

On Some Completion Problems for Various Subclasses of j_{pq} -inner Functions

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Several completion problems of the following type will be studied: Given meromorphic matrix-valued functions f and h , the question is to describe the set of all j_{pq} -inner functions W which have the block structure $W = \begin{pmatrix} * & f \\ * & h \end{pmatrix}$.

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0. Introduction

The starting point for the study of J -contractive matrix-valued functions was the famous paper [34] of V.P. Potapov, where the Riesz-Nevanlinna-Smirnov factorization theory of bounded holomorphic functions was generalized to meromorphic functions which are contractive with respect to some indefinite inner product generated by some signature matrix J . For several reasons, a particular subclass of J -contractive matrix-valued functions, namely the so-called J -inner functions, turned out to be very important. J -inner functions are immediately connected with lossless inverse scattering and Darlington synthesis (see, e.g., H. Dym [22] and the first author's papers [1-4]). Moreover, active research on matrix versions of classical interpolation problems has been indicated that under some non-degeneracy condition the set of solutions of such a problem can be described by a linear fractional transformation generated by some J -inner function which is only built from the original data (see, e.g., V.K. Dubovoj [17], H. Dym [22], I.V. Kovalishina [30] and the authors' papers [6-10, 13, 24, 25]). Naturally, this leads to so-called inverse problems for J -inner functions of the following type: Given a J -inner function W , one tries to find an appropriate matrix interpolation problem for which W is a resolvent matrix, i.e. for which the linear fractional transformation generated by W parametrizes the set of solutions of this matrix interpolation problem. In the context of generalized Schur-Nevanlinna-Pick interpolation, this topic was studied by the first author (see [6-10]). There are situations

in which only parts (blocks) of the resolvent matrix are known. Thus, there arises the problem of recovering the full resolvent matrix. This leads to various types of completion problems for J -inner functions. For the case of the special signature matrix j_{pq} , where $j_{pq} := \text{diag}(I_p, -I_q)$, such problems were first studied in the paper [5], which was motivated by a system-theoretical background and various questions of Darlington synthesis. Reflecting developments in matrix interpolation, the authors studied in [11] completion problems for various subclasses of j_{pq} -inner functions as j_{pq} -inner functions of Smirnov type, of inverse Smirnov type and A -singular j_{pq} -inner functions. Recent investigations indicated the important role of A -singular j_{pq} -inner functions in various contexts as matrix interpolation, factorization and completion problems (see V.E. Katsnelson [28, 29] and the authors' papers [6-10, 20]).

Completion problems for J -inner matrix-valued functions turn out to be intimately connected to various aspects of meromorphic pseudocontinuity. Meromorphic functions which admit a pseudocontinuation have been occurred in several areas: approximation of meromorphic functions by rational functions (see G.Ts. Tumarkin [36]), description of cyclic vectors of the backward shift operator in the Hardy space $H^2(\mathbb{D})$ (see R.G. Douglas, H.S. Shapiro and A.L. Shields [16]), factorization of non-negative Hermitian matrix-valued functions on the unit circle (see M. Rosenbaum and J. Rovnyak [35]) and realization of linear systems (see P.A. Fuhrmann [26]). In this paper, we will show that, similar as in [5] and [11], necessary and sufficient conditions for solvability of the completion problems under consideration can be expressed in terms of some properties of the pseudocontinuations of the given block functions. We will continue the investigations of [11]. In particular, we will treat the problem of determining all j_{pq} -inner functions with prescribed last block column (respectively, last block row), whereas in [11] we have only fixed the $p \times q$ block in the right upper corner (respectively, the $q \times p$ block in the left lower corner).

This paper is organized as follows. In Section 1, we will summarize basic facts on meromorphic matrix-valued functions in the unit disc. In Section 2, we will give a short survey on J -inner functions. Especially, we will turn our attention to the special case $J = j_{pq}$. Section 3 is the central one. It is aimed at studying the problem of describing all j_{pq} -inner functions which have a prescribed last block column (respectively, last block row). In Section 4, we will discuss restricted versions of the problems treated in Section 3. Hereby, we will concentrate on the subclass of all A -singular j_{pq} -inner functions. Finally, in Section 5, we will characterize the situation that a given matrix polynomial coincides with the last block column of some distinguished resolvent matrix associated with a non-degenerate matricial Schur problem. In particular, it will turn out that there is at most one resolvent matrix with the desired property.

1. Some basic facts on various classes of meromorphic matrix-valued functions in the unit disc

In this first section, we will summarize some facts on several classes of meromorphic functions. For a detailed treatment, we refer the reader to the monographs of R. Nevanlinna [31] and P.L. Duren [21]. We will start with some notations.

Throughout this paper, let p and q be positive integers. If m and n are non-negative integers with $m \leq n$, then \mathbb{N}_m^n stands for the set of all integers k with $m \leq k \leq n$, whereas \mathbb{N}_0 designates the set of all non-negative integers. We will use \mathbb{C} , \mathbb{D} , \mathbb{T} , $\hat{\mathbb{C}}$ and \mathbb{E}

to denote the set of complex numbers, the open unit disc, the unit circle, the extended complex plane and the exterior of the closed unit disc, respectively:

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}, \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}, \mathbb{E} := \hat{\mathbb{C}} \setminus (\mathbb{D} \cup \mathbb{T}).$$

Let $(\mathfrak{X}, \mathfrak{A}, \mu)$ be a measure space, and let \mathfrak{B} be the complex linear space of all $\mathfrak{A} - \mathfrak{B}_{p \times q}$ -measurable mappings $\Phi : \mathfrak{X} \rightarrow \mathbb{C}^{p \times q}$, where $\mathfrak{B}_{p \times q}$ is the σ -algebra of all Borelian subsets of $\mathbb{C}^{p \times q}$. Then

$$\mathfrak{Z} := \{\Phi \in \mathfrak{B} : \mu(\{\omega \in \mathfrak{X} : \Phi(\omega) \neq 0_{p \times q}\}) = 0\}$$

is a linear subspace of \mathfrak{B} . In the following, we will deal with the quotient space $\Omega := \mathfrak{B}/\mathfrak{Z}$. If $\Phi \in \mathfrak{B}$, then $\langle \Phi \rangle_\mu$ denotes that element of Ω which is generated by Φ . Obviously, $\langle \Phi \rangle_\mu = \langle \Psi \rangle_\mu$ if and only if $\Phi(\omega) = \Psi(\omega)$ for μ -almost all $\omega \in \mathfrak{X}$. We will mainly be concerned with the measure space $(\mathbb{T}, \mathfrak{B}, \frac{1}{2\pi} \tilde{\lambda})$, where \mathfrak{B} is the Borelian σ -algebra over \mathbb{T} , and where $\frac{1}{2\pi} \tilde{\lambda}$ is the normalized Lebesgue-Borel measure on \mathfrak{B} . We will shortly write $\langle \Phi \rangle$ for $\langle \Phi \rangle_{\tilde{\lambda}/2\pi}$.

Assume that G is a simply connected domain of $\hat{\mathbb{C}}$. Then let $\mathcal{NM}(G)$ be the Nevanlinna class of all functions which are meromorphic in G and which can be represented as quotient of two bounded holomorphic functions in G . Observe that $\mathcal{NM}(G)$ turns out to be a division algebra over \mathbb{C} . If $g \in \mathcal{NM}(\mathbb{D})$ (respectively, $g \in \mathcal{NM}(\mathbb{E})$), then a well-known theorem due to Fatou implies that there exist a Borelian subset \mathfrak{B}_0 of the unit circle \mathbb{T} with $\tilde{\lambda}(\mathfrak{B}_0) = 0$ and a Borel measurable function $\underline{g} : \mathbb{T} \rightarrow \mathbb{C}$ such that

$$\lim_{r \rightarrow 1-0} g(rz) = \underline{g}(z) \quad \left(\text{resp. } \lim_{r \rightarrow 1+0} g(rz) = \underline{g}(z) \right)$$

for all $z \in \mathbb{T} \setminus \mathfrak{B}_0$. In the following, we will continue to use the symbol \underline{g} to denote the boundary function of a function g which belongs to $\mathcal{NM}(\mathbb{D})$ or $\mathcal{NM}(\mathbb{E})$. The subalgebra of all $g \in \mathcal{NM}(G)$ which are holomorphic in G will be denoted by $\mathcal{N}(G)$. The class $\mathcal{N}(\mathbb{D})$ can be described as the set of all functions g which are holomorphic in \mathbb{D} and which fulfil

$$\sup_{r \in (0,1)} \frac{1}{2\pi} \int_{\mathbb{T}} \log^+ |g(rz)| \tilde{\lambda}(dz) < +\infty$$

where $\log^+ x := \max(\log x, 0)$ for each $x \in [0, \infty)$. A function $g \in \mathcal{N}(\mathbb{D})$ is said to be *outer* in $\mathcal{N}(\mathbb{D})$ if there exist a Borel measurable function $k : \mathbb{T} \rightarrow [0, \infty)$ and a number $\alpha \in \mathbb{T}$ such that the following two conditions are satisfied:

(i) $\frac{1}{2\pi} \int_{\mathbb{T}} |\log k| d\tilde{\lambda} < +\infty$.

