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On the Flow of a Temperature-Dependent Bingham Fluid in Non-Smooth Bounded Two-Dimensional Domains

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A result on the existence and smoothness of solutions for temperature-coupled Bingham problems in non-smooth bounded 2D-domains is proved, which complements the results of G. Duvaut and J. L. Lions [3] on this subject.

Key words: *Bingham fluids, non-smooth domains, temperature-coupled fluid flow*

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1. Introduction

Bingham fluids are usable in various technical and technological directions. For example coating processes may be considered as flows of Bingham fluids. Beside zones of viscous flow there exist so-called "plugs", that means zones where the derivative of the velocity vanishes.

Moreover, the boundary value problems arising from technical processes should be considered with changing types of boundary conditions. For example gas heated melting processes need three types of boundary conditions for the velocity: condition of adherence for solid walls, slip-conditions for uncovered fluid surfaces and conditions for in/out-stream surfaces.

Unlike the well-known existence results of G. Duvaut and J.L. Lions (cf.[3, 4]) the coupling between temperature and velocity by convection will be considered here. Whereas convection is essentially, the energy transport by radiation and convection of mass is negligible and the fluid flow may be regarded as stationary in many cases (e.g. the flow of liquids). Beyond this the material constants heat capacity and viscosity are considered to be temperature-dependend and — in a sence — unbounded.

The differences between the model considered here and that stated in [3, 4, 22, 23] are implying modified techniques for proving an existence result for the corresponding boundary value problem. Although the general scheme:

- proving an existence and uniqueness result for an — in a sence — linearized boundary value problem using variational inequality techniques
- proving an a priori estimate for the original non-linear problem
- using a fixed point theorem to prove the existence of a solution for the original non-linear problem

is used in our proof too, there are some differences. Caused by the models for heat capacity and viscosity as well as the temperature coupling by convection the space $W^{1,2}$ may not be used for fixed point considerations. Thus we need some regularity results for the "linearized" problem and therefore some results on isomorphisms for the Stokes as well as the Poisson problem in case of non-smooth boundary data. Moreover the proof of the a priori estimate is quite different to that used in the literature cited above.

2. Notations and definitions

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a $C^{0,1}$-boundary $\partial\Omega$. In the following we denote by D_i the partial derivative with respect to the i-th coordinate. The flow of a Bingham fluid is assumed to be incompressible, viscous and buoyant. Here we are especially interested in temperature-coupled flow. Thus the flow is described by velocity, pressure and temperature. For this the preservation of mass, momentum and energy results in the following differential equations in the domain Ω (D_i denotes the partial derivative with respect to the i-th coordinate):

$$\begin{aligned} -D_i\left(\sigma_{ij}(\vartheta,\underline{u},p)\right) + ku_iD_iu_j + K_j\vartheta &= f_j \qquad (j=1,2), && \text{(1.a)}\\ \operatorname{div}\underline{u} = D_iu_i &= 0, && \text{(1.b)}\\ -D_i(\kappa(\vartheta)D_i\vartheta) + u_iD_i\vartheta &= g. && \text{(1.c)} \end{aligned}$$

These are supplemented by the boundary conditions

$$\begin{aligned} -a\kappa(\vartheta)D_i\vartheta - b\vartheta &= c \qquad \text{on } \partial\Omega, && \text{(2.a)}\\ \underline{u}|_{R_1} &= \underline{\varphi}_1, && \text{(2.b)}\\ u_n|_{R_2} = \underline{u}\,\underline{n}|_{R_2} = \varphi_{2n}, \qquad \underline{t}\,\sigma(\vartheta,\underline{u},p)\underline{n}|_{R_2} &= \varphi_{2n}, && \text{(2.c)}\\ u_t|_{R_3} = \underline{u}\,\underline{t}|_{R_3} = \varphi_{3t}, \qquad \underline{n}\,\sigma(\vartheta,\underline{u},p)\underline{n}|_{R_3} &= -\varphi_{3n}. && \text{(2.d)} \end{aligned}$$

Here we have used the usual summation convention and the following notations:

$\underline{u}\ldots$ velocity
$\vartheta\ldots$ temperature
$\underline{K}\vartheta\ldots$ buoyancy force
$\kappa\ldots$ heat capacity
$\underline{\varphi}_1\ldots$ usually $=0$; i.e. one has adherence at R_1
$\varphi_{3t}\ldots$ tangential velocity at R_3
$\varphi_{3n}\ldots$ normal component of the area force at R_3
$k\ldots$ convection factor which depends on the boundary conditions for the velocity; $k=0$ if $R_3=\emptyset$ and $k=1$ elsewhere

$p\ldots$ pressure
$\underline{f}\ldots$ outer force
$\sigma(\vartheta,\underline{u},p)\ldots$ stress tensor
$g\ldots$ sources or sinks of heat
$\varphi_{2n}\ldots$ slip of the fluid at R_2
$\varphi_{2t}\ldots$ tangential stress at R_2
$\partial\Omega = R_1 \cup R_2 \cup R_3$
$a,b,c\ldots$ functions describing the heat transfer through $\partial\Omega$.

Moreover, underlined variables are denoting vectors in $\mathbb{R}^2$, $\underline{n}$ is the outward normal at $\partial\Omega$, $\underline{t}$ is the corresponding unit vector tangential to $\partial\Omega$ and $\underline{\alpha}\,\underline{\beta}$ is the scalar product of vectors $\underline{\alpha},\underline{\beta} \in \mathbb{R}^2$. Later on we denote by μ the viscosity. It seems to be more convenient to use ϑ as a normed temperature, i.e. to set $\vartheta = (T-T_B)/T_B$ with T absolute temperature and T_B a proper reference temperature (e.g. T_B melting temperature or mean value). If we do this, the notation of problem (1),(2) do not vary qualitatively and therefore in the following we identify ϑ with $(T - T_B)/T_B$.

In this paper we prove a theorem on the existence of a solution for the following special case of the problem written above — usually called Bingham fluid: The stress tensor $\boldsymbol{\sigma}$ is defined by the equations

$$\begin{aligned} \boldsymbol{\sigma}_{ij}(\vartheta, \underline{u}, p) &= -p\delta_{ij} + \tilde{\boldsymbol{\sigma}}_{ij}(\vartheta, \underline{u}) \\ \tilde{\boldsymbol{\sigma}}_{ij}(\vartheta, \underline{u}) &= \begin{cases} [\mu(\vartheta) + \tau \mathbf{D}_{II}(\underline{u})^{-1}]\mathbf{D}_{ij}(\underline{u}) & \text{for} \quad \mathbf{D}_{II}(\underline{u}) \neq 0 \\ \tilde{\boldsymbol{\sigma}}_{ij}(\underline{u}) & \text{for} \quad \mathbf{D}_{II}(\underline{u}) = 0 \end{cases} \end{aligned} \tag{3}$$

(i,j=1,2) where τ is a non-negative number. In the latter case we require $\tilde{\boldsymbol{\sigma}}_{II}(\underline{u}) \leq \tau$ and $\tilde{\boldsymbol{\sigma}}_{ij}(\underline{u}) = \tilde{\boldsymbol{\sigma}}_{ji}(\underline{u})$. Above we have used the abbreviations

$$\mathbf{D}_{ij}(\underline{u}) = 1/2(D_i u_j + D_j u_i) \qquad (i, j = 1, 2)$$

and

$$\mathcal{A}_{II}(\underline{u}) = \sqrt{\mathcal{A}_{ij}\mathcal{A}_{ij}} \text{ for any } \mathcal{A} \in \mathbb{R}^2 \times \mathbb{R}^2.$$

It should be remarked that in this case the boundary conditions at R_3 get the form

$$-\underline{n}\,\boldsymbol{\sigma}(\vartheta, \underline{u}, p)\underline{n}|_{R_3} = p - \underline{n}\,\tilde{\boldsymbol{\sigma}}(\vartheta, \underline{u})\underline{n}|_{R_3} = \varphi_{3n}.$$

Usually the vector of boundary stresses $\underline{S}(\vartheta, \underline{u}) = \underline{n}\,\tilde{\boldsymbol{\sigma}}(\vartheta, \underline{u})$ is used and hence the boundary conditions at R_2 and R_3 may be reformulated; we get

$$S_t(\vartheta, \underline{u})|_{R_2} = \varphi_{2t} \quad \text{and} \quad p - S_n(\vartheta, \underline{u})|_{R_3} = \varphi_{3n}.$$

