Zeitschrift für Analysis
und ihre Anwendungen
Vol. 11 (1992) 4, 455–488

A Study on the Geometry of Pairs of Positive Linear Forms, Algebraic Transition Probability and Geometrical Phase over Non-Commutative Operator Algebras (II)

PETER M. ALBERTI

The results of the first part [1] will be used to discuss and to investigate some extensions of geometrical notions, which recently have been found to be of interest in Mathematical Physics in context of the problems of the so-called geometrical phase. The concepts of the global phase, the phase group and holonomy group of a normal state of a vN-algebra will be introduced and discussed.

Key words: *Functional analysis, C^*-algebras, vN-algebras, non-commutative probability, non-commutative geometry*

AMS subject classifications: 46 L 05, 46 L 10, 46 L 50

8. Loops, global phases and the holonomy group (discrete case)

With this section the second part of the paper starts. The first part [1], consists of Sections 0 to 7 and the Appendix of technical tools. In all what follows it will be tacitly understood that by quotations in the text like Section 3, Theorem 5.1/(iv), (6.10) or Appendix 4 on refers to [1: Section 3], [1: Theorem 5.1/(iv)], [1: Formula (6.10)], and [1: Appendix 4], respectively, provided no other references have been mentioned explicitely.

Throughout this section we suppose that $A = M$ is a vN-algebra acting over some Hilbert space H, and that the positive linear forms considered are normal ones. For convenience let us also adopt all the suppositions and notations from the Sections 4 and 5 and the Appendix. Especially M is thought to act in standard form on H, with a cyclic and separating vector Ω and associated to $\{M, \Omega\}$ a natural positive cone P_Ω.

Let us consider finite sequences of normal positive linear forms over M of the following type. If $n \in \mathbb{N} \cup \{0\}$ and $\gamma = \{\omega_j : j = 0, 1, 2, \ldots, n\}$ is the sequence in question, then in case $n \neq 0$ it is required that, for any $\lambda \in \mathbb{R}_+$, $\omega_j \neq \lambda\omega_{j-1}$ and $\{\omega_j, \omega_{j-1}\}$ is «-minimal for any $j = 1, 2, \ldots, n$. A sequence γ of this specification will be referred to as a *path* (within M_{*+}). The case $n = 0$ is referred to as a *trivial path*. Thus, each element of M_{*+} can also be considered as constituting a trivial path. For a path γ (with $n \neq 0$) also the notation $\gamma\colon \omega_0 \to \omega_1 \to \ldots \to \omega_n$ will be in use. In this situation, the linear form ω_0 is referred to as the *initial form* of γ whereas ω_n is said to be the *final form* of γ. Let $\omega, \sigma \in M_{*+}$. In case that there exists a path γ with initial form ω and final form σ, this will be notified by $\omega \approx \sigma$, and γ in this situation is referred to as a *path connecting ω and σ.* Note that by

$\approx$ an equivalence relation on M_{*+} is established. In fact, reflexivity follows from our definition of a trivial path, and symmetry is a consequence of the fact that for a path γ: $\omega = \omega_0 \longrightarrow \omega_1 \longrightarrow \ldots \longrightarrow \omega_n = \sigma$ connecting ω and σ we have the inverse path γ^{-1}: $\sigma_0 \longrightarrow \sigma_1 \longrightarrow \ldots \longrightarrow \sigma_n$, with $\sigma_k = \omega_{n-k}$, for any $k = 0,1,\ldots,n$. That γ^{-1} is a path in our sense follows from the fact that if $\{\omega_j, \omega_{j-1}\}$ is $\ll$-minimal, then $\{\omega_{j-1}, \omega_j\}$ is $\ll$-minimal, too. Transitivity arises from the fact that paths γ: $\omega = \omega_0 \longrightarrow \omega_1 \longrightarrow \ldots \longrightarrow \omega_n = \sigma$ and γ': $\sigma = \sigma_0 \longrightarrow \sigma_1 \longrightarrow \ldots \longrightarrow \sigma_m = \mu$ can be joined together resulting in another path $\gamma'' = \gamma\gamma'$: $\omega = \omega_0 \longrightarrow \omega_1 \longrightarrow \ldots \longrightarrow \omega_{n+m} = \mu$, with $\omega_{n+j} = \sigma_j$ for $j = 0,1,\ldots,m$. The $\approx$-equivalence class of a normal positive linear form ω will be called the *ω-component*, for short. A path γ in M_{*+} connecting ω and σ is said to be an *ω-loop* or a *closed path at ω* if $\omega = \sigma$. Note that the only path containing the linear form 0 is the trivial path at 0. This follows since for a path γ: $\omega_0 \longrightarrow \omega_1 \longrightarrow \ldots \longrightarrow \omega_n$ such that $\omega_j = 0$ for some $j \in \{0,1,\ldots,n\}$, by the definition of the term *path* also the next neighbouring forms in the sequence had to vanish. This is so because $\{0, \nu\}$ is $\ll$-minimal if and only if $\nu = 0$. In the sequel this trivial case will be excluded from all considerations and the term *path* should be understood tacitely as connecting *non-vanishing* forms. We further note that for a path γ: $\omega_0 \longrightarrow \omega_1 \longrightarrow \ldots \longrightarrow \omega_n$ and given reals $\lambda_0,\ldots,\lambda_n \in \mathbb{R}_+\backslash\{0\}$ also γ': $\lambda_0\omega_0 \longrightarrow \lambda_1\omega_1 \longrightarrow \ldots \longrightarrow \lambda_n\omega_n$ is a path. This follows since for a non-trivial $\ll$-minimal pair $\{\omega, \sigma\}$ and given $\lambda, \mu \in \mathbb{R}_+\backslash\{0\}$ the pair $\{\lambda\omega, \mu\sigma\}$ is non-trivial and $\ll$-minimal, too, and the arising forms are positive multiples of each other if and only if this is the case for the original forms. Especially, if $\lambda_j = \omega_j(e)^{-1}$ is chosen for all j, we will arrive at a path γ' in the *normal state space* $S_0(M)$ of M. In all what follows it will be sufficient to consider only paths in $S_0(M)$. The results and effects which will be proved and discussed in case of paths in $S_0(M)$ will then persist to hold for arbitrary paths in M_{*+}.

Let ν be a normal state on M, and let $S(\nu)$ be the set of all vectors of H such that $\varphi \in S(\nu)$ implies $\nu(x) = \langle x\varphi, \varphi\rangle$, for all $x \in M$. By our suppositions $S(\nu)$ is a non-void and uniformly closed subset of the unit sphere of H. Thus $S(\nu)$ is a complete metric space when considered under the metric $d(\varphi, \psi) = \|\varphi - \psi\|$. Now, the structure of $S(\nu)$ will be analyzed from a more algebraic-geometrical point of view.

To start with, let us look on a path γ in $S_0(M)$, with initial state ω and final state σ. We are going to associate with γ a one-to-one mapping Φ_γ from $S(\omega)$ *onto* $S(\sigma)$. The map Φ_γ is constructed as follows. In case $\omega = \sigma$ and if γ is the trivial path we define $\Phi_\gamma = id$, where id denotes the identity map in $S(\omega)$. In the non-trivial case let γ be given as γ: $\omega = \omega_0 \longrightarrow \omega_1 \longrightarrow \ldots \longrightarrow \omega_n = \sigma$ and suppose that $\varphi \in S(\omega)$. Since $\{\omega, \omega_1\}$ is $\ll$-minimal, according to Theorem 5.1/(iv) there is a unique vector $\varphi_1 \in S(\omega_1)$ with $\varphi_1 \parallel \varphi$. By the same argument applied to the $\ll$-minimal pair $\{\omega_1, \omega_2\}$ and the vector $\varphi_1 \in S(\omega_1)$ we provide ourselves with a vector $\varphi_2 \in S(\omega_2)$, with $\varphi_2 \parallel \varphi_1$. This vector is uniquely determined by its *predecessor* $\varphi_1 \in S(\omega_1)$. Proceeding further in this way we finally arrive at a finite sequence $\varphi, \varphi_1, \varphi_2, \ldots, \varphi_n$ of vectors which are constructed according to the rule that from $\varphi_{j-1} \in S(\omega_{j-1})$ we get a *successor* $\varphi_j \in S(\omega_j)$ as the unique vector in $S(\omega_j)$ obeying $\varphi_j \parallel \varphi_{j-1}$. By our definition of the term *path* and by Theorem 5.1/(iv) this successive construction procedure works well. We define $\Phi_\gamma(\varphi) = \varphi_n$. For the image of $\varphi \in S(\omega)$ under the map Φ_γ the abbreviation $\varphi(\gamma)$ will be in use. About Φ_γ we get the following

Proposition 8.1: *For any path γ connecting ω and σ within $S_0(M)$ the map Φ_γ is a homeomorphism between $S(\omega)$ and $S(\sigma)$. For the inverse path $\gamma' = \gamma^{-1}$ one has $\Phi_{\gamma'}$.*

$=\Phi_\gamma^{-1}$. *Suppose* γ'' *is another path within* $S_0(M)$, *with initial state* σ *and final state* ν. *Then for the composite path* $\delta = \gamma\gamma''$ *of* γ *and* γ'' *connecting* ω *and* ν *one has* $\Phi_\delta = \Phi_{\gamma''}\Phi_\gamma$. *Suppose* $\varphi \in S(\omega)$ *and* $w \in M'$ *is a partial isometry with* $w^* w = p'(\varphi)$. *Then for any path* γ *connecting* ω *and* σ *within* $S_0(M)$ *the map* Φ_γ *fulfils* $\Phi_\gamma(w\varphi) = w\Phi_\gamma(\varphi)$.

Proof: For $\omega = \sigma$ and γ being the trivial path the assertions are valid. Suppose γ is a non-trivial path given as above, i.e. γ: $\omega = \omega_0 \longrightarrow \omega_1 \longrightarrow \ldots \longrightarrow \omega_n = \sigma$, for some $n \in \mathbb{N}$. First of all we remark that Φ_γ is injective due to the symmetry of $\parallel$ (cf. Remark 5.2/(1)). In fact, assume that $\Phi_\gamma(\varphi) = \Phi_\gamma(\psi)$ for vectors $\varphi, \psi \in S(\omega)$. Let $\varphi, \varphi_1, \varphi_2, \ldots, \varphi_n$ and $\psi, \psi_1, \psi_2, \ldots, \psi_n$ be the uniquely determined sequences of vectors with φ_j, $\psi_j \in S(\omega_j)$ constructed according to the above defining procedure for Φ_γ starting out of φ and ψ, respectively. By our assumption we have $\varphi_n = \psi_n$. Since $\varphi_n \parallel \varphi_{n-1}$ and $\psi_n \parallel \psi_{n-1}$, by symmetry and uniqueness (cf. Theorem 5.1/(iv)) for $\parallel$ we get $\varphi_{n-1} = \psi_{n-1}$. Now, this conclusion can be applied repeatedly and finally gives $\varphi = \psi$. This proves injectivity.

Let $\psi \in S(\sigma)$ be given and look on the inverse path $\gamma' = \gamma^{-1}$. Then $\psi(\gamma') \in S(\omega)$. Suppose that $\psi = \psi_0, \psi_1, \psi_2, \ldots, \psi_n = \psi(\gamma')$ is the sequence of vectors $\psi_j \in S(\omega_{n-j})$ constructed according to the defining procedure for $\Phi_{\gamma'}$ starting out of ψ. Then $\psi_j \parallel \psi_{j-1}$, for any j. By symmetry of $\parallel$ we have $\psi_j \parallel \psi_{j-1}$, for any j. In defining $\varphi_j = \psi_{n-j}$, for all j, we get a sequence $\varphi_0 = \psi(\gamma'), \varphi_1, \ldots, \varphi_n = \psi$ of vectors with $\varphi_j \in S(\omega_j)$ and $\varphi_j \parallel \varphi_{j-1}$, for any j, and the sequence in question belongs to the path γ. Hence, $\psi = \Phi_\gamma(\psi(\gamma')) = \Phi_\gamma(\Phi_{\gamma'}(\psi))$. Since $\psi \in S(\sigma)$ could have been chosen at will, $id = \Phi_\gamma \Phi_{\gamma'}$ on $S(\sigma)$ has to be followed. It is now evident that Φ_γ has to be surjective.

Analogously, one obtains $id = \Phi_\gamma \Phi_{\gamma'}$ on $S(\omega)$, with $\gamma' = \gamma^{-1}$. These conclusions can be drawn for each path γ within $S_0(M)$, with ω being the initial state of γ and σ being the final state of γ. Thus, $\Phi_{\gamma'}$ with $\gamma' = \gamma^{-1}$ has to be the inverse of Φ_γ.

From this also follows that, for a proof Φ_γ to be a homeomorphism, we may content with showing that Φ_γ is continuous for any path γ. We will do this. Let γ: $\omega = \omega_0 \longrightarrow \omega_1 \longrightarrow \ldots \longrightarrow \omega_n = \sigma$ be a path, and suppose that $\varphi = \varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_n$ is a sequence with $\varphi_j \in S(\omega_j)$ and $\varphi_j \parallel \varphi_{j-1}$ for all j. Assume that $\varphi^k = \varphi_0^k, \varphi_1^k, \varphi_2^k, \ldots, \varphi_n^k$, for $k \in \mathbb{N}$, are sequences with $\varphi_j^k \in S(\omega_j)$ and $\varphi_j^k \parallel \varphi_{j-1}^k$ for all j. Suppose that $\lim_k \varphi^k = \varphi$ in $S(\omega)$. What we have to prove is that also $\lim_k \varphi_n^k = \varphi_n$. The latter is certainly valid if we succeed in proving that for a $\ll$-minimal pair $\{\nu, \mu\}$ of normal states and given sequences $\{\xi^k\}$ and $\{\eta^k\}$, with $\xi^k \in S(\nu)$, $\eta^k \in S(\mu)$ and $\eta^k \parallel \xi^k$ for all $k \in \mathbb{N}$, from $\lim_k \xi^k = \xi$ and $\eta \in S(\mu)$ with $\eta \parallel \xi$ always follows $\lim_k \eta^k = \eta$. We are going to derive the last mentioned. Let $w_k \in M'$ be the partial isometry with $w_k^* w_k = p'(\xi)$ and $\xi^k = w_k \xi$. For any $x \in M$ we then see that $\lim_k w_k x\xi = \lim_k x w_k \xi = x(\lim_k w_k \xi) = x(\lim_k \xi^k) = x\xi = p'(\xi)x\xi$. Since $w_k p'(\xi) = w_k$, and because $\{x\xi : x \in M\}$ is uniformly dense in $p'(\xi)H$ and $\{w_k\}$ is a uniformly bounded (by one) sequence in M', st-$\lim_k w_k = p'(\xi)$ follows. Therefore we also have $\lim_k w_k \eta = p'(\xi)\eta$. By the assumption $\eta \parallel \xi$ and according to the definition of $\parallel$ (cf. (5.1)) we have $p'(\xi) = p'(\eta)$. Hence $\lim_k w_k \eta = \eta$. Note that $w_k \eta \in S(\mu)$. But then, by Theorem 5.1/ (iii) from $\eta \parallel \xi$ it follows that $w_k \eta \parallel w_k \xi$, i.e. $w_k \eta \parallel \xi^k$. From our assumption $\eta^k \parallel \xi^k$ and the uniqueness statement, Theorem 5.1/(iv), we then get $\eta^k = w_k \eta$, and thus $\lim_k \eta^k = \lim_k w_k \eta = \eta$ is seen. By our preliminary considerations this result then yields continuity of Φ_γ when applied successively to the pairs $\varphi_j^k \parallel \varphi_{j-1}^k$, $k \in \mathbb{N}$, and $\varphi_j \parallel \varphi_{j-1}$, for all j, i.e. in this situation $\lim_k \Phi_\gamma(\varphi^k) = \lim_k \varphi_n^k = \varphi_n = \Phi_\gamma(\varphi)$.

Finally, the validity of the last parts of the assertion follows from the construction procedure for the homeomorphisms in question. This construction for a given path γ guarantees that the defining sequence of vectors in the procedure is uniquely determined by the initial vector. As yet mentioned, this is a consequence of Theorem 5.1/(iii) and (iv). But then, by the construction two such sequences of vectors with the property that the final vector of the first is the initial vector of the second can be glued together and yield the unique defining sequence of the composite path, with initial vector of the first and ending with the final vector of the second sequence. This proves the composition property.

To see the last assertion, if $w \in M'$ is a partial isometry with $w^* w = p'(\varphi)$ and if $\varphi = \varphi_0, \varphi_1, \varphi_2, \dots, \varphi_n$ is the unique sequence with initial vector φ and $\varphi_j \in S(\omega_j)$ and $\varphi_j \| \varphi_{j-1}$, for all j, for the path γ: $\omega = \omega_0 \longrightarrow \omega_1 \longrightarrow \dots \longrightarrow \omega_n = \sigma$ in question, by the above reasoning $w\varphi, w\varphi_1, w\varphi_2, \dots, w\varphi_n$ has to be the uniquely determined sequence $w\varphi = \psi_0, \psi_1, \psi_2, \dots, \psi_n$ with $\psi_j \in S(\omega_j)$ and $\psi_j \| \psi_{j-1}$, for all j, which starts from $w\varphi$. Hence, $\Phi_\gamma(w\varphi) = w\Phi_\gamma(\varphi)$ ∎

Let us now suppose that γ is a loop at some state $\omega \in S_0(M)$. Then Φ_γ is an element of the group of homeomorphisms Γ_ω of the metric space $S(\omega)$. Since any two loops at ω can be joined together resulting in another loop at ω, in view to Proposition 8.1 we may conclude that $G_0(\omega) = \{ \Phi \in \Gamma_\omega : \exists\ \omega\text{-loop } \gamma \text{ with } \Phi = \Phi_\gamma \}$ is a subgroup of Γ_ω. In the sequel the group $G_0(\omega)$ will be referred to as the (*restricted*) *holonomy group* of the state ω. Note that we can also introduce an equivalence relation $\sim_\omega$ in the set of all ω-loops by the requirement that $\gamma \sim_\omega \gamma'$ if and only if $\varphi(\gamma) = \varphi(\gamma')$ for any $\varphi \in S(\omega)$. The set of equivalence classes $[\gamma]$ of ω-loops with respect to $\sim_\omega$ in a natural way is a group if we define $[\gamma][\gamma'] = [\gamma\gamma']$. Note that this group by Proposition 8.1 proves to be anti-isomorphic with $G_0(\omega)$.

Remark: The notions and notations introduced and used in this section seem to suggest some analogy with well-known and important notions and notations of geometry and topology. This is not at all an accident. As an example, $\|$ could be interpreted as giving a connection in the manifold $S_0(M)$(or in the unit sphere in H) and could be used for giving a law of parallel transport for some objects interpreted as *tangent vectors* etc. These interpretations of course are worth being discussed separetely. But such analogies and their consequences require to be handled with some care. We have to do there with dimensionally *infinite* and *non-commutative* phenomena which are intrinsically more complicated, both referring to C^*-algebraic as well as to topological and differential-geometrical aspects and reflecting their (attractive and desirable) common occurrence in one and the same context. For the case of finite-dimensional algebras (at least in the factor case) the problems are less difficult and the transport problem has been discussed by A. Uhlmann in [6] and [5] in terms of a naturally associated manifold. For some interesting differential-geometric problems related to this cf. also [3, 4]. In this paper we want to avoid to enter into a more principal and formal discussion of the global questions around non-commutative geometry. Instead we mainly want to elaborate on our approach of stating facts on the effects essentially caused by non-commutativity of the underlying mathematical objects and categories.

Suppose now $\Phi \in G_0(\omega)$ and fix a vector $\varphi \in S(\omega)$. Let M'_φ be the vN-algebra $M'_\varphi = p'(\varphi)M'p'(\varphi)$, and let $U(M'_\varphi)$ be the *unitary* group of M'_φ. Since $p'(\Phi(\varphi)) = p'(\varphi)$ and also $\Phi(\varphi) \in S(\omega)$ is fulfilled, there has to exist a unique unitary $w_\varphi(\Phi) \in U(M'_\varphi)$ such that $\Phi(\varphi) = w_\varphi(\Phi)\varphi$. We can associate such a unitary $w_\varphi(\Phi)$ to any vector $\varphi \in S(\omega)$ and any $\Phi \in G_0(\omega)$. According to the last result of Proposition 8.1, for any $v \in M'$ with

$v^* v = p'(\varphi)$ one has $v\Phi(\varphi) = \Phi(v\varphi) = w_{v\varphi}(\Phi)v\varphi$. Thus, we find the transformation law

$$w_\varphi(\Phi) = v^* w_{v\varphi}(\Phi) v \tag{8.1}$$

for any $\varphi \in S(\omega)$, $v \in M'$ with $v^* v = p'(\varphi)$, and all $\Phi \in G_O(\omega)$. Assume Φ' is another element of $G_O(\omega)$. Let $\Phi'' = \Phi\ \Phi'$. Then, inserting $v = w_\varphi(\Phi')$ into (8.1) we can conclude as follows :

$$\begin{aligned} w_\varphi(\Phi'')\varphi &= \Phi''(\varphi) = \big(\Phi\ \Phi'\big)(\varphi) = \Phi(w_\varphi(\Phi')\varphi) = w_{v\varphi}(\Phi)v\varphi \\ &= vv^* w_{v\varphi}(\Phi)v\varphi = v w_\varphi(\Phi)\varphi = w_\varphi(\Phi')w_\varphi(\Phi)\varphi . \end{aligned}$$

From this $w_\varphi(\Phi\Phi') = w_\varphi(\Phi'') = w_\varphi(\Phi')w_\varphi(\Phi)$ is obtained. Obviously, $w_\varphi(id) = p'(\varphi)$, and $p'(\varphi)$ is the unit of the group $U(M'_\varphi)$. Assume that $w_\varphi(\Phi) = p'(\varphi)$ for some $\Phi \in G_O(\omega)$. By definition of w_φ the latter means $\Phi(\varphi) = \varphi$. Argueing by means of the last part of Proposition 8.1 we get $\Phi(v\varphi) = v\varphi$, for any $v \in M'$ with $v^* v = p'(\varphi)$. For each fixed $\varphi \in S(\omega)$ we have

$$S(\omega) = \big\{\psi \in H:\ \psi = v\varphi \text{ for some } v \in M' \text{ with } v^* v = p'(\varphi)\big\}.$$

Hence, $\Phi(\psi) = \psi$ holds for any $\psi \in S(\omega)$, i.e. $\Phi(\varphi) = \varphi$ implies $\Phi = id$. We take together these steps and arrive at the following result.

