On a Full Discretization Scheme for a Hypersingular Boundary Integral Equation over Smooth Curves

R. Kieser, B. Kleemann and A. Rathsfeld

In this paper we consider a quadrature method for a hypersingular integral equation arising from a boundary integral formulation for the Neumann problem in a two-dimensional domain with smooth boundary. We prove that this method is equivalent to the trigonometric Galerkin method. Consequently, we obtain stability and asymptotic error estimates in the scale of Sobolev spaces. Finally, we present some numerical tests.

Key words: *Quadrature method, hypersingular integral equation*
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0. Introduction

Suppose the set $\Omega \subset \mathbf{R}^2$ is a bounded and simply connected domain with a smooth boundary $\Gamma := \partial\Omega$, i.e., Γ is a C^∞-curve homeomorphic to the unit circle $\mathbf{T} := \{x \in \mathbf{C} = \mathbf{R}^2 : |x| = 1\}$. Then the Neumann problem of Laplace's equation consists in seeking a function f on Ω such that

$$\Delta f = 0 \ \text{ in } \Omega, \quad \frac{\partial}{\partial\nu}f|_\Gamma = h \ \text{ on } \Omega, \tag{0.1}$$

where ν is the exterior normal to Ω at the points of Γ. Note that (0.1) admits a solution if and only if h satisfies

$$\int_\Gamma h(y)d_y\Gamma = 0. \tag{0.2}$$

We shall solve this problem and determine the boundary values $g = f|_\Gamma$ of f on Γ via a boundary integral method. In order to get a numerical solution of the integral equation, we shall apply the simplest discretization scheme, i.e., the trapezoidal quadrature rule and the corresponding quadrature method.

There are a lot of different boundary integral equations equivalent to the Neumann problem (0.1). The most popular variants lead to Fredholm integral equations of the second kind (cf., e.g., [6, 5, 3]). However, following this procedure, we have to compute integrals including a logarithmic kernel function. These logarithmic integrals appear either on the right-hand side of the integral equation or in the computation of g via the ansatz formula which includes the solution of the integral equation. For the quadrature of these integrals, the trapezoidal rule is not sufficient. A good approximation can be achieved by product integration rules or by some special subtraction methods (cf., e. g., [11, 12]). In order to avoid these troubles we prefer another direct variant of boundary integral equation.

We shall seek f in the form of the representation formula including the unknown function g. We differentiate the representation formula in the direction of the normal ν and consider the boundary values on Γ of both sides of the equation. Using the well-known jump relations, we obtain a first kind integral equation for the unknown function g, where the kernel of the integral operator is hypersingular. Thus the integral is to be understood as a finite part integral (for finite part integrals of boundary integral operators cf. [4]). The right-hand side of the boundary equation contains the adjoint double layer integral with density h. Note that this integral has a smooth kernel. The hypersingular integral operator is a pseudo-differential operator of order one.

We solve the hypersingular boundary equation using a quadrature rule based on the trapezoidal rule. Note that this rule converges with optimal order for smooth periodic functions under the integral. However, in the case of hypersingular functions we have to apply the subtraction technique up to the second term of the Taylor series expansion. In comparison to the method without subtraction a small modification term occurs. Without this term, however, the quadrature method would not converge. In a certain sense, the discretization of the boundary equation via quadrature and subtraction technique yields an optimal approximation. Namely, in case $\Gamma = T$ the method is equivalent to Galerkin's method, where the trial space is spanned by the first n eigenfunctions of the integral operator. The results of this paper are quite similar to those obtained for singular integral equations (cf. [10, 8, 9]).

In the first section we derive the boundary integral equation. We discuss its invertibility in the second section. In the third and fourth section we set up the quadrature method and prove stability as well as convergence. Finally, in the last section, we present numerical results.

1. Derivation of the boundary integral equation

The solution f of (0.1) is given by the well-known representation formula (cf., e.g., [6])

$$f(\tilde{x}) = \frac{1}{2\pi} \int_\Gamma \frac{\partial}{\partial \nu(y)} log|\tilde{x} - y|g(y)d_y\Gamma - \frac{1}{2\pi} \int_\Gamma log|\tilde{x} - y|h(y)d_y\Gamma, \ \tilde{x} \in \Omega, \qquad (1.1)$$

where $g := f|_\Gamma$ is the function we are looking for. We continue the field of normal vectors ν onto a neighbourhood of Γ, differentiate (1.1) in the direction of ν and get

$$\frac{\partial}{\partial \nu(\tilde{x})} f(\tilde{x}) = \frac{1}{2\pi} \int_\Gamma \frac{\partial}{\partial \nu(\tilde{x})} \frac{\partial}{\partial \nu(y)} log|\tilde{x} - y|g(y)d_y\Gamma - \frac{1}{2\pi} \int_\Gamma \frac{\partial}{\partial \nu(\tilde{x})} log|\tilde{x} - y|h(y)d_y\Gamma. \qquad (1.2)$$

Now let $\tilde{x} \in \Omega$ tend to $x \in \Gamma$. The well-known jump relations (cf., e.g., [6, 5, 2, 4, 3]) yield, for any $x \in \Gamma$,

$$h(x) = \frac{1}{2\pi} \int_\Gamma \frac{\partial}{\partial \nu(x)} \frac{\partial}{\partial \nu(y)} log|x - y|g(y)d_y\Gamma - \frac{1}{2\pi} \int_\Gamma \frac{\partial}{\partial \nu(x)} log|x - y|h(y)d_y\Gamma + \frac{1}{2}h(x). \qquad (1.3)$$