(ii) $g(w) = \alpha \cdot \exp \left[\frac{1}{2\pi} \int_{\mathbb{T}} \frac{z+w}{z-w} \log k(z) \tilde{\lambda}(dz) \right], \quad w \in \mathbb{D}$.

If $g \in \mathcal{N}(\mathbb{D})$ is outer, then $|g| = k$ a.e. on \mathbb{T} . For all $g \in \mathcal{N}(\mathbb{D})$, the inequality

$$\frac{1}{2\pi} \int_{\mathbb{T}} \log^+ |\underline{g}(z)| \tilde{\lambda}(dz) \leq \lim_{r \rightarrow 1-0} \frac{1}{2\pi} \int_{\mathbb{T}} \log^+ |g(rz)| \tilde{\lambda}(dz) \tag{1}$$

holds true. By the *Smirnov class* $\mathcal{N}_+(\mathbb{D})$ we will mean the set of all $g \in \mathcal{N}(\mathbb{D})$ for which equality holds true in (1). The class $\mathcal{N}_+(\mathbb{D})$ proves to be a subalgebra of $\mathcal{N}(\mathbb{D})$. If g is outer in $\mathcal{N}(\mathbb{D})$, then g necessarily belongs to $\mathcal{N}_+(\mathbb{D})$.

If $s \in (0, \infty)$, then by the *Hardy class* $H^s(\mathbb{D})$ we will mean the set of all functions $g : \mathbb{D} \rightarrow \mathbb{C}$ which are holomorphic in \mathbb{D} and which satisfy

$$\sup_{r \in (0,1)} \frac{1}{2\pi} \int_{\mathbb{T}} |g(rz)|^s \tilde{\lambda}(dz) < +\infty,$$

whereas $H^\infty(\mathbb{D})$ designates the set of all holomorphic and bounded functions $g : \mathbb{D} \rightarrow \mathbb{C}$. Note that the inclusions

$$H^s(\mathbb{D}) \subset H^t(\mathbb{D}) \subset \mathcal{N}_+(\mathbb{D}) \subset \mathcal{N}(\mathbb{D}) \subset \mathcal{NM}(\mathbb{D})$$

hold true where $0 < t < s \leq \infty$.

Let $g \in \mathcal{NM}(\mathbb{D})$. Then one says that g admits a pseudocontinuation into \mathbb{E} if there exists a function $g^\# \in \mathcal{NM}(\mathbb{E})$ such that the radial boundary values \underline{g} and $\underline{g^\#}$ of g and $g^\#$, respectively, coincide Lebesgue-almost everywhere on the unit circle \mathbb{T} . It is obvious that a function $g \in \mathcal{NM}(\mathbb{D})$ admits at most one pseudocontinuation. Note that if $g \in \mathcal{NM}(\mathbb{D})$ admits a pseudocontinuation $g^\#$ and if, additionally, g is analytically continuable through some open arc of \mathbb{T} , then the analytic continuation coincides with $g^\#$. Later we will use some properties of pseudocontinuation which can be found in N.K. Nikolskii [32, Lecture II] and R.G. Douglas, H.S. Shapiro and A.L. Shields [16]. In the following, the notation $\Pi(\mathbb{D})$ stands for the set of all functions $g \in \mathcal{NM}(\mathbb{D})$ which admit a pseudocontinuation. If $g \in \Pi(\mathbb{D})$, then the symbol $g^\#$ will be used to denote the pseudocontinuation of g .

If \mathfrak{X} is one of the classes $\mathcal{NM}(G)$, $\mathcal{N}(G)$, $\mathcal{N}_+(\mathbb{D})$, $\Pi(\mathbb{D})$ or $H^s(\mathbb{D})$, where $s \in (0, \infty]$, then $p \times q - \mathfrak{X}$ designates the class of all $p \times q$ matrix-valued functions each entry of which belongs to \mathfrak{X} .

If $g = (g_{jk})_{\substack{j=1,\dots,p \\ k=1,\dots,q}}$ belongs to $p \times q - \Pi(\mathbb{D})$, then we will also say that g admits a pseudocontinuation. In this case, we will write $g^\#$ for $(g_{jk}^\#)_{\substack{j=1,\dots,p \\ k=1,\dots,q}}$ and call $g^\#$ the pseudocontinuation of g .

For convenience of the reader, we will recall some facts on outer functions which belong to the matricial Smirnov class (see [7, 10]). A function $\Phi \in q \times q - \mathcal{N}_+(\mathbb{D})$ is called *outer* (in $q \times q - \mathcal{N}_+(\mathbb{D})$) if $\det \Phi$ is outer in $\mathcal{N}(\mathbb{D})$. A function $\Phi \in q \times q - \mathcal{N}_+(\mathbb{D})$ is outer in $q \times q - \mathcal{N}_+(\mathbb{D})$ if and only if there exist outer functions $\Phi_1 \in q \times q - H^\infty(\mathbb{D})$ and $\varphi_2 \in H^\infty(\mathbb{D})$ such that $\Phi = \frac{1}{\varphi_2} \Phi_1$. If Φ is an outer function in $q \times q - \mathcal{N}_+(\mathbb{D})$, then $\det \Phi(z) \neq 0$ for all $z \in \mathbb{D}$ and Φ^{-1} is also an outer function in $q \times q - \mathcal{N}_+(\mathbb{D})$. Conversely, if $\Phi \in q \times q - \mathcal{N}_+(\mathbb{D})$ satisfies $\det \Phi(z) \neq 0$ for all $z \in \mathbb{D}$ and if $\Phi^{-1} \in q \times q - \mathcal{N}_+(\mathbb{D})$, then Φ and Φ^{-1} are necessarily outer functions in $q \times q - \mathcal{N}_+(\mathbb{D})$. If $\Phi \in q \times q - \mathcal{N}_+(\mathbb{D})$ and $\Psi \in q \times q - \mathcal{N}_+(\mathbb{D})$ are outer, then the product $\Phi\Psi$ is also an outer function in $q \times q - \mathcal{N}_+(\mathbb{D})$. An outer function $\Phi \in q \times q - \mathcal{N}_+(\mathbb{D})$ is called *normalized* if $\Phi(0)$ is non-negative Hermitian. A $q \times q$ matrix-valued function θ which belongs to the Hardy class $q \times q - H^2(\mathbb{D})$ is outer if and only if $\det \theta$ is an outer function in $H^{2/q}(\mathbb{D})$. If $R \in (1, \infty)$, if Σ is a $q \times q$ matrix-valued function holomorphic in $K(0, R) := \{z \in \mathbb{C} : |z| < R\}$ and if $\det \Sigma(z) \neq 0$ for each $z \in K(0, R)$, then the restriction θ of Σ onto \mathbb{D} is an outer function in $q \times q - H^\infty(\mathbb{D})$. Let $\mathbb{C}_\geq^{q \times q}$ be the set of all $q \times q$ non-negative Hermitian matrices. If $\Lambda : \mathbb{T} \rightarrow \mathbb{C}_\geq^{q \times q}$ is Lebesgue integrable on \mathbb{T} and if Λ satisfies

$$\frac{1}{2\pi} \int_{\mathbb{T}} \log \det \Lambda \, d\tilde{\lambda} > -\infty,$$

then there exist unique normalized outer functions Φ and Ψ which belong to $q \times q - H^2(\mathbb{D})$ such that $\langle \Lambda \rangle = \langle \underline{\Phi} \Phi^* \rangle$ and $\langle \Lambda \rangle = \langle \Psi^* \underline{\Psi} \rangle$ (see, e.g., N. Wiener and P.R. Masani [37]).

A function $f : \mathbb{D} \rightarrow \mathbb{C}^{p \times q}$ is said to be a $p \times q$ Schur function if f is both holomorphic and contractive in \mathbb{D} . The class of all $p \times q$ Schur functions will be denoted by $S_{p \times q}(\mathbb{D})$. A function $f \in S_{q \times q}(\mathbb{D})$ is called *inner* if f has unitary boundary values a.e. on \mathbb{T} .

Let $f \in p \times q - \mathcal{NM}(\mathbb{D})$. An ordered pair $[B_1, B_2]$ consisting of a $p \times p$ inner function B_1 and a $q \times q$ inner function B_2 is said to be a *denominator* of f if $B_1 f B_2$ belongs to $p \times q - \mathcal{N}_+(\mathbb{D})$. A $p \times p$ inner function B_1 is called a *left denominator* of f if $[B_1, I_q]$ is a denominator of f , whereas a $q \times q$ inner function B_2 is said to be a *right denominator* of f if $[I_p, B_2]$ is a denominator of f .