Lemma 8.2: *The map w_φ: $G_O(\omega) \ni \Phi \mapsto w_\varphi(\Phi) \in U(M'_\varphi)$ provides a group anti-isomorphism from the (restricted) holonomy group $G_O(\omega)$ of the state ω onto some subgroup $U^O_\varphi(\omega) = \operatorname{im} w_\varphi$ of the unitary group $U(M'_\varphi)$.*

The unitary group $U^O_\varphi(\omega)$ is referred to as the (*restricted*) ω-*phase-group* at $\varphi \in S(\omega)$. By (8.1) we see that $U^O_\varphi(\omega)$ and $U^O_\psi(\omega)$ are mutually isomorphic for any two vectors $\varphi, \psi \in S(\omega)$. For an ω-loop γ in $S_O(M)$ the element $w_\varphi(\Phi_\gamma)$ is called *phase* of the ω-loop γ (to the initial vector φ). Note that the group of $\sim_\omega$-equivalence classes of ω-loops is isomorphic to $U^O_\varphi(\omega)$ for any $\varphi \in S(\omega)$. This follows from Proposition 8.1 and Lemma 8.2 since the map $[\gamma] \mapsto w_\varphi(\Phi_\gamma)$ can be composed of the two maps $[\gamma] \mapsto \Phi_\gamma$ and $\Phi_\gamma \mapsto w_\varphi(\Phi_\gamma)$ which both are anti-isomophisms from the group of $\sim_\omega$-equivalence classes of ω-loops onto $G_O(\omega)$ and from $G_O(\omega)$ onto $U^O_\varphi(\omega)$, respectively.

In the next step we want to establish the connections between relative phases as introduced in Section 6 and the elements of the ω-phase-group. Assume that γ: $\omega = \omega_0 \longrightarrow \omega_1 \longrightarrow \ldots \longrightarrow \omega_n = \omega$ is an ω-loop, and suppose that $\psi_k \in S(\omega_k)$, for $k = 0, 1, 2, \ldots, n-1$, are given vectors. Let $\varphi \in S(\omega)$, and suppose that $\psi_0 = \varphi$. We are going to construct from the given sequence $\psi_0, \psi_1, \ldots, \psi_n = \varphi$ the uniquely determined sequence $\varphi_0 = \varphi$, $\varphi_1, \ldots, \varphi_n$ of vectors with $\varphi_k \in S(\omega_k)$ and $\varphi_{k+1} \| \varphi_k$, for all k. By Proposition 6.12 we get $\varphi_1 = \delta(\varphi, \psi_1)\psi_1$ and, for $k > 1$, $\varphi_k = \delta(\varphi_{k-1}, \psi_k)\psi_k$. This is a recursive system which can be solved by means of successive application of Lemma 6.13 as follows:

$$\begin{aligned} \varphi_k = \delta(\varphi_{k-1}, \psi_k)\psi_k &= \delta(\,\delta(\varphi_{k-2}, \psi_{k-1})\psi_{k-2}\,, \psi_k)\psi_k \\ &= \delta(\varphi_{k-2}, \psi_{k-1})\,\delta(\psi_{k-2}\,, \psi_k)\psi_k \\ &= \delta(\,\delta(\varphi_{k-3}, \psi_{k-2})\psi_{k-2}\,, \psi_{k-1})\ \delta(\psi_{k-2}\,, \psi_k)\psi_k \end{aligned}$$

$$= \delta(\varphi_{k-3},\psi_{k-2})\,\delta(\psi_{k-2},\psi_{k-1})\,\delta(\psi_{k-2},\psi_k)\psi_k$$
$$\vdots$$
$$= \delta(\psi_0,\psi_1)\,\delta(\psi_1,\psi_2)\cdots\delta(\psi_{k-1},\psi_k)\psi_k .$$

The last member, i.e. for $k = n$, yields $\varphi(\gamma)$. Hence,

$$\varphi(\gamma) = \delta(\psi_0,\psi_1)\,\delta(\psi_1,\psi_2)\cdots\delta(\psi_{n-1},\psi_n)\psi_n.$$

Since $\psi_n = \varphi$ holds, we get

$$w_\varphi(\Phi_\gamma)\varphi = \varphi(\gamma) = \delta(\psi_0,\psi_1)\,\delta(\psi_1,\psi_2)\cdots\delta(\psi_{n-1},\psi_n)\varphi.$$

Because both $\delta(\psi_0,\psi_1)\delta(\psi_1,\psi_2)\cdots\delta(\psi_{n-1},\psi_n)$ and $w_\varphi(\Phi_\gamma)$ are partial isometries with the same initial projection $p'(\varphi)$ we finally conclude to

$$w_\varphi(\Phi_\gamma) = \delta(\psi_0,\psi_1)\,\delta(\psi_1,\psi_2)\cdots\delta(\psi_{n-1},\psi_n).$$

Note that according to our invariant definition of $w_\varphi(\Phi_\gamma)$ we have also proved that the product $\delta(\psi_0,\psi_1)\delta(\psi_1,\psi_2)\cdots\delta(\psi_{n-1},\psi_n)$ of the relative phases does not depend on the special choice of the representatives $\psi_k \in S(\omega_k)$, provided $\psi_0 = \varphi$ and $\psi_n = \varphi$ have been fixed. Having this structure in mind, we shall refer to the operator $w_\varphi(\Phi_\gamma)$ as the *global φ-phase* of the ω-loop γ. Summarizing we get

Theorem 8.3: *Let $\gamma : \omega = \omega_0 \longrightarrow \omega_1 \longrightarrow \ldots \longrightarrow \omega_n = \omega$ be an ω-loop in M_{*+}. For given fixed $\varphi \in S(\omega)$ the product*

$$\prod_{k=1}^{n} \delta(\psi_{k-1},\psi_k), \quad \textit{with } \psi_0 = \psi_n = \varphi,$$

is independent of the special choice of $\psi_k \in S(\omega_k)$ for $k = 1, 2, \ldots, n-1$. One has

$$w_\varphi(\Phi_\gamma) = \prod_{k=1}^{n} \delta(\psi_{k-1},\psi_k).$$

We remark that in this sense the global φ-phase $w_\varphi(\Phi_\gamma)$ is the operator-valued generalization of the invariant of V. Bargmann [2].

Next we want to analyze the behavior of Φ_γ in case that the path γ undergoes certain continuous deformations. We start with the following result on pairs of «-minimal positive linear forms.

Lemma 8.4: *Let $\{\omega_j,\sigma_j\}$ be a sequence of «-minimal pairs with $\omega_j \longrightarrow \omega$ and $\sigma_j \longrightarrow \sigma$. Suppose also that $\{\omega,\sigma\}$ is «-minimal, that $\{\varphi_j\}$ is a sequence of vectors with $\varphi_j \in S(\omega_j)$ and $\varphi_j \longrightarrow \varphi$, $\varphi \in S(\omega)$ and that $\xi_j \in S(\sigma)$ is chosen such that $\xi_j \parallel \varphi_j$, for any j. Then the sequence $\{\xi_j\}$ converges towards the uniquely determined $\xi \in S(\sigma)$ obeying $\xi \parallel \varphi$.*

Proof: Let $\psi_j \in S(\sigma_j)$ and $\psi \in S(\sigma)$ be defined such that $\psi_j, \psi \in P_\Omega$, for any j. Then we have $\psi_j \longrightarrow \psi$. Let us define forms $h, h_j \in M'_*$ by $h(\cdot) = \langle(\cdot)\psi,\varphi\rangle$ and $h_j(\cdot) = \langle(\cdot)\psi_j,\varphi_j\rangle$. Let $h = R_v|h|$ and $h_j = R_{v_j}|h_j|$ be the polar decompositions of h and h_j, respectively. Since both $\{\omega_j,\sigma_j\}$ and $\{\omega,\sigma\}$ are «-minimal, by Theorem 5.1/(iv) and Proposition 6.12 we have $\xi_j = v_j^*\psi_j$ and $\xi = v^*\psi$. Exactly the same arguments as those what we used in the proof of Theorem 4.6 apply in the situation at hand and show that $\xi_j \longrightarrow \xi$. (Note that the assumption $\varphi_j \in P_\Omega$ in the proof of Theorem 4.6 had been made there only in order to provide at least one example of a sequence $\{\varphi_j\}$ with $\varphi_j \in S(\omega_j)$ and $\varphi_j \longrightarrow \varphi \in S(\omega)$. Also, for the arguments used subsequently there and showing $v_j^* \psi_j \longrightarrow v^* \psi$, the special nature of the φ_j as elements of P_Ω was of no relevance; only the fact $\varphi_j \longrightarrow \varphi$ proved to be important there.) ■

Proposition 8.5: *Let $\gamma : \omega_0 \longrightarrow \omega_1 \longrightarrow \ldots \longrightarrow \omega_n$ be a path. Suppose that $\gamma^m : \omega_0^m \longrightarrow \omega_1^m \longrightarrow \ldots \longrightarrow \omega_n^m$ are paths with $\lim_m \omega_j^m = \omega_j$, for any j, and that $\varphi^m \in S(\omega_0^m)$, for any m, with $\varphi^m \longrightarrow \varphi \in S(\omega_0)$. Then $\lim_m \Phi_{\gamma^m}(\varphi^m) = \Phi_\gamma(\varphi)$.*

Proof: Let $\varphi^m = \varphi_0^m, \varphi_1^m, \ldots, \varphi_n^m = \Phi_{\gamma^m}(\varphi^m)$ and $\varphi = \varphi_0, \varphi_1, \ldots, \varphi_n = \Phi_\gamma(\varphi)$ be the uniquely determined sequences of vectors relating to γ^m and γ, respectively, with starting points φ^m and φ, and let $\varphi_j^m \| \varphi_{j-1}^m$ as well as $\varphi_j \| \varphi_{j-1}$ be fulfilled for any j. Since $\varphi^m \longrightarrow \varphi$ holds by assumption, the result now follows by successively applying Lemma 8.4 to the pairs of vectors $\{\varphi_j^m, \varphi_{j-1}^m\}$ and $\{\varphi_j, \varphi_{j-1}\}$, from $j = 1$ upwards.■

In case of $\omega_0 = \omega_n$ and $\omega_0^m = \omega_n^m$ both γ and γ^m are loops. In this case, let us define $w = w_\varphi(\Phi_\gamma)$ and $w_m = w_{\varphi^m}(\Phi_{\gamma^m})$. Then we have $\Phi_\gamma(\varphi) = w\varphi$ and $\Phi_{\gamma^m}(\varphi^m) = w_m \varphi^m$. By Propostion 8.5 we infer $\lim_m w_m \varphi^m = w\varphi$. Since $\varphi^m \longrightarrow \varphi$, and because $\{w_m\}$ is a uniformly bounded sequence, $\lim_m w_m \varphi = w\varphi$ can be followed. From this, in using our standard conclusions we deduce that st-$\lim_m w_m p'(\varphi) = w$. Assume in Proposition 8.5 the special situation with $\omega = \omega_0 = \omega_0^m$, for all m. We can choose $\varphi^m = \varphi$, for all m. Then $\lim_m \Phi_{\gamma^m}(\varphi) = \Phi_\gamma(\varphi)$. By the last part of Proposition 8.1 we then infer that

$$\begin{aligned}
&\sup\{\|\Phi_{\gamma^m}(\psi) - \Phi_\gamma(\psi)\| : \psi \in S(\omega)\} \\
&\quad = \sup\{\|\Phi_{\gamma^m}(w\varphi) - \Phi_\gamma(w\varphi)\| : w \in M', w^* w = p'(\varphi)\} \\
&\quad = \sup\{\|w(\Phi_{\gamma^m}(\varphi) - \Phi_\gamma(\varphi))\| : w \in M', w^* w = p'(\varphi)\} \\
&\quad = \|\Phi_{\gamma^m}(\varphi) - \Phi_\gamma(\varphi)\|.
\end{aligned}$$

Let us agree in using the abbreviation $\gamma^m \longrightarrow \gamma$ for the situation described in the assumptions of Proposition 8.5 . We will take together the previously derived results.

Lemma 8.6 : *Let $\{\gamma^j\}$ be a sequence of paths with $\gamma^j \longrightarrow \gamma$ and let ω be the initial form of all that paths. Then*

$$\lim_j \Phi_{\gamma^j} = \Phi_\gamma \quad \textit{uniformly on } S(\omega). \tag{8.2}$$

In case that γ^j and γ are ω^j- resp. ω-loops

$$\text{st-}\lim_j \; w_{\varphi^j}(\Phi_{\gamma^j}) p'(\varphi) = w_\varphi(\Phi_\gamma) \tag{8.3}$$

is valid. In this case $\varphi^j \in S(\omega^j)$, for any j, with $\varphi^j \longrightarrow \varphi \in S(\omega)$ is supposed. Especially, for γ^j and γ being ω-loops for any j one has

$$\text{st-}\lim_j \; w_\varphi(\Phi_{\gamma^j}) = w_\varphi(\Phi_\gamma), \tag{8.4}$$

for any $\varphi \in S(\omega)$.

Let now $\{\omega,\sigma\}$ be a «-minimal pair of normal positive linear forms. In case that $\sigma=\lambda\omega$ for some $\lambda\in\mathbb{R}_+$ it is clear that $G_0(\omega)\simeq G_0(\sigma)$, cf. the discussion at the beginning of this section and the meaning of the elements of the holonomy group. Assume that we are not in this trivial case. Let γ be an ω-loop. We associate with γ a σ-loop γ_σ in defining γ_σ: $\sigma\longrightarrow\gamma\longrightarrow\sigma$. Let $\varphi\in S(\omega)$ be given, and suppose ψ and ψ' to be the φ-relative and the $\varphi(\gamma)$-relative representatives of σ, respectively. Then we easily see that $\psi'=\psi(\gamma_\sigma)=w_\psi(\Phi_{\gamma_\sigma})\psi$. On the other hand we have $\varphi(\gamma)=w_\varphi(\Phi_\gamma)\varphi$, and because $\psi\parallel\varphi$ and $\psi'\parallel\varphi(\gamma)$ holds, according to Proposition 8.1 and the definition of the elements of the ω-phase group at φ we get for the special path γ'': $\omega\longrightarrow\sigma$ the conclusion

$$w_\psi(\Phi_{\gamma_\sigma})\psi = \psi' = \Phi_{\gamma''}(\varphi(\gamma)) = \Phi_{\gamma''}(w_\varphi(\Phi_\gamma)\varphi) = w_\varphi(\Phi_\gamma)\,\Phi_{\gamma''}(\varphi) = w_\varphi(\Phi_\gamma)\psi.$$

Due to $p'(\varphi)=p'(\psi)$ we have $M'_\varphi=M'_\psi$. Hence, $w_\psi(\Phi_{\gamma_\sigma}), w_\varphi(\Phi_\gamma)\in U(M'_\varphi)$ and the equation $w_\psi(\Phi_{\gamma_\sigma})\psi = w_\varphi(\Phi_\gamma)\psi$ implies $w_\psi(\Phi_{\gamma_\sigma}) = w_\varphi(\Phi_\gamma)$. Since γ could have been chosen arbitrarily in the set of ω-loops we have to follow that $U^0_\varphi(\omega)\subset U^0_\psi(\sigma)$. By symmetry ($\{\sigma,\omega\}$ is «-minimal since $\parallel$ is symmetric) we conclude that $U^0_\varphi(\omega)\supset U^0_\psi(\sigma)$, too. Hence we proved $U^0_\varphi(\omega)=U^0_\psi(\sigma)$. By Lemma 8.2 we then have $G_0(\omega)\simeq G_0(\sigma)$ for any «-minimal pair $\{\omega,\sigma\}$. Finally, let now $\omega'\approx\omega$, i.e. let the normal positive linear form ω' belong to the ω-component. By the definition of $\approx$ we have a path γ connecting ω with σ. Let γ: $\omega=\omega_0\longrightarrow\omega_1\longrightarrow\ldots\longrightarrow\omega_n=\sigma$. Since neighbouring forms in the sequence yield «-minimal pairs, we may conclude that $G_0(\omega_k)\simeq G_0(\omega_{k+1})$. Hence, $G_0(\omega)\simeq G_0(\sigma)$, and we can formulate

Theorem 8.7: *The (restricted) holonomy groups of two normal positive linear forms that can be joined by a path are mutually isomorhic.*

Let $GL(M)$ be the group of invertible elements of the vN-algebra M. For a given positive normal form ω we define $U_+(\omega)$ as follows :

$$U_+(\omega)=\left\{x\in GL(M):\ \begin{matrix}\exists\, n\in\mathbb{N},\ a_1,\ldots,a_n\in M_+,\ \text{invertible}\\ \text{with } x=a_n\cdots a_1,\ \omega^x=\omega(x^*(\cdot)x)=\omega\end{matrix}\right\}.$$

It is easy to see that $U_+(\omega)$ is a subgroup of $GL(M)$ and that $U_+(\omega)=U_+(\lambda\omega)$, for any $\lambda\in\mathbb{R}_+\backslash\{0\}$, is fulfilled. Therefore, and with view to our remarks from the beginning of this section, in all what follows we shall suppose all the linear forms considered to be states. We are going to define a map ι_ω: $U_+(\omega)\ni x\longmapsto\iota_\omega(x)\in G_0(\omega)$ by the following instructions. Let $\varphi\in S(\omega)$ be given, and assume $x\in U_+(\omega)$, $x=a_n\cdots a_1$, with invertible $a_k\in M_+$. We inductively define a finite sequence $\{\omega_0,\omega_1,\ldots,\omega_n\}$ of positive linear forms ω_k by the settings $\omega_0=\omega$, and $\omega_k=(\omega_{k-1})^{a_k}$ otherwise. We also define operators b_k as follows:

$$b_k=\begin{cases} e & \text{in case } \omega_k=\lambda\omega_{k-1} \text{ for some } \lambda\in\mathbb{R}_+\backslash\{0\},\\ \{\omega_{k-1}(e)/\omega_k(e)\}^{1/2}a_k & \text{otherwise.}\end{cases}$$

Note that, in case that $\omega_k=\lambda_k\omega_{k-1}$ occurs, $\{\omega_k,\omega_{k-1}\}$ in a trivial way is «-minimal (cf., e.g., Lemma 6.1). According to Example 6.2/(1) we must have both

$$\lambda_k^{1/2}a_{k-1}\cdots a_1\varphi\parallel a_{k-1}\cdots a_1\varphi \qquad\text{and}\qquad a_k a_{k-1}\cdots a_1\varphi\parallel a_{k-1}\cdots a_1\varphi.$$

By Theorem 5.1/(4) then $a_k a_{k-1} \cdots a_1 \varphi = \lambda_k^{1/2} a_{k-1} \cdots a_1 \varphi$ has to be followed, with $\lambda_k = \omega_k(e)/\omega_{k-1}(e)$. Also in the other remaining cases we let λ_k be defined by $\lambda_k = \omega_k(e)/\omega_{k-1}(e)$. Suppose $\{i_k\}$, with $i_1 < i_2 < \ldots < i_m$, is the set of all subscripts j such that $\omega_j \neq \lambda \omega_{j-1}$ for any $\lambda \in \mathbb{R}_+ \backslash \{0\}$. Then taking into account the preceding facts and using the definition of the b_k's we see that

$$x\varphi = a_n \cdots a_1 \varphi = \left(\Pi_{l \neq i_k} \lambda_l^{1/2}\right) a_{i_m} \cdots a_{i_1} \varphi = \left(\Pi_j \lambda_l^{1/2}\right) b_{i_m} \cdots b_{i_1} \varphi \ . \tag{8.5}$$

Let us define operators $c_k = b_{i_k}$, for $k = 1, 2, \ldots, m$. Then the sequence

$$\gamma_{\{a\}}: \omega = \sigma_0 \longrightarrow \sigma_1 \longrightarrow \ldots \longrightarrow \sigma_m = \sigma \quad \text{with } \sigma_k = (\sigma_{k-1})^{c_k} \text{ for } k \geq 1 \text{ , and } \sigma_0 = \omega,$$

is a path which even is an ω-loop in $S_0(M)$. For, if we define $x' = c_m \cdots c_1$, then $x' \in U_+(\omega)$ since $x\varphi = x'\varphi$ is fulfilled. The latter follows from (8.5) together with the fact that $\Pi_j \lambda_l^{1/2} = 1$. Thus $x'\varphi \in S(\omega)$. On the other hand $x'\varphi \in S(\sigma)$ by definition of the sequence $\gamma_{\{a\}}$. This proves $\omega = \sigma$. Note that by successively applying Example 6.2/(1) we get $\varphi(\gamma_{\{a\}}) = x'\varphi$. Hence, also $\varphi(\gamma_{\{a\}}) = x\varphi$ is fulfilled. We put now $\iota_\omega(x) = \Phi_{\gamma_{\{a\}}}$. This is a well defined setting. Because, for another possibly existing factorization of x, say $x = a'_r \cdots a'_1$, as above we arrive at a reduced element $x'' = c'_s \cdots c'_1$ by means of which the corresponding ω-loop $\gamma_{\{a'\}}$ had to be defined. As above we have $\varphi(\gamma_{\{a'\}}) = x''\varphi = x\varphi$, and $\varphi(\gamma_{\{a'\}}) = \varphi(\gamma_{\{a\}})$ had to be followed. From this we obtain $\Phi_{\gamma_{\{a\}}} = \Phi_{\gamma_{\{a'\}}}$. Thus, the definition of $\iota_\omega(x)$ actually proves to be independent from the factorization of x into finitely many positive operators $\{a_k\}$ of M. Moreover, for $x, y \in U_+(\omega)$ from the above construction of the mapping ι_ω it is easily inferred that $\iota_\omega(xy) = \Phi_{\gamma' \cdot \gamma}$ holds, provided $\iota_\omega(x) = \Phi_\gamma$ and $\iota_\omega(y) = \Phi_{\gamma'}$ are fulfilled. By Proposition 8.1 we then have $\iota_\omega(xy) = \Phi_{\gamma' \cdot \gamma} = \Phi_\gamma \Phi_{\gamma'} = \iota_\omega(x) \iota_\omega(y)$. The relation $\iota_\omega(e) = id$ is obviously valid. Hence, we can summarize the following result.