The non-smoothness of the boundary is described in the following way: For the set $\partial\Omega$ there exists a disjunct partition into subsets $\Gamma_1, \ldots, \Gamma_N$ such that $\partial\Omega = \cup_{j=1}^{N}\Gamma_j$, the subsets Γ_j are sufficiently smooth and for every $j \in \{1, \ldots, N\}$ there exists a unique $i \in \{1, 2, 3\}$ with $\Gamma_j \subset R_i$ and $a(\cdot) \geq a_0 > 0$ on Γ_j or $a(\cdot) \equiv 0$ on Γ_j. The partition $\{\Gamma_1, \ldots, \Gamma_N\}$ is assumed to be maximal in the following sence: if we enlarge any set Γ_j $(j = 1, \ldots, N)$ this set violates one of the last conditions. The points where the kind of the boundary conditions for the temperature or the velocity changes as well as the points where the boundary $\partial\Omega$ is non-smooth are of special interest; these are denoted by $\mathcal{O}_j$ $(j = 1, \ldots, N)$. This way we get

$$M = \{\mathcal{O}_1, \ldots, \mathcal{O}_N\} = \{x \in \partial\Omega : \exists i \neq j \in \{1, \ldots, N\} \text{ with } x = \Gamma_i \cap \Gamma_j\},$$

the set of all singular boundary points of problem (1), (2). By ω_j we denote the (inner) apex angle of Ω at the singular point $\mathcal{O}_j$ $(j = 1, \ldots, N)$ and for some sufficiently small $\varepsilon > 0$ and each point $\mathcal{O}_j$ of the set M we define a weight function ϱ_j,

$$\varrho_j(x) = \begin{cases} |x - \mathcal{O}_j| & \text{for } |x - \mathcal{O}_j| < \varepsilon/2 \\ \varepsilon & \text{for } |x - \mathcal{O}_j| \geq \varepsilon, \end{cases}$$

which near $\mathcal{O}_j$ reflects the distance between x and the singular point $\mathcal{O}_j$. Apart from $\mathcal{O}_j$ we assume this function to be sufficiently smooth. Later on we use the infinite cone $K \subset \mathbb{R}^2$ with vertex at zero and apex angle ω_o as a model domain to describe non-smoothness in $\mathbb{R}^2$-domains.

For the following considerations we need a number of function spaces. As usual the classical Sobolev spaces are denoted by $W^{s,p}(\Omega)$, $\overset{\circ}{W}{}^{s,p}(\Omega)$, $W^{s,p}(\partial\Omega)$ $(1 \leq p \leq \infty, s \in \mathbb{R})$. Besides this we need so-called weighted Sobolev spaces to describe the regularity of solutions for boundary value problems in case of non-smooth boundary data. For $l \in \mathbb{N} \cup \{0\}$, $p \in \mathbb{R}$ with $1 \leq p \leq \infty$ and $\vec{\beta} = (\beta_1, \ldots, \beta_N) \in \mathbb{R}^N$ we define the spaces $V^{l,p}_{\vec{\beta}}(\Omega, M)$ as the closure of

$$C^{\infty}(\Omega, M) = \{v \in C^{\infty}(\bar{\Omega}) : supp(v) \cap M = \emptyset \text{ , } supp(v) \text{ bounded}\}$$

relative to the norm

$$\left\| u \left| V^{l,p}_{\vec{\beta}}(\Omega, M) \right\| = \left(\sum_{|\alpha| \leq l} \left\| \left(\prod_{i=1}^{N} \varrho_i^{\beta_i - l + |\alpha|}(\cdot) \right) D^{\alpha} u \right| L_p \right\|^p \right)^{1/p} . \tag{4}$$

Obviously N is the cardinality of M. Similar, $\overset{\circ}{V}{}^{l,p}_{\vec{\beta}}(\Omega, M)$ is the closure of $C_{\circ}^{\infty}(\bar{\Omega})$ with respect to the norm (4). After that, weighted Sobolev spaces with negative order of derivation, i.e. for $l \in \mathbb{Z}$ with $l < 0$, and trace spaces of weighted Sobolev type may be defined by duality and as factor spaces, respectively, as this is known from the classical Sobolev spaces. The analoga of the above defined spaces for the infinite cone K are built in a similar way using $C^{\infty}(K, x_{\circ})$ instead of $C^{\infty}(\Omega, M)$ and $\varrho(x) = |x - x_{\circ}|$ instead of the functions ϱ_j. For further information on this topic see, e.g., [12, 13, 14]. In this context it should be pointed out that we often use the notation E instead of $E(\Omega)$ to describe a function space on Ω. For the norm of an element $x \in E$ we write synonymously $\|x|E(\Omega)\| = \|x|E\| = \|x\|$. Moreover, for E we use the abbreviation $\underline{E}$ instead of $E \times E$.

For technical and physical reasons we make use of the following basic

Assumption I: *Let $a(x) \geq a_0 > 0$, $b(x)/a(x) \geq 0$ and $|c(x)| \leq C_{\circ}|b(x)|$ for some $C_{\circ} \in \mathbb{R}_+$ and $x \in \Gamma_a = \{x \in \partial\Omega : a(x) \neq 0\}$, let $k = 0$ if $R_3 \neq \emptyset$ and assume that one of the following conditions is satiesfied:*

(i) $0 < m \leq \kappa(t) \leq M < \infty$ *for any* $t \in \mathbb{R}$ *and* $u_n|_{\Gamma_a} \geq 0$ *or* $g \equiv 0$ *in* Ω

(ii) $\kappa(t) > 0$ *for any* $t \in \mathbb{R}$, $u_n|_{\Gamma_a} \geq 0$ *and* $g \equiv 0$ *in* Ω

(iii) $\kappa(t) > 0$ *and* $(\kappa(s) - \kappa(t))(s - t) \geq 0$ *for any* $s, t \in \mathbb{R}$ *and* $g \equiv 0$ *in* Ω

Moreover, assume that the function c/b defined on $\Gamma_D = \partial\Omega \backslash \Gamma_a$ may be extended to an element of $W^{1/2,2}(\partial\Omega)$, that the functions $\underline{\varphi}_1, \underline{\varphi}_2, \underline{\varphi}_3$ fulfil appropriate compatibility conditions, which will be stated later (cf. (17)-(19)).

Remark 1: (i) The condition $u_n|_{\Gamma_a} \geq 0$ means

$$\Gamma_a \cap R_3 = \emptyset, \ \varphi_{1n}(x) \geq 0 \text{ for } x \in \Gamma_a \cap R_1 \text{ and } \varphi_{2n}(x) \geq 0 \text{ for } x \in \Gamma_a \cap R_2.$$

This means especially that at in/out-stream surfaces , which are part of R_3, we must have a Dirichlet boundary condition for the temperature. In practice we have usually $u_n|_{R_1 \cup R_2} = 0$.

(ii) The boundary conditions on R_2 describe the circumstances on an uncovered fluid surface and those on R_3 mean that an area force is acting.

(iii) The positivity of b/a on Γ_a is equivalent to the fact that the heat flux is directed from warmer to colder materials.

(iv) The condition $|c(x)| \leq C_\circ |b(x)|$ on Γ_a especially may be interpreted as follows: Constant heating or cooling through the walls is impossible if these are completely isolated, i.e. if the coefficient b is equal to zero and a is positive.

(v) The assumption that k vanishes whenever R_3 is non-empty is a technical one; otherwise a proof of existence for problem (1), (2) seems to be impossible.

(vi) Assumptions (i) — (iii) above signify especially that either the medium in flow is free of sources and sinks of heat or the heat capacity is everywhere bounded and strictly positive.

(vii) For heat capacity and viscosity the models $\nu(\vartheta) = \exp(-a_1\vartheta + a_2)$ and $\kappa(\vartheta) = \exp(b_1\vartheta + b_2)$ with some $a_1, b_1 \in \mathbb{R}_+$ and $a_2, b_2 \in \mathbb{R}$ are used in rheology.

3. Isomorphisms for the Stokes problem in non–smooth bounded domains

Because the results in this section are technical generalizations of well-known results we give only an outline of the considerations and omit the proofs. For details we refer to the conscious presentation of the material in case of elliptic operators (especially the Laplacian) in [14] and to author's thesis [9].

As pointed out in the introduction regularity results for some related Stokes problems are essentially for the proof of the existence of a solution for problem (1),(2). In the following let us consider the Stokes problem

$$-D_i(\nu(\cdot)(D_iu_j + D_ju_i)) + D_jp = f_j \quad (j = 1,2) \quad \text{in } \Omega, \tag{5.a}$$

$$\operatorname{div} \underline{u} = D_iu_i = f_3 \quad \text{in } \Omega, \tag{5.b}$$

$$\begin{aligned} \underline{u}|_{R_1} = \underline{\phi}_1, \quad u_n|_{R_2} = \phi_{2n}, \quad u_t|_{R_3} = \phi_{3t} \\ S_t(\underline{u})|_{R_2} = \phi_{2t} \quad p - S_n(\underline{u})|_{R_3} = \phi_{3n}. \end{aligned} \tag{5.c}$$

Here the viscosity $\nu : \Omega \to \mathbb{R}_+$ is a function fulfilling the inequality $0 < \nu_\circ \leq \nu(x) \leq \nu^\circ < \infty$ for $x \in \Omega$, $\underline{S} = \big((\nu(D_iu_j + D_ju_i))_{j=1}^2\big)$ denotes the vector of boundary stresses.