If $\varphi \in \mathcal{NM}(\mathbb{D})$, then $\varphi = g/h$ with some functions g and h which belong to $H^\infty(\mathbb{D})$. If $h = bh_0$ is an inner-outer factorization of h with some inner function b and some outer function h_0 , then $b\varphi$ belongs to $\mathcal{N}_+(\mathbb{D})$. Hence, one can easily see that, for each $f \in p \times q - \mathcal{NM}(\mathbb{D})$, there is a complex-valued inner function β such that βI_p and βI_q are a left denominator and a right denominator of f , respectively. In particular, each $f \in p \times q - \mathcal{NM}(\mathbb{D})$ has left and right denominators.

Let \mathfrak{x} be a non-empty subset of the extended complex plane $\hat{\mathbb{C}}$, and let $f : \mathfrak{x} \rightarrow \mathbb{C}^{p \times q}$. Then we will use the symbol \hat{f} for the function $\hat{f} : \mathfrak{y} \rightarrow \mathbb{C}^{q \times p}$ which is given by $\mathfrak{y} := \{z \in \hat{\mathbb{C}} : 1/\bar{z} \in \mathfrak{x}\}$ and $\hat{f}(z) := [f(1/\bar{z})]^*$.

Remark 1: (a) Let $f \in p \times q - \mathcal{NM}(\mathbb{D})$. Then $\hat{f} \in q \times p - \mathcal{NM}(\mathbb{E})$ and \underline{f}^* is a boundary function of \hat{f} . (b) Let $f \in p \times q - \mathcal{NM}(\mathbb{E})$. Then $\hat{f} \in q \times p - \mathcal{NM}(\mathbb{D})$ and f^* is a boundary function of \hat{f} .

The next lemma which is taken from [11] will play a key role in the following.

Lemma 1: Suppose that $f \in p \times q - \mathcal{NM}(\mathbb{D})$ admits a pseudocontinuation $f^\#$. Then:

- (a) The functions $\Delta_1 := I + f\hat{f}^\#$ and $\Delta_2 := I + \hat{f}^\# f$ belong to $p \times p - \mathcal{NM}(\mathbb{D})$ and $q \times q - \mathcal{NM}(\mathbb{D})$, respectively, and have non-identically vanishing determinants.
- (b) There are unique normalized outer functions $\varphi_1, \Psi_1 \in p \times p - \mathcal{N}_+(\mathbb{D})$ and $\varphi_2, \Psi_2 \in q \times q - \mathcal{N}_+(\mathbb{D})$ such that

$$\langle I + \underline{f} \underline{f}^* \rangle = \langle \underline{\varphi}_1^* \underline{\varphi}_1 \rangle, \quad \langle I + \underline{f} \underline{f}^* \rangle = \langle \underline{\Psi}_1 \underline{\Psi}_1^* \rangle \tag{2}$$

and

$$\langle I + \underline{f}^* \underline{f} \rangle = \langle \underline{\varphi}_2^* \underline{\varphi}_2 \rangle, \quad \langle I + \underline{f}^* \underline{f} \rangle = \langle \underline{\Psi}_2 \underline{\Psi}_2^* \rangle \tag{3}$$

- (c) Each of the functions $\varphi_1, \varphi_2, \Psi_1$ and Ψ_2 admits a pseudocontinuation, namely

$$\varphi_1^\# := \widehat{\varphi_1^{-1}} \widehat{\Delta_1}, \quad \Psi_1^\# := \widehat{\Delta_1} \widehat{\Psi_1^{-1}}, \quad \varphi_2^\# := \widehat{\varphi_2^{-1}} \widehat{\Delta_2}, \quad \Psi_2^\# := \widehat{\Delta_2} \widehat{\Psi_2^{-1}}.$$

2. On J -inner functions

Let m be a positive integer. Suppose that J is an $m \times m$ signature matrix, i.e. an $m \times m$ complex matrix with $J = J^*$ and $J^2 = I_m$. If $A \in \mathbb{C}^{m \times m}$ satisfies $A^* J A \leq J$ (respectively $A^* J A = J$), then A is called J -contractive (respectively, J -unitary).

If G is a simply connected domain of the extended complex plane $\hat{\mathbb{C}}$, then we will use $\mathfrak{P}_J(G)$ for the *Potapov class*, i.e. the set of all $m \times m$ matrix-valued functions W which satisfy the following three conditions:

- (i) W is meromorphic in G .
- (ii) $\det W$ does not identically vanish in G .
- (iii) $W(z)$ is J -contractive for all z which belong to the set $\mathbb{H}(W)$ of all points of analyticity of W .

Obviously, the class $\mathfrak{P}_J(G)$ is multiplicative, i.e. if W_1 and W_2 belong to $\mathfrak{P}_J(G)$, then the product W_1W_2 belongs to $\mathfrak{P}_J(\mathbb{D})$ as well. The Potapov class $\mathfrak{P}_J(\mathbb{D})$ is a subclass of $m \times m - \mathcal{NM}(\mathbb{D})$ (see, e.g., H. Dym [22, Corollary 2]). The boundary function \underline{W} of $W \in \mathfrak{P}_J(\mathbb{D})$ is J -contractive, i.e. $\underline{W}^*J\underline{W} \leq J$ holds true a.e. on \mathbb{T} . A function $W \in \mathfrak{P}_J(\mathbb{D})$ is called J -inner if \underline{W} is J -unitary, i.e. if $\underline{W}^*J\underline{W} = J$ is fulfilled a.e. on \mathbb{T} .

Remark 2: Every J -inner function W admits a pseudocontinuation $W^\#$ which is given by

$$W^\#(z) := J((W(1/\bar{z}))^*)^{-1}J$$

for all $z \in \mathbb{E}$ with $1/\bar{z} \in \mathbb{H}(W)$ and $\det W(1/\bar{z}) \neq 0$.

If W_1 and W_2 are J -inner, then the product $W_1 \cdot W_2$ is J -inner as well.

Later we will be concerned with some distinguished subclass of J -inner functions, namely the set of so-called J -elementary factors. An $m \times m$ matrix-valued function B , which is meromorphic in $\hat{\mathbb{C}}$, is called J -elementary factor if the following three conditions are satisfied:

- (i) B has exactly one pole $z_0 \in \hat{\mathbb{C}}$.
- (ii) For each $z \in \mathbb{D} \setminus \{z_0\}$, the matrix $B(z)$ is J -contractive.
- (iii) For each $z \in \mathbb{T} \setminus \{z_0\}$, the matrix $B(z)$ is J -unitary.

Obviously, conditions (ii) and (iii) mean that the restriction $\text{Rstr}_{\mathbb{D} \setminus \{z_0\}} B$ of B onto $\mathbb{D} \setminus \{z_0\}$ is a J -inner function. Furthermore, (i) implies that every J -elementary factor is a non-constant rational matrix-valued function. The following lemma, which is taken from [19, Proposition 4.2.1], summarizes some properties of J -elementary factors.

Lemma 2: Let B be a J -elementary factor with pole in $z_0 \in \hat{\mathbb{C}}$. Then:

- (a) For each $z \in \mathbb{C} \setminus \{z_0, 1/\bar{z}_0\}$,

$$B(z)JB^*(1/\bar{z}) = J, \det B(z) \neq 0 \quad \text{and} \quad B^{-1}(z) = JB^*(1/\bar{z})J.$$

- (b) B^{-1} is a $(-J)$ -elementary factor with pole in $1/\bar{z}_0$.

A J -inner function W is said to be *singular* if both W and W^{-1} are holomorphic in \mathbb{D} . The product of two singular J -inner functions is also a singular J -inner function.

Recent investigations of the first author (see [6-10]) indicated that there is an important subclass of singular J -inner functions which satisfy some growth conditions. This led him to the following object. A J -inner function W is said to be A -singular if both W and W^{-1} belong to the Smirnov class $m \times m - \mathcal{N}_+(\mathbb{D})$. Obviously, every A -singular J -inner function is singular. Note that a J -inner function W is A -singular if and only if W is an outer

function in $m \times m - \mathcal{N}_+(\mathbb{ID})$. If W_1 and W_2 are A -singular J -inner functions, then W_1W_2 is also A -singular. Moreover, we observe that in the cases $J = I_m$ and $J = -I_m$ every A -singular J -inner function is necessarily some J -unitary constant function. However, in the indefinite case $J \neq \pm I_m$ the class of A -singular J -inner functions is very rich (see V.E. Katsnelson [28, 29]). Observe that the first author used in [6 - 10] the notion "singular" instead of "A-singular". By a suggestion of V.E. Katsnelson it is now common to use the notion "A-singular".

Now we will specify the signature matrix. Namely, we will consider with the $(p + q) \times (p + q)$ signature matrix

$$j_{pq} := \text{diag} (I_p, -I_q) . \tag{4}$$

In the following, we will be concerned with functions W which belong to the Potapov class $\mathfrak{P}_{j_{pq}}(\mathbb{ID})$. In this case, we will work with the block partition

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \tag{5}$$

where W_{11} is a $p \times p$ matrix-valued function. The following proposition (see J.P. Ginzburg [27], P. Dewilde and H. Dym [15] and the first author's papers [2] and [4]) gives a summary of some basic facts of functions belonging to $\mathfrak{P}_{j_{pq}}(\mathbb{ID})$.