In order to substitute the variables in (1.3), we introduce a parametrization $\Gamma = \{\gamma(\tau), 0 \le \tau \le 2\pi\}$ such that $\gamma'(\tau) \ne 0, \gamma(0) = \gamma(2\pi)$ and γ is smooth. Furthermore, we suppose that Ω is on the left-hand side of Γ, i.e.,

$$\nu(\gamma(\tau)) = -i\gamma'(\tau)/|\gamma'(\tau)|. \tag{1.4}$$

We identify the functions g and h on Γ with their pull backs $g \circ \gamma$ and $h \circ \gamma$ on $[0, 2\pi]$ and write $g(\tau) = g(\gamma(\tau))$, $h(\tau) = h(\gamma(\tau))$. Setting $x = \gamma(\tau)$, $y = \gamma(\sigma)$ and multiplying (1.3) by $|\gamma'(\tau)|$, we arrive at

$$\int_0^{2\pi} k(\tau, \sigma) g(\sigma) d\sigma = \frac{1}{2}|\gamma'(\tau)|h(\tau) + \int_0^{2\pi} l(\tau, \sigma) h(\sigma) d\sigma, \tag{1.5}$$

$$k(\tau, \sigma) := \frac{1}{2\pi} \frac{\partial}{\partial \nu(x)} \frac{\partial}{\partial \nu(y)} log|x - y| \, |\gamma'(\sigma)| \, |\gamma'(\tau)|, \tag{1.6}$$

$$l(\tau, \sigma) := \frac{1}{2\pi} \frac{\partial}{\partial \nu(x)} log|x - y| \, |\gamma'(\sigma)| \, |\gamma'(\tau)|. \tag{1.7}$$

Here k takes the form $k = k_1 + k_2$ with

$$k_1(\tau, \sigma) := -\frac{1}{2\pi} \frac{\langle \nu(x), \nu(y) \rangle}{|x - y|^2} |\gamma'(\sigma)| \, |\gamma'(\tau)|, \tag{1.8}$$

$$k_2(\tau, \sigma) := \frac{1}{\pi} \frac{\langle x - y, \nu(x) \rangle \langle x - y, \nu(y) \rangle}{|x - y|^4} |\gamma'(\sigma)| \, |\gamma'(\tau)|. \tag{1.9}$$

The kernel l satisfies

$$l(\tau, \sigma) = \frac{1}{2\pi} \frac{\langle x - y, \nu(x) \rangle}{|x - y|^2} |\gamma'(\sigma)| \, |\gamma'(\tau)|. \tag{1.10}$$

As it is well known, the kernel l is the adjoint double layer kernel, k_2 is the product of l and the double layer kernel and l and k_2 are smooth. The kernel k_1 is hypersingular. Thus the integral on the left-hand side of (1.5) is to be understood as a finite part integral. Note that the substitution $y = \gamma(\sigma)$ in this finite part integral is justified since the integral arises from a boundary integral method for an elliptic partial differential equation (cf. [4]). To make the form of k more precise, we first consider the special case $\Gamma = \mathbf{T}$, $\gamma(\tau) = \gamma_0(\tau) = e^{i\tau}$. For this case, we denote the corresponding kernels k, k_1 and k_2 by $k_0, k_{0,1}$ and $k_{0,2}$, respectively. Taking into account that $\nu(x) = x$, $\nu(y) = y$ and $|\gamma_0'(\tau)| = |\gamma_0'(\sigma)| = 1$, we obtain

$$k_0(\tau, \sigma) = \frac{1}{2\pi} \frac{2\langle x - y, x \rangle \langle x - y, y \rangle - \langle x, y \rangle |x - y|^2}{|x - y|^4}. \tag{1.11}$$

The identity

$$\langle z_1, z_2 \rangle = Re \, z_1 \bar{z}_2 = Re \, (\frac{z_1}{y})(\overline{\frac{z_2}{y}}) = \langle \frac{z_1}{y}, \frac{z_2}{y} \rangle, \; y \in \mathbf{T} \tag{1.12}$$

implies

$$k_0(\tau, \sigma) = \frac{1}{2\pi} \frac{2\langle x/y - 1, x/y \rangle \langle x/y - 1, 1 \rangle - \langle x/y, 1 \rangle |x/y - 1|^2}{|x/y - 1|^4}$$

$$= \frac{1}{2\pi} \frac{2[1 - \langle x/y, 1 \rangle][\langle x/y, 1 \rangle - 1] - \langle x/y, 1 \rangle[1 + 1 - 2\langle x/y, 1 \rangle]}{|x/y - 1|^4}$$

$$
\begin{aligned}
&= \quad \frac{1}{2\pi} \frac{2\langle x/y, 1\rangle - 2}{|x/y - 1|^4} = -\frac{1}{2\pi} \frac{\langle x/y, x/y\rangle + \langle 1, 1\rangle - 2\langle x/y, 1\rangle}{|x/y - 1|^4} \\
&= \quad -\frac{1}{2\pi} \frac{1}{|x/y - 1|^2} = -\frac{1}{2\pi} \frac{1}{|x - y|^2}.
\end{aligned}
\tag{1.13}
$$

For an arbitrary curve Γ, we conclude

$$
k = k_0 + k_A + k_2 - k_{0,2}, \quad k_A := k_1 - k_{0,1}.
\tag{1.14}
$$

As we have mentioned above, the kernel functions k_2 and $k_{0,2}$ are smooth. The formulae (1.4), (1.8) lead to

$$
\begin{aligned}
k_A(\tau, \sigma) &= \quad -\frac{1}{2\pi} \left\{ \frac{\langle \gamma'(\sigma), \gamma'(\tau)\rangle}{|\gamma(\tau) - \gamma(\sigma)|^2} - \frac{\langle \gamma_0'(\sigma), \gamma_0'(\tau)\rangle}{|\gamma_0(\tau) - \gamma_0(\sigma)|^2} \right\} \\
&= \quad -\frac{1}{2\pi} \left\{ \frac{\langle \gamma'(\sigma), \gamma'(\tau)\rangle |\gamma_0(\tau) - \gamma_0(\sigma)|^2 - \langle \gamma_0'(\sigma), \gamma_0'(\tau)\rangle |\gamma(\tau) - \gamma(\sigma)|^2}{|\gamma(\tau) - \gamma(\sigma)|^2 |\gamma_0(\tau) - \gamma_0(\sigma)|^2} \right\}.
\end{aligned}
\tag{1.15}
$$

