0 \le x_5 - x_5^{12} \cdot 10^{-3}$  holds at least for  $0 \le t \le 945$ .  $[1+(n+1)\epsilon](t+D_1^{n-1}) \frac{a}{n! \epsilon}(t+154)/(t+D_1^{12})$  < 2.8.10<sup>-4</sup>.<br>
For  $n=1$  few computations give the unesaentially stronger inequality  $2t^3 + 311 \epsilon^2$ <br>  $-53404 t + 154 \cdot 0$ , so that  $A_{16}^{12} \cdot 2.8 \cdot 10^{-4}$  and, consequently,

**On the other side, in order to determine an integer** *N* **being as small as possi**start from the inequality  $(1+6t)(t+34)(t+154) \le 0.012t(t+\sum_{v=0}^{N} v!)$  implying (ii) with  $n = 5$ ,  $M = 6$ . If we put  $t = 100$  we obtain  $N \ge 11$ . Choosing  $N = 11$ . in fact the inequality is satisfied at least for  $0.02 \le t \le 208$ . By using (i) it can be shown as above that  $0 \le x_5 - x_5^{11} \le 10^{-3}$  for  $0 \le t \le 0.02$ . too. Remark that the present example has the constant solution  $x_n = 1/n!$   $(n=0,1,...)$ . By numerical computations one can obtain real upper bounds  $\delta_n$  for  $x_n - x_n^{12}(t)$  (0  $\le n \le 12$ ) holding in the intervall **1**  $0 \le t \le 10^3$ . They and the corresponding relative errors  $\delta_n / x_n = \delta_n n!$  are exhibited for some *n* in the following table. **for some n in the following table.** 



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