Proposition 1: *Let $W \in \mathfrak{P}_{j_{pq}}(\mathbb{ID})$. Then:*

- (a) *For each $z \in \mathbb{H}(W)$, the matrix $W_{22}(z)$ is non-singular.*
- (b) *The function*

$$S := \begin{pmatrix} W_{11} - W_{12}W_{22}^{-1}W_{21} & W_{12}W_{22}^{-1} \\ -W_{22}^{-1}W_{21} & W_{22}^{-1} \end{pmatrix} \tag{6}$$

belongs to $\mathcal{S}_{(p+q) \times (p+q)}(\mathbb{ID})$. In particular,

$$S_{11} := W_{11} - W_{12}W_{22}^{-1}W_{21} , \quad S_{12} := W_{12}W_{22}^{-1} \tag{7}$$

and

$$S_{21} := -W_{22}^{-1}W_{21} , \quad S_{22} := W_{22}^{-1} \tag{8}$$

are matrix-valued Schur functions, whereby S_{12} and S_{21} are strictly contractive. (A $p \times q$ Schur function S is called strictly contractive if $I - S(z)S^(z) > 0$ for each $z \in \mathbb{ID}$.)*

- (c) *The functions $\det S_{11}$ and $\det S_{22}$ do not identically vanish.*
- (d) $\mathbb{H}(W) = \{z \in \mathbb{ID} : \det S_{22}(z) \neq 0\}$.
- (e) *If W is j_{pq} -inner, then S is inner.*
- (f) *There are unique normalized outer functions $\Phi_1 \in \mathcal{S}_{p \times p}(\mathbb{ID})$ and $\Psi_1 \in \mathcal{S}_{p \times p}(\mathbb{ID})$ and unique inner functions $b_1 \in \mathcal{S}_{p \times p}(\mathbb{ID})$ and $c_1 \in \mathcal{S}_{p \times p}(\mathbb{ID})$ such that*

$$S_{11} = \Phi_1 b_1 \quad \text{and} \quad S_{11} = c_1 \Psi_1 . \tag{9}$$

(g) There are unique normalized outer functions $\Phi_2 \in \mathcal{S}_{q \times q}(\mathbb{D})$ and $\Psi_2 \in \mathcal{S}_{q \times q}(\mathbb{D})$ and unique inner functions $b_2 \in \mathcal{S}_{q \times q}(\mathbb{D})$ and $c_2 \in \mathcal{S}_{q \times q}(\mathbb{D})$ such that

$$S_{22} = \Phi_2 b_2 \quad \text{and} \quad S_{22} = c_2 \Psi_2. \tag{10}$$

(h) If W is j_{pq} -inner, then the pseudocontinuation $W^\#$ of W is given by

$$W^\# = \begin{pmatrix} \hat{S}_{11}^{-1} & -\hat{S}_{11}^{-1} \hat{S}_{21} \\ \hat{S}_{12} \hat{S}_{11}^{-1} & \hat{S}_{22} - \hat{S}_{12} \hat{S}_{11}^{-1} \hat{S}_{21} \end{pmatrix}$$

Note that the matrix-valued Schur function S defined by (6) is often called the *Potapov-Ginzburg transform* of the j_{pq} -inner function W .

The following theorem, which is taken from [11], characterizes two important subsets of the Potapov class $\mathfrak{P}_{j_{pq}}(\mathbb{D})$.

Theorem 1: Let $W \in \mathfrak{P}_{j_{pq}}(\mathbb{D})$, and let (5) be the block partition of W where W_{11} is a $p \times p$ block. Then:

(a) The following statements are equivalent:

- (i) $W \in (p + q) \times (p + q) - \mathcal{N}_+(\mathbb{D})$.
- (ii) $W_{22} \in q \times q - \mathcal{N}_+(\mathbb{D})$.
- (iii) W_{22} is an outer function in $q \times q - \mathcal{N}_+(\mathbb{D})$.
- (iv) W_{22}^{-1} is an outer function (in $\mathcal{S}_{q \times q}(\mathbb{D})$).

(b) The following statements are equivalent:

- (v) $W^{-1} \in (p + q) \times (p + q) - \mathcal{N}_+(\mathbb{D})$.
- (vi) $(W_{11} - W_{12}W_{22}^{-1}W_{21})^{-1} \in p \times p - \mathcal{N}_+(\mathbb{D})$.
- (vii) $(W_{11} - W_{12}W_{22}^{-1}W_{21})^{-1}$ is an outer function in $p \times p - \mathcal{N}_+(\mathbb{D})$.
- (viii) $W_{11} - W_{12}W_{22}^{-1}W_{21}$ is an outer function (in $\mathcal{S}_{p \times p}(\mathbb{D})$).

A j_{pq} -inner function W is said to be of *Smirnov type* (respectively, of *inverse Smirnov type*) if W (respectively, W^{-1}) belongs to $(p + q) \times (p + q) - \mathcal{N}_+(\mathbb{D})$. Thus, Theorem 1 provides necessary and sufficient conditions for the fact that a given j_{pq} -inner function W is of Smirnov type (respectively, of inverse Smirnov type). It should be mentioned that special j_{pq} -inner functions of Smirnov type have been investigated in P. Dewilde and H. Dym [14, 15]. However, these authors made not use of the fact that their objects are members of the Smirnov class.

Now we are able to characterize A -singular j_{pq} -inner functions.

Theorem 2: Let W be a j_{pq} -inner function, and let (5) be the block partition of W where W_{11} is a $p \times p$ block. Then the following statements are equivalent:

- (i) W is an A -singular j_{pq} -inner function.
- (ii) W is an outer function in $(p + q) \times (p + q) - \mathcal{N}_+(\mathbb{D})$.

(iii) $W_{11} - W_{12}W_{22}^{-1}W_{21}$ and W_{22}^{-1} are outer matrix-valued Schur functions.

Proof: (i) \Leftrightarrow (ii) is clear from the above mentioned properties of outer matrix-valued function on the Smirnov class. (i) \Leftrightarrow (iii) follows by application of Theorem 2 ■

Remark 3: Let A be a j_{pq} -unitary matrix, and let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ be the block partition of A with $p \times p$ block A_{11} . Then:

- (a) A_{11} and A_{22} are non-singular.
- (b) $A_{11} - A_{12}A_{22}^{-1}A_{21}$ and $A_{22} - A_{21}A_{11}^{-1}A_{12}$ are non-singular.
- (c) $(A_{11}^*)^{-1} = A_{11} - A_{12}A_{22}^{-1}A_{21}$, $(A_{22}^*)^{-1} = A_{22} - A_{21}A_{11}^{-1}A_{12}$.
- (d) $A_{21} = A_{22}A_{12}^*(A_{11}^*)^{-1} = A_{22}^*(A_{12}^*)^{-1}A_{11}$.
- (e) $A_{12} = (A_{11}^*)^{-1}A_{21}^*A_{22} = A_{11}A_{21}^*(A_{22}^*)^{-1}$.
- (f) $A_{22}^*A_{22} - A_{12}^*A_{12} = I$.
- (g) $A_{22}A_{22}^* - A_{21}A_{21}^* = I$.

Proposition 2: Let W be a j_{pq} -inner function with block partition (5) where W_{11} is a $p \times p$ block. Denote $W^\#$ the pseudocontinuation of W . Let $\Phi_1, \Psi_1, \Phi_2, \Psi_2$ and b_1, c_1, b_2, c_2 be the normalized outer Schur functions and inner functions, respectively, given in part (f) and (g) of Proposition 1. Then:

- (a) The functions $\varphi_1 := \Phi_1^{-1}$, $\psi_1 := \Psi_1^{-1}$ and $\varphi_2 := \Phi_2^{-1}$, $\psi_2 := \Psi_2^{-1}$ are normalized outer functions which belong to $p \times p - \mathcal{N}_+(\mathbb{D})$ and $q \times q - \mathcal{N}_+(\mathbb{D})$, respectively, and which satisfy the factorizations

$$\langle \underline{\varphi_1}^* \varphi_1 \rangle = \langle I_p + \underline{W_{12}} \underline{W_{12}^*} \rangle, \quad \langle \underline{\psi_1} \psi_1^* \rangle = \langle I_p + \underline{W_{21}^*} \underline{W_{21}} \rangle \tag{11}$$

$$\langle \underline{\varphi_2}^* \varphi_2 \rangle = \langle I_q + \underline{W_{12}^*} \underline{W_{12}} \rangle, \quad \langle \underline{\psi_2} \psi_2^* \rangle = \langle I_q + \underline{W_{21}} \underline{W_{21}^*} \rangle. \tag{12}$$

- (b) The functions $\varphi_1, \psi_1, \varphi_2$ and ψ_2 admit pseudocontinuations, which satisfy

$$\begin{aligned} W_{11} &= \widehat{\varphi_1^\#} b_1, & W_{11} &= c_1 \widehat{\psi_1^\#} \\ W_{22} &= \widehat{\varphi_2^\#} b_2, & W_{22} &= c_2 \widehat{\psi_2^\#}. \end{aligned}$$

- (c) The following identities hold true:

$$\begin{aligned} \widehat{W_{12}^\#} \varphi_1^{-1} b_1 &= W_{22}^{-1} W_{21}, & c_1 \psi_1^{-1} \widehat{W_{21}^\#} &= W_{12} W_{22}^{-1} \\ W_{12} \varphi_2^{-1} b_2 &= W_{12} W_{22}^{-1}, & c_2 \psi_2^{-1} W_{21} &= W_{22}^{-1} W_{21}. \end{aligned}$$

In particular, b_1 and b_2 are right denominators of $\widehat{W_{12}^\#} \varphi_1^{-1}$ and $W_{12} \varphi_2^{-1}$, respectively, whereas c_1 and c_2 are left denominators of $\psi_1^{-1} \widehat{W_{21}^\#}$ and $\psi_2^{-1} W_{21}$, respectively.