The weak formulation for problem (5) is given by

$$\begin{aligned} \tfrac{1}{2}\int_\Omega \nu(D_iu_j + D_ju_i)(D_iv_j + D_jv_i)\,dx \\ = \int_\Omega \underline{f}\,\underline{v}\,dx + \int_{R_2} \phi_{2t}v_t\,ds + \int_{R_3}\phi_{3n}v_n\,ds \end{aligned} \quad \forall \underline{v} \in X_\circ \tag{6.a}$$

$$D_iu_i = f_3,\ \underline{u}|_{R_1} = \underline{\phi}_1,\ u_n|_{R_2} = \phi_{2n},\ u_t|_{R_3} = \phi_{3t} \tag{6.b}$$

with the space $X_\circ = \{\underline{v} \in \underline{W}^{1,2}(\Omega) : \underline{v}|_{R_1} = \underline{0},\ v_n|_{R_2} = 0,\ v_t|_{R_3} = 0,\ \operatorname{div}\underline{v} = 0\}$.

A simple homogenization and Korn's second inequality (cf. [8]) yield

Lemma 2: *For problem (5) there exists a unique weak solution, i.e. a function $\underline{u} \in \underline{W}^{1,2}(\Omega)$ which fulfils the weak formulation (6).*

To describe the Fredholm properties of Stokes problems in non–smooth bounded domains we may use techniques of V.A. Kondratiev (cf. [10, 11]) and V.G. Maz'ya and B.A. Plamenevskii (cf. [17, 18, 19]). The usual localization argument led us to the model problem

$$\left.\begin{aligned} -\eta_\circ \Delta u_j + D_j p &= f_j \qquad (j=1,2) \\ D_i u_i &= f_3 \end{aligned}\right\} \text{ in } K \tag{7}$$

($\eta_\circ := \nu(0)$) with one of the following boundary conditions:

$$u_j|_{(\partial K)_i} = \phi_{ij} \qquad (i,j=1,2) \tag{8}$$

$$\begin{aligned} u_n|_{(\partial K)_1} &= \phi_{1n}, \\ S_t^\circ(\underline{u})|_{(\partial K)_1} &= \phi_{1t}, \end{aligned} \qquad u_j|_{(\partial K)_2} = \phi_{2j} \qquad (j=1,2) \tag{9}$$

$$\begin{aligned} u_t|_{(\partial K)_1} &= \phi_{1t}, \\ p - S_n^\circ(\underline{u}) + \eta_\circ \operatorname{div} \underline{u}|_{(\partial K)_1} &= \phi_{1n}, \end{aligned} \qquad u_j|_{(\partial K)_2} = \phi_{2j} \qquad (j=1,2) \tag{10}$$

$$\begin{aligned} u_t|_{(\partial K)_1} &= \phi_{1t}, & u_n|_{(\partial K)_2} &= \phi_{2n}, \\ p - S_n^\circ(\underline{u}) + \eta_\circ \operatorname{div} \underline{u}|_{(\partial K)_1} &= \phi_{1n}, & S_t^\circ(\underline{u})|_{(\partial K)_2} &= \phi_{2t}. \end{aligned} \tag{11}$$

The problems — denoted by (7),(8) - (7),(11) later on — are defined in the two-dimensional infinite cone K with vertex at zero, angle $\omega_\circ$ and sides $(\partial K)_i$ ($i = 1,2$). By $\underline{S}^\circ(\underline{u})$ we have denoted the boundary stress vector $\underline{S}^\circ = \left((\eta_\circ(D_i u_j + D_j u_i))_{j=1}^2\right)$ for the reduced problem.

Remark 3: Beside the four cases of boundary conditions above noted, there exist two other combinations which are out of physical interest. But they can be treated in the same manner.

Considering the model problems (7),(8) - (7),(11) the methods of Maz'ya and Plamenevskii [19, esp. Theorems 4.1 and 4.2] results

Theorem 4: *Assume $l \in \mathbb{N} \cup \{0\}$, $\beta \in \mathbb{R}$ and $1 < q < \infty$. The boundary value problems (7),(8) - (7),(11) define isomorphisms*

$$\underline{V}_\beta^{l+2,q}(K) \times V_\beta^{l+1,q}(K) \longrightarrow \mathcal{U}_\beta^{l,q}(K)$$

with

$$\mathcal{U}_\beta^{l,q}(K) = \underline{V}_\beta^{l,q}(K) \times V_\beta^{l+1,q}(K) \times \prod_{j=1}^{2} \left[V_\beta^{l+2-1/q,q}((\partial K)_j) \times V_\beta^{l+2-ms_j-1/q,q}((\partial K)_j) \right]$$

and

$$ms_j = \begin{cases} 0 & \text{for a Dirichlet boundary condition on } (\partial K)_j \\ 1 & \text{for an in/out-stream condition or a condition of an uncovered surface on } (\partial K)_j \end{cases}$$

if the line $l_h = \{\lambda \in \mathbb{C} \colon Im\lambda = h\}$ with $h = \beta - l - 2 + 2/q$ is free of solutions of the corresponding of the following equations:

$$\lambda^2 \sin^2 \omega_\circ - \sinh^2(\lambda\omega_\circ) = 0 \qquad (\lambda \neq 0) \tag{12}$$

for problem (7),(8),

$$\lambda \sin 2\omega_\circ - \sinh 2\lambda\omega_\circ = 0 \qquad (\lambda \neq 0) \tag{13}$$

for problem (7),(9),

$$\lambda \sin 2\omega_\circ + \sinh 2\lambda\omega_\circ = 0 \qquad (\lambda \neq 0) \tag{14}$$

for problem (7),(10) and

$$Re\lambda = 0, \quad Im\lambda \in \left\{ \frac{2k+1}{2\omega_o}\pi - 1,\ \frac{2k+1}{2\omega_o}\pi + 1 \right\}_{k\in\mathbb{Z}} \tag{15}$$

for problem (7),(11).

Moreover, for $1 < q_1 < \infty$, $l_1 \geq l$, $\beta_1 \in \mathbb{R}$ *with* $h_1 = \beta_1 - l_1 - 2 + 2/q_1 < h = \beta - l - 2 + 2/q$ *and* $\left((\underline{f}, f_3), \underline{\phi}_1, \underline{\phi}_2\right) \in \mathcal{U}^{l,q}_{\beta}(K) \cap \mathcal{U}^{l_1,q_1}_{\beta_1}(K)$ *one conclude that the solution of any of the problems (7),(8)-(7),(11) is an element of* $\underline{V}^{l_1+2,q_1}_{\beta_1}(K) \times V^{l_1+1,q_1}_{\beta_1}(K)$ *if the strip* $\{\lambda \in \mathbb{C}:\ h_1 \leq Im\lambda \leq h\}$ *is free of solutions of the corresponding one of the equations (12)-(15).*

Using duality and interpolation arguments —this way generalizing results of G. Wildenhain [26] and J. Rossmann [24] — we get moreover

Proposition 5: *The assertions of Theorem 4 behold true if we assume l to be less than zero.*

An outline of the proof in case of Dirichlet's problem for the Laplace operator is given in [14]. For an exact proof see author's thesis [9].

Summing up the results for the model problem in cones and the general Agmon-Douglis-Nirenberg results for elliptic boundary value problems in domains with smooth boundary data (see [1]) we may state the following theorem on Fredholm properties for the Stokes operator on corner domains in $\mathbb{R}^2$.