Theorem 8.8: *For any state ω the map ι_ω: $U_+(\omega) \ni x \longmapsto \iota_\omega(x) \in G_0(\omega)$ is a group homomorphism from $U_+(\omega)$ onto some subgroup of the (restricted) holonomy group.*

The group $U_+(\omega)$ will be analyzed in some important special cases. Assume that M admits tracial states. We are going to consider a situation where $\omega = \tau$ is such a tracial state. Let $x \in U_+(\tau)$. Hence, $\tau(\cdot) = \tau(x^*(\cdot)x) = \tau(xx^*(\cdot))$, the relation $\tau((e - xx^*)(\cdot)) = 0$ holds, and $\tau((e - xx^*)^2) = 0$ follows. Let $x = u|x|$ be the polar decomposition of x. Since $U_+(\omega)$ consists of invertible elements, $u \in U(M)$. Hence, and by unitary invariance of τ we also see that

$$\tau(x(\cdot)x^*) = \tau^{u^*}(|x|(\cdot)|x|) = \tau(|x|(\cdot)|x|) = \tau(u^*|x^*|u(\cdot)u^*|x^*|u)$$

$$= \tau(|x^*|u(\cdot)u^*|x^*|) = \tau(|x^*|^2 u(\cdot)u^*) = \tau(xx^*u(\cdot)u^*) = \tau(x^*u(\cdot)u^*x)$$

$$= \tau(u(\cdot)u^*) = \tau(\cdot),$$

i.e. with $x \in U_+(\tau)$ also $x^* \in U_+(\tau)$ follows. Especially, for $x \in U_+(\tau)$ we see that the relation $\tau((e - x^*x)^2) = 0$ is valid, too. Hence,

$$s(\tau) = xx^*s(\tau) = x^*x\,s(\tau) = s(\tau)xx^*s(\tau) = s(\tau)x^*x\,s(\tau).$$

Let us associate with each x of $U_+(\tau)$ the element $v_x = x s(\tau)$. Then, what we have seen is that v_x is unitary in $Ms(\tau)$, i.e. $U_+(\tau)s(\tau) \subset U(Ms(\tau))$ has to be fulfilled. On the other hand, let $a_2, \ldots, a_n \in Ms(\tau)_+$ be invertible positive operators in $Ms(\tau)$ and let $a_n \cdots a_2 = u\,|a_n \cdots a_2|$ be the polar decomposition of $a_n \cdots a_2$. In defining $a_1 = |a_n \cdots a_2|^{-1}$ we get a unitary u which is the product $u = a_n \cdots a_1$ of finitely many positive, invertible operators of the vN-algebra $Ms(\tau)$ (considered over the Hilbert space $s(\tau)H$). We define $b_k = a_k + s(\tau)^\perp$ for any k. Then $x = b_n \cdots b_1 = u + s(\tau)^\perp$ is a unitary in M, and $x \in U_+(\tau)$ due to unitary invariance of τ. Hence, we have proved that

$$U_+(\tau)s(\tau) = \left\{u \in U(Ms(\tau)) : u = a_n \cdots a_1 \text{ for invertible, positive } a_k \in Ms(\tau), n \in \mathbb{N}\right\}. \quad (8.6)$$

For a moment let us come back to the case of a general state ω. Since ι_ω is a group homomorphism, ker ι_ω is a normal subgroup of $U_+(\omega)$. By the construction of ι_ω we see that ker $\iota_\omega = \{x \in U_+(\omega) : x\varphi = \varphi\}$, with $\varphi \in S(\omega)$. Hence, $x \in$ ker ι_ω if and only if $xs(\omega) = s(\omega)$ holds. Let $[y] \in U_+(\omega)/\ker \iota_\omega$ where $[y]$ is the equivalence class of y modulo ker ι_ω in the factor group $U_+(\omega)/\ker \iota_\omega$. From the definition of the group $U_+(\omega)$ there follows that $xs(\omega) = s(\omega)xs(\omega)$ for any $x \in U_+(\omega)$. Hence, for $x, z \in U_+(\omega)$ we have $(xs(\omega))(ys(\omega)) = xys(\omega)$. The latter implies that $U_+(\omega)s(\omega)$ is a subgroup of the group $GL(s(\omega)Ms(\omega))$. Since $x \in [y]$ implies $x^{-1}y\, s(\omega) = s(\omega)$, we have $ys(\omega) = xs(\omega)$. Therefore, by

$$\eta : U_+(\omega)/\ker \iota_\omega \ni [y] \longmapsto \eta([y]) = ys(\omega) \in U_+(\omega)s(\omega)$$

we have given a well-defined surjective map. Since $ys(\omega) = xs(\omega)$ implies $[y] = [x]$, we see injectivity of η. Since η is a homomorphism, we finally get

$$U_+(\omega)s(\omega) \simeq U_+(\omega)/\ker \iota_\omega. \quad (8.7)$$

In case of a tracial state τ, according to (8.6) and (8.7) we may now conclude that $U_+(\tau)s(\tau) \lhd U(Ms(\tau))$ ($G_1 \lhd G_2$ indicates that G_1 is a *normal* subgroup of G_2) and

$$\begin{aligned} G_0(\omega) &\supset \iota_\tau(U_+(\tau)) \\ &\simeq \left\{u \in U(Ms(\tau)) : u = a_n \cdots a_1 \textit{ for invertible, positive } a_k \in Ms(\tau)\right\}. \end{aligned} \quad (8.8)$$

Let us now look on a τ-loop γ which is given by $\gamma: \tau = \sigma_0 \longrightarrow \sigma_1 \longrightarrow \ldots \longrightarrow \sigma_m = \tau$. By Theorem 5.1/(1) and Definition 5.0 (cf. (5.1)) we know that $s(\sigma_k) \sim s(\sigma_{k-1})$ for all k. Thus, $s(\sigma_k) \sim s(\tau)$ for any k. Since $s(\tau)$ is a central orthoprojection, from this also $s(\sigma_k) \leq s(\tau)$ follows. Since the support of a tracial state is a finite projection, $s(\sigma_k) \leq s(\tau)$ and $s(\sigma_k) \sim s(\tau)$ imply $s(\sigma_k) = s(\tau)$, for any k. Let a be an invertible, positive element of M, and suppose that $\omega \in S_0(M)$ with $s(\omega) = s(\tau)$ is given. Then also $s(\tau^a) = s(\tau) = s(\omega)$. From Lemma 6.1 we infer that $\{\tau^a, \omega\}$ is «-minimal. Let $\varepsilon > 0$ be a real. Then there exists an invertible $b \in M_+$ such that $\|\tau^{ab} - \omega\|_1 \leq \varepsilon$. This follows either from a special case of Theorem 6.4 or, more simply, is a consequence of elementary facts about finite vN- algebras. According to this, we provide ourselves successively by positive, invertible elements $a_1, \ldots, a_{n-1}$ of M such that $\|\tau^{a_k \cdots a_1} - \sigma_k\|_1 \leq \varepsilon$, for $k \leq n-1$. Let us define a_n as $a_n = |a_1 \cdots a_{n-1}|^{-1}$. Then, $u_\varepsilon = a_n \cdots a_2 a_1$ is a unitary (the adjoint of the unitary from the polar decomposition of $a_1 \cdots a_{n-1}$), and therefore $\tau = \tau^{u_\varepsilon} = \tau^{a_n \cdots a_1}$. Let γ_ε be defined as the sequence $\gamma_\varepsilon : \tau \longrightarrow \tau^{a_1} \longrightarrow \ldots \longrightarrow \tau^{a_k \cdots a_1} \longrightarrow \ldots \longrightarrow \tau^{a_n \cdots a_1} = \tau$. This is a τ-loop for all ε which are sufficiently small. According to Example 6.2/(1), $\varphi(\gamma_\varepsilon) = a_n \cdots a_2 a_1 \varphi = u_\varepsilon \varphi$, for any $\varphi \in S(\tau)$. By the definition of $U_+(\tau)$, $u_\varepsilon \in U_+(\tau)$. Hence, by Theorem 8.8 $\Phi_{\gamma_\varepsilon} = \iota_\tau(u_\varepsilon)$. Note that, in the notations from Lemma 8.6, $\gamma_\varepsilon \longrightarrow \gamma$ provided $\varepsilon \longrightarrow 0$. Thus, Lemma 8.6 gives $\lim_\varepsilon \Phi_{\gamma_\varepsilon} = \Phi_\gamma$, uniformly on $S(\tau)$. This shows that $\iota_\tau(U_+(\tau))$ is a dense subgroup of $G_0(\tau)$. With view to (8.8) we may now summarize.

Theorem 8.9 : *Let τ be a tracial state on M. Then the normal subgroup $SU(M,\tau) \triangleleft U(Ms(\tau))$ given by*

$$SU(M,\tau) = \left\{ u \in U(Ms(\tau)) : u = a_n \cdots a_1 \text{ for invertible, positive } a_k \in Ms(\tau);\ n \in \mathbb{N} \right\}$$

is isomorphic to some dense subgroup of the (restricted) holonomy group $G_0(\tau)$.

We remark that Theorem 8.9 is a key tool in order to accomplish the task of identifying the holonomy groups. What one has to do essentially is to identify the group $SU(M,\tau)$ of all unitaries which are finite products of positive, invertible operators in the vN- algebra $Ms(\tau)$. Especially, the situation is clear if M is a finite-dimensional algebra. In this case, for a tracial state τ and any other state ω with $s(\omega) = s(\tau)$, we find $\omega = \tau^a$ for some invertible, positive $a \in M$. Let σ be another state with $s(\sigma) = s(\tau)$ and suppose that $\sigma = \tau^b$, with invertible, positive $b \in M$. Let an invertible positive element c be defined as $c = a^{-1} | ba | a^{-1}$. Then $\sigma = \omega^c$. Now, if γ is a τ-loop γ: $\tau = \sigma_0 \longrightarrow \sigma_1 \longrightarrow \ldots \longrightarrow \sigma_m = \tau$, due to our above discussion $s(\sigma_k) = s(\tau)$, for any k. Hence, there are positive, invertible c_k such that $\sigma_k = (\sigma_{k-1})^{c_k}$, for any k. According to Example 6.2/(1) we obtain $\varphi(\gamma) = c_m c_{m-1} \cdots c_1 \varphi$ for $\varphi \in S(\tau)$. Since $u = c_m c_{m-1} \cdots c_1 \in U_+(\tau)$, the construction of ι_τ shows that $\Phi_\gamma = \iota_\tau(u)$ has to hold. Hence, ι_τ is surjective in this case. Also, since on a finite -dimensional vN-algebra a faithful tracial state τ_0 exists and therefore $\{\tau_0, \omega\}$ is $\ll$-minimal for any other faithful state ω on M, by means of Theorems 8.7 and 8.9 and the facts about the structure of $SU(M,\tau)$ from above, we finally can conclude as follows.

Corollary 8.10 : *For a tracial state τ on a finite-dimensional vN-algebra M we have $G_0(\tau) \simeq SU(M,\tau)$. For every faithful state ω over M one has*

$$G_0(\omega) \simeq \left\{ u \in M : u = a_n \cdots a_1 | a_n \cdots a_1 |^{-1}, \text{ with invertible, positive } a_k \in M,\ n \in \mathbb{N} \right\}.$$

The next simplest case is a normal positive linear form ω which is given as $\omega = \tau^p$, with a normal tracial state τ and some projection $p \in M$. We will suppose that $s(\tau)$ is the central support of p (the remaining cases can be reduced to such situation). Under these suppositions the following result holds.

Proposition 8.11 : *For given $\Phi \in G_0(\omega)$ and every $\varepsilon > 0$ there is $\Phi_\varepsilon \in G_0(\tau)$ such that, for any $\varphi \in S(\tau)$, the limit $\Phi(p\varphi) = \lim_\varepsilon p\, \Phi_\varepsilon(\varphi)$ exists. Moreover, Φ_ε can be chosen as $\Phi_\varepsilon \in \iota_\tau(U_+(\tau))$.*

Proof: Suppose that $\Phi = \Phi_\gamma$, with the ω-loop $\gamma : \omega = \omega_0 \longrightarrow \omega_1 \longrightarrow \ldots \longrightarrow \omega_n = \omega$. Then, $s(\omega_k)\, s(\omega_0)$ for any k (cf. Definition 5.0, formula (5.1) and Theorem 5.1/(i)) and since $s(\omega_0) = p \le s(\tau)$ and $s(\tau) \in M \cap M'$ is fulfilled, also $s(\omega_k) \le s(\tau)$ holds. Let us define $q_k = s(\tau) - s(\omega_k)$. For any $\varepsilon > 0$ and k we put $\omega_k[\varepsilon] = \omega_k + \varepsilon\tau^{q_k}$. Then, for all ε sufficiently small, we get an $\omega_0[\varepsilon]$-loop γ_ε given as γ_ε: $\omega[\varepsilon] = \omega_0[\varepsilon] \longrightarrow \omega_1[\varepsilon] \longrightarrow \ldots \longrightarrow \omega_n[\varepsilon] = \omega[\varepsilon]$. We note that $s(\omega_k[\varepsilon]) = s(\tau)$, for any k, and that $\gamma_\varepsilon \longrightarrow \gamma$ in the sense of Lemma 8.6 as $\varepsilon \longrightarrow 0$. Let us define $\Psi_\varepsilon = \Phi_{\gamma_\varepsilon}$. The arguments raised below (8.8) assure that some invertible $c_k \in M_+$ exist such that $\omega_k[\varepsilon]^{c_{k+1}} = \omega_{k+1}[\varepsilon]$, for all k. We may suppose that $c_k\, s(\tau)^\perp = s(\tau)^\perp$ in this context. We define $v_\varepsilon = c_n \cdots c_1$. Then $v_\varepsilon \in U_+(\omega[\varepsilon])$. By definition of $\omega[\varepsilon]$ it follows that $m_\varepsilon^2 = v_\varepsilon m_\varepsilon^2 v_\varepsilon^*$, with m_ε given as $m_\varepsilon = p + \varepsilon^{1/2} p^\perp$. We define $u_\varepsilon = m_\varepsilon^{-1} v_\varepsilon m_\varepsilon$. Then u_ε is unitary and $u_\varepsilon \in U_+(\tau)$ because u_ε is the product of finitely many positive invertible operators of M. Let $\varphi \in S(\tau)$. Then $\varphi_\varepsilon = m_\varepsilon \varphi \in S(\omega[\varepsilon])$, and $\varphi_\varepsilon \longrightarrow p\varphi \in S(\omega)$ as $\varepsilon \longrightarrow 0$. From Lemma 8.6 and (8.3), st$-\lim_\varepsilon w_{\varphi_\varepsilon}(\Psi_\varepsilon)\, p'(p\varphi) = w_{p\varphi}(\Phi_\gamma) = w_{p\varphi}(\Phi)$ can be followed. Therefore we get

$$\lim_\varepsilon w_{\varphi_\varepsilon}(\Psi_\varepsilon)\, p\varphi = w_{p\varphi}(\Phi)\, p\varphi \ . \tag{8.8}$$

From the definiton of m_ε we follow that $p\varphi = \varphi_\varepsilon - \varepsilon^{1/2} p^\perp \varphi$. We thus can transfer relation (8.8) into the form

$$\lim_\varepsilon w_{\varphi_\varepsilon}(\Psi_\varepsilon)\, \varphi_\varepsilon = w_{p\varphi}(\Phi)\, p\varphi \ . \tag{8.9}$$

Remind that $w_{\varphi_\varepsilon}(\Psi_\varepsilon)\, \varphi_\varepsilon = \Psi_\varepsilon(\varphi_\varepsilon) = v_\varepsilon \varphi_\varepsilon = m_\varepsilon u_\varepsilon m_\varepsilon^{-1} \varphi_\varepsilon = m_\varepsilon u_\varepsilon \varphi$. Since $m_\varepsilon \longrightarrow p$ as $\varepsilon \longrightarrow 0$, from (8.9) we finally infer

$$\Phi(p\varphi) = w_{p\varphi}(\Phi)\, p\varphi = \lim_\varepsilon p u_\varepsilon \varphi\, , \quad \text{with } u_\varepsilon \in U_+(\tau), \text{ for any } \varepsilon > 0. \tag{8.10}$$

We now define $\Phi_\varepsilon = \iota_\tau(u_\varepsilon)$. Then $\Phi_\varepsilon \in G_0(\tau)$, and as we know in this situation, for any $\varepsilon > 0$ we have $\Phi_\varepsilon(\varphi) = u_\varepsilon \varphi$. Relation (8.10) now yields the result $\Phi(p\varphi) = \lim_\varepsilon p\, \Phi_\varepsilon(\varphi)$ ∎

Corollary 8.12 : *Suppose that $U_+(\tau)s(\tau)$ is compact. Then $G_0(\omega)$ is isomorphic to some subgroup of $G_0(\tau)$. Especially, to each $\Phi \in G_0(\omega)$ there exists a unique $\Phi_0 \in G_0(\tau)$ such that $\Phi(p\varphi) = p\,\Phi_0(\varphi)$.*

Proof: According to (8.10) $\Phi(p\varphi) = w_{p\varphi}(\Phi)\, p\varphi = \lim_\varepsilon p u_\varepsilon \varphi$, with $u_\varepsilon \in U_+(\tau)$, for all $\varepsilon > 0$. By compactness of $U_+(\tau)s(\tau)$ the net $\{u_\varepsilon s(\tau)\}$ has a cluster point $u\, s(\tau)$, with $u \in U_+(\tau)$. Let $\Phi_0 \in G_0(\tau)$ be given by $\Phi_0 = \iota_\tau(u)$. Then $\Phi_0(\varphi) = u\varphi$ and $\Phi(p\varphi) = \lim_\varepsilon p u_\varepsilon \varphi = pu\varphi = p\Phi_0(\varphi)$. This proves the last assertion. On the other hand, we can write $\Phi(p\varphi) = w_{p\varphi}(\Phi)\, p\varphi = p\,\Phi_0(\varphi) = p\, w_\varphi(\Phi_0)\, \varphi = w_\varphi(\Phi_0)\, p\varphi$. Since $w_\varphi(\Phi_0)^* w_\varphi(\Phi_0) = p'(\varphi)$ and $p'(p\varphi) = w_{p\varphi}(\Phi)^* w_{p\varphi}(\Phi)$ hold, due to $p'(p\varphi) = p'(\varphi)$ (which is a consequence of our supposition that $s(\tau)$ should be the central support of p) the relation $w_{p\varphi}(\Phi) = w_\varphi(\Phi_0)$ can be inferred from $w_{p\varphi}(\Phi)\, p\varphi = w_\varphi(\Phi_0)\, p\varphi$. According to Lemma 8.2 , for $\Phi \in G_0(\tau)$ from $w_{p\varphi}(\Phi) = w_\varphi(\Phi_0)$ we can draw the conclusion that $\Phi_0 \in G_0(\tau)$ is the unique solution Φ' within $G_0(\tau)$ of the equation $w_{p\varphi}(\Phi) = w_\varphi(\Phi')$. On the other hand, by the same result we also see that the map $\Phi \longrightarrow \Phi_0$ obtained in this way has to be a group homomorphism from $G_0(\omega)$ into $G_0(\tau)$ ∎

9. Continuous loops

In this section the notions introduced previously will find a very natural extension. We will start in explaining what continuous paths and continuous loops in $S_O(M)$ could be. To this sake we have to place at our disposal some auxiliary arrangements.

Let $I=[s,t]\subset \mathbb{R}_+$ be an interval of the non-negative reals. A *partition* τ of the interval I is a finite-ordered subset of reals $\tau=\{t_0,t_1,\ldots,t_n\}$, $n\in \mathbb{N}\bigcup\{0\}$, such that $t_0=s$, $t_n=t$, and $t_j>t_{j-1}$, for any j. The set of all partitions of I will be notified as Λ_I or Λ_{st}. The set of all partitions Λ_I in a natural manner becomes a directed set if for two partitions τ and τ' we define the notation $\tau\geq\tau'$ indicating that the set of points of the partion τ' is contained in the set of points of the partition τ. Also, for two partitions τ and τ' there is a third partition $\tau\vee\tau'$ such that $\tau\vee\tau'\geq\tau$ and $\tau\vee\tau'\geq\tau'$, where $\tau\vee\tau'$ is the partition which contains only those reals which belong to at least one of the sets of points of τ or τ'. Suppose now we are given two neighbouring intervals I and J, with $I=[s,t]$ and $J=[t,u]$. For partitions $\tau\in\Lambda_I$ and $\tau'\in\Lambda_J$ we define $\tau\oplus\tau'\in\Lambda_{I\cup J}$ as the partition of $I\bigcup J$ which contains all the points of both τ *and* τ'. We note that

$$\left\{\tau\oplus\tau'\in\Lambda_{I\cup J}: \forall\ \tau\in\Lambda_I,\ \tau'\in\Lambda_J\right\}=\left\{\tau''\in\Lambda_{I\cup J}: \forall\ \tau''\geq\tau_0\right\},$$

with $\tau_0\in\Lambda_{I\cup J}$ being the partition which contains exactly the endpoints s,t,u of all the intervals. It should be clear how this generalizes to more than two "summands".

Let $I=[\alpha,\beta]$ be some non-trivial interval, and suppose $\gamma: I\ni t\longmapsto \omega_t\in S_O(M)$ is a continuous map from I into the normal state space. Let $\delta\in I$, and let be defined $I_\delta=[\alpha,\delta]$. Suppose that $\tau=\{t_0,t_1,\ldots,t_n\}$, $n\in\mathbb{N}\bigcup\{0\}$, is a partition of $\Lambda_\delta=\Lambda_{I_\delta}$. Then $\gamma(\tau)$ is defined as the sequence $\gamma(\tau): \omega_{t_0}\longrightarrow\omega_{t_1}\longrightarrow\ldots\longrightarrow\omega_{t_n}$.