(d) The j_{pq} -inner function W admits the representations

$$W = \text{diag} (I_p, b_2^{-1}) \cdot \begin{pmatrix} \widehat{\varphi_1^\#} & W_{12} \\ \widehat{\varphi_2 W_{12}^\# \varphi_1^{-1}} & \varphi_2 \end{pmatrix} \cdot \text{diag} (b_1, I_q)$$

$$W = \text{diag} (c_1, I_q) \cdot \begin{pmatrix} \widehat{\psi_1^\#} & \psi_1^{-1} \widehat{W_{21}^\#} \psi_2 \\ W_{21} & \psi_2 \end{pmatrix} \cdot \text{diag} (I_p, c_2^{-1}) .$$

(e) If W is A -singular, then

$$\begin{aligned} W_{12} \varphi_2^{-1} &\in \mathcal{S}_{p \times q}(\mathbb{D}), & \psi_2^{-1} W_{21} &\in \mathcal{S}_{q \times p}(\mathbb{D}) \\ \widehat{W_{12}^\#} \varphi_1^{-1} &\in \mathcal{S}_{q \times p}(\mathbb{D}), & \psi_1^{-1} \widehat{W_{21}^\#} &\in \mathcal{S}_{p \times q}(\mathbb{D}) \\ W_{12} &\in p \times q - \mathcal{N}_+(\mathbb{D}), & \widehat{W_{12}^\#} &\in q \times p - \mathcal{N}_+(\mathbb{D}) \\ W_{21} &\in q \times p - \mathcal{N}_+(\mathbb{D}), & \widehat{W_{21}^\#} &\in p \times q - \mathcal{N}_+(\mathbb{D}). \end{aligned}$$

A proof of Proposition 2 can be found in [11].

3. Some completion problems for j_{pq} -inner functions with prescribed block column or block row

Now we are going to study some specified completion problems for j_{pq} -inner functions. First we note that if W is a j_{pq} -inner function with block partition (5) where W_{11} is a $p \times p$ block, then

$$W_{11} \in p \times p - \mathcal{NM}(\mathbb{D}), \quad W_{12} \in p \times q - \mathcal{NM}(\mathbb{D})$$

and

$$W_{21} \in q \times p - \mathcal{NM}(\mathbb{D}), \quad W_{22} \in q \times q - \mathcal{NM}(\mathbb{D}).$$

The central completion problems we will discuss in our paper can be formulated as follows.

(CPC) Let $f \in p \times q - \mathcal{NM}(\mathbb{D})$, and let $h \in q \times q - \mathcal{NM}(\mathbb{D})$. Describe the set $\mathcal{J}_c(f, h)$ of all j_{pq} -inner functions W such that $W_{12} = f$ and $W_{22} = h$. In particular, characterize the case that $\mathcal{J}_c(f, h)$ is non-empty.

(CPR) Let $g \in q \times p - \mathcal{NM}(\mathbb{D})$, and let $h \in q \times q - \mathcal{NM}(\mathbb{D})$. Describe the set $\mathcal{J}_r(g, h)$ of all j_{pq} -inner functions W such that $W_{21} = g$ and $W_{22} = h$. In particular, characterize the case that $\mathcal{J}_r(g, h)$ is non-empty.

We are immediately able to give some necessary conditions for the existence of solutions of Problems (CPC) and (CPR).

Proposition 3: *The following statements are true.*

(a) *Let $f \in p \times q - \mathcal{NM}(\mathbb{D})$ and $h \in q \times q - \mathcal{NM}(\mathbb{D})$ be such that $\mathcal{J}_c(f, h)$ is non-empty. Then:*

- (i) *det h does not identically vanish.*
- (ii) *$h^{-1} \in \mathcal{S}_{q \times q}(\mathbb{D})$, $fh^{-1} \in \mathcal{S}_{p \times q}(\mathbb{D})$.*
- (iii) *f and h admit pseudocontinuations $f^\#$ and $h^\#$ which satisfy*

$$I + \widehat{f^\#} f = \widehat{h^\#} h. \tag{13}$$

(b) *Let $g \in q \times p - \mathcal{NM}(\mathbb{D})$ and $h \in q \times q - \mathcal{NM}(\mathbb{D})$ be such that $\mathcal{J}_r(g, h)$ is non-empty. Then:*

- (iv) *det h does not identically vanish.*
- (v) *$h^{-1} \in \mathcal{S}_{q \times q}(\mathbb{D})$, $h^{-1}g \in \mathcal{S}_{q \times p}(\mathbb{D})$.*
- (vi) *g and h admit pseudocontinuations $g^\#$ and $h^\#$ which satisfy*

$$I + g \widehat{g^\#} = h \widehat{h^\#}. \tag{14}$$

Proof: Statements (i), (ii), (iv) and (v) are immediate consequences of Proposition 1. Because of Remark 2, the functions f , h and g have pseudocontinuations. If $W \in \mathcal{J}_c(f, h)$ (respectively, $W \in \mathcal{J}_r(g, h)$), then W has j_{pq} -unitary boundary values a.e. on \mathbf{T} . Taking into account that all the functions $I + \widehat{f^\#} f$, $I + \widehat{h^\#} h$, $I + g \widehat{g^\#}$ and $h \widehat{h^\#}$ belong to $q \times q - \mathcal{NM}(\mathbb{D})$, then the application of parts (f) and (g) of Remark 3 and Remark 1 provides (iii) and (vi) ■

Remark 4: In view of Remark 1 and the fact that the Nevanlinna class $\mathcal{NM}(\mathbb{D})$ is an algebra one can easily verify the following:

(a) Let $f \in p \times q - \mathcal{NM}(\mathbb{D})$ and $h \in q \times q - \mathcal{NM}(\mathbb{D})$ be such that the following two conditions are satisfied:

- (i) f admits a pseudocontinuation $f^\#$.
- (ii) $\langle \underline{h^*} \underline{h} \rangle = \langle I + \underline{f^*} \underline{f} \rangle$.

Then h admits a pseudocontinuation $h^\#$, namely $h^\# = \hat{h}^{-1}(I + \hat{f}f^\#)$.

(b) Let $g \in q \times p - \mathcal{NM}(\mathbb{D})$ and $h \in q \times q - \mathcal{NM}(\mathbb{D})$ be such that the following conditions are satisfied:

- (i) g admits a pseudocontinuation $g^\#$.
- (ii) $\langle \underline{h} \underline{h^*} \rangle = \langle I + \underline{g} \underline{g^*} \rangle$.

Then h admits a pseudocontinuation $h^\#$, namely $h^\# = (I + g^\# \hat{g}) \hat{h}^{-1}$.

The following lemma due to the first author (see [2]) will turn out to play a key role in our considerations.

Lemma 3: Let $W \in (p + q) \times (p + q) - \mathcal{NM}(\mathbb{D})$ be such that

$$\langle j_{pq} - \underline{W}^* j_{pq} \underline{W} \rangle = \langle 0 \rangle, \tag{15}$$

and let (5) be the block partition of W where W_{11} is a $p \times p$ matrix-valued function. Then $\det W_{22}$ does not identically vanish. If S given by (6) belongs to $(p + q) \times (p + q) - \mathcal{N}_+(\mathbb{D})$, then W is a j_{pq} -inner function.

Now we are able to give a complete answer to Problem (CPS).

Theorem 3: Let $f \in p \times q - \mathcal{NM}(\mathbb{D})$, and let $h \in q \times q - \mathcal{NM}(\mathbb{D})$. Then:

(a) $\mathcal{J}_c(f, h)$ is non-empty if and only if the functions f and h satisfy

$$\langle \underline{h}^* \underline{h} \rangle = \langle I + \underline{f}^* \underline{f} \rangle \tag{16}$$

$$h^{-1} \in q \times q - \mathcal{N}_+(\mathbb{D}) \tag{17}$$

$$fh^{-1} \in p \times q - \mathcal{N}_+(\mathbb{D}) \tag{18}$$

$$f \in p \times q - \Pi(\mathbb{D}). \tag{19}$$

(b) Suppose that $\mathcal{J}_c(f, h)$ is non-empty. Let ϕ_1 be the unique normalized outer function belonging to $p \times p - \mathcal{N}_+(\mathbb{D})$ such that

$$\langle \underline{\phi}_1^* \underline{\phi}_1 \rangle = \langle I + \underline{f} \underline{f}^* \rangle. \tag{20}$$

Then a $(p + q) \times (p + q)$ matrix-valued function W which is meromorphic in \mathbb{D} belongs to $\mathcal{J}_c(f, h)$ if and only if there exists a right denominator b_1 of $\widehat{f^\# \phi_1^{-1}}$ such that

$$W = \begin{pmatrix} \widehat{\phi_1^\#} & f \\ h \widehat{f^\# \phi_1^{-1}} & h \end{pmatrix} \cdot \text{diag} (b_1, I_q). \tag{21}$$

In this case, b_1 is unique.