Obviously, the last function is smooth for $\tau \neq \sigma$. By $O(|\tau - \sigma|^r)$ we denote a function of order $|\tau - \sigma|^r$ for $|\tau - \sigma| \to 0$. Then the Taylor series expansion gives

$$
\begin{aligned}
\gamma'(\sigma) &= \quad \gamma'(\tau) + \gamma''(\tau)(\sigma - \tau) + \frac{\gamma'''(\tau)}{2}(\sigma - \tau)^2 + O(|\tau - \sigma|^3), \\
\gamma(\sigma) &= \quad \gamma(\tau) + \gamma'(\tau)(\sigma - \tau) + \frac{\gamma''(\tau)}{2}(\sigma - \tau)^2 + \frac{\gamma'''(\tau)}{6}(\sigma - \tau)^3 + O(|\tau - \sigma|^4).
\end{aligned}
\tag{1.16}
$$

Substituting this into (1.15), we get

$$
\begin{aligned}
k_A(\tau, \sigma) &= \quad -\frac{1}{2\pi} \frac{\langle \gamma'(\tau), \gamma'(\tau) + \gamma''(\tau)(\sigma - \tau) + \frac{\gamma'''(\tau)}{2}(\sigma - \tau)^2\rangle}{|\gamma'(\tau)|^2 |\gamma_0'(\tau)|^2 |\sigma - \tau|^4 + O(|\tau - \sigma|^5)} \\
&\quad \times \left| \gamma_0'(\tau)(\sigma - \tau) + \frac{\gamma_0''(\tau)}{2}(\sigma - \tau)^2 + \frac{\gamma_0'''(\tau)}{6}(\sigma - \tau)^3 \right|^2 \\
&\quad -\frac{1}{2\pi} \frac{-\langle \gamma_0'(\tau), \gamma_0'(\tau) + \gamma_0''(\tau)(\sigma - \tau) + \frac{\gamma_0'''(\tau)}{2}(\sigma - \tau)^2\rangle}{|\gamma'(\tau)|^2 |\gamma_0'(\tau)|^2 |\sigma - \tau|^4 + O(|\tau - \sigma|^5)} \\
&\quad \times \left| \gamma'(\tau)(\sigma - \tau) + \frac{\gamma''(\tau)}{2}(\sigma - \tau)^2 + \frac{\gamma'''(\tau)}{6}(\sigma - \tau)^3 \right|^2 \\
&\quad -\frac{1}{2\pi} \frac{O(|\tau - \sigma|^5)}{|\gamma'(\tau)|^2 |\gamma_0'(\tau)|^2 |\sigma - \tau|^4 + O(|\tau - \sigma|^5)} \\
&= \quad -\frac{1}{2\pi} \frac{\{\frac{1}{6}\langle \gamma'(\tau), \gamma'''(\tau)\rangle - \frac{1}{4}|\gamma''(\tau)|^2 + \frac{5}{12}|\gamma'(\tau)|^2\}|\tau - \sigma|^4 + O(|\tau - \sigma|^5)}{|\gamma'(\tau)|^2 |\gamma_0'(\tau)|^2 |\sigma - \tau|^4 + O(|\tau - \sigma|^5)}.
\end{aligned}
\tag{1.17}
$$

From this formula and the smoothness of the terms $O(|\tau - \sigma|^5)$ we conclude that the kernel $k_A(\tau, \sigma)$ is smooth for $\tau \to \sigma$, too, and the only non-smooth part of k (cf. (1.14)) is the kernel k_0.

Finally, let us give explicit formulae for the kernels k and l. From (1.10), (1.4), and (1.16), we deduce

$$l(\tau,\sigma) = \begin{cases} -\frac{1}{2\pi}|\gamma'(\sigma)|\dfrac{\langle\gamma(\tau)-\gamma(\sigma),i\gamma'(\tau)\rangle}{|\gamma(\tau)-\gamma(\sigma)|^2} & \text{if} \quad \sigma \neq \tau \\ \frac{1}{4\pi}\dfrac{\langle\gamma''(\tau),i\gamma'(\tau)\rangle}{|\gamma'(\tau)|} & \text{if} \quad \sigma = \tau. \end{cases} \tag{1.18}$$

The kernel k is given by (1.14), where k_0, k_A, k_2 and $k_{0,2}$ take the following forms: From (1.13), we get

$$k_0(\tau,\sigma) = -\frac{1}{4\pi}\frac{1}{1-\cos(\tau-\sigma)} \tag{1.19}$$

for $\tau \neq \sigma$. Using (1.15), (1.17) as well as $\gamma_0(\tau) = e^{i\tau}$, we obtain

$$k_A(\tau,\sigma) = \begin{cases} -\frac{1}{4\pi}\dfrac{2(1-\cos(\tau-\sigma))\langle\gamma'(\sigma),\gamma'(\tau)\rangle - \cos(\tau-\sigma)|\gamma(\tau)-\gamma(\sigma)|^2}{(1-\cos(\tau-\sigma))|\gamma(\tau)-\gamma(\sigma)|^2} & \text{if} \quad \sigma \neq \tau \\ -\frac{1}{2\pi}\dfrac{\frac{1}{6}\langle\gamma'(\tau),\gamma'''(\tau)\rangle - \frac{1}{4}|\gamma''(\tau)|^2 + \frac{5}{12}|\gamma'(\tau)|^2}{|\gamma'(\tau)|^2} & \text{if} \quad \sigma = \tau. \end{cases} \tag{1.20}$$

