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The SILP–Relaxation Method in Optimal Control: General Boundary Conditions II

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In the first part of this paper the measure–theoretical approach to classical control problems, based on ideas of Young in variational calculus and developed by Rubio for control problems, was slightly extended by choosing a semi-infinite approach instead of a finite one. This results in a lower bound for the global minimum and an approximation for the optimal solution. It was still an open question, whether Rubio's Approximation Theorem holds in the semi-infinite case and for more general boundary conditions. The second part of the paper deals with the discussion of the approximation properties and gives as an example the numerical treatment of a nice geometric extremal problem by Focke.

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1. The Control Problem

The control problem we shall study is of the following type:
Minimize the integral

$$\int_0^T r(t,x,u)\,dt \tag{1.1}$$

with respect to the state equation

$$\dot{x} = g(t,x,u), \tag{1.2a}$$

with coupled boundary conditions

$$x(T) = Cx(0) \tag{1.2b}$$

and control resp. state constraints

$$u(t) \in U, \quad x(t) \in X. \tag{1.3}$$

The following analytical assumptions about data and solution functions of the control problem (1.1)–(1.3) have to be fulfilled:

r and g are Lipschitz over $[0,T] \times X \times U$
C is an regular (n,n) matrix
$x(\cdot) = [x_1(\cdot),\ldots,x_n(\cdot)]^\top$ is an n–vector of absolutely continuous state functions
$u(\cdot) = [u_1(\cdot),\ldots,u_m(\cdot)]^\top$ is an m–vector of bounded measurable control functions.

The state equation (1.2a) is to be satisfied almost everywhere over $[0,T]$ in the sense of Carathéodory.

Problem (1.1)–(1.3) is assumed to be consistent, that is, there exists at least one admissible pair (admissible process) (x,u), which satisfies all the constraints in (1.1)–(1.3).
We will use all the notations from Part I of the paper.

2. The Relaxed Problem

We have to study the so-called *relaxed problem* to problem (1.1)–(1.3) as derived in Section 2 of Part I:

$$\langle r, \mu_p \rangle \longrightarrow \min_{\mu_p : p \in P} \tag{2.1}$$

$$\text{(R)} \qquad \langle \varphi^g, \mu_p \rangle - \Delta\varphi = 0, \qquad \varphi \in C^1(J) \tag{2.2}$$

$$\mu_p \geq 0, \tag{2.3}$$

where $\varphi = \varphi(t, \xi) : J \longrightarrow R$ are the testing functions from the variational description.

In fact one has to mention that problem (R) in our case of coupled boundary values is not a "pure" linear program (LP) over the measure space $M(Z)$, the dual to $C(Z)$, because the difference terms $\Delta\varphi = \varphi(T, x(T)) - \varphi(0, x(0)) = \varphi(T, Cx(0)) - \varphi(0, x(0))$ contain the unknown initial values $x(0) =: \xi_0$ as additional variables. Therefore problem (R) is a "mixture" of a LP over $M(Z)$ and a nonlinear programming problem with respect to the variable $\xi_0 \in R^n$; this fact causes some changes in the numerical method to solve problem (R), but not any difficulty from the theoretical point of view.

Proposition 1. *Let the analytical assumptions from Section 1 to be fulfilled. Then problem (R) has an optimal solution.*

Proof. Problem (R) can be considered as an abstract nonlinear programming problem over the space of variables $M(Z) \times R^n$, in which the feasible region is weak* compact due to the compactness assumptions about X and U. The objective is a continuous linear functional, which therefore is also weak* continuous, so that from the generalized WEIERSTRASS Theorem (compare [5]) the existence of an optimal pair $(\mu^*, \xi_0^*) \in M(Z) \times R^n$ follows ∎

The question, whether RUBIO's Approximation Theorem (compare Part I) in our case holds, can also be answered.

Proposition 2. *Let (μ^*, ξ_0^*) be an optimal solution of problem (R). Then there exists a sequence of pairs $p^j = (x^j, u^j)$ with*

$$\dot{x}^j = g(t, x^j, u^j), \quad x^j(0) = \xi_0^*, \quad \lim_{j \to \infty} x^j(T) = C\xi_0^*,$$

$$u^j(t) \in U, \quad \lim_{j \to \infty} \rho(x^j(t), X) = 0, \quad t \in [0, T]$$

such that

$$\lim_{j \to \infty} \int_0^T r(t, x^j, u^j)\, dt = Min(\mathrm{R}) \leq Inf(\mathrm{P}),$$

i.e. $p^j = (x^j, u^j)$ is an almost feasible (asymptotic feasible) global minimizing sequence for problem (1.1)-(1.3).