Definition 9.1: The continuous map $\gamma: I\ni t\longmapsto\omega_t\in S_O(M)$ is said to be a *continuous path in $S_O(M)$, with initial state $\omega=\omega_\alpha$ and final state $\sigma=\omega_\beta$*, if for any $\delta\in I$ the following conditions are fulfilled :

(1) *there exists $\tau_\delta\in\Lambda_\delta$ such that, for $\tau\geq\tau_\delta$, $\gamma(\tau)$ is a path;*
(2) *there exists $\varphi\in S(\omega)$ such that* $\lim_{\tau\geq\tau_\delta}\varphi(\gamma(\tau))=\varphi_\delta(\gamma)$ *exists ;*
(3) $p'(\varphi)=p'(\varphi_\delta(\gamma))$.

We will discuss some consequences of this definition. Let γ be a continuous path in the sense of our definition (the existence of non-trivial continuous paths will be shown below), and fix a $\delta\in I$. First of all we remark that (in the notations from above) whenever the limit

$$\lim_{\tau\geq\tau_\delta}\varphi(\gamma(\tau))=\varphi_\delta(\gamma)$$

exists for one $\varphi\in S(\omega)$ (see (2)), then this limit has to exist for any $\varphi\in S(\omega)$. In fact, let ψ be another vector of $S(\omega)$, and let $w\in M'$ be the partial isometry with $w^*w=p'(\varphi)$ and $\psi=w\varphi$. As a consequence of Proposition 8.1 we have $(w\varphi)(\gamma(\tau))=w\varphi(\gamma(\tau))$. Hence, from $\lim_{\tau\geq\tau_\delta}\varphi(\gamma(\tau))=\varphi_\delta(\gamma)$ we may conclude to $\lim_{\tau\geq\tau_\delta}\psi(\gamma(\tau))=w\varphi_\delta(\gamma)$, i.e. the limit exists. Moreover, we even infer that the relation $\psi_\delta(\gamma)=(w\varphi)_\delta(\gamma)=w\varphi_\delta(\gamma)$ between the corresponding limits exists, from which also $p'(\psi)=p'(\psi_\delta)$ follows.

Let now Φ be the map from $S(\omega)$ into $S(\omega_\delta)$ which is given by $\psi\longmapsto\Phi(\psi)=\psi_\delta(\gamma)$. As yet mentioned we have $\psi_\delta(\gamma)=w\varphi_\delta(\gamma)$ for any $\psi\in S(\omega_\delta)$ and $w\in M'$ with $w^*w=p'(\varphi)$ and

$\psi = w\varphi$. This shows that Φ is a surjective map.

Since both $\varphi(\gamma(\tau))$ and $\varphi_\delta(\gamma)$ belong to $S(\omega_\delta)$, for any $\tau \geq \tau_\delta$ we find a partial isometry $w_\tau \in M'$ with $w_\tau^* w_\tau = p'(\varphi)$, $\varphi(\gamma(\tau)) = w_\tau \varphi_\delta(\gamma)$. Therefore, for any $x \in M$ we get

$$\|(p'(\varphi) - w_\tau)x\,\varphi_\delta(\gamma)\| = \|x(p'(\varphi) - w_\tau)\varphi_\delta(\gamma)\| = \|x(\varphi_\delta(\gamma) - \varphi(\gamma(\tau)))\|,$$

where we used that $p'(\varphi) = p'(\varphi_\delta(\gamma))$. Since $w_\tau\, p'(\varphi) = w_\tau$, for all $\tau \geq \tau_\delta$, and the set of all these w_τ is uniformly bounded, from $\lim_{\tau \geq \tau_\delta} \varphi(\gamma(\tau)) = \varphi_\delta(\gamma)$ we obtain the relation st-$\lim_{\tau \geq \tau_\delta} w_\tau = p'(\varphi)$. Let ξ be an arbitrary element of H. We are considering the expression

$$\|w_\tau^* \xi - p'(\varphi)\xi\|^2 = \langle w_\tau w_\tau^* \xi, \xi\rangle + \langle p'(\varphi)\xi, \xi\rangle - \langle \xi, w_\tau p'(\varphi)\xi\rangle - \langle w_\tau p'(\varphi)\xi, \xi\rangle .$$

By our assumptions and Definition 9.1/(3), $w_\tau w_\tau^* = p'(\varphi_\delta(\gamma)) = p'(\varphi)$. Then from $w_\tau\, p'(\varphi) = w_\tau$ and st-$\lim_{\tau \geq \tau_\delta} w_\tau = p'(\varphi)$ we conclude that $\|w_\tau^* \xi - p'(\varphi)\xi\|^2 \longrightarrow 0$ in $\tau \geq \tau_\delta$. Hence we see that

$$\text{st-}\lim_{\tau \geq \tau_\delta} w_\tau^* = \text{st-}\lim_{\tau \geq \tau_\delta} w_\tau = p'(\varphi). \tag{9.1}$$

It is easy to see from above and by means of Proposition 8.1 that, for a partial isometry $w \in M'$ with $w^* w = p'(\varphi)$, we get

$$(w\varphi)_\delta(\gamma)(\gamma(\tau)^{-1}) = \big(w\varphi_\delta(\gamma)\big)(\gamma(\tau)^{-1}) = w\big(\varphi_\delta(\gamma)(\gamma(\tau)^{-1})\big)$$
$$= w\big(w_\tau^* \varphi(\gamma(\tau))\big)(\gamma(\tau)^{-1}) = w\, w_\tau^* \varphi(\gamma(\tau))(\gamma(\tau)^{-1}) = w\, w_\tau^* \varphi .$$

By (9.1) then

$$\lim_{\tau \geq \tau_\delta} (w\varphi)_\delta(\gamma)(\gamma(\tau)^{-1}) = w\varphi$$

is obtained. Since any $\psi \in S(\omega)$ can be represented in the form $\psi = w\varphi$ for some $w \in M'$ with $w^* w = p'(\varphi)$, we have arrived at the result

$$\lim_{\tau \geq \tau_\delta} \psi_\delta(\gamma)(\gamma(\tau)^{-1}) = \psi, \quad \forall\ \psi \in S(\omega).$$

In other words we have

$$\lim_{\tau \geq \tau_\delta} \Phi(\psi)(\gamma(\tau)^{-1}) = \psi, \quad \forall\ \psi \in S(\omega). \tag{9.2}$$

Suppose $\psi, \psi' \in S(\omega)$ are such that $\Phi(\psi) = \Phi(\psi')$. Then from (9.2) $\psi = \psi'$ follows. Hence, the map Φ from $S(\omega)$ onto $S(\omega_\delta)$ which is given by $\psi \longmapsto \Phi(\psi) = \psi_\delta(\gamma)$ has also to be injective. But Φ is also a continous map. To see this assume $\varphi_n \in S(\omega)$, for any $n \in \mathbb{N}$, with $\varphi_n \longrightarrow \varphi$. Let $v_n \in M'$, with $v_n^* v_n = p'(\varphi)$, and such that $\varphi_n = v_n \varphi$ for any $n \in \mathbb{N}$. Arguments like those which we used in order to show that $w_\tau \longrightarrow p'(\varphi)$ strongly in τ, also apply in our situation and show that st-$\lim_n v_n = p'(\varphi)$. Because of $\Phi(\varphi_n) = (v_n\varphi)_\delta(\gamma) = v_n \varphi_\delta(\gamma)$ and since $p'(\varphi) = p'(\varphi_\delta(\gamma))$ we see that $\lim_n \Phi(\varphi_n) = \lim_n v_n \varphi_\delta(\gamma) = \varphi_\delta(\gamma) = \Phi(\varphi)$. Let now $\varphi_n' \in S(\omega_\delta)$, for $n \in \mathbb{N}$, with $\varphi_n' \longrightarrow \varphi'$. There are partial isometries $v_n' \in M'$ with $v_n'^* v_n' = p'(\varphi_\delta(\gamma)) = p'(\varphi)$ and such that $\varphi_n' = v_n' \varphi'$ for any $n \in \mathbb{N}$. The relation st-$\lim_n v_n' = p'(\varphi)$ follows. Suppose that $\varphi' = \Phi(\varphi) = \varphi_\delta(\gamma)$. From above we infer that $\varphi_n' = v_n'\varphi' = v_n'\varphi_\delta(\gamma) = (v_n'\varphi)_\delta(\gamma) = \Phi(v_n'\varphi)$. Hence $v_n'\varphi = \Phi^{-1}(\varphi_n')$, for all $n \in \mathbb{N}$, and $\lim_n \Phi^{-1}(\varphi_n') = \lim_n v_n'\varphi = \varphi = \Phi^{-1}(\varphi')$, with $\varphi_n' \in S(\omega_\delta)$ such that $\varphi_n' \longrightarrow \varphi'$. This proves Φ^{-1} to be a continuous map, too. Thus, Φ is a homeomorphism between $S(\omega)$ and $S(\omega_\delta)$.

The properties of the continuous path γ derived so far have been obtained in using a fixed, but arbitrarily chosen $\delta \in I$. The previously analyzed homeomorphism from $S(\omega)$ onto $S(\omega_\delta)$ will be denoted by Φ^δ. Then for our continuous path γ we have a whole family $\{\Phi^\delta\colon \delta \in I\}$ of homeomorphisms. Let $J = [s, u]$ be a non-trivial subinterval of I. We are going to show that the restriction $\gamma' = \gamma_{|J}$ of γ to the subinterval J is a continuous path on its own rights. To see this, let us fix $\delta \in J$. First we note that there exists $\tau_{2\delta} \in \Lambda_{s\delta}$ such that for any $\tau_2 \in \Lambda_{s\delta}$ with $\tau_2 \geq \tau_{2\delta}$ the sequence $\gamma'(\tau)$ is a path. In fact, let us take τ_δ supplemented with the point s. Let us call this new partition τ_δ'. We then have $\tau_\delta' = \tau_{\delta 1} \oplus \tau_{\delta 2}$, with $\tau_{\delta 1} \in \Lambda_{0s}$, $\tau_{\delta 2} \in \Lambda_{s\delta}$. Because of $\tau_\delta' \geq \tau_\delta$ for any $\tau \geq \tau_\delta'$ the sequence $\gamma(\tau)$ is a path, and $\gamma(\tau) = \gamma(\tau_1)\gamma'(\tau_2)$ holds if $\tau = \tau_1 \oplus \tau_2$ with $\tau_1 \in \Lambda_{0s}$, $\tau_2 \in \Lambda_{s\delta}$ is supposed. If $\tau \geq \tau_\delta'$, then also $\tau_2 \geq \tau_{\delta 2}$ has to hold. Hence $\gamma'(\tau_2)$ has to be a path whenever $\tau_2 \geq \tau_{\delta 2}$. We show that

$$\lim_{\tau_2 \geq \tau_{2\delta}} \varphi_s(\gamma)(\gamma'(\tau_2)) = \varphi_\delta(\gamma). \tag{9.3}$$

Assume that this were not the case. Then we could find $\varepsilon > 0$ such that to any τ_2 with $\tau_2 \geq \tau_{\delta 2}$ there were $\tau_2' \geq \tau_2$ with

$$\|\varphi_\delta(\gamma) - \varphi_s(\gamma)(\gamma'(\tau_2'))\| > \varepsilon. \tag{9.4}$$

From $\lim_{\tau \geq \tau_\delta} \varphi(\gamma(\tau)) = \varphi_\delta(\gamma)$ we infer the existence of τ_ε with $\tau_\varepsilon \geq \tau_\delta$ such that

$$\|\varphi_\delta(\gamma) - \varphi(\gamma(\tau))\| < \varepsilon/2, \tag{9.5}$$

for any τ with $\tau \geq \tau_\varepsilon$. Let $\tau_\varepsilon = \tau_{\varepsilon 1} \oplus \tau_{\varepsilon 2}$, with $\tau_{\varepsilon 1} \in \Lambda_{0s}$ and $\tau_{\varepsilon 2} \in \Lambda_{s\delta}$. Let τ_2' be chosen to given $\tau_{\varepsilon 2}$ such that $\tau_2' \geq \tau_{\varepsilon 2}$ and with (9.4) fulfilled. Assume $\tau_1 \geq \tau_{\varepsilon 1}$. We know that the relation $\gamma(\tau_1 \oplus \tau_2') = \gamma(\tau_1)\gamma'(\tau_2')$ is true. Hence we have

$$\varphi(\gamma(\tau_1 \oplus \tau_2')) = \varphi(\gamma(\tau_1)\gamma'(\tau_2')) = \varphi(\gamma(\tau_1))(\gamma'(\tau_2')) = \Phi_{\gamma'(\tau_2')}(\gamma(\tau_1)).$$

By the continuity of $\Phi_{\gamma'(\tau_2')}$ (cf. Proposition 8.1) and since $\lim_{\tau_1 \geq \tau_{1\varepsilon}} \varphi(\gamma(\tau_1)) = \varphi_s(\gamma)$ holds, we find $\tau_1' \geq \tau_{\varepsilon 1}$ such that $\|\varphi(\gamma(\tau_1' \oplus \tau_2')) - \Phi_{\gamma'(\tau_2')}(\varphi_s(\gamma))\| < \varepsilon/2$ is fulfilled. With other words, for $\tau' = \tau_1 \oplus \tau_2 \in \Lambda_\delta$

$$\|\varphi(\gamma(\tau')) - \varphi_s(\gamma)(\gamma'(\tau_2'))\| < \varepsilon/2 \tag{9.6}$$

has to be fulfilled. By construction $\tau' \geq \tau_\varepsilon$. Thus (9.5) and (9.6) have to be valid simultaneously and result in the inequality $\|\varphi_\delta(\gamma) - \varphi_s(\gamma)(\gamma'(\tau_2'))\| < \varepsilon$. By the choice of τ_2' this were in contradiction with (9.4). Consequently, (9.3) has to be true. A repetition of the arguments from the start of our discussion below Definition 9.1 makes clear that (9.3) can be extended to see that

$$\lim_{\tau_2 \geq \tau_{2\delta}} \psi_s(\gamma)(\gamma'(\tau_2)) = \psi_\delta(\gamma), \quad \forall\, \psi \in S(\omega). \tag{9.7}$$

We know that $p'(\psi) = p'(\psi_s(\gamma)) = p'(\psi_\delta(\gamma))$ and $\psi_s(\gamma) = \Phi^s(\psi)$ covers the whole $S(\omega_s)$

when ψ is running through $S(\omega)$. Hence, all the requirements of Definiton 9.1 with γ' instead of γ have been verified. A submap γ' of a continuous map γ over the interval I will be referred to as a *connected submap* of γ if it is the restriction of γ to some subinterval J of I.

Hence, what we have shown so far is that our definition of the term *continuous path* is *selfconsistent* in the sense that any connected submap of a continuous path gives rise to a continuous subpath. This has some important consequences.

To discuss this more in detail, let $\{\Phi'^{\delta}: \delta \in J\}$ be the family of homeomorphisms between $S(\omega_s)$ and $S(\omega_\delta)$ corresponding to γ'. Then by (9.7) it is evident that $\Phi'^{\delta} = \Phi^{\delta}(\Phi^{s})^{-1}$ for any $\delta \in J$. This suggests to introduce the following notion: in case of a continuous path γ in $S_0(M)$, with initial state ω and final state σ, we define Φ_γ to be the homeomorphism acting from $S(\omega)$ onto $S(\sigma)$ which is given by $\Phi_\gamma = \Phi^{\beta}$ (the notations of our discussion have been adopted tacitly). In this way we get a map $\gamma \mapsto \Phi_\gamma$ from the set of continuous paths into the homeomorphisms between the spaces of vectors realizing the respective endpoints of γ. By our approximation procedure of the continuous path γ through discrete paths $\gamma(\tau)$ relating to partitions τ which get finer and finer, we are now able to rewrite Definition 9.1/(1) also into the following form:

$$\lim_{\tau \geq \tau_\gamma} \Phi_{\gamma(\tau)} = \Phi_\gamma , \tag{9.8}$$

where τ_γ is a partition such that $\tau \geq \tau_\gamma$ always implies $\gamma(\tau)$ to be a path in the sense of Section 8. In fact, from Definition 9.1/(1) we learn that, for some $\varphi \in S(\omega)$,

$$\lim_{\tau \geq \tau_\gamma} \Phi_{\gamma(\tau)}(\varphi) = \lim_{\tau \geq \tau_\gamma} \varphi(\gamma(\tau)) = \varphi_\beta(\gamma) = \Phi_\gamma(\varphi). \tag{9.9}$$

Let ψ be another vector arbitrarily chosen from $S(\omega)$. Take the partial isometry $w \in M'$, with $w^*w = p'(\varphi)$ and $\psi = w\varphi$. According to the last part of Proposition 8.1 and due to our discussion below Definition 9.1 we know that $\Phi_{\gamma(\tau)}(\psi) = \Phi_{\gamma(\tau)}(w\varphi) = w\Phi_{\gamma(\tau)}(\varphi)$ as well as $\Phi_\gamma(w\varphi) = w\Phi_\gamma(\varphi)$. From (9.9) then $\lim_{\tau \geq \tau_\gamma} \Phi_{\gamma(\tau)}(\psi) = \Phi_\gamma(\psi)$ is followed. The latter is true for any $\psi \in S(\omega)$, which means that (9.8) holds.

In all what follows we want to agree in using the convention that a *continuous* path is a continuous map from an interval $I \subset \mathbb{R}_+$, with left endpoint 0, into $S_0(M)$ obeying the conditions of Definition 9.1. Let us denote by I_α the interval $[0, \alpha]$. Let γ, γ' be a continuous paths, with $\gamma: I_\alpha \ni t \mapsto \omega_t$ and $\gamma': I_\beta \ni t \mapsto \sigma_t$. Suppose that γ has the initial state ω and the final state σ, whereas γ' has the initial state σ and final state ν. Then let us define the *composition* $\gamma'' = \gamma\,\gamma'$ of the two given continuous paths by setting $\gamma'': I_{\alpha+\beta} \ni t \mapsto \omega_t$, with ω_t defined for $t \in [\alpha, \beta]$ by $\omega_t = \sigma_{t-\alpha}$. Clearly γ'' is a continuous map. We are going to deduce that γ'' is a continuous path. Let us interpret γ and γ' as submaps of the map γ''. By assumption there are partitions $\tau_1 \in \Lambda_\alpha$ and $\tau_2 \in \Lambda_{\alpha\beta}$ such that $\gamma(\tau')$ as well as $\gamma'(\tau'')$ are paths for $\tau' \geq \tau_1$ and $\tau'' \geq \tau_2$, respectively. Assume that $\varphi \in S(\omega)$ and $\Phi = \Phi_{\gamma'} \cdot \Phi_\gamma$. Suppose that $\tau \in \Lambda_{\alpha+\beta}$ and $\tau = \tau' \oplus \tau''$. Then we have

$$\begin{aligned}\varphi(\gamma''(\tau)) &= \varphi(\gamma(\tau')\gamma'(\tau'')) = \big(\varphi(\gamma(\tau'))\big)\big(\gamma'(\tau'')\big) \\ &= \Phi_{\gamma'(\tau'')}\big(\varphi(\gamma(\tau'))\big) = \Phi_{\gamma'(\tau'')}\Phi_{\gamma(\tau')}(\varphi).\end{aligned}$$

Since $\Phi_{\gamma(\tau')}(\varphi)$ and $\Phi_\gamma(\varphi)$ belong to $S(\sigma)$, we find partial isometries $w_{\tau'} \in M'$ with

$$w_{\tau'}^* w_{\tau'} = p'(\varphi) = p'(\Phi_\gamma(\varphi)) = p'(\Phi_{\gamma(\tau')}(\varphi)) \quad \text{and} \quad \Phi_{\gamma(\tau')}(\varphi) = w_{\tau'} \Phi_\gamma(\varphi).$$

By standard arguments (remind the derivation of (9.1) in an analoguous situation) we will get

$$\text{st} - \lim_{\tau' \geq \tau_1} w_{\tau'} = p'(\varphi). \tag{9.10}$$

In using the last part of Proposition 8.1 we get

$$\varphi(\gamma''(\tau)) = \Phi_{\gamma'(\tau'')} \Phi_{\gamma(\tau')}(\varphi) = \Phi_{\gamma'(\tau'')}\Big(w_{\tau'} \Phi_\gamma(\varphi)\Big) = w_{\tau'} \Phi_{\gamma'(\tau'')}(\Phi_\gamma(\varphi)).$$

By (9.9) we have $\lim_{\tau'' \geq \tau_2} \Phi_{\gamma'(\tau'')}(\Phi_\gamma(\varphi)) = \Phi_{\gamma'} \Phi_\gamma(\varphi)$, and if we take into account the relation (9.10) and $p'(\varphi) = p'(\Phi_\gamma(\varphi)) = p'(\Phi_{\gamma'} \Phi_\gamma(\varphi))$, we finally obtain

$$\lim_{\tau \geq \tau_1 \oplus \tau_2} \varphi(\gamma''(\tau)) = \Phi_{\gamma'} \Phi_\gamma(\varphi) . \tag{9.11}$$

Therefore, all the conditions of Definition 9.1 are satiesfied by the map γ'', i.e. γ'' is a continuous path. It is now evident from (9.11) that

$$\Phi_{\gamma''} = \Phi_{\gamma'} \Phi_\gamma \tag{9.12}$$

holds for the corresponding homeomorphisms. Finally we note that, for a given continuous path γ: $I_\alpha \ni t \longmapsto \omega_t$, we could define γ^{-1}: $I_\alpha \ni t \longmapsto \sigma_t$, with $\sigma_t = \omega_{\alpha-t}$. It comes out that γ^{-1} is a continuous path, and $\Phi_{\gamma^{-1}} = \Phi_\gamma^{-1}$ is satisfied. This follows easily from (9.12) and (9.8) together with the fact that $\Phi_{\gamma^{-1}(\tau)} = \Phi_{\gamma(\tau)}{}^{-1}$ (cf. Proposition 8.1) holds for all $\tau \in \Lambda$ sufficiently fine. The path γ^{-1} will be referred to as the *inverse* of the continuous path γ. Let us summarize in the following

Theorem 9.2: *Let γ be a continuous path in $S_0(M)$. γ^{-1} and let every connected submap γ' of γ be a continuous path in its own. Each continuous path γ connecting ω and σ determines a homeomorphism Φ_γ between $S(\omega)$ and $S(\sigma)$ such that $\Phi_{\gamma^{-1}} = \Phi_\gamma^{-1}$ and, if $\gamma = \gamma'\gamma''$ is a continuous map that is the composition of two continuous paths γ' and γ'', then γ is also a continuous path and $\Phi_\gamma = \Phi_{\gamma''} \Phi_{\gamma'}$ holds. Let $\varphi \in S(\omega)$. Whenever $w \in M'$ is a partial isometry with $w^* w = p'(\varphi)$, then $\Phi_\gamma(w\varphi) = w\Phi_\gamma(\varphi)$.*

It remains to show that sufficiently many non-trivial continuous paths exist.