Formulae (1.9), (1.16), and (1.4) lead to

$$k_2(\tau,\sigma) = \begin{cases} \frac{1}{\pi}\dfrac{\langle\gamma(\tau)-\gamma(\sigma),i\gamma'(\tau)\rangle\langle\gamma(\tau)-\gamma(\sigma),i\gamma'(\sigma)\rangle}{|\gamma(\tau)-\gamma(\sigma)|^4} & \text{if} \quad \sigma \neq \tau \\ -\frac{1}{4\pi}\dfrac{\langle\gamma''(\tau),i\gamma'(\tau)\rangle^2}{|\gamma'(\tau)|^4} & \text{if} \quad \sigma = \tau. \end{cases} \tag{1.21}$$

From this and $\gamma = \gamma_0$ for $\Gamma = \mathbf{T}$, we conclude $k_{0,2}(\tau,\sigma) = -1/4\pi$.

2.   The invertibility of the hypersingular integral operator

Let K stand for the integral operator defined by the left-hand side of (1.5) and K_0 for the corresponding operator K over the special curve $\mathbf{T}$. By H^s ($s \in \mathbf{R}$) we denote the Sobolev space of order s over the periodic interval $[0,2\pi]$. If $\hat{f}_k$ stands for the k-th Fourier coefficient of f, then one norm in H^s is given by

$$\|f\|_{H^s} := \left\{|\hat{f}_0|^2 + \sum_{0\neq k\in\mathbf{Z}} |k|^{2s}|\hat{f}_k|^2\right\}^{1/2}. \tag{2.1}$$

Theorem 2.1 (cf., e.g., [2, 3]):   *For K and K_0, we have the following assertions :*

i)   The operator K acting in L^2 is an unbounded selfadjoint operator.

ii)   The functions $\tau \mapsto e^{ik\tau}, k \in \mathbf{Z}$ are just the eigenfunctions of K_0. The eigenvalue corresponding to $\tau \mapsto e^{ik\tau}$ is equal to $|k|/2$.

iii)   Let $s \in \mathbf{R}$ be arbitrary and consider K as an operator from H^s to H^{s-1}. Then K is Fredholm with index zero. The null space of K is spanned by the constant functions, so the image space of K is orthogonal to the space of constant functions.

Proof: Let us start with ii). Of course, the sequence of functions $\{\tau \mapsto e^{ik\tau}\}$ is an orthogonal basis of H^s. In order to show that $\tau \mapsto e^{ik\tau}$ is an eigenfunction of K_0, let us set $f(x) := x^k$ if $k \geq 0$ and $f(x) := \overline{(x^k)}$ if $k < 0$. Obviously, f satisfies (0.1) with

$$h(x) = \frac{\partial}{\partial \nu(x)} f(x) = \frac{\partial}{|x|}\left\{ f\left(\frac{x}{|x|}\right)|x|^{|k|}\right\} = |k|\left\{ f\left(\frac{x}{|x|}\right)|x|^{|k|-1}\right\}, \; x \in \mathbf{T}. \tag{2.2}$$

Hence, $h(\tau) := h(e^{i\tau}) = |k|e^{ik\tau}$. Furthermore, we have $g(\tau) := f|_\Gamma(e^{i\tau}) = e^{i\tau}$. Formula (1.18) for $\gamma = \gamma_0$ implies $l_0(\tau,\sigma) = 1/4\pi$. Consequently, formula (1.5) with $k = k_0$ and $l = l_0$ yields ii). Moreover, from (2.1) we deduce that $K_0 : H^s \to H^{s-1}$ is a bounded Fredholm operator with index zero.

As we have seen in Section 1, the kernel function k is the sum of k_0 and a smooth function. Hence, K is a compact perturbation of K_0 and the Fredholm property as well as Ind $K = 0$ follows. Assertion i) is a consequence of $k(\tau,\sigma) = \overline{k(\sigma,\tau)}$ (cf. (1.6)). Thus it remains to prove that the null space of K is the set of constant functions. Surely, the constant functions are contained in the null space. Indeed, this follows if we set $f \equiv g \equiv const.$ and $h \equiv 0$ in (1.5). In other words, all what we have to show is that any function of the null space is constant.

Suppose g satisfies $Kg = 0$. Since $K = K_0 + K_R$ with a smoothing operator K_R, we get that $K_0 g = -K_R g$ is a smooth function. From ii) we deduce that g is smooth, too. Let f denote the solution of the Dirichlet problem $\triangle f = 0$ in Ω, $f|_\Gamma = g$ and denote the smooth function $(\partial f/\partial \nu)|_\Gamma$ by h. Then (1.3) and $Kg = 0$ imply

$$\frac{1}{2}h(x) + \frac{1}{2\pi}\int_\Gamma \frac{\partial}{\partial\nu(x)} log|x - y|h(y)d_y\Gamma = 0. \tag{2.3}$$

The integral operator on the right-hand side is a well-known invertible boundary operator (cf. [6, 5]). Thus $h \equiv 0$ and the solution of the Neumann problem $\triangle f = 0$ in Ω, $(\partial f/\partial \nu)|_\Gamma = h$ is constant. The boundary value $g = f|_\Gamma$ is constant, too. ∎

For a numerical computation, we need an operator equation with invertible operator. Therefore, we replace (1.5) by the following modification:

$$\cdot\int_0^{2\pi} k(\tau,\sigma)g(\sigma)d\sigma + \alpha = \frac{1}{2}|\gamma'(\tau)|h(\tau) + \int_0^{2\pi} l(\tau,\sigma)h(\sigma)d\sigma, \quad \int_0^{2\pi} g(\sigma)d\sigma = 0. \tag{2.4}$$

Here $(g,\alpha) \in H^s \bigoplus \mathbf{C}$ is unknown. The operator $\underline{K}$ on the left-hand side of the equations (2.4) is a Fredholm operator with index zero which maps $H^s \bigoplus \mathbf{C}$ into $H^{s-1} \bigoplus \mathbf{C}$. From Theorem 2.1 iii) we conclude that the null space of $\underline{K}$ is trivial. Hence, $\underline{K} : H^s \bigoplus \mathbf{C} \to H^{s-1} \bigoplus \mathbf{C}$ is invertible. Thus, for any right-hand side, there exists exactly one solution of (2.4). Moreover, if (g,α) is the solution of (2.4) and h satisfies (0.2), then $\alpha = 0$ and g is just that solution of (1.5) which satisfies the second equation of (2.4). The general solution of (1.5) is $g + C$, where C is a constant.