Proof. We consider problem (1.1)-(1.3), but with fixed boundary values, namely $x(0) = \xi_0^*$, $x(T) = C\xi_0^*$, where ξ_0^* is the R^n-component of the optimal solution of problem (R). If we now repeat the relaxation procedure, we obtain as the relaxed problem to the classical control problem with fixed boundary values our problem (R); the $\Delta\varphi_i$ in (2.2) now has to be computed for the given boundary values. Therefore RUBIO's Theorem is applicable and this completes the proof ∎

3. A Geometric Extremal Problem with Orbiforms

We want to illustrate the numerical method by an old geometric problem in convex geometry, namely the determination of a so–called n–orbiform with minimal area. FOCKE [3] has treated the symmetric case and KLÖTZLER [6] gave a proof for optimality of FOCKE's symmetric orbiforms even in the general nonsymmetric case.

In a geometric language the problem can be stated as follows: Let be given a regular n–polygon $\mathfrak{P}_n$. An n–orbiform $\mathfrak{O}_n$ is an inscribed convex curve, which can be rotated in $\mathfrak{P}_n$, such that all edges of $\mathfrak{P}_n$ are tangent to $\mathfrak{O}_n$. Find an n–orbiform with minimal area !

The problem has been treated in the case $n = 4$ by LEBESGUE [7] and BLASCHKE [1], and for $n = 3$ by FUJIWARA and KAKEYA [4]. FOCKE [3] has given the following analytical formulation of the problem in the general case:

Let us denote by $h(\varphi)$ the support function and by $\rho(\varphi)$ the curvature radius of the orbiform $\mathfrak{O}_n$ with respect to the polar angle φ, $0 \le \varphi \le 2\pi$. The connection between $h(\varphi)$ and $\rho(\varphi)$ is given by the differential equation

$$\ddot{h} + h = \rho, \tag{3.1}$$

and a closed convex curve will be described if and only if

$$\rho(\varphi) \ge 0 \quad \text{for all } \varphi, \qquad h(0) = h(2\pi), \quad \dot{h}(0) = \dot{h}(2\pi). \tag{3.2}$$

Here the periodicity for h and $\dot{h}$ describes the closedness of the curve. If $\delta = \frac{2\pi}{n}$ is the interior angle of $\mathfrak{P}_n$, then for the curvature radius there holds the difference equation

$$\rho(\varphi - \delta) + \rho(\varphi + \delta) - 2\rho(\varphi)\cos\delta = s\sin\delta, \tag{3.3}$$

where s is the length of the edges of $\mathfrak{P}_n$. With $\varphi = j\delta + t, \quad 0 \le t \le \delta, \quad j = 0, 1, \ldots, n-1$ and $\rho_j(t) := \rho(j\delta + t) = \rho(\varphi)$ we obtain the solution of the difference equation (3.1) as

$$\rho_j(t) = r + u_1(t)\cos j\delta + u_2(t)\sin j\delta \tag{3.4}$$

with two arbitrary functions u_1 and u_2, where r denotes the radius of the interior circle of $\mathfrak{P}_n$. Then the equivalence

$$\rho_j(t) \ge 0 \qquad \text{if and only if} \qquad u(t) = [u_1(t), u_2(t)]^T \in \mathfrak{P}_n \tag{3.5}$$

holds. The area of $\mathfrak{O}_n$ is given by the formula

$$F = \frac{1}{2}\int_0^{2\pi} [h^2(\varphi) - \dot{h}^2(\varphi)]\, d\varphi. \tag{3.6}$$