Proof: If $\mathcal{J}_c(f, h) \neq \emptyset$, then Proposition 3 and Remark 1 yield that (16)-(19) hold true. Conversely, now suppose that (16)-(19) are satisfied. Equation (16) implies that $\det h$ does not identically vanish. Thus, conditions (17) and (18) have sense. We want to apply Lemma 3. Let $f^\#$ be the pseudocontinuation of f , and let b_1 be a right denominator of $\widehat{f^\# \phi_1^{-1}}$. In view of Lemma 1, by the formula

$$W = \begin{pmatrix} \widehat{\phi_1^\#} & f \\ h \widehat{f^\# \phi_1^{-1}} & h \end{pmatrix} \cdot \text{diag} (b_1, I_q). \tag{22}$$

a well-defined function W is given. Since $\mathcal{NM}(\mathbb{D})$ is an algebra over \mathbb{C} , W belongs to $(p + q) \times (p + q) - \mathcal{NM}(\mathbb{D})$. Let (5) be the block partition of W where W_{11} is a $p \times p$ block. Then we see immediately that $W_{12} = f$ and $W_{22} = h$. Now we will show that W has j_{pq} -unitary boundary values a.e. on \mathbb{T} . Let

$$X := \underline{W}^* j_{pq} \underline{W},$$

and let

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

be the block partition of X where X_{11} is a $p \times p$ block. From (16) and (20) we obtain

$$\langle \underline{f} \underline{h}^* \underline{h} \underline{f}^* \rangle = \langle \underline{f} (I + \underline{f}^* \underline{f}) \underline{f}^* \rangle = \langle \underline{f} \underline{f}^* (I + \underline{f} \underline{f}^*) \rangle = \langle \underline{f} \underline{f}^* \underline{\phi}_1^* \underline{\phi}_1 \rangle .$$

Hence,

$$\begin{aligned} \langle X_{11} \rangle &= \left\langle (\underline{b}_1^* \underline{\phi}_1) (\underline{\phi}_1^* \underline{b}_1) - (\underline{b}_1^* ((\underline{\phi}_1)^{-1})^* \underline{f} \underline{h}) (\underline{h}^* \underline{f}^* (\underline{\phi}_1)^{-1} \underline{b}_1) \right\rangle \\ &= \left\langle \underline{b}_1^* ((\underline{\phi}_1)^{-1})^* (\underline{\phi}_1^* \underline{\phi}_1 \underline{\phi}_1^* - \underline{f} \underline{h} \underline{h}^* \underline{f}^* (\underline{o}_1)^{-1}) \underline{b}_1 \right\rangle \\ &= \left\langle \underline{b}_1^* ((\underline{\phi}_1)^{-1})^* [(I + \underline{f} \underline{f}^*) \underline{\phi}_1^* - \underline{f} \underline{f}^* \underline{\phi}_1^-] \underline{b}_1 \right\rangle \\ &= \langle \underline{b}_1^* \underline{b}_1 \rangle = \langle I_p \rangle . \end{aligned}$$

Similarly,

$$\begin{aligned} \langle X_{12} \rangle &= \left\langle \underline{b}_1^* \underline{\phi}_1 \underline{f} - \underline{b}_1^* ((\underline{\phi}_1)^{-1})^* \underline{f} \underline{h}^* \underline{h} \right\rangle \\ &= \left\langle \underline{b}_1^* ((\underline{\phi}_1)^{-1})^* (\underline{\phi}_1^* \underline{\phi}_1 \underline{f} - \underline{f} \underline{h}^* \underline{h}) \right\rangle \\ &= \left\langle \underline{b}_1^* ((\underline{\phi}_1)^{-1})^* [(I + \underline{f} \underline{f}^*) \underline{f} - \underline{f} (I + \underline{f}^* \underline{f})] \right\rangle \\ &= \langle 0_{p \times q} \rangle \end{aligned}$$

and

$$\langle X_{22} \rangle = \langle \underline{f}^* \underline{f} - \underline{h}^* \underline{h} \rangle = \langle -I_q \rangle .$$

Thus, (15) is fulfilled. Let $S, S_{11}, S_{12}, S_{21}$ and S_{22} be given by (6), (7) and (8). We will show that S belongs to $(p + q) \times (p + q) - \mathcal{N}_+(\mathbb{ID})$. From (17) and (18) we see that S_{22} and S_{12} are functions which belong to the Smirnov class. Part (c) of Lemma 1 provides $\widehat{\phi}_1^\# = (I + \underline{f} \underline{f}^\#) \phi_1^{-1}$. Therefore, $S_{11} = (\widehat{\phi}_1^\# - \underline{f} \underline{f}^\# \phi_1^{-1}) b_1 = \phi_1^{-1} b_1$. Since ϕ_1 is an outer function in $p \times p - \mathcal{N}_+(\mathbb{ID})$, ϕ_1^{-1} is also an outer function which belongs to $p \times p - \mathcal{N}_+(\mathbb{ID})$. Since b_1 is a $p \times p$ Schur function and since $\mathcal{V}_+(\mathbb{ID})$ is an algebra, we get $S_{11} \in p \times p - \mathcal{N}_+(\mathbb{ID})$. Since b_1 is a right denominator of $\widehat{f}^\# \phi_1^{-1}$, we have

$$S_{21} = -\widehat{f}^\# \phi_1^{-1} b_1 \in q \times p - \mathcal{N}_+(\mathbb{ID}) .$$

Hence, $S \in (p + q) \times (p + q) - \mathcal{N}_+(\mathbb{ID})$. Using Lemma 3 it follows that W is a j_{pq} -inner function. Consequently, $W \in \mathcal{J}_c(f, h)$. In particular, $\mathcal{J}_c(f, h)$ is non-empty.

Now we suppose that W is an arbitrary function which belongs to $\mathcal{J}_c(f, h)$. Then $W_{12} = f$ and $W_{22} = h$. Let S_{11} be defined by (7), and let b_1 be the unique inner function occurring in part (f) of Proposition 1. We obtain from parts (c) and (d) of Proposition 2 that b_1 is a right denominator of $\widehat{f}^\# \phi_1^{-1}$ and that identity (21) is satisfied. The unicity of b_1 is clear ■

Similarly, as Theorem 3 gives an answer to Problem (CPC), the following theorem describes the solution set $\mathcal{J}_r(g, h)$ of Problem (CPR).

Theorem 4: Let $g \in q \times p - \mathcal{NM}(\mathbb{D})$, and let $h \in q \times q - \mathcal{NM}(\mathbb{D})$. Then:

(a) $\mathcal{J}_r(g, h)$ is non-empty if and only if the functions g and h satisfy

$$\langle \underline{h} \underline{h}^* \rangle = \langle I + \underline{g} \underline{g}^* \rangle \tag{23}$$

$$h^{-1} \in q \times q - \mathcal{N}_+(\mathbb{D}) \tag{24}$$

$$h^{-1}g \in q \times p - \mathcal{N}_+(\mathbb{D}) \tag{25}$$

$$g \in q \times p - \Pi(\mathbb{D}). \tag{26}$$

(b) Suppose that $\mathcal{J}_r(g, h)$ is non-empty. Let ψ_1 be the unique normalized outer function belonging to $p \times p - \mathcal{N}_+(\mathbb{D})$ such that

$$\langle \underline{\psi}_1 \underline{\psi}_1^* \rangle = \langle I + \underline{g}^* \underline{g} \rangle. \tag{27}$$

Then a $(p + q) \times (p + q)$ matrix-valued function W which is meromorphic in \mathbb{D} belongs to $\mathcal{J}_r(g, h)$ if and only if there exists a left denominator c_1 of $\widehat{\psi_1^{-1}g^\#}$ such that

$$W = \text{diag}(c_1, I_q) \cdot \begin{pmatrix} \widehat{\psi_1^\#} & \psi_1^{-1} \widehat{g^\#} h \\ g & h \end{pmatrix}. \tag{28}$$

In this case, c_1 is unique.