Finally, let us introduce some notation. Define $\lambda \in L(H^s, \mathbf{C})$ and $L \in L(H^{s-1})$ by

$$\lambda(g) := \int_0^{2\pi} g(\sigma)d\sigma, \quad Lh(\tau) := \int_0^{2\pi} l(\tau,\sigma)h(\sigma)d\sigma. \tag{2.5}$$

Setting $\underline{g} := (g,\alpha)^T$ and $\underline{h} := (h,0)^T$, the equations (2.4) take the form $\underline{K}\,\underline{g} = \underline{L}\,\underline{h}$, where

$$\underline{K} = \begin{bmatrix} K & Id \\ \lambda & 0 \end{bmatrix}, \quad \underline{L}\,\underline{h} = \begin{bmatrix} \frac{1}{2}|\gamma'|h + Lh \\ 0 \end{bmatrix}. \tag{2.6}$$

3.   The discretization of the integral equation

Suppose n is an even and positive integer. We set $\sigma_j := \sigma_j^{(n)} := 2\pi j/n$. Then the trapezoidal rule for the approximation of an integral is given by

$$\int_0^{2\pi} f(\sigma)d\sigma \sim \frac{2\pi}{n}\sum_{j=0}^{n-1} f(\sigma_j). \tag{3.1}$$

Beside this quadrature, we use also the rule

$$\int_0^{2\pi} f(\sigma)d\sigma \sim \frac{4\pi}{n}\sum_{j\in I(k)} f(\sigma_j),\quad I(k):=\{j\in \mathbf{Z}:\ 0\le j\le n-1,\ j\equiv k+1 \bmod 2\}, \tag{3.2}$$

where $k \in \{0,1,\ldots,n-1\}$ and the mesh width is $4\pi/n$. This quadrature applies if the function f is singular at σ_k. In other words, we shall use (3.1) for the integrals of the functions g, $l(\tau,\cdot)h$ and $k_R(\tau,\cdot)g$, where $k_R := k - k_0 = k_A + k_2 - k_{0,2}$. For the integral of $k_0(\tau,\cdot)g$, we apply (3.2). Note that we could have used (3.2) also for the integrals of $l(\tau,\cdot)h$ and $k_R(\tau,\cdot)g$ in order to avoid the computations of γ'', γ''' (cf. (1.20), (1.21), and (1.18)).

Now the discretization of the integral of $k_0(\sigma_k,\cdot)g$ via (3.2) and subtraction techniques goes as follows: First note that if $g'(\sigma)$ is the derivative of $\sigma \mapsto g(\sigma)$, then $(-i)e^{-i\sigma}g'(\sigma)$ is the derivative of $e^{i\sigma} \mapsto g(\sigma)$. Thus we can write (cf. (1.13))

$$\begin{aligned}
\int_0^{2\pi} k_0(\sigma_k,\sigma)g(\sigma)d\sigma &= -\frac{1}{2\pi}\int_0^{2\pi}\frac{g(\sigma)-g(\sigma_k)-(-i)e^{-i\sigma_k}g'(\sigma_k)(e^{i\sigma}-e^{i\sigma_k})}{|e^{i\sigma}-e^{i\sigma_k}|^2}d\sigma \\
&\quad +g(\sigma_k)(-\frac{1}{2\pi})\int_0^{2\pi}|e^{i\sigma}-e^{i\sigma_k}|^{-2}d\sigma \\
&\quad +(-i)e^{-i\sigma_k}g'(\sigma_k)(-\frac{1}{2\pi})\int_0^{2\pi}(e^{i\sigma}-e^{i\sigma_k})|e^{i\sigma}-e^{i\sigma_k}|^{-2}d\sigma.
\end{aligned} \tag{3.3}$$

From Theorem 2.1 ii) we deduce

$$\begin{aligned}
\int_0^{2\pi} k_0(\sigma_k,\sigma)g(\sigma)d\sigma &= -\frac{1}{2\pi}\int_0^{2\pi}\frac{g(\sigma)-g(\sigma_k)-(-i)e^{-i\sigma_k}g'(\sigma_k)(e^{i\sigma}-e^{i\sigma_k})}{|e^{i\sigma}-e^{i\sigma_k}|^2}d\sigma \\
&\quad +\frac{1}{2}(-i)g'(\sigma_k).
\end{aligned} \tag{3.4}$$

Applying (3.2), we arrive at

$$\int_0^{2\pi} k_0(\sigma_k,\sigma)g(\sigma)d\sigma \sim \frac{1}{2}(-i)g'(\sigma_k)$$

$$+\frac{4\pi}{n}(-\frac{1}{2\pi})\sum_{j\in I(k)}\frac{g(\sigma_j)-g(\sigma_k)-(-i)e^{-i\sigma_k}g'(\sigma_k)(e^{i\sigma_j}-e^{i\sigma_k})}{|e^{i\sigma_j}-e^{i\sigma_k}|^2}$$

$$\sim \quad (-i)g'(\sigma_k)\left\{\frac{1}{2} + \frac{2}{n}\sum_{j\in I(k)}\frac{e^{i(\sigma_j-\sigma_k)}-1}{|e^{i\sigma_j}-e^{i\sigma_k}|^2}\right\}$$

$$+g(\sigma_k)\left\{\frac{2}{n}\sum_{j\in I(k)}\frac{1}{|e^{i\sigma_j}-e^{i\sigma_k}|^2}\right\} - \frac{2}{n}\sum_{j\in I(k)}\frac{g(\sigma_j)}{|e^{i\sigma_j}-e^{i\sigma_k}|^2}. \tag{3.5}$$