Lemma 9.3: *Let γ be a (discrete) path, with initial form ω and final form σ. There exists a differentiable, continuous path γ' connecting the endpoints of γ such that $\Phi_\gamma = \Phi_{\gamma'}$.*

Proof: For the trivial path the assertion is true. Let γ: $\omega = \omega_0 \longrightarrow \omega_1 \longrightarrow \ldots \longrightarrow \omega_n = \sigma$ be a path, and suppose $0 = t_0 < t_1 < t_2 < \ldots < t_n = \alpha$ to be a given sequence of reals. Let $\varphi = \varphi_0, \varphi_1, \ldots, \varphi_n = \varphi(\gamma) = \Phi_\gamma(\varphi)$ be the sequence of vectors with $\varphi_j \in S(\omega_j)$, $\varphi_j \parallel \varphi_{j-1}$, for any j, which is uniquely determined by the initial vector φ. We define non-negative real-valued functions over $\mathbb{R}_+$ as follows :

$$\begin{aligned} \zeta_j(t) &= 1/2 + (1/2)\cos\left(\pi(t-t_{j-1})(t_j-t_{j-1})^{-1}\right) \\ \eta_j(t) &= -\zeta_j(t)P_M(\omega_j,\omega_{j-1})^{1/2} + \left(1-\zeta_j(t)^2\{1-P_M(\omega_j,\omega_{j-1})\}\right)^{1/2} \end{aligned} \qquad (j=1,2,\dots,n).$$

Look on the map $I_\alpha \ni t \longmapsto \varphi_t \in H$ which is given as follows :

$$\text{if } t\in[t_{j-1},t_j], \text{ then } \quad \varphi_t = \zeta_j(t)\,\varphi_{j-1} + \eta_j(t)\varphi_j \quad . \tag{9.13}$$

If we take into account that $\langle \varphi_{j-1}, \varphi_j \rangle = P_M(\omega_j, \omega_{j-1})^{1/2}$, then since for the derivatives of ζ_j and η_j at the points t_j the right "boundary" conditions

$$\dot{\zeta}_j(t_{j-1}) = \dot{\eta}_j(t_{j-1}) = \dot{\zeta}_j(t_j) = \dot{\eta}_j(t_j) = 0 \ , \ \forall\, j\,,$$

are fulfilled (the dot indicates the derivative), it is easily verified that $\|\varphi_t\| = 1$, for any $t\in I_\alpha$, and that φ_t is differentiable in the whole interval I_α. Let us define $\omega_t(\cdot) = \langle(\cdot)\varphi_t, \varphi_t\rangle$ over M. Then γ': $I_\alpha \ni t \longmapsto \omega_t \in S_0(M)$ is a differentiable map. Since ζ_j and η_j are non-negative functions which do not vanish simultaneously, Example 6.9 applies and shows that, for any j,

$$\varphi_t \parallel \varphi_s\,, \quad \forall\, s,t \in [t_{j-1}, t_j]\,. \tag{9.14}$$

We have $\varphi_{t_j} = \varphi_j$, for any j. Hence, the partition $\tau_0 = \{t_0, t_1, t_2, \dots, t_n\}$ belongs to a path $\gamma'(\tau_0)$ from the very beginning, and $\gamma'(\tau)$ for $\tau \geq \tau_0$ is a path due to (9.14). Moreover, (9.14) tells us also that for $\tau \geq \tau_0$ we have $\varphi(\gamma'(\tau)) = \varphi_n = \varphi(\gamma) = \Phi_\gamma(\varphi)$. Hence the limit $\lim_{\tau \geq \tau_0} \varphi(\gamma'(\tau)) = \Phi_\gamma(\varphi)$ exists in a trivial way. Since for the path γ we have $p'(\varphi) = p'(\Phi_\gamma(\varphi))$, all the properties of Definition 9.1 are satisfied for the map γ'. This proves that γ' is a continuous path and shows, at the same time, that $\Phi_\gamma(\varphi) = \Phi_{\gamma'}(\varphi)$ has to be valid. Since, by Theorem 9.2, we know that $\Phi_\gamma(w\varphi) = w\Phi_\gamma(\varphi)$ as well as $\Phi_{\gamma'}(w\varphi) = w\Phi_{\gamma'}(\varphi)$ is fulfilled for any partial isometry $w \in M'$ with $w^*w = p'(\varphi)$, we get that $\Phi_\gamma = \Phi_{\gamma'}$ has to hold ∎

Let $\omega, \sigma \in S_0(M)$. In case that there exists a continuous path γ with initial state ω and final state σ, this will be notified by $\omega \approx \sigma$, and γ in this situation is referred to as a continuous path *connecting* ω and σ. Note that due to Definition 9.1/(1) and Lemma 9.3, by this definition of $\approx$ exactly the same equivalence relation on $S_0(M)$ arises as that one introduced in the preceding section and referring to (discrete) paths. Hence, the ω-*component* of a normal state ω is the set of all normal states that can be connected with ω through some path. A continuous path γ in $S_0(M)$ connecting ω and σ is referred to as a *continuous ω-loop* or a *continuous closed path at* ω if $\omega = \sigma$. Note that Theorem 9.2 is in complete analogy to Proposition 8.1 if the term *continuous path* is substituted for the term *path*. Thus, all the facts introduced and deduced in consequence of Proposition 8.1 are of relevance in case of continuous paths, too, and they can be literally taken over and then compared. Especially, for a state $\omega \in S_0(M)$ the subgroup $G(\omega)$ of the group of homeomorphisms Γ_ω of the metric space $S(\omega)$ given by

$$G(\omega) = \{\, \Phi \in \Gamma_\omega \colon \exists \text{ continuous } \omega\text{-loop } \gamma \text{ with } \Phi = \Phi_\gamma \,\}$$

contains $G_0(\omega)$ as a dense subgroup. This is a consequence of Lemma 9.3 and (9.8).

$G(\omega)$ will be called *holonomy group* of ω. Also if we introduce the equivalence relation $\sim_\omega$ in the the set of all continuous ω-loops by the requirement that $\gamma \sim_\omega \gamma'$ if and only if $\varphi(\gamma) = \varphi(\gamma')$ for any $\varphi \in S(\omega)$, then the set of equivalence classes $[\gamma]$ of continuous ω-loops with respect to $\sim_\omega$ is a *group* in a natural way if we define $[\gamma]\,[\gamma'] = [\gamma\gamma']$. This group is anti-isomorphic with $G(\omega)$ via the map $[\gamma] \mapsto \Phi_\gamma$ (cf. Theorem 9.2).

Note that in this continuous case the class $[\gamma]$ contains many elements that are equivalent in a very trivial sense, they differ only by their respective *parametrization.* By this notion the following will be meant. We will say the continuous paths $\gamma: I_\alpha \ni t \mapsto \omega_t \in S_0(M)$ and $\gamma': I_\beta \ni t \mapsto \sigma_t \in S_0(M)$ *differ only by their respective parametrization* if there exists an order preserving homeomorphism f from I_α onto I_β such that $\sigma_{f(t)} = \omega_t$, for any $t \in I_\alpha$. This can be used to attribute to the term *continuous path* the more improved understanding as a *class* up to a certain kind of parametrization (in our case only continuity is considered). Also note that in the set of all continuous paths connecting some normal state ω with some normal state σ an equivalence relation $\sim$ can be introduced by requiring that $\gamma \sim \gamma'$ if $\varphi(\gamma) = \varphi(\gamma')$ for some (and therefore any) $\varphi \in S(\omega)$. Also in this case the class of γ will be abbreviated by $[\gamma]$.

Suppose now that $\Phi \in G(\omega)$ and fix a vector $\varphi \in S(\omega)$. Let M'_φ be the vN-algebra $M'_\varphi = p'(\varphi) M' p'(\varphi)$, and let $U(M'_\varphi)$ be the unitary group of M'_φ. Since $p'(\Phi(\varphi)) = p'(\varphi)$ and also $\Phi(\varphi) \in S(\omega)$ is fulfilled, there has to exists a unique unitary $w_\varphi(\Phi) \in U(M'_\varphi)$ such that $\Phi(\varphi) = w_\varphi(\Phi)\varphi$. We can associate such a unitary $w_\varphi(\Phi)$ to any vector $\varphi \in S(\omega)$ and to any $\Phi \in G(\omega)$. Then the result for continuous paths corresponding Lemma 8.2 reads as follows:

> w_φ: $G(\omega) \ni \Phi \mapsto w_\varphi(\Phi) \in U(M'_\varphi)$ *is a group anti-isomorphism from the holonomy group* $G(\omega)$ *of the state* ω *onto some subgroup* $U_\varphi(\omega) = \mathrm{im}\, w_\varphi$ *of the unitary group* $U(M'_\varphi)$.

The unitary group $U_\varphi(\omega)$ will be called *ω-phase-group* at $\varphi \in S(\omega)$. By (9.8) and since $\Phi_{\gamma(\tau)}(\varphi) = w_\varphi(\Phi_{\gamma(\tau)})\varphi$ holds, we obtain

$$\lim_{\tau \geq \tau_0} w_\varphi(\Phi_{\gamma(\tau)})\varphi = \lim_{\tau \geq \tau_0} \Phi_{\gamma(\tau)}(\varphi) = \Phi_\gamma(\varphi) = w_\varphi(\Phi_\gamma)\varphi.$$

From this by standard conclusions $\text{st-}\lim_{\tau \geq \tau_0} w_\varphi(\Phi_{\gamma(\tau)}) = w_\varphi(\Phi_\gamma)$ follows. This, together with $G_0(\omega) \subset G(\omega)$, implies that the *restricted ω-phase group* $U^0_\varphi(\omega)$ at $\varphi \in S(\omega)$ is a strongly dense subgroup of the ω-phase-group $U_\varphi(\omega)$ at $\varphi \in S(\omega)$. Since (8.1) remains also valid in the case of continuous paths, $U_\varphi(\omega)$ and $U_\psi(\omega)$ are mutually isomorphic for any two vectors $\varphi, \psi \in S(\omega)$. The term *global φ-phase* for an element $w_\varphi(\Phi_\gamma)$ is extended to the case of a continuous *ω-loop.*

Let us assume now that $\gamma: I_\alpha \ni t \mapsto \omega_t$ is a continuous ω-loop. Fix $\varphi \in S(\omega)$. Suppose $\psi_t \in S(\omega_t)$ to be choosen such that $\psi_0 = \varphi$ and $\psi_\alpha = \varphi$. Let $\tau \in \Lambda_\alpha$ be a partition of the interval I_α. Assume that $\tau = \{0 = t_0, t_1, t_2, \dots, t_{n-1}, t_n = \alpha\}$, with $t_k < t_{k+1}$, for all k. Then, $\prod_\tau \delta(\psi_t, \psi_{t'})$ will be used as an abbreviation for the ordered product of the relative phases (cf. Section 6, below Propostion 6.12) $\delta(\varphi, \psi_{t_1})\,\delta(\psi_{t_1}, \psi_{t_2}) \cdots \delta(\psi_{t_{n-1}}, \varphi)$. Since we know that $\text{st-}\lim_{\tau \geq \tau_0} w_\varphi(\Phi_{\gamma(\tau)}) = w_\varphi(\Phi_\gamma)$, according to Theorem 8.3 we can take for established the following result.

Theorem 9.4 : *Suppose that* $\gamma: I_\alpha \ni t \mapsto \omega_t$ *is a continuous ω-loop. For any map* Ψ: $I_\alpha \ni t \mapsto \psi_t \in H$ *such that* $\psi_t \in S(\omega_t)$ *and* $\psi_0 = \varphi = \psi_\alpha$ *the limit* $\text{st-}\lim_{\tau \geq \tau_0} \prod_\tau \delta(\psi_t, \psi_{t'})$ *exists* *and depends only on φ and γ. The limit is the global φ-phase of the loop :*

$$w_\varphi(\Phi_\gamma) = \text{st}-\lim_{\tau \geq \tau_0} \prod_\tau \delta(\psi_t, \psi_{t'}). \tag{9.15}$$

Remark 9.5: The formula (9.15) is of some practical importance. In fact, suppose that we are given a continuous map $\Psi: I_\alpha \ni t \longmapsto \psi_t \in H$ such that $\psi_\alpha = w\psi_0$ for some $w \in M'$ with $w^* w = p'(\psi_0)$. If

$$v = \text{st}-\lim_{\tau \geq \tau_0} \prod_\tau \delta(\psi_t, \psi_{t'}) w^*$$

exists and is unitary in $p'(\psi_0) M' p'(\psi_0)$ we can be sure that the map $\gamma: I_\alpha \ni t \longmapsto \omega_t$, with ω_t defined by the condition $\psi_t \in S(\omega_t)$, is a *continuous ω-loop*. This follows from Definition 9.1 and from the discussions below the definition together with Lemma 6.13. Note that according to the assertion on independence in Theorem 9.4 the products $\prod_\tau \delta(\psi_t; \psi_{t'}) w^*$ do *not* depend from the special nature of the map in the inner of the interval (provided the expectation values remain unaffected).

Finally, also the classes of continuous ω-loops are isomorpic to $U_\varphi(\omega)$ for any $\varphi \in S(\omega)$. This follows since the map $[\gamma] \longmapsto w_\varphi(\Phi_\gamma)$ can be composed of the two maps $[\gamma] \longmapsto \Phi_\gamma$ and $\Phi_\gamma \longmapsto w_\varphi(\Phi_\gamma)$ which both are anti-isomophisms from the group of classes of continuous ω-loops onto $G(\omega)$ and from $G(\omega)$ onto $U_\varphi(\omega)$, respectively. Also note that due to the last part of the assertion of Theorem 9.2, which is a continuous equivalent of the last part of the assertion of Propoition 8.1, the idea of the proof of Theorem 8.7 also works in the continuous case, i.e. $G(\omega) \simeq G(\sigma)$ for $\omega, \sigma \in M_{*+}$ provided there is a continuous path γ with initial form ω and final form σ.

We also note that according to our definition of the term *continuous path* it is of no relevance whether the corresponding map γ has its range in $S_0(M)$ or in M_{*+}. This follows since a continuous path with range in M_{*+} contains the 0-form if and only if it is the trivial map into 0 (see Definition 9.1/(2) and our discussions of the analogous question for the case of paths in Section 8). This case will always be excluded. But then, for a continuous map $\gamma: I \ni t \longmapsto \omega_t \in M_{*+}$ with the properties (1)-(3) of Definition 9.1 fulfilled (and M_{*+} instead of $S_0(M)$), we can be assured to find some $\eta > 0$ with $\| \omega_t \|_1 \geq \eta$ for any $t \in I$. Hence, $I \ni t \longmapsto \| \omega_t \|_1$ is a strictly positive continuous function on I. But then the map $\gamma': I \ni t \longmapsto \| \omega_t \|_1^{-1} \omega_t \in S_0(M)$ is a continuous path in the sense discussed above. Let γ connect ω with σ. Then it is not hard to see that $\Phi_\gamma(\cdot) = \| \sigma \|_1^{1/2} \Phi_{\gamma'}(\| \omega \|_1^{-1/2}(\cdot))$ is the corresponding homeomorphismus. Hence, if $\omega \in S_0(M)$ and γ is a continuous ω-loop in M_{*+}, then γ' is a continuous ω-loop in $S_0(M)$ such that $\Phi_\gamma = \Phi_{\gamma'}$. Therefore, both the holonomy group at ω and the ω-phase group do not depend of whether or not the loops or continuous loops arising from and ending in the state ω are staying in $S_0(M)$, exclusively. This independence can simplify further discussions and will be referred to tacitly if by the situation this will be allowed.

Let γ be a continuous map $\gamma: I \ni t \longmapsto \omega_t \in M_{*+}$. Suppose that there is another continuous map $\Psi: I \ni t \longmapsto \psi_t \in H$ with $\psi_t \in S(\omega_t)$, for any $t \in I$. Assume that Ψ is of *bounded variation*, i.e. $V(\Psi) = \sup \left\{ \sum_{t \in \tau_j} \| \psi_{t_j} - \psi_{t_{j+1}} \| : \tau \in \Lambda_I \right\} < \infty$. Then γ is said to be of *bounded variation* and $V(\gamma) = \inf_\Psi V(\Psi)$ is called *variation of* γ if the infimum extends over all Ψ which refer to the same map γ. Suppose that $I = [0, \alpha] \ni t, s$ and $t > s$. Then by $V_{ts}(\Psi)$ and $V_{ts}(\gamma)$ the variation of the restriction of the map Ψ resp. γ onto the interval $[s, t]$ is denoted. We also use the abbreviations $V_t(\Psi) = V_{t0}(\Psi)$ and $V_t(\gamma) = V_{t0}(\gamma)$. It is evident that both $V_t(\Psi)$ and $V_t(\gamma)$ are increasing functions of t and $V_{ts}(\Psi) = V_t(\Psi) - V_s(\Psi)$

holds. As in the classical calculus of the functions of bounded variation from the continuity of Ψ also continuity of $V_t(\Psi)$ in t follows. Suppose that $\omega = \omega_0$ and let $\varphi \in S(\omega)$ be chosen. Let us assume that γ is a continuous path. We are going to look for continuity properties of the map Φ: $I \ni t \longmapsto \varphi_t(\gamma) \in H$.

Lemma 9.6 : *Suppose that γ is a continuous path of bounded variation. For any $\varphi \in S(\omega)$, $\Phi : I \ni t \longmapsto \varphi_t(\gamma) \in H$ is a continuous map, and for any $t, s \in I$, $t \geq s$, we have*

$$V_{ts}(\Phi) = V_{ts}(\gamma) = \inf_{\Psi} V_{ts}(\Psi) = \inf_{\Psi'} V(\Psi'), \tag{9.16}$$

where Ψ extends over all continuous maps with Ψ: $I \ni t \longmapsto \psi_t \in H$ such that $\psi_t \in S(\omega_t)$ for any $t \in I$, and Ψ' extends over all continuous maps with Ψ': $[s, t] \ni r \longmapsto \psi'_r \in H$ such that $\psi'_r \in S(\omega_r)$ for any $r \in [s, t]$.