The proof of Theorem 4 can be done in a similar manner as in the proof of Theorem

4. Some completion problems for subclasses of j_{pq} -inner functions

This section is aimed to study completion problems for some subclasses of j_{pq} -inner functions. In particular, we will look for A -singular j_{pq} -inner functions in the solution sets $\mathcal{J}_c(f, h)$ and $\mathcal{J}_r(g, h)$ of Problems (CPC) and (CPR), respectively. To be more precise, we will treat the following questions:

(CPCA) Let $f \in p \times q - \mathcal{NM}(\mathbb{D})$, and let $h \in q \times q - \mathcal{NM}(\mathbb{D})$. Describe the set $\mathcal{A}_c(f, h)$ of all functions $W \in \mathcal{J}_c(f, h)$ which are A -singular.

(CPRA) Let $g \in q \times p - \mathcal{NM}(\mathbb{D})$, and let $h \in q \times q - \mathcal{NM}(\mathbb{D})$. Describe the set $\mathcal{A}_r(g, h)$ of all functions $W \in \mathcal{J}_r(g, h)$ which are A -singular.

From the definition of A -singular j_{pq} -inner functions it seems to be reasonable to decompose each of the just formulated problems into two completion problems for j_{pq} -inner functions of Smirnov type and inverse Smirnov type, respectively:

(CPCS) Let $f \in p \times q - \mathcal{NM}(\mathbb{D})$, and let $h \in q \times q - \mathcal{NM}(\mathbb{D})$. Describe the set $\mathcal{N}_c(f, h)$ of all functions $W \in \mathcal{J}_c(f, h)$ which are of Smirnov type.

(CPRS) Let $g \in q \times p - \mathcal{NM}(\mathbb{D})$, and let $h \in q \times q - \mathcal{NM}(\mathbb{D})$. Describe the set $\mathcal{N}_r(g, h)$ of all functions $W \in \mathcal{J}_r(g, h)$ which are of Smirnov type.

(CPCIS) Let $f \in p \times q - \mathcal{NM}(\mathbb{ID})$, and let $h \in q \times q - \mathcal{NM}(\mathbb{ID})$. Describe the set $\mathcal{M}_c(f, g)$ of all functions $W \in \mathcal{J}_c(f, h)$ which are of inverse Smirnov type.

(CPRIS) Let $g \in q \times p - \mathcal{NM}(\mathbb{ID})$, and let $h \in q \times q - \mathcal{NM}(\mathbb{ID})$. Describe the set $\mathcal{M}_r(g, h)$ of all functions $W \in \mathcal{J}_r(g, h)$ which are of inverse Smirnov type.

In view of the results in Section 3, we get immediately a description of the solution sets of (CPCS) and (CPRS).

Theorem 5: Let $f \in p \times q - \mathcal{NM}(\mathbb{ID})$, and let $h \in q \times q - \mathcal{NM}(\mathbb{ID})$. Then:

(a) $\mathcal{N}_c(f, h)$ is non-empty if and only if the functions f and h satisfy conditions (16) - (19) and

$$h \in q \times q - \mathcal{N}_+(\mathbb{ID}). \tag{29}$$

(b) Suppose that $\mathcal{N}_c(f, h)$ is non-empty. Let φ_1 be the unique normalized outer function belonging to $p \times p - \mathcal{N}_+(\mathbb{ID})$ such that (20) holds. Then a $(p + q) \times (p + q)$ matrix-valued function W which is meromorphic in \mathbb{ID} belongs to $\mathcal{N}_c(f, h)$ if and only if there exists a right denominator b_1 of $\widehat{f\#}\varphi_1^{-1}$ such that (21) is fulfilled. In this case, b_1 is unique. In particular, $\mathcal{N}_c(f, h) = \mathcal{J}_c(f, h)$.

Proof: First we observe $\mathcal{N}_c(f, h) \subseteq \mathcal{J}_c(f, h)$. Thus, Theorem 3 yields that $\mathcal{N}_c(f, h) \neq \emptyset$ implies (16) - (19). Furthermore, we see from Theorem 1 that $h \in q \times q - \mathcal{N}_+(\mathbb{ID})$ is also necessary for $\mathcal{N}_c(f, g) \neq \emptyset$. Conversely, now suppose that (16) - (19) and (29) are fulfilled. Combining Theorems 1 and 3 we obtain immediately the rest of the assertion ■

Theorem 6: Let $g \in q \times p - \mathcal{NM}(\mathbb{ID})$, and let $h \in q \times q - \mathcal{NM}(\mathbb{ID})$. Then:

(a) $\mathcal{N}_r(g, h)$ is non-empty if and only if the functions g and h satisfy conditions (23) - (26) and (29).

(b) Suppose that $\mathcal{N}_r(g, h)$ is non-empty. Let ψ_1 be the unique normalized outer function belonging to $p \times p - \mathcal{N}_+(\mathbb{ID})$ such that (27) holds. Then a $(p + q) \times (p + q)$ matrix-valued function W which is meromorphic in \mathbb{ID} belongs to $\mathcal{N}_r(g, h)$ if and only if there exists a left denominator c_1 of $\psi_1^{-1}\widehat{g\#}$ such that (28) is fulfilled. In this case, c_1 is unique. In particular, $\mathcal{N}_r(g, h) = \mathcal{J}_r(g, h)$.

Theorem 6 can be proved in a similar manner as Theorem 5. We omit the details.

Now we are going to turn our attention to Problems (CPCIS) and (CPRIS). We will see that the study of these problems requires more work in comparison with Problems (CPCS) and (CPRS).

Theorem 7: Let $f \in p \times q - \mathcal{NM}(\mathbb{ID})$, and let $h \in q \times q - \mathcal{NM}(\mathbb{ID})$. Then:

(a) $\mathcal{M}_c(f, h)$ is non-empty if and only if f and h satisfy conditions (16) - (19) and

$$\widehat{f\#} \in q \times p - \mathcal{N}_+(\mathbb{ID}). \tag{30}$$

(b) Suppose that $\mathcal{M}_c(f, h)$ is non-empty. Let ϕ_1 be the unique normalized outer function belonging to $p \times p - \mathcal{N}_+(\mathbb{D})$ such that (20) is satisfied. Then a $(p + q) \times (p + q)$ matrix-valued function W belongs to $\mathcal{M}_c(f, h)$ if and only if there exists a $p \times p$ unitary matrix b_1 such that (21) is satisfied. In this case, b_1 is unique.

Proof: First suppose $\mathcal{M}_c(f, h) \neq \emptyset$. Because of $\mathcal{M}_c(f, h) \subseteq \mathcal{J}_c(f, h)$, Theorem 3 yields that (16) - (19) hold. From Theorem 14 in [11] (30) follows. Conversely, now suppose that (16) - (19) and (30) are fulfilled. Since the functions $\widehat{\varphi}_1^\# \varphi_1$ and $I + f\widehat{f}^\#$ belong to $p \times p - \mathcal{NM}(\mathbb{D})$, from Remark 1 and (20) we obtain

$$\widehat{\varphi}_1^\# \varphi_1 = I + f\widehat{f}^\# . \tag{31}$$

Assume that W is a $(p + q) \times (p + q)$ matrix-valued function which is meromorphic in \mathbb{D} and which admits a representation (21) with some $p \times p$ unitary matrix b_1 . Then Theorem 3 implies $W \in \mathcal{J}_c(f, h)$, whereas (31) provides that the function

$$\Omega := \left[\widehat{\varphi}_1^\# - f h^{-1} \left(h \widehat{f}^\# \varphi_1^{-1} \right) \right] b_1$$

fulfils

$$\Omega = \widehat{\varphi}_1^\# b_1 - f \widehat{f}^\# \varphi_1^{-1} b_1 = \left[\widehat{\varphi}_1^\# \varphi_1 - \left(\widehat{\varphi}_1^\# \varphi_1 - I \right) \right] \varphi_1^{-1} b_1 = \varphi_1^{-1} b_1 . \tag{32}$$

The function $\varphi_1^{-1} b_1$ is obviously an outer function in $p \times p - \mathcal{N}_+(\mathbb{D})$. Consequently, we infer from part (b) of Theorem 1 and (21) that $W \in \mathcal{M}_c(f, h)$. Now we consider an arbitrary function W which belongs to $\mathcal{M}_c(f, h)$. Then $W \in \mathcal{J}_c(f, h)$. Theorem 3 shows that W admits representation (21) with some right denominator b_1 of $\widehat{f}^\# \varphi_1^{-1}$. Hence, from Theorem 1/(b) we get that Ω is an outer function in $\mathcal{S}_{p \times p}(\mathbb{D})$. From (32) we have $b_1 = \varphi_1 \Omega$, i.e. the inner function b_1 coincides with the outer function $\varphi_1 \Omega \in p \times p - \mathcal{N}_+(\mathbb{D})$. Therefore, $\varphi_1 \Omega \in \mathcal{S}_{p \times p}(\mathbb{D}) \subseteq p \times p - H^2(\mathbb{D})$. Using the inner-outer factorization theorem in $p \times p - H^2(\mathbb{D})$ (see P.R. Masani [33]), we see that b_1 is a constant unitary matrix ■

Theorem 8: Let $g \in q \times p - \mathcal{NM}(\mathbb{D})$, and let $h \in q \times q - \mathcal{NM}(\mathbb{D})$. Then:

(a) $\mathcal{M}_r(g, h)$ is non-empty if and only if the functions g and h satisfy conditions (23) - (26) and

$$\widehat{g}^\# \in p \times q - \mathcal{N}_+(\mathbb{D}) . \tag{33}$$

(b) Suppose that $\mathcal{M}_r(g, h)$ is non-empty. Let ψ_1 be the unique normalized outer function belonging to $p \times p - \mathcal{N}_+(\mathbb{D})$ such that (27) is satisfied. Then a $(p + q) \times (p + q)$ matrix-valued function W meromorphic in \mathbb{D} belongs to $\mathcal{M}_r(g, h)$ if and only if there exists a $p \times p$ unitary matrix c_1 such that (28) holds. In this case, c_1 is unique.