The conditions (3.1)–(3.5) are the side conditions for feasible orbiforms, and the objective of the optimization problem, which has to be minimized, is given by (3.6). Following FOCKE [3] we introduce a complex variable

$$w(t) = u_1(t) + iu_2(t). \tag{3.7}$$

Then the equation (3.4) transforms into

$$\rho_j(t) = \mathrm{Re}\,(w(t)e^{-ij\delta}) + r \tag{3.8}$$

and because for $h(\varphi) = h(j\delta + t) = h_j(t)$ an analogous relation

$$h_j(t) = U(t)\cos j\delta + V(t)\sin j\delta + r = \mathrm{Re}\,(W(t)e^{-ij\delta}) + r \tag{3.9}$$

holds, the relations (3.1), (3.2) and (3.5) can be transformed into

$$W''(t) + W(t) = w(t) \in \mathfrak{P}_n \quad , \tag{3.10}$$

$$W(\delta) = W(0)e^{-i\delta}, \quad W'(\delta) = W'(0)e^{-i\delta}. \tag{3.11}$$

Finally the area formula (3.6) in this terms can be written as

$$F = \frac{1}{4n}\int_0^\delta [|W|^2 - |\dot{W}|^2]\,dt + \pi r^2. \tag{3.12}$$

By introducing state variables $x_1(t), \ldots, x_4(t)$ and setting

$$W(t) = x_1(t) + ix_2(t), \qquad W'(t) = x_3(t) + ix_4(t) \tag{3.13}$$

and using $u_1(t)$, $u_2(t)$ from (3.7) as controls, we get from (3.10)–(3.12) the optimal control problem

$$F = F(x,u) = \frac{1}{4n}\int_0^\delta [(x_1^2 - x_3^2) + (x_2^2 - x_4^2)]\,dt + \pi r^2 \longrightarrow \text{Min!} \tag{3.14}$$

with respect to the state equations

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= -x_1 + u_1 \\ \dot{x}_4 &= -x_2 + u_2, \end{aligned} \tag{3.15}$$

boundary conditions

$$\begin{aligned} x_1(\delta) &= x_1(0)\cos\delta + x_2(0)\sin\delta \\ x_2(\delta) &= -x_1(0)\sin\delta + x_2(0)\cos\delta \\ x_3(\delta) &= x_3(0)\cos\delta + x_4(0)\sin\delta \\ x_4(\delta) &= -x_3(0)\sin\delta + x_4(0)\cos\delta, \end{aligned} \tag{3.16}$$

and control constraints

$$u(t) = [u_1(t), u_2(t)]^T \in \mathfrak{P}_n, \qquad t \in [0,\delta]. \tag{3.17}$$

The boundary conditions (3.16) are in the coupled form (1.2b).

Remark. Problem (3.14)-(3.17) does not contain any state constraint, such that the compactness assumption from Section 1 does not hold automatically; nevertheless there could be introduced state constraints by the geometrical nature of the state variables as values of the support function resp. its derivative, such that the theory is applicable.

4. The Numerical Model

We now shall describe the numerical model, which corresponds to the semi-infinite problem $(\mathrm{R})_M$ from Part I, Section 3. For this purpose we choose a finite set of functions $\{\varphi_i : i = 1, \ldots, M\}$, which we divide into three groups:

$\{\varphi_i : i \in I_\varphi\}$ as monomials in $t, \xi_1, \ldots, \xi_4$, corresponding to the boundary conditions (3.16);

$\{\theta_j : j \in J_\theta\}$ as trigonometric polynomials in t with coefficients in $\xi_1, \ldots, \xi_4$, corresponding to the state equations (3.15) (compare part (ii) from the proof of the Theorem in Part I, Section 2);

$\{\chi_k : k \in K_\chi\}$ as indicator functions of subintervals Δ_k of the time interval; this group of side conditions reflects the uniform distribution of the measure μ with respect to t.

Let $z = (t, x_1, \ldots, x_4, u_1, u_2) \in Z$, and let $G = \{z^l : l = 1, \ldots, N\} \subset Z$ be a grid, $\mu = \sum_{l=1}^{N} \alpha_l \varepsilon_{z^l}$ a discrete measure. The measure μ is feasible in $(\mathrm{R})_M$ if and only if

$$\sum_{l=1}^{N} \alpha_l \varphi_i^g(z^l) = \Delta\varphi_i, \qquad i \in I_\varphi \tag{4.1}$$

$$\sum_{l=1}^{N} \alpha_l \theta_j(z^l) = 0, \qquad j \in J_\theta \tag{4.2}$$

$$\sum_{l=1}^{N} \alpha_l \chi_k(z^l) \geq \tau\delta_k, \qquad k \in K_\chi \tag{4.3}$$