Proof: Let γ' be a continuous path of bounded variation with initial form ω'. We firstly show that, for any $\varphi' \in S(\omega')$ and any continuous map Ψ': $I' \ni t \longmapsto \psi'_t \in H$ with $\psi'_t \in S(\omega'_t)$, the following inequality is true:

$$\| \varphi'(\gamma') - \varphi' \| \leq V(\gamma') \leq V(\Psi'). \tag{9.17}$$

Let us assume ψ'' and φ'' to be vectors in H such that $\varphi'' \in S(\omega'')$ and $\psi'' \in S(\sigma'')$ for certain normal positive linear forms over M. Assume that $\varphi''_0 \in S(\omega'')$ and $\psi''_0 \in S(\sigma'')$ fulfil $\psi''_0 \parallel \varphi''_0$ (i.e. $\{\omega'', \sigma''\}$ is «-minimal, by Theorem 5.1/(i)). Following Remark 5.2/(2) we have $\langle \psi''_0, \varphi''_0 \rangle = P_M(\omega'', \sigma'')^{1/2}$. By definition of P_M we also see that

$$P_M(\omega'', \sigma'')^{1/2} \geq |\langle \psi'', \varphi'' \rangle| \geq \mathrm{Re} \langle \psi'', \varphi'' \rangle.$$

Hence $-\mathrm{Re} \langle \psi'', \varphi'' \rangle \geq -\langle \psi''_0, \varphi''_0 \rangle$, and we can conclude as follows :

$$\begin{aligned} \| \psi''_0 - \varphi''_0 \|^2 &= \| \psi''_0 \|^2 + \| \varphi''_0 \|^2 - 2 \langle \psi''_0, \varphi''_0 \rangle \\ &\leq \| \psi'' \|^2 + \| \varphi'' \|^2 - 2\, \mathrm{Re} \langle \psi'', \varphi'' \rangle = \| \psi'' - \varphi'' \|^2 . \end{aligned}$$

Let $\tau = \{t_k\}$ be a partition of I' such that $\gamma'(\tau)$ is a path. Assume that $\varphi'_k \in S(\omega'_{t_k})$ and $\varphi'_k \parallel \varphi'_{k-1}$ for any k, with starting vector $\varphi'_0 = \varphi'$. Then the final vector of the sequence is $\varphi'(\gamma'(\tau))$, and we have $\| \varphi'(\gamma'(\tau)) - \varphi' \| \leq \sum_k \| \varphi'_k - \varphi'_{k-1} \|$. Because of $\varphi'_k \parallel \varphi'_{k-1}$, and since φ'_k, $\psi'_{t_k} \in S(\omega'_{t_k})$, φ'_{k-1}, $\psi'_{t_{k-1}} \in S(\omega'_{t_{k-1}})$ is fulfilled, we are in a situation as described above with ψ'', φ'', ψ''_0, φ''_0. The conclusion is that, for all k, $\| \varphi'_k - \varphi'_{k-1} \| \leq \| \psi'_{t_k} - \psi'_{t_{k-1}} \|$. Hence

$$\| \varphi'(\gamma'(\tau)) - \varphi' \| \leq \sum_k \| \varphi'_k - \varphi'_{k-1} \| \leq \sum_k \| \psi'_{t_k} - \psi'_{t_{k-1}} \| \leq V(\Psi').$$

Taking the τ-limit yields $\| \varphi'(\gamma') - \varphi' \| \leq V(\Psi')$. This has to hold for any continuous vector-valued map Ψ' referring to γ' (cf. above). Therefore also $\| \varphi'(\gamma') - \varphi' \| \leq \inf_{\Psi'} V(\Psi') \leq V(\Psi')$ holds true. This proves (9.17). For given $t, s \in I$, $t \geq s$, we put $\gamma' = \gamma|_{[s,t]}$. By Theorem 9.2 we know that γ' is a continuous path and $\varphi_t(\gamma) = \varphi_s(\gamma)(\gamma')$. Moreover, γ' is of bounded variation if γ is supposed to be of bounded variation. For a given continuous vector-valued map Ψ referring to γ the restriction Ψ' of Ψ to $[s, t]$ is a corresponding map for γ'. It is evident that $V_{ts}(\Psi) = V(\Psi')$. As we remarked $V_{ts}(\Psi) = V_t(\Psi) - V_s(\Psi)$

holds in case that Ψ is of bounded variation. We can specify $\varphi' = \varphi_s(\gamma)$ in (9.17) and obtain in this case $\| \varphi_t(\gamma) - \varphi_s(\gamma) \| \le V_t(\Psi) - V_s(\Psi)$. Continuity of $V_t(\Psi)$ then implies continuity of $\varphi_t(\gamma)$. Note also that the inequality $\| \varphi_t(\gamma) - \varphi_s(\gamma) \| \le V_t(\Psi) - V_s(\Psi)$, for $t, s \in I$ with $t > s$ and for any admissible Ψ, implies that the continuous map $\Phi : I \ni t \mapsto \varphi_t(\gamma) \in H$ is of bounded variation, with the property that $V(\Phi) \le V(\Psi)$. Hence, $V(\Phi) \le \inf_\Psi V(\Psi) = V(\gamma)$. On the other hand, by definition of $V(\gamma)$ we have $V(\gamma) \le V(\Phi)$. Hence, equality has to be followed. What we have proved is that $V(\Phi) = \inf_\Psi V(\Psi) = V(\gamma)$. This holds for any continuous path of bounded variation. Especially we may apply the conclusion to a subpath of a given path γ. If the subpath refers to the subinterval $[s, t]$, we get in this way $V(\Phi') = \inf_{\Psi'} V(\Psi') = V_{ts}(\gamma)$, where $\Psi' : [s, t] \ni r \mapsto \psi'_r \in H$ is such that $\psi'_r \in S(\omega_r)$ for any $r \in [s, t]$, and Φ' is the map $\Phi' : [s, t] \ni r \mapsto \varphi'_r(\gamma') \in H$. If we choose $\varphi'_s = \varphi_s(\gamma)$, by Theorem 9.2 we have $\varphi'_r(\gamma') = \varphi_r(\gamma)$. Hence, $\Phi|_{[s,t]} = \Phi'$. Therefore $V(\Phi') = V_{ts}(\Phi)$, and thus $V_{ts}(\Phi) = V(\Phi') = \inf_{\Psi'} V(\Psi') = \inf_\Psi V_{ts}(\Psi) = V_{ts}(\gamma)$ ∎

Remark 9.7: (1) Lemma 9.6 also proves that, for a continuous path of bounded variation, $V_{ts}(\gamma) = V_t(\gamma) - V_t(\gamma)$ is fulfilled for any $t \ge s$, and $V_t(\gamma)$ depends continuously on t.

(2) Suppose that the map $\Psi: I \ni t \mapsto \psi_t \in H$ is continuously differentiable. Then Ψ obeys some Lipschitz condition on I, i.e. for some $c > 0$ we have $\| \psi_t - \psi_s \| \le c|t - s|$, for any $t, s \in I$. From this $V(\Psi) < \infty$ follows. Moreover, the map $\gamma: I \ni t \mapsto \omega_t \in M_{*+}$ with $\omega_t(\cdot) = \langle (\cdot)\psi_t, \psi_t \rangle$ is continuous. Therefore, in case that γ is a continuous path, it provides an example of a continuous path of bounded variation.

(3) The class of continuous paths described in the proof of Lemma 9.3 obeys the requirements of (2). It is also not hard to see that any continuous map $\gamma: I \ni t \mapsto \omega_t \in M_{*+}$, with the property that, for some $\tau_0 \in \Lambda_I$, $\tau \ge \tau_0$ always implies $\gamma(\tau)$ to be a path (cf. Definition 9.1/(1)), can be uniformly (over I) approximated by continuous paths of this class. Hence, the set of continuous paths of bounded variation is uniformly dense in the set of all continuous maps obeying Definition 9.1/(1) and mapping from closed bounded intervals into M_{*+}.

Let $\{\omega, \sigma\}$ be a «-minimal pair of normal states. Then according to a special case of Lemma 9.3 we find a continuous path γ in $S_0(M)$ with initial state ω and final state σ such that, for any given $\varphi \in S(\omega)$, $\varphi(\gamma)$ is the φ-relative representative of σ. By the special construction of the example in the proof of Lemma 9.3 (cf. also Remark 9.7/(3)) we can be sure that such γ with $V(\gamma) < \infty$ exists. Let us associate with the «-minimal pair $\{\omega, \sigma\}$ of normal states a positive real $V(\omega, \sigma)$ which is defined as

$$V(\omega,\sigma) = \inf\left\{ V(\gamma) : \gamma \subseteq S_0(M) \textit{ with } \varphi(\gamma) \in S(\sigma) \textit{ for } \varphi \in S(\omega) \textit{ such that } \varphi(\gamma) \parallel \varphi \right\}.$$

By the above we know that $V(\omega, \sigma) < \infty$. Let γ be a continuous path with initial state ω and final state σ such that $V(\gamma) < \infty$ and $\varphi(\gamma) \parallel \varphi$. Let us fix $\varphi \in S(\omega)$, and suppose for definiteness of the discussion that γ refers to the parameter interval $I = [0, 1]$. Then, by Lemma 9.6, $\Phi : I \ni t \mapsto \varphi_t(\gamma) \in H$ is a continuous map with $\varphi_0 = \varphi$ and $\varphi_1(\gamma) = \varphi(\gamma) = \psi$, where ψ is the φ-relative representative of σ. We then have $V(\omega, \sigma) \le V(\Phi)$. In any case there are many of such paths. We are going to give an estimation of $V(\omega, \sigma)$ from above by continuous paths of the following construction. Let $f: I \ni t \mapsto f(t) \in I$ be a continuous map from I onto I with $f(0) = 1$ and $f(1) = 0$. We define for $t \in I$ unit vectors $\varphi_t[f] \in H$ by

$$\varphi_t[f] = f(t)\varphi + \left\{ -f(t) P_M(\omega,\sigma)^{1/2} + \left(f(t)^2 P_M(\omega,\sigma) + (1 - f(t)^2) \right)^{1/2} \right\} \psi. \qquad (9.18)$$

We associate to f the map $\Phi_f : I \ni t \mapsto \varphi_t[f] \in H$, and require the state $\omega_t[f]$ to obey $\varphi_t[f] \in S(\omega_t[f])$. Then, according to Example 6.9, $\gamma_f : I \ni t \mapsto \omega_t[f]$ is a continuous path with initial state ω and final state σ. In fact, $\varphi_t[f] \| \varphi_s[f]$ holds for all $t, s \in I$. The latter implies $\varphi(\gamma_f(\tau)) = \varphi_1[f]$ for any finite partition τ of I. Hence, according to Definition 9.1 and Theorem 5.1/(i), $\varphi(\gamma_f) = \varphi_1[f]$ exists and $\varphi_t(\gamma_f) = \varphi_t[f]$, $t \in I$. In our situation we see $\varphi(\gamma_f) = \psi$, and in view of Lemma 9.6 we have $V(\Phi_f) = V(\gamma_f)$ and we get the estimate

$$V(\omega,\sigma) \leq \inf V(\Phi_f) = \inf V(\gamma_f), \tag{9.19}$$

where the infimum extends over all functions f of the class which was admitted in the construction of the family of vectors given in (9.18). Let f be one of these functions. Suppose for the moment, we had inner points $t, s \in I$, $t < s$, such that $f(t) = f(s)$. We define another function f' by $f'(u) = f(u)$ for $u \leq t$, and $f'(u) = f(\alpha u + \beta)$ for $u \geq t$, with α and β given by $\alpha = (1-s)(1-t)^{-1}$, $\beta = (s-t)(1-t)^{-1}$. Then also f' belongs to the class of continuous functions admitted in (9.18). Also we have $V(\gamma_{f'}) = V_t(\gamma_f) + V(\gamma_f) - V_s(\gamma_f) = V(\gamma_f) - V_{st}(\gamma_f)$, i.e. $V(\gamma_{f'}) \leq V(\gamma_f)$. Hence, in order to calculate (9.19) we may content with extending the infimum over all monotoneously decreasing functions of the class in question. In line with this, let f be such a continuous, positive function which also decreases monotoneously from one to zero. Then by means of a little calculation from (9.18) we get (Stieltjes integral)

$$V(\gamma_f) = V(\Phi_f) = [1 - P_M(\omega,\sigma)]^{1/2} \int_I \left\{1 - \left(1 - P_M(\omega,\sigma)\right) f^2\right\}^{-1/2} |\mathrm{d}f| .$$

The result is

$$V(\gamma_f) = V(\Phi_f) = \frac{\pi}{2} - \arccos\,[1 - P_M(\omega,\sigma)]^{1/2} .$$

This establishes a relation between the Bures distance $d_M(\omega,\sigma) = 2[1 - P_M(\omega,\sigma)]^{1/2}$ (cf. formula (1.2)) between the states ω, σ of our minimal pair $\{\omega,\sigma\}$ and the total variation $V(\gamma_f)$ of the special class of paths parametrized according to (9.18) by any continuous, positive function which decreases monotoneously from one to zero. The relation reads as

$$V(\gamma_f) + \arccos \frac{1}{2} d_M(\omega,\sigma) = \frac{\pi}{2}, \quad \text{for all } f. \tag{9.20}$$

On the other hand we get the upper bound for $V(\omega,\sigma)$ we were aiming at:

$$V(\omega,\sigma) \leq \frac{\pi}{2} - \arccos\,[1 - P_M(\omega,\sigma)]^{1/2} = \frac{\pi}{2} - \arccos \frac{1}{2} d_M(\omega,\sigma). \tag{9.21}$$

Let us now introduce a unit vector $\varphi' = \| \psi - P_M(\omega,\sigma)^{1/2}\varphi \|^{-1} \{ \psi - P_M(\omega,\sigma)^{1/2}\varphi \}$. Then $\varphi' \perp \varphi$, and the vector ψ can be written as $\psi = P_M(\omega,\sigma)^{1/2}\varphi + [1 - P_M(\omega,\sigma)]^{1/2}\varphi'$. Substituting ψ within (9.18) yields $\varphi_t[f] = g(t)\varphi + (1 - g(t)^2)^{1/2}\varphi'$, with a continuous function g which depends on f and obeys the boundary conditions $g(0) = 1$, $g(1) = P_M(\omega,\sigma)^{1/2}$. Note that in all what follows we will fix some function f which is monotoneously decreasing. Then (9.20) is fulfilled and the function g proves to be monotoneously decreasingly, too. Let now γ be another continuous path connecting ω with σ within $S_0(M)$ such that $\varphi(\gamma) \| \varphi$, for a given $\varphi \in S(\omega)$. Suppose that γ is of bounded variation. By Lemma 9.6 we then know that $\varphi_t(\gamma)$ is continuously depending on t (we will assume that t extends over the unit interval; this special choice of parametrization has no effect on the value of the variation). We will analyze the behavior of $V(\gamma)$. Since $\varphi_0(\gamma)=\varphi$ and $\varphi_1(\gamma)=\psi=P_M(\omega,\sigma)^{1/2}\varphi+[1-P_M(\omega,\sigma)]^{1/2}\varphi'$, we find a representation of $\varphi_t(\gamma)$ in the form

$$\varphi_t(\gamma)=f_0(t)\varphi+f_1(t)\varphi'+\xi_t\ , \tag{9.22}$$

with f_0, $f_1\in C(I)$ and $f_1(0)=0$, $f_1(1)=[1-P_M(\omega,\sigma)]^{1/2}$, $f_0(0)=1$, $f_0(1)=P_M(\omega,\sigma)^{1/2}$. Note that $t\longmapsto\xi_t$ is a continuous map into H with $\xi_t\perp\varphi$, $\xi_t\perp\varphi'$, for any t, and $\xi_0=\xi_1=0$. Let φ'' be an auxiliary unit vector of H such that $\varphi''\perp\varphi$, $\varphi''\perp\varphi'$. Let us associate with $\{\varphi_t(\gamma)\}$ another family $\{\psi_t\}$ of unit vectors which is defined as a map

$$\Psi:\ I\ni t\ \longmapsto\ \psi_t=|f_0(t)|\varphi+|f_1(t)|\varphi'+\|\xi_t\|\varphi''\ . \tag{9.23}$$

Since, for all $t,s\in I$,

$$\begin{aligned}\|\psi_t-\psi_s\|^2 &= \big(|f_0(t)|-|f_0(s)|\big)^2+\big(|f_1(t)|-|f_1(s)|\big)^2+|\,\|\xi_t\|-\|\xi_s\|\,|^2\\ &\le \big(f_0(t)-f_0(s)\big)^2+\big(f_1(t)-f_1(s)\big)^2+\|\xi_t-\xi_s\|^2\\ &= \|\varphi_t(\gamma)-\varphi_s(\gamma)\|^2\end{aligned}$$

holds, we have to conclude that $V(\Psi)\le V(\gamma)<\infty$ holds. Let us consider the unit sphere S^2 in $\mathbb{R}^3$. The map $\Psi':I\ni t\longmapsto\{|f_0(t)|,|f_1(t)|,\|\xi_t\|\}\in S^2$ gives a continuous one-parameter curve connecting the points $\{1,0,0\}$ and $\{P_M(\omega,\sigma)^{1/2},[1-P_M(\omega,\sigma)]^{1/2},0\}$. Obviously, this curve is rectificable and for their Euclidean length $l(\Psi')$ we have $l(\Psi')=V(\Psi)$. On the other hand, also $\Phi':\ I\ni t\longmapsto\{g(t),(1-g(t)^2)^{1/2},\ 0\}\in S^2$ gives a continuous one-parameter curve on S^2 connecting the same two points. According to (9.20) their Euclidean length $l(\Phi')$ is given by $l(\Phi')=V(\Phi)=\frac{\pi}{2}-\arccos\frac{1}{2}d_M(\omega,\sigma)$. By our assumptions on Φ the latter curve is order homeomorphic to the set of points which completely fills the smaller one of the arcs of the great circle on S^2 which passes through the points $\{1,0,0\}$ and $\{P_M(\omega,\sigma)^{1/2},[1-P_M(\omega,\sigma)]^{1/2},0\}$. From Euclidean geometry, however, we then have to follow that $l(\Phi')\le l(\Psi')$, necessarily. Hence, we may draw the conclusion that $V(\gamma)\ge\frac{\pi}{2}-\arccos\frac{1}{2}d_M(\omega,\sigma)$, and by the definition of $V(\omega,\sigma)$ we have to follow that $V(\omega,\sigma)\ge\frac{\pi}{2}-\arccos\frac{1}{2}d_M(\omega,\sigma)$. From (9.21) we then obtain as an exact value

$$V(\omega,\sigma)=\frac{\pi}{2}-\arccos\frac{1}{2}d_M(\omega,\sigma). \tag{9.24}$$

Note that (9.24) establishes a relation between a seemingly global quantity, namely the Bures distance between the states of a «-minimal pair, and the locally determined quantity of the total variation along some path connecting these two states. We can even learn more from the facts demonstrated above. Suppose we had a continuous path connecting ω with σ within $S_0(M)$ such that $\varphi(\gamma)\parallel\varphi$, for a given $\varphi\in S(\omega)$. Then $\varphi_t(\gamma)$ has a form as given in (9.22). Assume $V(\gamma)=V(\omega,\sigma)$. To obey this, the corresponding Ψ' has to satisfy $l(\Phi')=l(\Psi')$. The latter can occure if and only if $\|\xi_t\|=0$, for any t, and $|f_0|$ is a monotoneously decreasing function such that $f_0(0)=1$, $f_0(1)=P_M(\omega,\sigma)^{1/2}$. For the function $|f_1|$ we then have $|f_0|^2+|f_1|^2=1$. Let us now look on the map

$$\Psi'':\ I\ni t\ \longmapsto\{\,\mathrm{Re}\ f_0(t),\ \mathrm{Re}\ f_1(t),\ \mathrm{Im}\ f_0(t),\ \mathrm{Im}\ f_1(t)\,\}\in S^3$$

into the unit sphere of $\mathbf{R}^4$. Then Ψ'' gives a continuous one-parameter curve on S^3 connecting the two points $\{1,0,0,0\}$ and $\{P_M(\omega,\sigma)^{1/2}, [1-P_M(\omega,\sigma)]^{1/2},0,0\}$. By assumption this curve has Euclidean length $l(\Psi'') = \frac{\pi}{2} - \arccos \frac{1}{2} d_M(\omega,\sigma) = l(\Phi')$, with the curve Φ' defined as above and supplemented by 0 in the fourth coordinate. Since the curve Φ' gives the shortest connection between the two points mentioned above, we have to conclude from $l(\Psi'') = l(\Phi')$ that $\mathrm{Im}\, f_0(t) = \mathrm{Im}\, f_1(t) = 0$, $t \in I$. Hence, both f_0 and f_1 are real functions and f_0 is monotoneously decreasing from 1 to $P_M(\omega,\sigma)^{1/2}$. This proves that the infimum considered is attained exactly for those continuous paths γ, which belong to the class γ_f, with monotoneously decreasing, continuous f obeying the boundary values $f(0)=1$, $f(1)=0$ (cf.(9.18)).

Theorem 9.8 : *Let $\{\omega,\sigma\}$ be a $\ll$-minimal pair of normal states over M and $\varphi \in S(\omega)$. Let ψ be the φ-relative representative of σ. For any continuous path γ within $S_0(M)$, with initial state ω and final state σ, such that $\psi = \varphi(\gamma)$ we have for the variation of γ*

$$V(\gamma) \geq \frac{\pi}{2} - \arccos\, [1-P_M(\omega,\sigma)]^{1/2} = \frac{\pi}{2} - \arccos \frac{1}{2} d_M(\omega,\sigma).$$

Equality is attained if and only if $\gamma : I \ni t \mapsto \omega_t \in S_0(M)$ is of the following form: there exist a positive, continuous decreasing function p, with $p(0)=1$ and $p(1)=0$, and another positive continuous function q such that

$$\omega_t = p(t)\omega + q(t)\sigma + 2\{p(t)\, q(t)\}^{1/2}\, \mathrm{Re}\, I(\omega,\sigma), \tag{9.25}$$

with the skew form $I(\omega,\sigma)$ of $\{\omega,\sigma\}$.

Proof: The previous discussion shows that the infimum is attained, and by (9.24) the exact value is given. We have also shown that in order to get the infimum for a continuous path γ, the family $\varphi_t(\gamma)$ necessarily is of the shape

$$\varphi_t(\gamma) = f(t)\varphi + \left\{- f(t) P_M(\omega,\sigma)^{1/2} + \left(f(t)^2 P_M(\omega,\sigma) + (1 - f(t)^2)\right)^{1/2}\right\}\psi,$$

with f being a continuous, decreasing function with $f(0)=1$ and $f(1)=0$. We now define $p = f^2$, and find out q as the solution of the equation $1 \equiv p + q + 2\{pq\}^{1/2} P_M(\omega,\sigma)^{1/2}$. Then

$$q = \{-p^{1/2} P_M(\omega,\sigma)^{1/2} + [p P_M(\omega,\sigma) + (1-p)]^{1/2}\}^2,$$

and since $\psi \parallel \varphi$ holds, by Theorem 5.1/(ii) we get $I(\omega,\sigma) = \langle(\cdot)\psi,\varphi\rangle$. The form (9.25) now follows by inserting the expression of $\varphi_t(\gamma)$ into $\omega_t(\cdot) = \langle(\cdot)\varphi_t(\gamma), \varphi_t(\gamma)\rangle$. Let us assume that (9.25) holds. Then, by Lemma 4.1/(i), $\gamma: I \ni t \mapsto \omega_t$ is a state-valued, continuous map, with $\omega_0 = \omega$ and $\omega_1 = \sigma$. Let $\varphi \in S(\omega)$ and suppose that ψ is the φ-relative representative of σ. By Theorem 5.1/(ii) then $I(\omega,\sigma) = \langle(\cdot)\psi,\varphi\rangle$ holds. From this on gets $\psi_t = p(t)^{1/2}\varphi + q(t)^{1/2}\psi \in S(\omega_t)$, for any t. Since p and q are positive functions, according to Example 6.9 $\psi_t \parallel \psi_s$, for $t,s \in I$, and $\psi_0 = \varphi$, $\psi_1 = \psi$. From Definition 9.1 $\varphi_t(\gamma) = \psi_t$ follows. This shows that γ is a continuous path connecting ω and σ such that $\varphi(\gamma) \parallel \varphi$. Since $\varphi_t(\gamma) = \psi_t$ has the form as asserted in (9.18), according to (9.24) $V(\gamma)$ is the infimum ∎

10. Relative representatives and the skew phase

All what we have proved up to now can be considered also for the vN-algebra M' instead of M. Especially, two vectors $\varphi, \psi \in H$ can be considered in context of both the vN-algebras M or M'. If the latter situation occurs, then φ, ψ will be considered as representing vectors for two normal positive linear forms ω', σ' over M'. Suppose that $\varphi \in S(\omega'), \psi \in S(\sigma')$. It can happen that ψ is the uniquely determined vector of $S(\sigma')$ that is φ-associated with respect to M'. Then the pair $\{\omega', \sigma'\}$ is «-minimal over M'. In order to avoid confusion with the $\Vert$-relation on M, the notation $\psi \Vert' \varphi$ will be used if the vectors are considered in context of M'. There is an interesting relationship between $\Vert$ and $\Vert'$ the consequences of which will be analyzed subsequently.