Theorem 8 can be proved analogously to Theorem 7.

Combining Theorems 5 and 7 (respectively, Theorems 6 and 8) we get an immediate answer to Problems (CPCA) and (CPRA).

Theorem 9: Let $f \in p \times q - \mathcal{NM}(\mathbb{D})$, and let $h \in q \times q - \mathcal{NM}(\mathbb{D})$. Then:

- (a) $\mathcal{A}_c(f, h) \neq \emptyset$ if and only if the functions f and h satisfy conditions (16) - (19), (29) and (30).
- (b) If $\mathcal{A}_c(f, h)$ is non-empty, then $\mathcal{A}_c(f, h) = \mathcal{M}_c(f, h)$.

Proof: Obviously,

$$\mathcal{A}_c(f, h) = \mathcal{N}_c(f, h) \cap \mathcal{M}_c(f, h). \tag{34}$$

Thus, Theorems 5 and 7 provide that conditions (16) - (19), (29) and (30) are necessary for $\mathcal{A}_c(f, h) \neq \emptyset$. Now we suppose conversely that (16) - (19), (29) and (30) hold true. Theorem 7 shows that $\mathcal{M}_c(f, h) \neq \emptyset$. Let $W \in \mathcal{M}_c(f, h)$. Because of (29) we obtain from Theorem 5 that W belongs to $\mathcal{N}_c(f, h)$. In view of (34), $W \in \mathcal{A}_c(f, h)$ follows. In particular, $\mathcal{M}_c(f, h) \subseteq \mathcal{A}_c(f, h) \neq \emptyset$. Clearly, (34) implies then $\mathcal{A}_c(f, h) = \mathcal{M}_c(f, h)$ ■

Theorem 10: Let $g \in q \times p - \mathcal{NM}(\mathbb{D})$, and let $h \in q \times q - \mathcal{NM}(\mathbb{D})$. Then:

- (a) $\mathcal{A}_r(g, h)$ is non-empty if and only if the functions g and h satisfy conditions (23) - (26), (29) and (33).
- (b) If $\mathcal{A}_r(g, h)$ is non-empty, then $\mathcal{A}_r(g, h) = \mathcal{M}_r(g, h)$.

Theorem 10 can be analogously proved as Theorem 9.

5. A completion problem for full-rank A -normalized j_{pq} -elementary factors

In this section, we will consider distinguished j_{pq} -elementary factors which can be conceived as convenient normalized resolvent matrix associated with some non-degenerate matricial Schur problem. The matricial Schur problem consists of the following question. Given a non-negative integer n and $p \times q$ complex matrices A_0, A_1, \dots, A_n , the problem is to describe the set $\mathcal{S}_{p \times q}[A_0, A_1, \dots, A_n]$ of all $p \times q$ Schur functions with first $n + 1$ Taylor coefficients A_0, A_1, \dots, A_n in their Taylor series representation around the origin. It is known that $\mathcal{S}_{p \times q}[A_0, A_1, \dots, A_n]$ is non-empty if and only if the block Toeplitz matrix

$$S_n = \begin{pmatrix} A_0 & 0 & \dots & 0 \\ A_1 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_n & A_{n-1} & \dots & A_0 \end{pmatrix}$$

is contractive (see, e.g., [19, Section 3]). If the problem is non-degenerate, i.e. if S_n is strictly contractive, then the set $\mathcal{S}_{p \times q}[A_0, A_1, \dots, A_n]$ can be parametrized via linear fractional transformations of the Schur class $\mathcal{S}_{p \times q}(\mathbb{D})$ generated by appropriately constructed j_{pq} -inner matrix polynomials. Choosing convenient normalizations there is a one-to-one correspondence between non-degenerate Schur problems and particular subclasses of j_{pq} -inner polynomials (see [18] and [23]). In order to define the special subclass of j_{pq} -inner functions we will deal with in this section, it seems to be useful to recall the notion of reciprocal matrix polynomial and some facts on j_{pq} -elementary factors.

If P_0, P_1, \dots, P_m are $p \times q$ complex matrices and if the matrix polynomial P is given by

$$P(z) = \sum_{k=0}^m P_k z^k, \quad z \in \mathbb{C},$$

then the reciprocal matrix polynomial $\check{P}^{[m]}$ of P (with respect to the unit circle \mathbb{T} and the formal degree m) is defined by

$$\check{P}^{[m]}(z) := \sum_{k=0}^m P_{m-k}^* z^k, \quad z \in \mathbb{C}.$$

Obviously, $\check{P}^{[m]}(z) = z^m P^*(1/\bar{z})$ for all $z \in \mathbb{C} \setminus \{0\}$.

Let $n \in \mathbb{N}_0$, and let D be a j_{pq} -elementary factor with pole of order $n + 1$ at $z_0 = \infty$ (see Section 2). Clearly, D is a $(p + q) \times (p + q)$ matrix polynomial of formal degree $n + 1$. It can be easily shown that the rank of the leading coefficient matrix of D is not greater than p (see V.K. Dubovoj [17], or [19, Lemma 4.4.8]). If this rank is equal to p , then we will say that D is a full-rank j_{pq} -elementary factor.

In [10] (see also [23]) a general concept of normalizing of j_{pq} -inner function has been developed. For the case of a full-rank j_{pq} -elementary factor with pole of order $n + 1$ at $z = \infty$, it can be equivalently formulated as follows (see [23, Theorem 70]): A full-rank j_{pq} -elementary factor \mathcal{A} with pole of order $n + 1$ at $z = \infty$ is called *A-normalized* if the following three conditions are satisfied where $\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{pmatrix}$ is the block partition of \mathcal{A} with $p \times q$ block \mathcal{A}_{11} :

- (i) $\check{\mathcal{A}}_{11}^{[n+1]}(0)$ is positive Hermitian.
- (ii) $\mathcal{A}_{22}(0)$ is positive Hermitian.
- (iii) $\mathcal{A}_{21}(0) = 0_{q \times p}$.

In [23] it was shown that there is a one-to-one correspondence between non-degenerate matricial Schur problems with $n + 1$ given coefficients and *A-normalized* full-rank j_{pq} -elementary factors with pole of order $n + 1$ at $z = \infty$. The origin of the notion *A-normalized* full-rank j_{pq} -elementary factor with pole at $z = \infty$ lies in the concrete resolvent matrix associated with a non-degenerate Schur problem which originates in [13] and which was alternately refound in [22] and [25, Part IV].

In [12, Theorem 31] we showed that there is a bijective correspondence between the set of *A-normalized* full-rank j_{pq} -elementary factors with pole of order $n + 1$ at $z = \infty$ and the set $p \times q - \mathcal{P}_n$ of all $p \times q$ matrix polynomials of formal degree not greater than n . Using this connection we can immediately characterize the situation that a given matrix polynomial is the last block column of some *A-normalized* full-rank j_{pq} -elementary factor:

Theorem 11: *Let $n \in \mathbb{N}_0$, let $P \in p \times q - \mathcal{P}_n$, and let $R \in q \times q - \mathcal{P}_n$. Suppose that m is an integer with $m \geq n$. Then there is an *A-normalized* full-rank j_{pq} -elementary factor θ_{m+1} with pole of order $m + 1$ at $z = \infty$ such that*

$$\theta_{m+1} = \begin{pmatrix} * & P \\ * & R \end{pmatrix}$$

if and only if the following conditions are satisfied:

- (i) $\det R(z) \neq 0$ for all $z \in \mathbb{D}$.
- (ii) For each $z \in \mathbb{T}$, $R^*(z)R(z) = I + P^*(z)P(z)$.
- (iii) $R(0)$ is non-negative Hermitian.

In this case, θ_{m+1} is unique and can be expressed by

$$\theta_{m+1} = \begin{pmatrix} \dot{P}_1^{[m+1]} & P \\ R\dot{P}_1^{[m+1]}P_1^{-1} & R \end{pmatrix}$$

where P_1 is the unique $p \times p$ matrix polynomial which fulfils the following three conditions:

- (α) $\det P_1(z) \neq 0$ for all $z \in \mathbb{D}$.
- (β) For each $z \in \mathbb{T}$, $P_1(z)P_1^*(z) = -I + P(z)P^*(z)$.
- (γ) $P_1(0)$ is non-negative Hermitian.

For the **Proof** use [12: Theorem 31] ■

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