We get

$$\left\{\frac{1}{2} + \frac{2}{n}\sum_{j\in I(k)}\frac{e^{i(\sigma_j-\sigma_k)}-1}{|e^{i\sigma_j}-e^{i\sigma_k}|^2}\right\}$$

$$= \frac{1}{2} + \frac{2}{n}\sum_{j\in I(k)}\frac{e^{i\sigma_j}-1}{(e^{i\sigma_j}-1)(e^{-i\sigma_j}-1)} \qquad = \frac{1}{2} + \frac{2}{n}\sum_{j\in I(k)}\frac{1}{(e^{-i\sigma_j}-1)}\frac{e^{-i\sigma_j n/2}-1}{-2}$$

$$= \frac{1}{2} + \frac{2}{n}\left(-\frac{1}{2}\right)\sum_{j\in I(k)}\sum_{k=0}^{n/2-1}e^{-i2\pi jk/n} \qquad = \frac{1}{2} - \frac{1}{n}\sum_{k,l=0}^{n/2-1}e^{-i2\pi(2l+1)k/n}$$

$$= \frac{1}{2} - \frac{1}{2}\sum_{k=0}^{n/2-1}e^{-i2\pi k/n}\frac{1}{n/2}\sum_{l=0}^{n/2-1}e^{-i2\pi lk/(n/2)} \quad = \quad 0$$

$$\tag{3.6}$$

and

$$\left\{\frac{2}{n}\sum_{j\in I(k)}\frac{1}{|e^{i\sigma_j}-e^{i\sigma_k}|^2}\right\}$$

$$= \frac{2}{n}\sum_{j\in I(k)}\frac{1}{(e^{i\sigma_j}-1)(e^{-i\sigma_j}-1)} = \frac{2}{n}\sum_{j\in I(k)}\frac{e^{i\sigma_j n/2}-1}{e^{i\sigma_j}-1}\frac{e^{-i\sigma_j n/2}-1}{e^{-i\sigma_j}-1}\frac{1}{(-2)^2}$$

$$= \frac{1}{2n}\sum_{j\in I(k)}\sum_{l=0}^{n/2-1}e^{i2\pi jl/n}\sum_{r=0}^{n/2-1}e^{-i2\pi jr/n} = \frac{1}{2n}\sum_{r,l=0}^{n/2-1}\sum_{u=0}^{n/2-1}e^{i2\pi(2u+1)(l-r)/n} \tag{3.7}$$

$$= \frac{1}{4}\sum_{l=0}^{n/2-1}\sum_{r=0}^{n/2-1}e^{i2\pi(l-r)/n}\frac{1}{n/2}\sum_{u=0}^{n/2-1}e^{i2\pi u(l-r)/(n/2)} = \frac{1}{4}\sum_{l=0}^{n/2-1}1 = \frac{n}{8}.$$

Substituting (3.6), (3.7) into (3.5) leads to

$$\int_0^{2\pi}k_0(\sigma_k,\sigma)g(\sigma)d\sigma \sim \frac{n}{8}g(\sigma_k) - \frac{2}{n}\sum_{j\in I(k)}\frac{g(\sigma_j)}{|e^{i\sigma_j}-e^{i\sigma_k}|^2}. \tag{3.8}$$

This is the usual (3.2) quadrature rule supplied with the simple correction term $ng(\sigma_k)/8$. Now let $g_n(\sigma_j)$ and α_n stand for the approximate values of $g(\sigma_j)$ and α, respectively. These values are to be determined by the following linear system of equations which is obtained by approximating the integrals in (2.4) with the help of the quadrature rules (3.1) and (3.8):

$$\frac{n}{8}g_n(\sigma_k) - \frac{1}{n}\sum_{j\in I(k)}\frac{g_n(\sigma_j)}{1-\cos(\sigma_j-\sigma_k)} + \frac{2\pi}{n}\sum_{j=0}^{n-1}k_R(\sigma_k,\sigma_j)g_n(\sigma_j) + \alpha_n$$

$$= \frac{1}{2}|\gamma'(\sigma_k)|h(\sigma_k) + \frac{2\pi}{n}\sum_{j=0}^{n-1}l(\sigma_k,\sigma_j)h(\sigma_j), \quad k=0,1,\ldots,n-1,$$

$$\frac{2\pi}{n} \sum_{j=0}^{n-1} g_n(\sigma_j) \quad = \quad 0.$$ (3.9)

Note that if we multiply the second equation of (3.9) by $n/2\pi$, then we obtain a system of equations with a symmetric matrix. This matrix is even positive definite in the case of the unit circle **T**.