Proposition 10.1: *Let $\varphi, \psi \in H$, and assume $\psi \Vert \varphi$. Suppose $\omega, \sigma \in M_{*+}$ and $\omega', \sigma' \in M'_{*+}$ are normal positive linear forms over M and M', respectively, such that $\varphi \in S(\omega)$, $\psi \in S(\sigma)$ and $\varphi \in S(\omega')$, $\psi \in S(\sigma')$, respectively. Let $u(\omega, \sigma)$ be the ω,σ-skew phase. Then $\{\omega', \sigma'\}$ is «-minimal over M' and $u(\omega, \sigma)\psi \Vert' \varphi$.*

Proof: Due to our assumptions and according to (5.1), $p'(\varphi) = p'(\psi)$. Hence $s(\omega') = s(\sigma')$ in the vN-algebra M'. Hence, by Lemma 6.1, $\{\omega', \sigma'\}$ is «-minimal over M'. Let $h \in M_* = (M')'_*$ be defined by $h(\cdot) = \langle(\cdot)\psi, \varphi\rangle$, and let be $h = R_u|h|$ the polar decomposition of h. By assumptions $\psi \Vert \varphi$ and $\varphi \in S(\omega)$, $\psi \in S(\sigma)$. Thus, Theorem 5.1/(ii) applies and gives $h = I(\omega, \sigma)$. By definition of the ω, σ-skew phase in Section 3, $h = I(\omega, \sigma) = R_u|I(\omega, \sigma)|$ with $u = u(\omega, \sigma)^*$. The assertion now follows from Proposition 6.12 in application to the vN-algebra M' ∎

Remark 10.2: Note that the result of Proposition 10.1 can be considered as an appropriate generalization of the characterization of the ω,σ-skew phase in Lemma 6.3. In fact, let us adopt the situation $\sigma = \omega^a$, with $a \in M_+$ such that $\{\omega, \sigma\}$ is «-minimal. Assume that $\psi \Vert \varphi$ and $\varphi \in S(\omega)$, $\psi \in S(\sigma)$. According to Example 6.2, $\psi = a\varphi$, necessarily. Hence, on M' we have $\sigma'(\cdot) = \langle(\cdot)a\varphi, a\varphi\rangle \le \|a\|^2\omega'(\cdot)$, with $\omega'(\cdot) = \langle(\cdot)\varphi, \varphi\rangle$. Let $t \in M'_+$ be Sakai's uniquely determined Radon-Nikodym operator such that $\sigma' = \omega'^{\,t}$. By Proposition 10.1 we know that $\{\omega', \sigma'\}$ is «-minimal. This and Example 6.2, but the last now applying over M', yields $\psi' = t\varphi$ as the uniquely determined vector in $S(\sigma')$ such that $\psi' \Vert' \varphi$. According to Proposition 10.1 and respecting the uniqueness result of Theorem 5.1/(iv) we get in the situation at hand over M' that $u(\omega, \sigma)a\varphi = u(\omega, \sigma)\psi = \psi' = t\varphi$. This is the assertion of Lemma 6.3, exactly.

The result of Proposition 10.1 can be extended from pairs of «-minimal, normal positive linear forms to paths. To explain this suppose that $\gamma: \omega = \omega_0 \to \omega_1 \to \ldots \to \omega_n = \sigma$ is a path in $S_0(M)$ and that $\varphi \in S(\omega)$ is fixed. We are going to associate to the pair $\{\gamma, \varphi\}$ another path γ_φ within $S_0(M')$ as follows. Let $\varphi = \varphi_0, \varphi_1, \ldots, \varphi_n = \varphi(\gamma)$ be the uniquely determined sequence of vectors of H to the initial vector φ such that $\varphi_k \in S(\omega_k)$ and $\varphi_k \Vert \varphi_{k-1}$, for all k. We choose $\omega_k' \in S_0(M')$ such that $\varphi_k \in S(\omega_k')$, for any k. By the same argument as this one used to see that $\{\omega', \sigma'\}$ is «-minimal in the proof of Proposition 10.1 we get that also $\{\omega_k', \omega_{k-1}'\}$ is «-minimal for any k. The path γ_φ arises from the sequence $\omega' = \omega_0' \to \omega_1' \to \ldots \to \omega_n' = \sigma'$ through omitting all multiple (neighbouring) occurences of states in the sequence (this reduction is necessary due to our definition of the term *path* and because we know that $\{\omega_k', \omega_{k-1}'\}$ is «-minimal, indeed, but this does not exclude at all that $\omega_k' = \omega_{k-1}'$ could happen for some k). The resulting path in

$S_0(M')$ then fulfils

Lemma 10.3 : *Let $\gamma : \omega = \omega_0 \longrightarrow \omega_1 \longrightarrow \dots \longrightarrow \omega_n = \sigma$ be a path over M, $\varphi \in S(\omega)$, and let γ_φ be the path over M' associated with $\{\gamma, \varphi\}$. Then*

$$\varphi(\gamma_\varphi) = u(\omega, \omega_1) u(\omega_1, \omega_2) \cdots u(\omega_{n-1}, \sigma) \varphi(\gamma). \tag{10.1}$$

Proof: In the above notations and in using Proposition 10.1 , from $\varphi_k \parallel \varphi_{k-1}$, for all k, $u(\omega_{k-1}, \omega_k)\varphi_k \parallel' \varphi_{k-1}$ follows for any k. Moreover, for a given k and any $u \in M = (M')'$ with $u^* u = s(\omega_{k-1})$ we have $u\, u(\omega_{k-1}, \omega_k)\varphi_k \parallel' u\varphi_{k-1}$. This is a consequence of Theorem 5.1/(iii) in application to the M'-context. From Definition 5.0, cf. especially (5.1), we know that

$$u(\omega_k, \omega_{k-1})^* u(\omega_k, \omega_{k-1}) = s(\omega_{k-1}) \quad \text{and} \quad u(\omega_k, \omega_{k-1}) u(\omega_k, \omega_{k-1})^* = s(\omega_{k-1}).$$

Hence, in chosing $u = u(\omega, \omega_1) u(\omega_1, \omega_2) \cdots u(\omega_{k-2}, \omega_{k-1})$ we have $u^* u = s(\omega_{k-1})$, and

$$u(\omega, \omega_1) u(\omega_1, \omega_2) \cdots u(\omega_{k-1}, \omega_k)\varphi_k \parallel' u(\omega, \omega_1) u(\omega_1, \omega_2) \cdots u(\omega_{k-2}, \omega_{k-1})\varphi_{k-1}$$

can be followed. Let us define, for $k = 1, 2, \dots, n$,

$$\psi_k = u(\omega, \omega_1) u(\omega_1, \omega_2) \cdots u(\omega_{k-2}, \omega_{k-1}) u(\omega_{k-1}, \omega_k)\varphi_k \tag{10.2}$$

and $\psi_0 = \varphi$. Then $\psi_0, \psi_1, \dots, \psi_n$ is a sequence of vectors such that $\psi_k \in S(\omega_k')$ and $\psi_k \parallel' \psi_{k-1}$. Suppose the case $\omega_k' = \omega_{k-1}'$ for some subscript k. By the uniqueness assertion of Theorem 5.1/(iv), in this case $\psi_k = \psi_{k-1}$ has to be followed from $\psi_k \parallel' \psi_{k-1}$. By the way, using (10.2) we see that this case occurs if and only if $u(\omega_{k-1}, \omega_k)\varphi_k = \varphi_{k-1}$. Hence, up to possibly multiple occurences of vectors, the vectors of the sequence $\psi_0, \psi_1, \dots, \psi_n$ are exactly those which also constitute the defining sequence for the path γ_φ with starting vector φ. Especially, the last vector of this sequence has to equal ψ_n. Since $\varphi(\gamma_\varphi) = \psi_n$ and $\varphi(\gamma) = \varphi_n$, the equation (10.1) now follows from (10.2) for $k = n$ ∎

Let $\gamma : \omega = \omega_0 \longrightarrow \omega_1 \longrightarrow \dots \longrightarrow \omega_n = \sigma$ be a path in $S_0(M)$. We define a partial isometry $u(\gamma)$ in M by $u(\gamma) = u(\omega, \omega_1) u(\omega_1, \omega_2) \cdots u(\omega_{n-1}, \sigma)$. Due to Lemma 10.3, the final state of γ_φ is given by $\sigma'(\cdot) = \langle (\cdot)\varphi(\gamma), \varphi(\gamma)\rangle = \omega'(w_\varphi(\Phi_\gamma)^*(\cdot)\, w_\varphi(\Phi_\gamma))$. Thus, even if γ is an ω-loop, γ_φ is not a loop in general. In case $\gamma \sim_\omega \{\omega\}$ ($\{\omega\}$ stands for the trivial ω-loop), however, $\Phi_\gamma = id_\omega$ implies that γ_φ is an ω'-loop in $S_0(M')$. In this case we learn from Lemma 10.3 that $w_\varphi(\Phi_{\gamma_\varphi})\varphi = \varphi(\gamma) = u(\gamma)\varphi$. Since $w_\varphi(\Phi_{\gamma_\varphi})^* w_\varphi(\Phi_{\gamma_\varphi}) = s(\omega) = u(\gamma)^* u(\gamma)$ is valid, we can be assured that the following is true :

$$w_\varphi(\Phi_{\gamma_\varphi}) = u(\gamma), \quad \text{for any } \gamma \text{ with } \gamma \sim_\omega \{\omega\} \text{ and each } \varphi \in S(\omega). \tag{10.3}$$

By the definition of $u(\gamma)$, the subset $\Theta(\omega) = \{u(\gamma) : \gamma \text{ with } \gamma \sim_\omega \{\omega\}\}$ of $U(s(\omega)Ms(\omega))$ is a subgroup. In fact, if γ' is another ω-loop with $\gamma' \sim_\omega \{\omega\}$, then

$$\gamma\gamma' \sim_\omega \{\omega\}, \quad u(\gamma)u(\gamma') = u(\gamma\gamma') \tag{10.4}$$

by construction of $u(\cdot)$. Due to Lemma 4.1/(ii) we have $u(\gamma)^* = u(\gamma^{-1})$. Since $\gamma \sim_\omega \{\omega\}$ always implies $\gamma^{-1} \sim_\omega \{\omega\}$, we further have $u(\gamma)^* \in \Theta(\omega)$. The element $u(\gamma)^*$ is the inverse of $u(\gamma)$ in $U(s(\omega)Ms(\omega))$. This proves that $\Theta(\omega)$ is a subgroup of $U(s(\omega)Ms(\omega))$.

We will refer to $\Theta(\omega)$ as the *torsion group* of ω. For a given $\varphi \in S(\omega)$, let us consider the subset $\Delta(\varphi)$ of the full group of equivalence classes of ω'-loops in $S_0(M')$, with $\omega' \in S_0(M')$ defined by $\omega'(\cdot) = \langle (\cdot)\varphi, \varphi \rangle$, which is given as

$$\Delta(\varphi) = \left\{ [\gamma_\varphi] : \forall\ \omega\text{-loops } \gamma \sim_\omega \{\omega\} \right\}.$$

Then $\Delta(\varphi)$ is a subgroup which is isomorphic with the torsion group of ω. This follows easily from (10.3) and (10.4) in using the facts yet proven and telling us that $\Theta(\omega)$ is a group and both the maps $[\gamma] \longmapsto \Phi_\gamma$ and $\Phi \longmapsto w_\varphi(\Phi)$ are anti-isomorhisms of groups (cf. Section 8).

11. Examples (the finite - dimensional case)

In this section we want to discuss examples for some of the results we have proved up to now. We start with considering an ω-loop for a normal, pure state ω (i.e. we suppose to exist such a non-trivial state ω over M). Our aim will be to calculate $\varphi(\gamma)$ if $\varphi \in S(\omega)$ is given. In doing this we will follow the line as indicated in Section 0, essentially. We start with some preliminary considerations.

Firstly, let us suppose $\sigma \in S_0(M)$ and $\{\omega, \sigma\}$ to be «-minimal. Since ω is an extremal state, $p'(\varphi)$ has to be a minimal projection of M'. Let $\psi \in S(\sigma)$ be the unique vector representing σ such that $\psi \parallel \varphi$ (cf. Theorem 5.1/(iv)). By the definition of $\parallel$ (cf. (5.1)) we have also $p'(\psi) = p'(\varphi)$. Hence, $p'(\psi)$ is minimal. This implies σ to be an extremal state. Therefore, also the second state of our «-minimal pair is a normal, pure state. Suppose that $\psi_1 \in S(\sigma)$ is given, and assume that $\psi' = p'(\varphi)\psi_1 = p'(\psi)\psi_1$ is not vanishing. Then since $p'(\psi)$ is minimal we have $p'(\psi') = p'(\varphi)$. Because of $\| x p'(\varphi)\psi_1 \| = \| p'(\varphi) x \psi_1 \| \leq \| x\psi_1 \|$, for any $x \in M$, and since σ is extremal, in the case of $\psi' \neq 0$ we have $\psi'' = \| \psi' \|^{-1} \psi' \in S(\sigma)$, $p'(\psi'') = p'(\varphi)$. Let $h \in M'_*$ be defined by $h(\cdot) = \langle (\cdot)\psi'', \varphi \rangle$. Since $s(|h|) \leq p'(\varphi)$ and $s(|h^*|) \leq p'(\psi'') = p'(\varphi)$ and $p'(\varphi)$ is minimal, $s(|h|) = p'(\varphi) = s(|h^*|)$ has to hold. Let $h = R_v |h|$ be the polar decomposition of h. Then, $v^* v = s(|h|) = p'(\varphi) = s(|h^*|) = v v^*$. Hence, $v = e^{i\vartheta} p'(\varphi)$ for some $\vartheta \in [0, 2\pi)$, and by Proposition 6.12 we may conclude to

$$\psi = v^* \psi'' = e^{-i\vartheta} \| p'(\varphi)\psi_1 \|^{-1} p'(\varphi)\psi_1.$$

It remains to calculate the real ϑ. To this sake let us consider

$$|h|(e) = \langle \psi, \varphi \rangle = e^{-i\vartheta} \| p'(\varphi)\psi_1 \|^{-1} \langle p'(\varphi)\psi_1, \varphi \rangle.$$

This has to be a real number. Since $\{\omega, \sigma\}$ is «-minimal, $|h|(e) = P_M(\omega, \sigma)^{1/2} \neq 0$ (if not we had $\omega = \sigma = 0$, by Remark 2.2, which were in contradiction to $\omega, \sigma \in S_0(M)$). Thus, $\langle p'(\varphi)\psi_1, \varphi \rangle \neq 0$ has to be fulfilled in consequence of $p'(\psi)\psi_1 \neq 0$, especially. Hence we get $e^{i\vartheta} = \langle p'(\varphi)\psi_1, \varphi \rangle | \langle p'(\varphi)\psi_1, \varphi \rangle |^{-1} = \langle \psi_1, \varphi \rangle | \langle \psi_1, \varphi \rangle |^{-1}$, and therefore

$$\psi = \| p'(\varphi)\psi_1 \|^{-1} \langle \varphi, \psi_1 \rangle | \langle \varphi, \psi_1 \rangle |^{-1} p'(\varphi)\psi_1 \tag{11.1}$$

follows. This formula will now be used to calculate $\varphi(\gamma)$. Assume that our ω-loop γ in $S_0(M)$ has length n, i.e. $\gamma: \omega = \omega_0 \longrightarrow \omega_1 \longrightarrow \ldots \longrightarrow \omega_n = \omega$. By our definitions of the terms *path* resp. *loop* (cf. Section 8) and our preliminary discussions we can be sure that all states of the loop are normal pure states if only ω is a normal pure state. Assume that $\psi_k \in S(\omega_k)$ are given, with $\psi_0 = \psi_n = \varphi$ and $p'(\varphi)\psi_k \neq 0$, for all k. Let $\varphi = \varphi_0, \varphi_1, \dots, \varphi_n = \varphi(\gamma)$ be the uniquely determined sequence of vectors with $\varphi_k \in S(\omega_k)$ and $\varphi_k \parallel \varphi_{k-1}$, for any k. In using formula (11.1) and respecting $p'(\varphi_k) = p'(\varphi)$, for any k we get $\varphi_k = \| p'(\varphi)\psi_k \|^{-1} \times \langle \varphi_{k-1}, \psi_k \rangle |\langle \varphi_{k-1}, \psi_k \rangle|^{-1} p'(\varphi)\psi_k$. By this recursive formula we finally arrive at the following result.

Lemma 11.1 : *Let $\gamma : \omega = \omega_0 \longrightarrow \omega_1 \longrightarrow \dots \longrightarrow \omega_n = \omega$ be an ω-loop, with ω being a normal, pure state over M. Suppose $\varphi \in S(\omega)$ has been fixed, and $\psi_k \in S(\omega_k)$, for $k = 1, 2, \dots, n-1$, are vectors with $p'(\varphi)\psi_k \neq 0$. Then for $w_\varphi(\Phi_\gamma) \in U_\varphi^0(\omega)$ we find (with $\psi_0 = \psi_n = \varphi$)*

$$w_\varphi(\Phi_\gamma) = \prod_{k=1}^{n} \langle p'(\varphi)\psi_{k-1}, \psi_k \rangle |\langle p'(\varphi)\psi_{k-1}, \psi_k \rangle| \, p'(\varphi), \tag{11.2}$$

and therefore

$$\varphi(\gamma) = \prod_{k=1}^{n} \langle p'(\varphi)\psi_{k-1}, \psi_k \rangle |\langle p'(\varphi)\psi_{k-1}, \psi_k \rangle|^{-1} \varphi \ . \tag{11.3}$$

Note that by our preliminary discussions we can be assured that $\langle p'(\varphi)\psi_{k-1}, \psi_k \rangle \neq 0$ whenever only the ψ_k are chosen in accordance with $p'(\varphi)\psi_k \neq 0$, for any k. As discussed above, the very reason for this is that two neighbouring states in a path cannot be mutually orthogonal because this would violate the condition of «-minimality (cf. Lemma 4.1./(vi)). We also remark that the expression (with $\psi_n = \psi_0 = \varphi$)

$$\Delta(\omega_0, \omega_1, \dots, \omega_{n-1}) = \prod_{k=1}^{n} \langle p'(\varphi)\psi_{k-1}, \psi_k \rangle |\langle p'(\varphi)\psi_{k-1}, \psi_k \rangle|^{-1}$$

which is the multiplier of φ in (11.3), is the straightforward (normed) extension of the invariant factor Δ introduced and used by V.Bargmann in [2] (cf. our discussion of the case $B(H)$ in Section 0). The term *invariance* in this context means that the value does not depend on the special choice of the representatives ψ_k of the pure normal states ω_k, provided $p'(\psi_0)\psi_k \neq 0$ is fulfilled and any two neighbouring states in the sequence $\omega_0, \omega_1, \dots, \omega_{n-1}, \omega_0$ are non-orthogonal to each other. We also note that the restriction $p'(\psi_0)\psi_k \neq 0$ is only necessary in order to get an explicite representation of $\Delta(\omega_0, \omega_1, \dots, \omega_{n-1})$ in terms of scalar products of the given vectors. The relative phases in the sense of Section 6 are defined also without this restriction, and according to Theorem 8.3 the ordered product of these factors (which are partial isometries) yields the same $w_\varphi(\Phi_\gamma)$ as we calculated for a system of representatives of the same states with this restriction fulfilled.

In the next step we want to identify the ω-phase group $U_\varphi(\omega)$ and the holonomy group $G(\omega)$ of a pure normal state ω. By Lemma 11.1 and Lemma 8.2 we see that both $U_\varphi^0(\omega)$ and $G_0(\omega)$ are isomorhic to some subgroup of the group $U(1) = \{ z \in \mathbb{C} : |z| = 1 \}$.

Theorem 11.2 : *Let ω be a normal, pure state over the vN-algebra M. Then $G_0(\omega) = G(\omega)$ and $G(\omega) \simeq \{id\}$ in case that ω is a tracial state, and $G(\omega) \simeq U(1)$ otherwise. The torsion group $\Theta(\omega)$ is trivial.*

Proof: We use the notations from above. Let ω be a normal, pure state over the vN-algebra M. Suppose $\varphi \in S(\omega)$ is fixed. Let $K = p'(\varphi)H$. Since ω is both pure and normal, $Mp'(\varphi)$ has to be irreducible and weakly closed if considered as an algebra of operators over K. Hence, $Mp'(\varphi)$ can be identified with $B(K)$. Two cases have to be considered : $\dim_H K = 1$, or $\dim_H K > 1$.

The first case $\dim_H K = 1$ occurs if and only if $Mp'(\varphi) = \mathbb{C}p'(\varphi)$, and in this case $xy\varphi = yx\varphi$ follows for any two operators $x, y \in M$. Moreover, if $x\varphi = \lambda\varphi$ and $y\varphi = \mu\varphi$, with $\lambda, \mu \in \mathbb{C}$, then $xy\varphi = \lambda\mu\varphi$, and $\omega(x) = \lambda$, $\omega(y) = \mu$ and $\omega(xy) = \lambda\mu$ follow. Hence, $\omega(xy) = \omega(yx) = \omega(x)\omega(y)$ for any $x, y \in M$. Thus ω is tracial in this case. But then, for the vector $\varphi_k \in S(\omega_k)$ we have $p'(\varphi_k) = p'(\varphi)$. Thus, also $Mp'(\varphi_k) = \mathbb{C}p'(\varphi_k)$ follows, i.e. ω_k obeys the same conditions as ω. Since on $\mathbb{C}$ there exists only one state, $\omega_k(x) = \omega(x)$ has to hold for any $x \in M$. Hence, the ω-loop γ can only be the trivial one, i.e. $\Phi_\gamma = id_\omega$, and therefore the groups $G_0(\omega)$ resp. $U_\varphi^0(\omega)$ have only one element. The same then also holds for $G(\omega)$ resp. $U_\varphi(\omega)$.