Theorem 6: *Assume that* $\nu(x) \in V^{k+1,p_1}_{\underline{k+1-2/p_1}}(\Omega, M)$ *with* $p_1 > \max(2-k, q)$ *is fulfilled for* $k \geq 0$, *that* $l \in \mathbb{Z}, l \leq k$ *and that the lines* $l_{h_j} = \{\lambda \in \mathbb{C} : Im\lambda = h_j\}$ *with* $h_j = \beta_j - l - 2 + 2/q$ *are free of solutions of that equation of (12)-(15), which corresponds to the boundary conditions at the singular point* O_j, *i.e.*

- *that* l_{h_j} *is free of solutions of equation (12) if we have Dirichlet boundary conditions on both sides of* O_j,
- *that* l_{h_j} *is free of solutions of equation (13) if we have a Dirichlet boundary condition on one and a condition of type* R_2 *on the other side of* O_j,
- *that* l_{h_j} *is free of solutions of equation (14) if we have a Dirichlet boundary condition on one and a condition of type* R_3 *on the other side of* O_j *and*
- *that* l_{h_j} *is free of elements of (15) if we have a condition of type* R_2 *on one and a condition of type* R_3 *on the other side of* O_j

for $j = 1, \dots, N$. *Then the Stokes problem (5) defines a Fredholmian operator*

$$\underline{V}^{l+2,q}_{\vec{\beta}}(\Omega, M) \times V^{l+1,q}_{\vec{\beta}}(\Omega, M) \longrightarrow \mathcal{U}^{l,q}_{\vec{\beta}}(\Omega, M)$$

with

$$\mathcal{U}^{l,q}_{\vec{\beta}}(\Omega, M) = \underline{V}^{l,q}_{\vec{\beta}}(\Omega, M) \times V^{l+1,q}_{\vec{\beta}}(\Omega, M) \times \prod_{j=1}^{N} \Big[V^{l+2-1/q,q}_{(\beta_j,\beta_{j+1})}(\Gamma_j, \{O_j, O_{j+1}\}) \times V^{l+2-m_{s_j}-1/q,q}_{(\beta_j,\beta_{j+1})}(\Gamma_j, \{O_j, O_{j+1}\}) \Big]$$

and

$$m_{Sj} = \begin{cases} 0 & \text{for the case of a Dirichlet boundary condition on } \Gamma_j \\ 1 & \begin{array}{l}\text{for the case of an in/out-stream condition or a condition} \\ \text{of an uncovered surface on } \Gamma_j\end{array} \end{cases} .$$

Once again we remark that the proof is a simple generalization of that given in [14] for elliptic operators.

Together with the existence and uniqueness of a weak solution we can now state

Corollary 7: *The weak solution of problem (5) is an element of the space* $\underline{V}_{\vec{\beta}}^{l+2,q}(\Omega, M) \times V_{\vec{\beta}}^{l+1,q}(\Omega, M)$ *if the right-hand sides of the differential equations are fulfilling the smoothness assumptions of Theorem 6, if the condition* $\nu \in C^{\ell,\zeta}(\bar{\Omega})$ *with* $\zeta \in (0,1)$ *and* $\ell = \max(0, l)$ *or* $\nu \in V^{k,p_2}_{\underline{k-2/p_2}}(\Omega, M)$ *with* $k \geq \max(1, l-1)$ *and* $p_2 > \max(3-k, q)$ *holds for the coefficient* $\nu(\cdot)$ *and if for each* $j \in \{1, \dots, N\}$ *the strip* $\beta_j - l - 2 + 2/q \leq Im\lambda \leq \varepsilon_j$ *of the complex plane is free of solutions of that equation of (12)-(15) which corresponds to the kind of boundary conditions near the singular point* O_j*. Here* $\varepsilon_j > 0$ $(j = 1, \dots, N)$ *are sufficiently small.*

Up to now we have considered the Stokes problem (5) in weighted Sobolev spaces. In difference to the classical ones the elements of these spaces must vanish at the singular boundary points by definition. On the basis of the considerations of P.Grisvard [6] we try to answer wether a generalization of the regularity results to the case of classical Sobolev spaces is possible or not. In keeping with the scope of this paper we restrict our consideration to the case of spaces with first order of derivation and summing exponents $q \geq 2$. A generalization to other cases is possible, but this involves some technical difficulties, which are avoidable here. To get an idea what we have to do, we assume that the right-hand sides of (5) are sums of a $W^{s,p}$- and a $V_{\vec{\beta}}^{s,p}$-part, i.e.

$$\begin{pmatrix} \underline{f} \\ f_3 \\ \underline{\phi}_1 \\ (\phi_{2n}, \phi_{2t}) \\ (\phi_{3n}, \phi_{3t}) \end{pmatrix} \in \begin{array}{l} [\underline{W}^{-1,q}(\Omega) \oplus \underline{V}_{\vec{\beta}}^{-1,q}(\Omega, M)] \times \\ [L_q(\Omega) \oplus L_{q,\vec{\beta}}(\Omega, M)] \times \\ [\underline{W}^{1-1/q,q}(R_1) \oplus \underline{V}_{\vec{\beta}}^{1-1/q,q}(R_1, M)] \times \\ [W^{1-1/q,q}(R_2) \oplus V_{\vec{\beta}}^{1-1/q,q}(R_2, M)] \times [W^{-1/q,q}(R_2) \oplus V_{\vec{\beta}}^{-1/q,q}(R_2, M)] \times \\ [W^{-1/q,q}(R_3) \oplus V_{\vec{\beta}}^{-1/q,q}(R_3, M)] \times [W^{1-1/q,q}(R_3) \oplus V_{\vec{\beta}}^{1-1/q,q}(R_3, M)]. \end{array}$$

The last space we denote by $\tilde{\mathcal{U}}_{\vec{\beta}}^{1,q}(\Omega, M)$. If we use a function $\underline{u}_o \in \underline{W}^{1,q}(\Omega)$ with

$$\begin{aligned} \underline{u}_o|_{R_1} &= \underline{\phi}_1^{(w)} \in \underline{W}^{1-1/q,q}(R_1), \\ u_{on}|_{R_2} &= \phi_{2n}^{(w)} \in W^{1-1/q,q}(R_2), \\ u_{ot}|_{R_3} &= \phi_{3t}^{(w)} \in W^{1-1/q,q}(R_3), \end{aligned} \tag{16}$$

i.e. a function which homogenizes the $W^{s,p}$-part of the non-natural boundary conditions and if we assume $\beta_i \geq 0$ $(i = 1, \dots, N)$, problem (5) may be transformed into a similar one with right-hand side $(\tilde{\underline{f}}, \tilde{f}_3, \tilde{\underline{\phi}}_1, \tilde{\phi}_{2n}, \tilde{\phi}_{2t}, \tilde{\phi}_{3n}, \tilde{\phi}_{3t}) \in \mathcal{U}_{\vec{\beta}}^{1,q}(\Omega, M)$ using the imbedding theorems for weighted Sobolev spaces into the classical ones (cf. [14]).

The existence of $\underline{u}_o$ is proved by using P. Grisvard's trace and continuation theorems for Sobolev spaces on domains with singular boundary points (cf. [6, Theorem 1.6.1.4]). We construct $\underline{u}_o = (D_1\xi, D_2\xi)+(D_2\zeta, -D_1\zeta)$ with $\xi,\zeta \in W^{2,q}(\Omega)$. In this case the boundary conditions (16) get the form (D_n and D_t denote normal and tangential derivative respectively)

$$D_n\xi = \begin{cases} \phi_{1n}^{(w)} - \chi_1 \\ \phi_{2n}^{(w)} \\ \chi_5 \end{cases}, \quad D_t\xi = \begin{cases} \chi_2 \\ \chi_3 \\ 0 \end{cases}, \quad D_n\zeta = \begin{cases} \phi_{1t}^{(w)} - \chi_2 \\ \chi_4 \\ \phi_{3t}^{(w)} \end{cases}, \quad D_t\zeta = \begin{cases} \chi_1 & \text{on } R_1 \\ 0 & \text{on } R_2 \\ \chi_6 & \text{on } R_3 \end{cases}$$

with $\chi_i \in W^{1-1/q,q}(R_k)$ for $i = 1,\ldots,6$. P. Grisvard's compatibility conditions for ξ and ζ yield the following conditions (assume $\{\ell_1,\ell_2\} = \{j-1,j\}$):

(i) The right-hand sides of (16) should fulfil the equation

$$\phi_{1t_{\ell_1}}^{(w)}(O_j) = (\sin\omega_j - \cos\omega_j)\phi_{2n_{\ell_2}}^{(w)}(O_j) \tag{17}$$

if at O_j a Dirichlet condition and a condition of type R_2 intersect and ω_j is an integral multiple of $\pi/2$.

(ii) The right-hand sides of (16) should fulfil the equation

$$\phi_{1n_{\ell_1}}^{(w)}(O_j) = (\sin\omega_j - \cos\omega_j)\phi_{3t_{\ell_2}}^{(w)}(O_j) \tag{18}$$

if at O_j a Dirichlet condition and a condition of type R_3 intersect and ω_j is an integral multiple of $\pi/2$.

(iii) The right-hand sides of (16) should fulfill the equation

$$\phi_{2n_{\ell_1}}^{(w)}(O_j) = (\sin\omega_j)\phi_{3t_{\ell_2}}^{(w)}(O_j) \tag{19}$$

if at O_j a condition of type R_2 and a condition of type R_3 intersect and ω_j is an odd multiple of $\pi/2$.

It is easily seen that functions $\underline{\phi}_1, \phi_{2n}, \phi_{3t}$ defined by $\underline{u}_o \in \underline{W}^{1,q}(\Omega)$ in sence of (16) fulfill the conditions (17)–(19).