4. The stability and convergence of the quadrature method

The main point in the convergence analysis of method (3.9) is to show that the latter is equivalent to the Galerkin method with trigonometric trial functions. Thus let us denote the space of trigonometric-polynomials $\text{span}\{e^{ik\tau} : k = -n/2, -n/2+1, \ldots, n/2 - 1\}$ by T_n. Let R_n stand for the interpolation projection onto T_n, i.e., R_n is given by

$$R_n f := \sum_{j=0}^{n-1} f(\sigma_j)\psi_j, \qquad \psi_j(\tau) := \psi_j^{(n)}(\tau) := \frac{1}{n} \sum_{k=-n/2}^{n/2-1} e^{-i\sigma_j k} e^{ik\tau}.$$ (4.1)

Now we identify the set of approximate values $\{g_n(\sigma_k) : k = 0, \ldots, n-1\}$ with the interpolant $g_n := \sum_{j=0}^{n-1} g_n(\sigma_j)\psi_j$. Taking into account this identification, the system of equations (3.9) can be considered as an operator equation in the space $T_n \bigoplus \mathbf{C}$. We denote the operators on the left-hand and right-hand side of (3.9) by $\underline{K_n}$ and $\underline{L_n}$, respectively, and set $\underline{g_n} := (g_n, \alpha_n)^T$, $\underline{h_n} := (h_n, 0)^T$ as well as $h_n := R_n h$. In other words, (3.9) is equivalent with $\underline{K_n}\,\underline{g_n} = \underline{L_n}\,\underline{h_n}$, where we get

$$\underline{K_n} = \begin{bmatrix} K_n & Id \\ \lambda_n & 0 \end{bmatrix}, \quad \underline{L_n}\,\underline{h_n} = \begin{bmatrix} \frac{1}{2}R_n|\gamma'|h_n + L_n h_n \\ 0 \end{bmatrix}$$ (4.2)

analogously to (2.6). We call the method (3.9) *stable* if the operators $\underline{K_n} \in L(T_n \bigoplus \mathbf{C})$ are invertible for n large enough and if the norms of their inverse operators are uniformly bounded with respect to n. Here the norm of $\underline{K_n}^{-1} \in L(T_n \bigoplus \mathbf{C})$ is the operator norm induced by the norm of $L(H^{s-1} \bigoplus \mathbf{C}, H^s \bigoplus \mathbf{C})$. Note that the stability implies that the condition numbers of the matrices $\underline{K_n}$ are of order n. Finally, let us denote the operators $\underline{K}$, $\underline{K_n}$ and K_n for the special choice $\Gamma = \mathbf{T}$ by $\underline{K_0}$, $\underline{K_{0,n}}$ and $K_{0,n}$, respectively. Note that K_0 maps T_n into T_n by Theorem 2.1 ii).

Theorem 4.1 : *For $\underline{K_0}$ and $\underline{K_{0,n}}$, we get :*
i) The approximate operator $\underline{K_{0,n}}$ is just the restriction of $\underline{K_0}$ to the space $T_n \bigoplus \mathbf{C}$.
ii) The method (3.9) is stable for any $s \in \mathbf{R}$.
iii) Let $r > 3/2$, $r \geq s \geq 1$ and suppose $h \in H^{r-1}$. If $\underline{g_n} = (g_n, \alpha_n)^T$ is the solution of (3.9) and $\underline{g} = (g, \alpha)^T$ the exact solution of (2.4), then $g_n \rightarrow g$ ($n \rightarrow \infty$) in H^r and $\alpha_n \rightarrow \alpha$ ($n \rightarrow \infty$). Furthermore, we get

$$\|g_n - g\|_{H^s} \leq Cn^{s-r}\|h\|_{H^{r-1}}, \quad |\alpha_n - \alpha| \leq Cn^{s-r}\|h\|_{H^{r-1}},$$ (4.3)

where the constant C is independent of h and n.

Proof: Assertion i) follows if we prove that all the quadrature rules used in the discretization of (2.4) are exact. Obviously, (3.1) is exact for the functions from $\mathrm{span}\{e^{ik\tau} : k = -n+1, -n+2, \ldots, n-1\}$ and (3.2) for functions from $\mathrm{span}\{e^{ik\tau} : k = -n/2+1, -n/2+2, \ldots, n/2-1\}$. Hence, λ_n is the restriction of λ to T_n and it remains to show that there arises no error when we discretize (3.4). Obviously, $\tilde{g}(\sigma) := g(\sigma) - g(\sigma_k) - (-i)e^{-i\sigma_k}g'(\sigma_k)(e^{i\sigma} - e^{i\sigma_k})$ belongs to T_n and $\tilde{g}(\sigma_k) = (-i)e^{-i\sigma_k}\tilde{g}'(\sigma_k) = 0$. Consequently, $\tilde{g}$ takes the form $(e^{i\sigma} - e^{i\sigma_k})^2 \sum_{j=0}^{n-2} \eta_j e^{i(j-n/2)\sigma}$. We get

$$
\begin{aligned}
\frac{\tilde{g}(\sigma)}{|e^{i\sigma} - e^{i\sigma_k}|^2} &= \sum_{j=0}^{n-2} \eta_j e^{i(j-n/2)\sigma}(e^{i\sigma} - e^{i\sigma_k})^2 (e^{i\sigma} - e^{i\sigma_k})^{-2}(-1)e^{i\sigma}e^{i\sigma_k} \\
&= \sum_{j=0}^{n-2} \{-\eta_j e^{i\sigma_k}\} e^{i(j+1-n/2)\sigma},
\end{aligned}
$$

and the quadrature (3.2) is exact for $f(\sigma) = \tilde{g}(\sigma)|e^{i\sigma} - e^{i\sigma_k}|^{-2}$.

Now we observe that $\underline{K}_n = \underline{K}_{0,n} + \underline{K}_{R,n}$, where $\underline{K}_{R,n} := \underline{K}_n - \underline{K}_{0,n}$ is the usual discretization of a smoothing operator. The main part $\underline{K}_{0,n}$ of $\underline{K}_n$ is the restriction of $\underline{K}_0$ to the space $T_n \oplus \mathbf{C}$. Hence, $\underline{K}_{0,n}$ is equal to the approximate operator of the trigonometric Galerkin or collocation method applied to $\underline{K}_0$. The main part of the operator is discretized by collocation and the smoothing part by quadrature. Thus the assertions ii) and iii) follow from Theorem 4.3 in [1] (or cf. the Chapters 1, 3, and 10 in [8]). ∎

Corollary 4.1 : *Suppose all the assumptions of Theorem 4.1 are satisfied and $\epsilon > 0$. Then we get*

$$
\sup_{k=0,1,\ldots,n-1} |g_n(\sigma_k) - g(\sigma_k)| \leq C n^{1/2+\epsilon-r}\|h\|_{H^{r-1}}. \tag{4.4}
$$

Finally, let us note that the $1/2$ appearing in the last equation is due to the Sobolev embedding theorem. In order to get better estimates in the supremum norm one has to replace the Sobolev spaces by Hölder-Zygmund spaces (cf. [7]).