In situation of the second case $\dim_H K > 1$, let $\varphi^\perp \in K$ be a normalized to one vector which is orthogonal to φ. Let us define $\psi_0 = \varphi$, $\psi_1 = \alpha^{1/2}\varphi + (1-\alpha)^{1/2}\varphi^\perp$, $\psi_2 = \alpha^{1/2}e^{i\varepsilon}\varphi + (1-\alpha)^{1/2}\varphi^\perp$, with $0 < \alpha < 1$ and some fixed, small ε , say $0 < \varepsilon < \pi/2$. The sequence of vectors $\psi_0 = \varphi$, ψ_1, ψ_2, $\psi_3 = \varphi$ corresponds to an ω-loop $\gamma(\alpha)$. We have $\langle \psi_0, \psi_1 \rangle = \alpha^{1/2}$, $\langle \psi_1, \psi_2 \rangle = 1 + \alpha(e^{-i\varepsilon} - 1)$ and $\langle \psi_2, \psi_0 \rangle = \alpha^{1/2}e^{i\varepsilon}$. Since all the vectors belong to K, according to Lemma 11.1 and (11.1), we have to calculate $\Pi_{k=1}^3 \langle \psi_{k-1}, \psi_k \rangle |\langle \psi_{k-1}, \psi_k \rangle|^{-1}$ in order to get the multiplier in (11.2) as a function of α. Call this function $z(\alpha)$. The result of the calculation is

$$z(\alpha) = e^{i(\varepsilon + \vartheta(\alpha))}, \quad \text{with } \tan\vartheta(\alpha) = \left(1 + (\alpha^{-1} - 1)(\cos\varepsilon)^{-1}\right)^{-1}\tan\varepsilon .$$

Hence, the range of $\vartheta(\alpha)$ contains some open interval if α varies continuously, and therefore the corresponding values $z(\alpha)$ generate the full group $\mathrm{U}(1)$. Since $z(\alpha)p'(\varphi)$ belongs to $U_\varphi^0(\omega)$, the assertion $U_\varphi^0(\omega) \simeq \mathrm{U}(1)$ follows. Since $U_\varphi^0(\omega)$ is a dense subgroup of $U_\varphi(\omega)$, also $U_\varphi(\omega) \simeq \mathrm{U}(1)$. The remainder now follows by Lemma 8.2 and the corresponding result for continuous loops (cf. Section 9). To see the last assertion, note that minimality of $p'(\varphi)$ for $\varphi \in S(\omega)$ means $p'(\varphi)M'p'(\varphi) = \mathbb{C}p'(\varphi)$. According to the discussion at the beginning of this section, the same also holds for the vectors φ_k of the ω-loop γ with $\gamma : \omega = \omega_0 \longrightarrow \omega_1 \longrightarrow \ldots \longrightarrow \omega_n = \omega$ (cf. the notations used in the discussions at the beginning of this section), i.e. $p'(\varphi_k)M'p'(\varphi_k) = \mathbb{C}p'(\varphi_k)$ for any k. According to the definition of ω_k' in Section 10 we then get $\omega' = \omega_0' = \omega_k'$, for any k. Hence, the loop γ_φ is the trivial ω'- loop in any case. This implies $\Theta(\omega) = \{s(\omega)\}$ ∎

Let us now come to another special case. We are going to identify the groups $G_0(\omega)$ and $G(\omega)$ for a faithful state ω over a finite-dimensional vN-algebra M. Such an algebra has the form $M = \sum_{k=0}^m Mz_k$, with a family of mutually orthogonal, central projections z_j which sum up to e and for which $Mz_k \simeq M_{n_k}$ is fulfilled. By M_m the full algebra of $m \times m$-matrices with complex entries is meant. As usual, the group of all unitary $m \times m$-matrices with determinant equal to one is denoted by $\mathrm{SU}(m)$. With these notations we have the following result.

Theorem 11.3: $G(\omega) = G_0(\omega) \simeq \mathrm{SU}(n_1) \times \mathrm{SU}(n_2) \times \cdots \times \mathrm{SU}(n_m)$.

Proof: Following our remarks on the central decomposition of M and according to Corollary 8.10 it is sufficient to consider the cases $M \simeq M_n$, $n \in \mathbb{N}$. In these cases we shall prove that

$$\mathrm{SU}(n) = \left\{u \in M_n: u = a_r \cdots a_1 |a_r \cdots a_1|^{-1}, \text{ with invertible, positive } a_k \in M_n,\ r \in \mathbb{N}\right\}. \quad (11.4)$$

Note that the group on the right-hand side of (11.4) in the terminology of Theorem 8.9 is $SU(M_n, \tau_n)$, where τ_n is the unique tracial state of M_n. We will proceed inductively. Assume $n=2$. Since there exist positive, invertible $a, b \in M_2$ such that $ab \neq ba$, we can be sure that $u = ab\,|ab|^{-1} \neq \pm e$. Since $u \in SU(M_2, \tau_2)$, we see that $SU(M_2, \tau_2)$ is a non-trivial subgroup of $\mathrm{U}(2) = U(M_2)$. By Theorem 8.9 we know that $SU(M_2, \tau_2) \lhd \mathrm{U}(2)$. Note that, by the definition of $SU(M_2, \tau_2)$, $\det u = 1$ for any $u \in SU(M_2, \tau_2)$. Hence, we even know that $SU(M_2, \tau_2) \lhd SU(2)$. Therefore, we may suppose $u = \exp i\varepsilon I$, with $I = I^*$, $I^2 = e$, $\mathrm{tr}\, I = 0$, and $(-\pi) < \varepsilon < \pi$, $\varepsilon \neq 0$. Since $SU(M_2, \tau_2)$ is a normal subgroup of SU(2), each of the elements $u(\vartheta) = u^* v_\vartheta^* u v_\vartheta$ has also to belong to $SU(M_2, \tau_2)$, with $\vartheta \in \mathbb{R}$ and $v_\vartheta = \exp i\vartheta I'$ for some I' with $I' = I'^*$, $I'^2 = e$, $\mathrm{tr}\, I' = 0$. Now, for a given ϑ there has to exist another I'' with $I'' = I''^*$, $I''^2 = e$, $\mathrm{tr}\, I'' = 0$ such that $u(\vartheta) = \exp i\Theta(\vartheta) I''$. Since $u(\vartheta) = e \cos\Theta(\vartheta) + i I'' \sin\Theta(\vartheta)$ holds, we obtain the expression $\Theta(\vartheta) = \arccos(\mathrm{tr}\, u(\vartheta)/2)$ for $\Theta(\vartheta)$. The explicite calculation gives

$$\Theta(\vartheta) = \arccos\left(1 - (1 - 2^{-1}\,\mathrm{tr}(II')^2)\sin^2\varepsilon \sin^2\vartheta\right). \quad (11.5)$$

Since $SU(M_2, \tau_2) \lhd \mathrm{U}(2)$ we may conclude to

$$SU(M_2, \tau_2) = \left\{u \in \mathrm{U}(2): \mathrm{spec}(u) = \{e^{i\Theta(\vartheta)}, e^{-i\Theta(\vartheta)}\}, \forall \vartheta \in \mathbb{R}\right\},$$

where $\mathrm{spec}(u)$ means the spectrum of u. Note that to given I it is always possible to choose I' with the properties mentioned such that $II' \neq \pm e$. Then $\mathrm{tr}(II')^2 \neq 2$, and according to (11.5) the range of Θ contains a whole subinterval of reals. Argueing by $SU(M_2, \tau_2) \lhd \mathrm{U}(2)$ once more again we infer that for any two reals ϑ, ϑ' there exist $u \in SU(M_2, \tau_2)$ such that $\mathrm{spec}(u) = \{e^{i(\Theta(\vartheta) + \Theta(\vartheta'))}, e^{-i(\Theta(\vartheta) + \Theta(\vartheta'))}\}$. With other words, the spectrum forms a group, too. Since Θ ranges over a set which contains a non-trivial subinterval, we have to conclude to

$$SU(M_2, \tau_2) = \left\{u \in \mathrm{U}(2): \mathrm{spec}(u) = \{e^{i\Theta}, e^{-i\Theta}\}, \forall \Theta \in \mathbb{R}\right\}.$$

Hence, $SU(M_2, \tau_2) = \mathrm{SU}(2)$ is seen. The general case of $n > 2$ dimensions will be reduced to the 2×2-case as follows. Assume that $\Theta_1, \ldots, \Theta_{n-1} \in \mathbb{R}$. Let us define $\delta_k = \sum_{j=0}^{k} \Theta_j$, for $k = 1, 2, \ldots, n-1$. By the 2×2-case we can be sure that positive, invertible 2×2-matrices $a_1(j), \ldots, a_{m_j}(j)$ exist such that $a_1(j) \cdots a_{m_j}(j) = \mathrm{diag}[e^{i\delta_j}, e^{-i\delta_j}]$. Let us define the $n \times n$-matrix $A_k(j)$ as the bloc matrix given by

$$A_k(j) = \mathrm{diag}[1, \ldots, 1]_{j-1} \oplus a_k(j) \oplus \mathrm{diag}[1, \ldots, 1]_{n-j-1},$$

where $\mathrm{diag}[1, \ldots, 1]_s$ is ignored for $s = 0$. Then $A_k(j) \in M_n$ are invertible elements and for any j each of the products

$$A_1(j) \cdots A_{m_j}(j) = \mathrm{diag}[1, \ldots, 1]_{j-1} \oplus \mathrm{diag}[e^{i\delta_j}, e^{-i\delta_j}] \oplus \mathrm{diag}[1, \ldots, 1]_{n-j-1}$$

belongs to $SU(M_n, \tau_n)$. From this we infer

$$A_1(1)\cdots A_{m_1}(1)\cdots A_1(n-1)\cdots A_{m_{n-1}}(n-1)$$
$$=\mathrm{diag}[e^{i\delta_1}, e^{-i\delta_1+i\delta_2},\ldots, e^{-i\delta_{n-2}+i\delta_{n-1}}, e^{-i\delta_{n-1}}]$$
$$=\mathrm{diag}[e^{i\Theta_1}, e^{i\Theta_2},\ldots, e^{i\Theta_{n-1}}, e^{-i\Sigma\Theta_j}] \in SU(M_n,\tau_n).$$

This is true for any choice of $\Theta_1, \ldots, \Theta_{n-1} \in \mathbb{R}$. Finally, taking $SU(M_n,\tau_n) \triangleleft U(n)$ into account we see that $SU(M_n,\tau_n) = SU(n)$. By our remarks from the beginning of the proof $G_0(\omega) \simeq SU(n_1) \times SU(n_2) \times \cdots \times SU(n_m)$ is now evident. The fact $G(\omega) = G_0(\omega)$ follows from the discussion below Lemma 9.3. As we have shown there $G_0(\omega)$ is a uniformly dense subgroup of $G(\omega)$. We may content with treating the case of $M \simeq M_n$ and the state τ_n. In line with this, assume $\Phi \in G(\omega)$ and suppose that the sequence $\{\Phi_k\} \subseteq G_0(\omega)$ tends to Φ. Then for any $\varphi \in S(\tau_n)$ we have $\Phi_k(\varphi) = u_k\varphi$, where $u_k \in SU(n)$ and $\Phi_k = \iota_{\tau_n}(u_k)$ (cf. Theorem 8.8 and the definition of the map ι). Since $SU(n) \subset M_n$ is compact we find a converging subsequence of $\{u_k\}$. Suppose we have $u_{k_j} \to u$. Then u is unitary and $\Phi(\varphi) = u\varphi$ follows. Since $u \in SU(n)$ we find some homeomorphism $\Phi' \in G_0(\tau_n)$ such that

$$\Phi' = \iota_{\tau_n}(u), \text{ and } \Phi'(\varphi) = u\varphi = \lim_j u_{k_j}\varphi = \lim_j \Phi_{k_j}(\varphi) = \Phi(\varphi)$$

for any $\varphi \in S(\tau_n)$. Hence, $\Phi = \Phi' \in G_0(\omega)$. Since Φ has been chosen arbitrarily from $G(\omega)$, $G(\omega) = G_0(\omega)$ is seen ∎

We want to continue in analyzing the cases of finite type-I-factors $M \simeq M_n$, $n \in \mathbb{N}$, but now we are going to consider the case of a positive linear form $\omega = \tau_n^p$, with a projection p of relative dimension $\dim p = m < n$. This case corresponds to the suppositions of Proposition 8.11. We will work within M_n and we also shall use the notations and conventions from above. We will identify the group $U_+(\omega)s(\omega) = U_+(\omega)p$ up to isomorphy. By (8.7) we know that $U_+(\omega)p$ is isomorphic to some subgroup of $G_0(\omega)$. Hence we can hope to get some information on the group $G_0(\omega)$ by describing the type of $U_+(\omega)p$. We may also suppose that $p = \mathrm{diag}[1,1,\ldots,1]_m \oplus \mathrm{diag}[0,\ldots,0]_{n-m}$, for simplicity. Note first that, by Theorem 11.3, for any given $\Theta_1, \ldots, \Theta_{m-1} \in \mathbb{R}$ we find positive invertible $m \times m$ - matrices $a_1, \ldots, a_r$ such that

$$a_1\cdots a_r = \mathrm{diag}\left[e^{i\Theta_1}, e^{i\Theta_2},\ldots, e^{i\Theta_{m-1}}, e^{-i\Sigma\Theta_j}\right]_m.$$

We define, for any subscript k, $a'_k = a_k \oplus \mathrm{diag}[1,\ldots,1]_{n-m}$. Let $\Theta \in \mathbb{R}$ be arbitrarily given. By (11.4) we know that positive invertible 2×2 - matrices $b_1,\ldots,b_s$ exist such that $b_1\cdots b_s = \mathrm{diag}[e^{i\Theta}, e^{-i\Theta}]$. Hence, we have

$$b_1\cdots b_s\,\mathrm{diag}[1,0] = \mathrm{diag}[1,0]\,b_1\cdots b_s\,\mathrm{diag}[1,0] = \mathrm{diag}[e^{i\Theta},0].$$

We define $b'_k = \mathrm{diag}[1,\ldots,1]_{m-1} \oplus b_k \oplus \mathrm{diag}[1,\ldots,1]_{n-m-1}$. We then have

$$a'_1\cdots a'_r\, b'_1\cdots b'_s\, p = \mathrm{diag}\left[e^{i\Theta_1}, e^{i\Theta_2},\ldots, e^{i\Theta_{m-1}}, e^{-i\Sigma\Theta_j}e^{i\Theta}\right]_m \oplus \mathrm{diag}[0,\ldots,0]_{n-m},$$

from which $a'_1\cdots a'_r\, b'_1\cdots b'_s\, p\, b'_s\cdots b'_1\, a'_r\cdots a'_1 = p$ follows. This, on the one hand, proves that $a'_1\cdots a'_r\, b'_1\cdots b'_s \in U_+(\omega)$, with $\omega = \tau_n^p$. On the other hand it shows that

$\operatorname{diag}[e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_{m-1}}, e^{-i\Sigma\theta_j} e^{i\theta}]_m \oplus \operatorname{diag}[0, \ldots, 0]_{n-m} \in U_+(\omega)p$

for *any* choice of $\theta_1, \ldots, \theta_{m-1} \in \mathbb{R}$ and $\theta = \theta_m \in \mathbb{R}$. From this, and since $U_+(\omega)p \lhd U(pM_n p) \oplus \operatorname{diag}[0, \ldots, 0]_{n-m}$ has to hold (this follows because $\omega|_{pMp}$ is *tracial*), we obtain the relation $U_+(\omega)p = U(pM_n p) \oplus \operatorname{diag}[0, \ldots, 0]_{n-m}$. We may summarize.

Proposition 11.4 : *Assume M is a factor of type I. For a normal state ω such that* dim $s(\omega) = m < \infty$ (*and* $m < n$ *if* $M \simeq M_n$ *for some* $n \in \mathbb{N}$) *we have*

$$U_+(\omega)s(\omega) \simeq U(s(\omega)Ms(\omega)) \simeq \mathrm{U}(m). \tag{11.6}$$

For $n < \infty$ and $\omega = \tau_n^p$ with dim $p = m$ the proof is now complete by our preceding considerations and the case $m = 1$ has been considered in detail at the beginning of this section. The validity for a state ω with dim $s(\omega) = m$ follows by Theorem 8.7 and the fact that $\{\omega, \tau_n^{s(\omega)}\}$ is «-minimal (cf. Lemma 6.1). The latter remark also applies in the case $n = \infty$. Note that, for a projection p with dim $p = m < \infty$ and any projection q with dim $q = n > m$, for the state $\omega(\cdot) = \operatorname{tr} p(\cdot)$ we have that $\omega|_{qMq}$ corresponds to $\tau_n^{q'}$ with m-dimensional projection q' of M_n. Hence, and according to the above proven,

$$\mathrm{U}(m) \simeq U_+(\tau_n^{q'})q' \lesssim U_+(\omega)p \lesssim U(pMp) \simeq \mathrm{U}(m),$$

where $A \lesssim B$ indicates that B contains a subgroup isomorphic to A. From this $U_+(\omega)p \simeq U(pMp) \simeq \mathrm{U}(m)$ is obtained also in case that $n = \infty$.

Note that due to Theorem 11.3 in the cases $M \simeq M_n$, $n \in \mathbb{N}$, the suppositions under which Proposition 8.12 holds are given. Hence, following Proposition 8.12, Theorem 11.3 and Proposition 11.4 we may take notice of the following inclusions in case of a state ω with dim $s(\omega) = m \leq n$:

$$\mathrm{U}(m) \lesssim G_0(\omega) \lesssim \mathrm{SU}(n) \ . \tag{11.7}$$

As we know from a special case of Theorem 11.3, in the situation $m = n$ we have $G_0(\omega) \simeq \mathrm{SU}(n)$. On the other hand, from Theorem 11.2 in case of $m = 1 < n$ it is known that $G_0(\omega) \simeq \mathrm{U}(1)$. We are going to show that in the case dim $s(\omega) = m < n$ the left-hand side inclusion in (11.7) is not a proper one, actually, i.e. $\mathrm{U}(m) \simeq G_0(\omega)$ also occurs if $m > 1$. To prove this we remind Lemma 8.2 from which there follows that $G_0(\omega)$ is anti-isomorphic to some subgroup of $U(M'_\psi)$, where M'_ψ is the vN-algebra $M'_\psi = p'(\psi)M'p'(\psi)$ and $\psi \in S(\omega)$. Now, let us identify M_n with $M = M_n \otimes e$ over the Hilbert space $H = \mathbb{C}^n \otimes \mathbb{C}^n$. Then $M' = e \otimes M_n$. As usual we may content with treating the case of a $\omega = \tau_n^p$ over M_n, with dim $p = m$. Let $\{\varphi_1, \ldots, \varphi_m\}$ be a complete orthonormal system in $p\,\mathbb{C}^n$. Then the vector $\psi = m^{-1/2}\{\varphi_1 \otimes \varphi_1 + \ldots + \varphi_m \otimes \varphi_m\}$ belongs to $S(\omega)$ in the representation of M_n ($x \mapsto x \otimes e$ is the representation of M_n in which we work). Obviously $p'(\psi) = e \otimes p$, i.e. the relative dimension of $p'(\psi)$ with respect to M' is m, too. Therefore, $U(M'_\psi) \simeq \mathrm{U}(m)$, since $M'_\psi \simeq M_m$ in this case. On the other hand, as remarked above $G_0(\omega)$ is anti-isomorphic to some subgroup of $U(M'_\psi)$. But then there is also a group isomorhism from $G_0(\omega)$ into $U(M'_\psi)$ (remind that the "anti" in Lemma 8.2 comes only due to the special map considered there), and we have $G_0(\omega) \lesssim \mathrm{U}(m)$. By (11.7) now $\mathrm{U}(m) \simeq G_0(\omega)$ follows. It should be clear that this result persists to hold also in the case of an arbitrary type-I-factor (cf. the argumentation in the reasoning of Theorem 11.4). By our discussions of Section 9 we know that $U_\psi^O(\omega) \subset U_\psi(\omega)$ is a subgroup of $U_\psi(\omega)$. Finally, because of $U_\psi(\omega) \subset U(M'_\psi)$ and $U(M'_\psi) \simeq \mathrm{U}(m) \simeq G_0(\omega) \simeq U_\psi(\omega)$ the relation $G(\omega) \simeq G_0(\omega) \simeq U(m)$ follows.

Theorem 11.5 : *Assume that M is a factor of type I. For a normal state ω with* dim $s(\omega) = m < \infty$ *we have*

$$G(\omega) \simeq G_0(\omega) \begin{cases} \simeq \mathrm{U}(m) & \text{in case of } m < \dim e \\ \simeq \mathrm{SU}(m) & \text{in case of } m = \dim e \end{cases}$$

We remark that, by Theorem 11.5 and by means of the considerations which led us to Theorem 11.3, the complete classification of the holonomy group of a state ω on a finite dimensional vN-algebra can be described. In the notations of Theorem 11.3 the result is

$$G(\omega) = G_0(\omega) \simeq (\mathrm{S})\mathrm{U}(k_1) \times (\mathrm{S})\mathrm{U}(k_2) \times \cdots \times (\mathrm{S})\mathrm{U}(k_m), \tag{11.8}$$

with the convention $(\mathrm{S})\mathrm{U}(k_j) = \begin{cases} \mathrm{SU}(k_j) \text{ if } \dim s(\omega^{z_j}) = k_j = n_j \\ \mathrm{U}(k_j) \text{ if } \dim s(\omega^{z_j}) = k_j < n_j \end{cases}$ with $\dim s(\omega) = \sum_{j=1}^{m} k_j$.

ACKNOWLEDGEMENTS: For general discussions and hints as to the relevant literature I am very indebted especially to *Dr. Bernd Crell*, and also to *Prof. A. Uhlmann*, both from Physics Department of Leipzig University. Special discussions with *Dipl.-Phys. Matthias Hübner*, Physics Department, and with *Dipl.-Math. Volker Heinemann* and *Dr. Jochen Dittmann* from the Mathematical Department of Leipzig University are kindly acknowledged.

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Received 20.01.1992

Doz. Dr. Peter M. Alberti
Institut für Mathematik der Universität Leipzig
Augustusplatz 10
D (*Ost*) - 7010 Leipzig