Using the imbedding theorems between weighted and classical Sobolev space ones again we conclude from Theorem 6 and Corollary 7 the following

Corollary 8: *Let the conditions of Corollary 7 be fulfilled and assume $\omega_j \notin \{\pi/2, 3\pi/2\}$ for any O_j, where boundary conditions of type R_2 and type R_3 intersect. Then for any real number q with $2 < q < 2/(1+max\{s_j : j = 1,\ldots,N\})$ the operator of problem (5) defines an isomorphism between $\underline{W}^{1,q}(\Omega) \times L_q(\Omega)$ and the subspace of*

$$\underline{W}^{-1,q}(\Omega) \times L_q(\Omega) \times \underline{W}^{1-1/q,q}(R_1) \times W^{1-1/q,q}(R_2) \times W^{-1/q,q}(R_2) \\ \times W^{-1/q,q}(R_3) \times W^{1-1/q,q}(R_3),$$

which is defined by the conditions (17)-(19). Therein the numbers s_j denote:

(i) $s_j = \max\{-1, \sup\{s \in I\!R_- : \lambda = t+is$ is a solution of $(12)\}\}$ *in the case of intersecting Dirichlet boundary conditions at O_j.*

(ii) $s_j = \max\{-1, \sup\{s \in \mathbb{R}_- : \lambda = t+is$ is a solution of (13)$\}\}$ *if at* O_j *a Dirichlet boundary condition and a condition of type* R_2 *intersect.*

(iii) $s_j = \max\{-1, \sup\{s \in \mathbb{R}_- : \lambda = t+is$ is a solution of (14)$\}\}$ *if at* O_j *a Dirichlet boundary condition and a condition of type* R_3 *intersect.*

(iv) $s_j = \max\{-1, \sup\{s \in \mathbb{R}_- : \lambda = t+is$ is a solution of (15)$\}\}$ *if at* O_j *a condition of type* R_2 *and a condition of type* R_3 *intersect.*

4. The Bingham equation without convection of mass for a fixed temperature

After substituting for ϑ any $\theta \in L_\infty(\Omega)$ and neglecting the convection term $u_i D_i u_j$ we consider the equations (1.a),(1.b) and the corresponding boundary conditions (2.b)–(2.d). It is well known (cf. [3, 23]) that this problem implies the variational inequality

$$(\underline{f}, \underline{v} - \underline{u}) + \Phi_2(\underline{v} - \underline{u}) + \Phi_3(\underline{v} - \underline{u}) \leq a(\underline{u} + \underline{h}, \underline{v} - \underline{u}) + \Psi(\underline{v} + \underline{h}) - \Psi(\underline{u} + \underline{h}) \quad (20)$$

for all $\underline{v} \in \mathcal{W}$, where we have used the notations

$$a(\underline{u}, \underline{v}) = \int_\Omega \mu(\theta) \mathbf{D}_{ij}(\underline{u}) \mathbf{D}_{ij}(\underline{v})\, dx, \qquad \Psi(\underline{u}) = \int_\Omega \tau \mathbf{D}_{II}(\underline{u})\, dx,$$

$$\Phi_2(\underline{u}) = \int_{R_2} \tilde{\varphi}_{2t} u_t\, ds, \qquad \Phi_3(\underline{u}) = \int_{R_3} \tilde{\varphi}_{3n} u_n\, ds,$$

$$\mathcal{W} = \{\underline{v} \in \underline{W}^{1,2}(\Omega) : \quad \operatorname{div} \underline{v} = 0,\ \underline{v}|_{R_1} = 0,\ v_n|_{R_2} = 0,\ v_t|_{R_3} = 0\}$$

and the function $\underline{h}$ denotes any element of $\underline{W}^{1,2}(\Omega)$ fulfilling the conditions

$$\operatorname{div} \underline{h} = 0,\ \underline{h}|_{R_1} = \underline{\varphi}_1,\ h_n|_{R_2} = \varphi_{2n},\ h_t|_{R_3} = \varphi_{3t}$$

and

$$\tilde{\varphi}_{2t} = \varphi_{2t} - S_t(\underline{h})|_{R_2}, \qquad \tilde{\varphi}_{3n} = \varphi_{3n} - S_n(\underline{h})|_{R_3}.$$

It is easily seen that the above written variational inequality has a unique solution in the space $\mathcal{W}$. The proof is based on the existence result for variational inequalities with pseudo-monotone operators given in [16] (cf. also [3, 23]). The monotonicity of the operator A defined by $(A(\underline{u}), \underline{v}) = a(\underline{u}, \underline{v})$ is obvious. Moreover, we have the a priori estimate

$$\begin{aligned} \|\underline{u}\,|\underline{W}^{1,2}(\Omega)\| &\leq \tfrac{C}{m(\theta)}\{\|\underline{f}\,|\underline{W}^{-1,2}(\Omega)\| + \|\varphi_{2t}\,|W^{-1/2,2}(R_2)\| \\ &\qquad + \|\varphi_{3n}\,|W^{-1/2,2}(R_3)\| + \|\underline{h}\,|\underline{W}^{1,2}(\Omega)\|\} \\ &\leq \tfrac{C}{m(\theta)}\{\|\underline{f}\,|\underline{W}^{-1,2}(\Omega)\| + \textstyle\sum_{\substack{k=2,3 \\ l=n,t}} \|\varphi_{kl}\,|W^{-1/2-m_{kl},2}(R_k)\|\} \end{aligned}$$

where $0 < m(\theta) \leq \kappa(\theta(\cdot))$ a.e. in Ω and $m_{kl} = \begin{cases} 0 & \text{if } k=2,\ l=n \text{ or } k=3,\ l=t \\ 1 & \text{if } k=2,\ l=t \text{ or } k=3,\ l=n \end{cases}$.

Let us remark that another variational formulation of the Bingham problem is given by

$$(\underline{f}, \underline{v} - \underline{u}) + \Phi_2(\underline{v} - \underline{u}) + \Phi_3(\underline{v} - \underline{u}) \leq a(\underline{u}, \underline{v} - \underline{u}) + \Psi(\underline{v}) - \Psi(\underline{u}) \qquad \forall \underline{v} \in \mathcal{W} \quad (21.a)$$

with

$$\operatorname{div} \underline{u} = 0, \ \underline{u}|_{R_1} = \underline{\varphi}_1, \ u_n|_{R_2} = \varphi_{2n}, \ u_t|_{R_3} = \varphi_{3t}. \tag{21.b}$$

We use the regularity results for weak solutions of Stokes problems with changing types of boundary conditions in non-smooth bounded domains given in Section 3 to describe the regularity of the solutions of (21). Recalling the inequality (21.a) we substitute there $\pm\lambda\underline{v}$ for $\underline{v}$ with $\lambda \geq 0$. When λ tends to zero we get the system

$$|a(\underline{u},\underline{v}) - (\underline{f},\underline{v}) + \Phi_2(\underline{v}) + \Phi_3(\underline{v})| \leq \Psi(\underline{v}), \tag{22.a}$$

$$-a(\underline{u},\underline{u}) + (\underline{f},\underline{u}) - \Phi_2(\underline{u}) - \Phi_3(\underline{u}) = \Psi(\underline{u}), \tag{22.b}$$

$$\operatorname{div} \underline{u} = 0, \ \underline{u}|_{R_1} = \underline{\varphi}_1, \ u_n|_{R_2} = \varphi_{2n}, \ u_t|_{R_3} = \varphi_{3t}. \tag{22.c}$$

Introducing the space $\Xi = (L_1(\Omega))^4$ with the norm $\|\xi|\Xi\| = \int_\Omega \tau \left(\sum_{i,j=1}^2 \xi_{ij}^2\right)^{1/2} dx$ ($\tau > 0$) and an operator $\pi : \mathcal{W} \longrightarrow \Xi, \underline{v} \longmapsto \left((\mathbf{D}_{ij}(\underline{v}))_{i,j=1}^2\right)$ and denoting $M(\underline{v}) = a(\underline{u},\underline{v}) - (\underline{f},\underline{v}) - \Phi_2(\underline{v}) - \Phi_3(\underline{v})$, we see that (22.a) is equivalent to $|M(\underline{v})| \leq \|\pi(\underline{v})|\Xi\|$. By the Hahn–Banach theorem it follows that there exists $\left((m_{ij})_{i,j=1}^2\right) \in \Xi^* = (L_\infty(\Omega))^4$ with $m_{ij} = m_{ji}$ such that

$$M(\underline{v}) = \sum_{i,j=1}^{2} \int_\Omega m_{ij} \mathbf{D}_{ij}(\underline{v})\, dx \tag{23}$$

and

$$\|m|\Xi^*\| = \operatorname{ess\,sup} \left(\sum_{i,j=1}^{2} m_{ij}^2\right)^{1/2} \leq 1 \tag{24}$$

hold. Because of (22.b) we have

$$M(\underline{u}) + \Psi(\underline{u}) = \int_\Omega \left[\sum_{i,j=1}^{2} m_{ij}\mathbf{D}_{ij}(\underline{u}) + \left(\sum_{i,j=1}^{2} \mathbf{D}_{ij}^2(\underline{u})\right)^{1/2}\right] dx = 0$$

and, with (24),

$$\sum_{i,j=1}^{2} m_{ij}\mathbf{D}_{ij}(\underline{u}) + \left(\sum_{i,j=1}^{2} \mathbf{D}_{ij}^2(\underline{u})\right)^{1/2} = 0 \qquad \text{a.e. in } \Omega.$$