	$\rho = 1(\Gamma_1 = \mathbf{T})$		$\rho = 2$		$\rho = 10$	
n	err_n	ord_n	err_n	ord_n	err_n	ord_n
4	$4.69 \cdot 10^{-1}$		$5.15 \cdot 10^{-1}$			
8	$5.11 \cdot 10^{-2}$	-3.20	$2.47 \cdot 10^{-1}$	-1.06		
16	$7.51 \cdot 10^{-4}$	-6.09	$2.89 \cdot 10^{-2}$	-3.98	$8.68 \cdot 10^{0}$	-0.08
32	$8.87 \cdot 10^{-7}$	-9.72	$1.24 \cdot 10^{-4}$	-6.98	$1.86 \cdot 10^{0}$	-2.22
64	$3.57 \cdot 10^{-12}$	-17.9	$3.21 \cdot 10^{-8}$	-11.91	$5.99 \cdot 10^{-2}$	-4.96
128			$2.56 \cdot 10^{-14}$	-20.26	$2.81 \cdot 10^{-4}$	-7.73
256					$1.41 \cdot 10^{-7}$	-10.96
512					$1.13 \cdot 10^{-11}$	-13.60

Table 1: f of (5.1), different ρ of $\Gamma_\rho, \rho = 1, 2$ and 10.

n	$\delta = 3/2$		$\delta = 5/2$		$\delta = 7/2$	
	err_n	ord_n	err_n	ord_n	err_n	ord_n
4	$3.75 \cdot 10^{-1}$		$7.33 \cdot 10^{0}$		$4.50 \cdot 10^{0}$	
8	$1.92 \cdot 10^{-1}$	-0.97	$5.02 \cdot 10^{-2}$	-3.87	$5.35 \cdot 10^{-2}$	-6.39
16	$8.88 \cdot 10^{-2}$	-1.11	$8.71 \cdot 10^{-3}$	-2.53	$2.46 \cdot 10^{-3}$	-4.44
32	$3.91 \cdot 10^{-2}$	-1.18	$1.72 \cdot 10^{-3}$	-2.34	$1.96 \cdot 10^{-4}$	-3.64
64	$1.67 \cdot 10^{-2}$	-1.23	$3.48 \cdot 10^{-2}$	-2.30	$1.82 \cdot 10^{-5}$	-3.43
128	$6.91 \cdot 10^{-3}$	-1.27	$7.05 \cdot 10^{-5}$	-2.30	$1.76 \cdot 10^{-6}$	-3.37
256	$2.81 \cdot 10^{-3}$	-1.30	$1.42 \cdot 10^{-5}$	-2.31	$1.73 \cdot 10^{-7}$	-3.35
512	$1.12 \cdot 10^{-3}$	-1.32	$2.81 \cdot 10^{-6}$	-2.33	$1.70 \cdot 10^{-8}$	-3.35

Table 2: f_δ of (5.4), $\delta = 3/2, 5/2$ and $7/2$.

5. Numerical results

As a first example we consider the ellipse $\Gamma_\rho := \{\gamma_\rho(\tau) := (\cos \tau, \rho \sin \tau) : 0 \le \tau \le 2\pi\}$ and the harmonic function

$$f(x,y) := \log\{(x-2)^2 + y^2\} - 4\pi(\log 2)\left\{\int_{\Gamma_\rho} |dz|\right\}^{-1}. \tag{5.1}$$

For the corresponding Neumann data $h := \frac{\partial}{\partial \nu} f$ over Γ_ρ, we have computed $g := f|_{\Gamma_\rho}$ using the quadrature method (3.9). In Table 1 the error

$$err_n := \sup_{k=0,1,\dots,n-1} |g(\sigma_k) - g_n(\sigma_k)| \tag{5.2}$$

as well as the convergence order

$$ord_n := \frac{\log(e_n) - \log(e_{n/2})}{\log(n) - \log(n/2)} \tag{5.3}$$

is given for $\rho = 1$, $\rho = 2$ and $\rho = 10$, respectively. All computations are made with a maximum accuracy of $2 \cdot 10^{-16}$. Note that a minimum value of ord_n is obtained when the error err_n is near to $2 \cdot 10^{-16}$. For n higher than this critical value, ord_n increases again. However, we only list the computations, where ord_n is decreasing. In accordance with (4.3), the order of convergence tends to infinity.

Finally, we consider Γ_1 and the analytic function

$$f_\delta(x,y) := \{(x-1)^2 + y^2\}^{\delta/2} \cos\left[\delta \arctan \frac{-y}{1-x}\right] - 1 \tag{5.4}$$

with $g := f|_{\Gamma_1} \in H^s$ for $s < \delta + 1/2$. For $\delta := 3/2, 5/2, 7/2$, the errors and orders of convergence are given in Table 2. Again, the orders tend to the values δ appearing in (4.4).

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B. Kleemann and Dr. A. Rathsfeld   Dr. R. Kieser
Institut für Angewandte Analysis   Universität Stuttgart
und Stochastik   Mathematisches Institut A
Hausvogteiplatz 4-5   Pfaffenwaldring 57
D-O 1086 Berlin   D-W 7000 Stuttgart 80