$$\alpha_l \geq 0, \qquad l = 1, \ldots, N. \tag{4.4}$$

where $\delta_k = \operatorname{mes} \Delta_k$ and $\tau \approx 0.5 \ldots 0.9$ is some relaxation parameter, which improves the consistency of the discretized problem $(\mathrm{R})_M$ for a given grid. For getting a starting solution for the semi-infinite simplex method (SISM) we use the finite simplex method for solving (4.1)–(4.4) with the objective

$$\sum_{l=1}^{N} \alpha_l r(z^l) \longrightarrow \text{Min!} \tag{4.5}$$

The resulting discrete measure $\bar{\mu} = \sum_{l=1}^{M} \bar{\alpha}_l \varepsilon_{\bar{z}^l}$ is of type (3.2) from Part I, Section 3, and by means of SISM there will be generated a sequence $\{\bar{\mu}^{(n)}\}$ of basic feasible solutions of the semi-infinite problem $(\mathrm{R})_M$ with the same discrete structure, but changing support points $\{\bar{z}^l : l = 1, \ldots, M\}$. The algorithm stops, if the difference between primal objective value $\langle r, \bar{\mu}^{(n)} \rangle$ and a computed dual bound S_M (compare Part I, Section 3, formula (3.3)) is less than a given Epsilon.

Results [2]: We give some numerical results for the case $n = 4, r = 1$. Here we have $\delta = \frac{\pi}{2}$ and the control problem is

$$F = \int_0^{\frac{\pi}{2}} [(x_1^2 - x_3^2) + (x_2^2 - x_4^2)]\, dt + \pi \longrightarrow \text{Min!}$$

$$\begin{array}{rclcrcl}
\dot{x}_1 &=& x_3 & \qquad & x_1(\frac{\pi}{2}) &=& x_2(0) \\
\dot{x}_2 &=& x_4 & & x_2(\frac{\pi}{2}) &=& -x_1(0) \\
\dot{x}_3 &=& -x_1 + u_1 & & x_3(\frac{\pi}{2}) &=& x_4(0) \\
\dot{x}_4 &=& -x_2 + u_2, & & x_4(\frac{\pi}{2}) &=& -x_3(0),
\end{array}$$

$$|u_1(t)| \leq 1, \qquad |u_2(t)| \leq 1, \qquad t \in [0, \frac{\pi}{2}].$$

The optimal solution is already known; it is the so-called Reuleaux triangle, which can be constructed from a regular triangle by drawing circular arcs from each vertex through the opposite vertices. The radius of each arc is $\bar{r} = 2$, the length of an edge of the square. The area of the Reuleaux triangle is $F = 2(\pi - \sqrt{3}) = 2.819084\ldots$. The functions φ_i in the numerical model are: $\varphi_1 = t$, $\varphi_{1+i} = x_i$, $\varphi_{5+i} = tx_i$, $\varphi_{9+i} = tx_i^2$ $(i = 1,\ldots,4)$. The θ_j are of type $\theta(t,x) = x_i\psi_k(t)$ with $\psi_k(t) = \sin(4kt)$ resp. $\psi_k(t) = 1 - \cos(4kt)\,(k = 1,\ldots,4)$, $i = 1,\ldots,4$, such that for $t = 0$ and $t = \frac{\pi}{2}$ the functions $\psi_j(t)$ are vanishing. Table 1 shows some numerical results where N is the number of grid points, M is the number of equations, τ is a relaxation parameter, *Tol* is the maximal tolerance between the endpoints of the computed trajectory $\bar{x}$ and the (known) optimal solution, and *Bound* denotes a dual bound.

Table 1.

N	M	τ	*Tol*	$F(\bar{x},\bar{u})$	*Bound*
448	22	0.5	0.007	2.818996	2.797925
...	..	0.8	0.010	2.826115	2.756907
576	28	0.8	0.010	2.802072	2.771784
...	..	0.9	0.010	2.802741	2.785253
960	34	0.5	0.017	2,819092	2.755042
...	..	0.8	0.016	2.819883	2.787095

The pairs $\bar{p} = (\bar{x},\bar{u})$ corresponding to Proposition 2 are almost feasible; their objective values are near to the optimal value. The best computed bound $S_M = 2.797925$ gives some additional information, available without knowledge about the optimal trajectory: there cannot be an admissible process with a better objective than S_M.

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