With the definition of the operator M and (23) we get the Stokes problem

$$\begin{aligned}
-D_i(\mu\mathbf{D}_{ij}(\underline{u}) - m_{ij} - p\delta_{ij}) &= f_j \quad (j=1,2) && \text{in } \Omega\\
D_i u_i &= 0 && \text{in } \Omega\\
\underline{u} &= \underline{\varphi}_1 && \text{on } R_1\\
u_n = \varphi_{2n} \quad , \quad (\mu\mathbf{D}_{ij}(\underline{u}) - m_{ij})n_i t_j &= \varphi_{2t} && \text{on } R_2\\
u_t = \varphi_{3t}\,, \ (p\delta_{ij} - \mu\mathbf{D}_{ij}(\underline{u}) - m_{ij})n_i n_j &= \varphi_{3n} && \text{on } R_3
\end{aligned}$$

which is equivalent to (21). Because m_{ij} is essentially bounded in Ω we get

$$\left((D_i m_{ij})_{j=1}^2\right) \in \underline{W}^{-1,p}(\Omega)\,,\ m_{ij}n_i t_j|_{R_2} \in W^{-1/p,p}(R_2)\,,\ m_{ij}n_i n_j|_{R_3} \in W^{-1/p,p}(R_3)$$

for every $p \in (1,\infty)$. Therefore it is possible to use the regularity theorems for the Stokes operator with non-smooth boundary data in two-dimensional domains (cf. Section 3) to get

Theorem 9: *Assume $q \geq 2$. The solution of problem (21) is an element of the space*

$$\mathcal{H}^{1,q}(\Omega) = \underline{W}^{1,q}(\Omega) \times L_q(\Omega) \tag{25}$$

if the right-hand sides of the differential equations and the boundary conditions are elements of

$$\mathcal{U}^{1,q}(\Omega) = \underline{W}^{-1,q}(\Omega) \times \prod_{j=1}^{N} \left[W^{1-1/q,q}(\Gamma_j) \times W^{1-m_{S_j}-1/q,q}(\Gamma_j)\right] \tag{26}$$

$$m_{S_j} \ldots \text{ order of the boundary condition on } \Gamma_j,$$

if the functions $\underline{\varphi}_1$, $\underline{\varphi}_{2n}$, $\underline{\varphi}_{3t}$ fulfil the compatibility conditions (17)-(19), if $\mu(\theta) \in W^{1,s}(\Omega)$ with $s > \max(2,q)$ holds and if for each $j \in \{1,\ldots,N\}$ the strip $\beta_j - l - 2 + 2/q \leq Im\lambda \leq \varepsilon_j$ of the complex plane is free of solutions of that equation of (12)-(15) which corresponds to the kind of boundary conditions given at the sides of the singular point O_j. Therein ε_j $(j = 1,\ldots,N)$ may be any positive numbers.

Remark 10: Obviously, at the line $\{\lambda \in \mathbb{C} : Im\lambda = 0\}$ there are situated only solutions of (15). These correspond to the apex angles $\pi/2$ and $3\pi/2$. Therefore if the domain under consideration has no corner with apex angle $\pi/2$ or $3\pi/2$ and intersecting boundary conditions of type R_2 and type R_3, if $\theta \in W^{1,p_3}(\Omega)$ for some $p_3 > 2$ and if the coefficient function $\mu(\cdot)$ is sufficiently smooth, then there always exists a number $q > 2$ fulfilling the assumptions of Theorem 9. Moreover the excluded case seems to be unreasonable by physical arguments.

By another point of view we define formally an operator

$$\mathbf{A} := A + \Psi' : \mathcal{H}^{1,q}(\Omega) \longmapsto \mathcal{U}^{1,q}(\Omega)$$

with the spaces $\mathcal{H}^{1,q}(\Omega)$, $\mathcal{U}^{1,q}(\Omega)$ as defined in Theorem 9 and

$$(A(\mathbf{u}),\mathbf{v}) = a(\underline{u},\underline{v}) \text{ and } (\Psi'(\mathbf{u}),\mathbf{v}) = \int_\Omega \tau \frac{\mathbf{D}_{ij}(\underline{u})\mathbf{D}_{ij}(\underline{v})}{\mathbf{D}_{II}(\underline{u})}\,dx$$

($\mathbf{u}$ is defined to be $(\underline{u},p)$) and a sequence $\{\mathbf{A}_\varepsilon\}_{\varepsilon>0}$ of operators approximating $\mathbf{A}$ by $\mathbf{A}_\varepsilon = \mathbf{A} + \Psi'_\varepsilon$ with

$$(\Psi'_\varepsilon(\mathbf{u}),\mathbf{v}) = \int_\Omega \tau \frac{\mathbf{D}_{ij}(\underline{u})\mathbf{D}_{ij}(\underline{v})}{\mathbf{D}_{II}(\underline{u})^{1-\varepsilon}}\,dx.$$

It is easily seen that $\mathbf{A}_\varepsilon$ tends to $\mathbf{A}$ with respect to uniform convergence if ε tends to zero and that $\mathbf{A}_\varepsilon : \mathcal{H}^{1,q}(\Omega) \longmapsto \mathcal{U}^{1,q}(\Omega)$ is an isomorphism for any $\varepsilon > 0$. Moreover for $\theta \in W^{1,s}(\Omega)$ with $s > 2$ we have

$$\left\|\mathbf{A}_\varepsilon^{-1}\left(\underline{f},\underline{\varphi}_1,\underline{\varphi}_2,\underline{\varphi}_3\right)\Big|\mathcal{H}^{1,q}(\Omega)\right\| \leq C\left(\|\theta|W^{1,s}(\Omega)\|\right)\left\|\left(\underline{f},\underline{\varphi}_1,\underline{\varphi}_2,\underline{\varphi}_3\right)\Big|\mathcal{U}^{1,q}(\Omega)\right\|,$$

where $C(\cdot)$ is a constant depending on the norm of θ but not on ε. This may be proved in two steps. First we deduce this inequality for $q = 2$ from the inequality of coercivity for $a(\underline{u},\underline{v})$, where we use the monotonicity of Ψ'_ε on $\mathcal{H}^{1,2}(\Omega)$ to prove the coercivity of $\mathbf{A}_\varepsilon$. With the usual arguments (cf. [1, 15]) we get then the asserted inequality for $q > 2$.

5. The linearized energy equation

First we consider now the variational equation

$$\begin{aligned} \int_\Omega \kappa(\vartheta) D_i\vartheta D_i\eta\, dx + \int_\Omega v_i(D_i\vartheta)\eta\, dx + \int_{\Gamma_a} \tfrac{b}{a}\vartheta\eta\, ds + \int_{\Gamma_a} \tfrac{c}{a}\eta\, ds = \int_\Omega g\eta\, dx \\ \vartheta = -c/b \quad \text{on } \Gamma_D = \partial\Omega\backslash\Gamma_a \end{aligned} \tag{27}$$

for any $\underline{v} \in \underline{L}^{p_4}(\Omega)$ ($p_4 > 2$), which corresponds to (1.c) with fixed velocity. For this problem we get the following weak maximum principle.

Lemma 11: *Assume that the conditions*

(i) $\frac{b(x)}{a(x)} \geq 0$ *on* Γ_a,
(ii) $|c(x)| \leq C_o|b(x)|$ *on* Γ_a,
(iii) $\kappa(t) > 0$ *for all* $t \in \mathbb{R}$,
(iv) $|\Gamma_D| > 0$, *that means that Dirichlet boundary conditions are given on a set with positive measure and*
(v) *(a)* $v_n \geq 0$ *on* Γ_a, $\operatorname{div}\underline{v} = 0$ *and* $g(\cdot) \equiv 0$ *in* Ω *or*
(b) $(\kappa(s) - \kappa(t))(s - t) \geq 0$ *for all* $s, t \in \mathbb{R}$ *and* $g(\cdot) \equiv 0$ *in* Ω *or*
(c) $0 < m \leq \kappa(t) \leq M < \infty$ *for all* $t \in \mathbb{R}$, $\sum_{i=1,2} |v_i(x)| \leq m^2\zeta^2$ *a.e. in* Ω *and* $g \in L_r(\Omega)$ *for some* $r > 1$

are fulfilled. Then any weak solution $\vartheta \in W^{1,2}(\Omega)$ *of problem (27) fulfils the weak maximum principle*

$$\min\left\{-C_o, \inf_{\Gamma_D}\left(\frac{c}{b}\right)^-\right\} - C_1K_r \leq \inf_\Omega \vartheta \leq \sup_\Omega \vartheta \leq \max\left\{C_o, \sup_{\Gamma_D}\left(\frac{c}{b}\right)^+\right\} + C_2K_r$$

with $C_i = C_i(r, \zeta, |\Omega|)$ $(i = 1, 2)$ *and* $K_r = \|g\,|L_r(\Omega)\|\,/m$. *By* $(h)^+$ *and* $(h)^-$ *we denote the positive and negative part of a real-valued function* h, *respectively.*

The assertion of Lemma 11 is a generalization of other well-known statements of the weak maximum principle (see, e.g., [5, 15]). An exact prove is given in author's thesis [9, Section 5.5].

Remark 12: If condition (v)(c) of Lemma 11 holds the assumption (ii) may be skipped, if we assume that the function c/a defined on Γ_a may be extended to an element of $W^{1/2,2}(\partial\Omega)$. We get in this case the maximum principle

$$\inf_{\Gamma_D}\left(\frac{c}{b}\right)^- - C_1\tilde{K}_r \leq \inf_\Omega \vartheta \leq \sup_\Omega \vartheta \leq \sup_{\Gamma_D}\left(\frac{c}{b}\right)^+ + C_2\tilde{K}_r$$

with $\tilde{K}_r = K_r + \|c/a|W^{1/2,2}(\partial\Omega)\|/m$.

Let us now consider the problem consisting of the equation

$$- D_i(\tilde{\kappa} D_i\vartheta) + v_i D_i\vartheta = g \qquad \text{in } \Omega \tag{28.a}$$

and the boundary condition

$$- a\tilde{\kappa} D_n\vartheta - b\vartheta = c \qquad \text{on } \partial\Omega, \tag{28.b}$$

where $\tilde{\kappa}$ is a function possibly depending on $x \in \partial\Omega$ but not on ϑ. For this we state

Lemma 13: *A unique weak solution $\vartheta \in W^{1,2}(\Omega)$ of (28) exists if the conditions*

(i) $|\Gamma_D| = |\partial\Omega \backslash \Gamma_a| > 0,$

(ii) the function c/b defined on Γ_D may be extended to an element of $W^{1/2,2}(\partial\Omega)$,

(iii) $0 < m \le \tilde{\kappa}(\cdot) \le M < \infty$ *a.e. in Ω,*

(iv) $\underline{v} \in \underline{W}^{1,2}(\Omega)$ *with* $\operatorname{div} \underline{v} = 0,$

(v) $b(x)/a(x) \ge 0$ *on Γ_a,*

(vi) $g \in L_r(\Omega)$ for some $r > 1$ and

(vii) the function c/a defined on Γ_a may be extended to an element of $W^{1/2,2}(\partial\Omega)$

are fulfilled.

Proof: The operator

$$E : W^{1,2}(\Omega) \longrightarrow W^{-1,2}(\Omega) \times \prod_{j=1}^{N} [W^{1/2-m_{P_j},2}(\Gamma_j)]$$

with

$$(E(\vartheta), \eta) = \int_\Omega \tilde{\kappa} D_i\vartheta D_i\eta \, dx + \int_\Omega v_i(D_i\vartheta)\eta \, dx + \int_{\Gamma_a} \frac{b}{a}\vartheta\eta \, ds$$

and

$$m_{P_j} \ldots \text{order of the boundary condition on } \Gamma_j$$

is pseudo–monotone. Moreover, any weak solution of (28) fulfils an a priori estimate. This is seen by using Lemma 11 and the inequality

$$\begin{aligned}
\int_\Omega v_i(D_i\vartheta)\vartheta \, dx + \int_{\Gamma_a} \frac{b}{a}\vartheta^2 \, ds \;&\ge\; -\int_{\Gamma_a} |\frac{1}{2}v_n|\vartheta^2 \, ds \\
&\ge\; -C(\Gamma_a)\, \|\vartheta| L_\infty\| \, \|\underline{v}| \underline{W}^{1,2}\| \, \|\vartheta| W^{1,2}\| .
\end{aligned}$$

(The coercivity of the principle part of the operator E is obvious.) Hence the main theorem on pseudo–monotone operators (cf. [27]) ensures the existence of a weak solution of problem (28). Using the assertion of Lemma 11 once again we conclude the uniqueness of the solution ∎

In connection with the consideration of the Bingham equation for a fixed temperature (cf. Section 4) we have assumed that $\theta \in L_\infty(\Omega)$. Therefore we state now a result on the regularity of the weak solution of problem (28).

To this end we need some information on the data of the boundary value problem near a corner. The general theory on elliptic problems in non-smooth bounded domains results that the following numbers are characteristic with respect to the regularity of the solution near a singular boundary point (cf. [14]):

$\left\{\frac{ik\pi}{\omega_j}\right\}_{k\in\mathbb{Z}}$	if at $\mathcal{O}_j$ two Newton conditions intersect, i.e., if $a(x) \geq a_0 > 0$ along Γ_{j-1} and Γ_j;
$\left\{i\frac{(2k+1)\pi}{2\omega_j}\right\}_{k\in\mathbb{Z}}$	if at $\mathcal{O}_j$ a Newton and a Dirichlet condition intersect, i.e., if $a(x) \equiv 0$ either along Γ_{j-1} or along Γ_j;
$\left\{\frac{ik\pi}{\omega_j}\right\}_{k\in\mathbb{Z}\setminus\{0\}}$	if at $\mathcal{O}_j$ two Dirichlet conditions intersect, i.e., if $a(x) \equiv 0$ along Γ_{j-1} and Γ_j.

(Here i denotes the imaginary unit.) We get

Proposition 14: *Assume the conditions of Lemma 13 be fulfilled. The weak solution of problem (28) is an element of the space $W^{2,t}(\Omega)$ if for any $j \in \{1,\ldots,N\}$ the strip $\{\lambda \in \mathbb{C}:\ 2/t - 2 \leq Im\lambda \leq \varepsilon_j\}$ is free of the respectitive of the above listed numbers for some $\varepsilon_j > 0$, if $\underline{v} \in \underline{W}^{1,2}$ and if the vector built of the right-hand sides of the differential equation and the boundary condition is an element of the space*

$$L_t(\Omega) \times \prod_{j=1}^{N} \left[W^{2-m_{P_j}-1/t,t}(\Gamma_j)\right]$$

where $m_{P_j} = 0$ if we have a Dirichlet boundary condition on Γ_j and $m_{P_j} = 1$ else.

The **proof** of the last assertion is a direct consequence of V.A. Kondratiev's and P. Grisvard's regularity theory (cf. [6, 10, 14])

Remark 15: It is easily seen that for every boundary configuration there exists a number $t > 1$ fulfilling the assumptions of Proposition 14. We may choose $t = 2$ if the inner apex angles at the singular boundary points are less than

(i) π for two intersecting Dirichlet or Newton conditions at $\mathcal{O}_j$ and
(ii) $\pi/2$ if at $\mathcal{O}_j$ a Newton and a Dirichlet condition intersect.

6. A priori estimates for the solutions of the non-linear problem (1),(2)

In this section we proof the following

Theorem 16: *If Assumption I is fulfilled, then for a solution of problem (1),(2) we get the estimate*

$$\left\|(\underline{u},p,\vartheta)\right|\mathcal{H}^{1,q} \times W^{2,t}\right\| \leq C(\underline{f},\underline{K},g,a,b,c,\underline{\varphi},\Omega).$$

The space $\mathcal{H}^{1,q}(\Omega)$ is defined in (25).

Proof: First we remark that, because of the weak maximum principle (cf. Lemma 11), we have a universal bound in the L_∞-norm for the ϑ-component of the solution.

This we use secondly to prove an a priori estimate for the velocity components of the solution in terms of $W^{1,2}$-norms. As noted above we set $k = 0$ if $R_3 \neq \emptyset$ and therefore we can use the well known technique of estimating weak solutions of Navier-Stokes (for $k \neq 0$) or Stokes (for $k = 0$) problems (cf. [25] in both cases) to get an a priori estimate for the velocity components of the solution.

Third we state

Lemma 17: *Let $\underline{u} \in \underline{W}^{1,2}(\Omega)$ and $\vartheta_{00} \in W^{1,2}(\Omega)$. Then there exists $\vartheta_0 \in W^{1,2}(\Omega)$ with $\mathrm{Tr}_\circ\vartheta_0 = \mathrm{Tr}_\circ\vartheta_{00}$ and*

$$\int_\Omega |u_i(D_i\vartheta_0)\eta|\, dx \leq \frac{1}{2}\, \|\vartheta_{00}|\, W^{1,2}(\Omega)\|\, \|\eta|\, W^{1,2}(\Omega)\|$$

for all $\eta \in W^{1,2}(\Omega)$ such that $\eta = 0$ on $\Gamma_D = \{x \in \partial\Omega :\ a(x) = 0\}$. Here $\mathrm{Tr}_\circ$ denotes the usual trace operator. Moreover, for this function ϑ_0 we get the estimate

$$\left|\int_\Omega \tilde{\kappa} D_i\vartheta_0 D_i\eta\, dx\right| \leq C\, \|\tilde{\kappa}|\, L_\infty(\Omega)\|\, \|\vartheta_{00}|\, W^{1,2}(\Omega)\|\, \|\eta|\, W^{1,2}(\Omega)\|$$

for all $\eta \in W^{1,2}(\Omega)$ and any coefficient $\tilde{\kappa}$. Therein the constant C is independent of $\underline{u}$.

Proof: Near smooth parts of the boundary we use E. Hopf's function (cf. [25, p.175]), which describes the distance between a point of Ω and the boundary in a smooth way, and R. Temam's [25] construction of vectors homogenizing the boundary conditions of Navier-Stokes problems.

Near non-smooth parts of the boundary we define a function ξ_ε analogous to E. Hopf's using polar-coordinates (r, ω). Therefore once again we use the standard cone K with apex angle $\omega_\circ$ defined in Section 2. The function in request should only depend on ω. We split the interval $[0, \omega_\circ]$ into five sub-intervals symmetrically. In the outer of this intervals the function ξ_ε is required to be equal to one, in the inner sub-interval we demand ξ_ε to be equal to zero and in the intermediate intervals we interpolate between zero and one smoothly. To be more precise; the function ξ_ε is defined by

$$\xi_\varepsilon(\omega) = \begin{cases} 1 & \text{for} \quad 0 \leq \omega \leq \varepsilon \\ p((\omega - \varepsilon)/\varepsilon) & \text{for} \quad \varepsilon \leq \omega \leq 2\varepsilon \\ 0 & \text{for} \quad 2\varepsilon \leq \omega \leq \omega_\circ - 2\varepsilon \\ p((\omega_\circ - \varepsilon - \omega)/\varepsilon) & \text{for} \quad \omega_\circ - 2\varepsilon \leq \omega \leq \omega_\circ - \varepsilon \\ 1 & \text{for} \quad \omega_\circ - \varepsilon \leq \omega \leq \omega_\circ \end{cases}$$

where p is a polynomial in the interval $[0, 1]$ with

$$p(0) = 1,\ p(1) = p'(1) = p''(1) = 0 \text{ and } p'(0) = p''(0) = 0,$$

which guarantees that the interpolation between zero and one in the intermediate intervalls is twice continuously differentiable. (The simplest polynomial fulfilling these conditions is $p(t) = -6t^5 + 15t^4 - 10t^3 + 1$.) The definition of ξ_ε shows that there exists a number $B \in \mathbb{R}_+$ such that

$$|\xi_\varepsilon(x)|, |D_k\xi_\varepsilon(x)|, |D_{ik}\xi_\varepsilon(x)| \leq B \qquad (x \in K;\ i, k = 1, 2).$$

Therein D_{ik} denotes the second partial derivative. The construction of a homogenizing function now follows the line of R. Temam's proof ∎

Inserting this homogenizing function in (27) for the case that ϑ_{00} is a continuation of the function $-c/b$ defined on Γ_D and using the weak maximum principle (cf. Lemma 11) we may proof an a priori estimate in $W^{1,2}$ for the temperature component in the usual way. The strong monotonicity of the principle part of the respective variational equation is obvious and the assumption $u_n|_{\Gamma_a} \geq 0$ together with $\operatorname{div} \underline{u} = 0$ ensure that $\int_\Omega u_i(D_i\vartheta)\vartheta\, dx$ is positive. In the case $u_n \not\geq 0$ on Γ_a we use the weak maximum principle for ϑ to get an a priori estimate in $\underline{W}^{1,2}$ for $\underline{u}$, which is independent of ϑ. This estimate may be used to prove an estimate for ϑ in terms of the $W^{1,2}$-norm, which only depends on the L_∞-bound for ϑ, the geometry of Ω and the right-hand sides of the equations.

After that we use the estimates for ϑ and $\underline{u}$ and a result of K. Gröger [7, Theorem 1] to improve the a priori estimate for ϑ as follows:

Lemma 18: *There exists a number $s > 2$ (depending only on the geometry) such that the weak solution ϑ of*

$$\begin{aligned} -D_i(\tilde\kappa D_i\vartheta) + u_i D_i\vartheta &= g && \text{in } \Omega \\ -a\tilde\kappa D_n\vartheta - b\vartheta &= c && \text{on } \partial\Omega, \end{aligned}$$

with $0 < m \leq \tilde\kappa \leq M < \infty$, is an element of $W^{1,s}(\Omega)$ and the estimate

$$\begin{aligned} \|\vartheta|\, W^{1,s}\| \leq C(m,M)\Big\{ &\|g|\, W^{-1,s}\|^2 + \big\|\tfrac{c}{a}\big|\, W^{-1/s,s}(\Gamma_a)\big\|^2 + \big\|\tfrac{c}{b}\big|\, W^{1-1/s,s}(\Gamma_D)\big\|^2 \\ &+ \textstyle\sum_{k=1}^3 \big\|\underline{\varphi}\big|\, W^{1/2,2}(R_k) \times W^{1/2-m_{sk},2}(R_k)\big\|^2 + \big\|\underline{f}\big|\, W^{-1,2}\big\|^2 \Big\} \end{aligned}$$

holds, if the functions a, b, c, g are sufficiently smooth on their supports.

For the **proof** we only remark that because of

$$\begin{aligned} \left|\int_\Omega u_i(D_i\vartheta)\eta\, dx\right| &\leq C\textstyle\sum_{i=1}^2 \|u_i|\, L_s\|\, \|D_i\vartheta|\, L_2\|\, \|\eta|\, L_t\| \\ &\leq C\big\|\underline{u}\big|\, \underline{W}^{1,2}\big\|\, \|\vartheta|\, W^{1,2}\|\, \big\|\eta\big|\, W^{1,\bar s'}\big\| \end{aligned}$$

with $\frac{1}{s} + \frac{1}{t} = \frac{1}{2}$, $\frac{1}{\bar s} + \frac{1}{\bar s'} = 1$ the inequality $\big\|u_i D_i\vartheta\big|\, W^{-1,\bar s}\big\| \leq C\big\|\underline{u}\big|\, \underline{W}^{1,2}\big\|\, \|\vartheta|\, W^{1,2}\|$ holds. ∎

The last assertion means that there exists a number $s > 2$ such that

$$\big\|\vartheta\big|\, W^{1,s}\big\| \leq C(m, M, g, a, b, c, \underline{\varphi}, \underline{f}, \underline{K})$$

because we have $0 < m \leq \kappa(\vartheta(x)) \leq M < \infty$ a.e. in Ω by the maximum principle. Using Theorem 9 and the "bootstrapping" argument known from the considerations of Navier–Stokes problems in the "smooth" case we get an estimate for $(\underline{u}, p)$ in $\mathcal{H}^{1,q}(\Omega)$-norms for some $q > 2$, namely

$$\begin{aligned} &\|(\underline{u}, p)|\, \mathcal{H}^{1,q}\| \\ \leq\ & C(m,M)C(\|\vartheta|\, W^{1,s}\|)\Big\{\big\|\underline{f}\big|\, \underline{W}^{-1,q}\big\| + \big\|\underline{K}\vartheta\big|\, \underline{W}^{-1,q}\big\| \\ &\qquad + \textstyle\sum_{k=1}^3 \big\|\underline{\varphi}\big|\, W^{1-1/q,q}(R_k) \times W^{1-m_{sk}-1/q,q}(R_k)\big\|\Big\} \\ \leq\ & C(m, M, g, a, b, c, \underline{\varphi}, \underline{f}, \underline{K}, \Omega), \end{aligned}$$

because for any $s_1 > 2q/(q-2)$ the term $\|\underline{K}\vartheta|\, \underline{W}^{-1,q}\|$ may be estimated by $\|\underline{K}|\, \underline{L}_{s_1}\|$ $\|\vartheta|\, W^{1,2}\|$. Once again we improve the estimate for ϑ, now using the last inequality and Proposition 14, and we get

$$\|(\underline{u},p,\vartheta)|\, \mathcal{H}^{1,q}\times W^{2,t}\| \le C(m,M,g,a,b,c,\underline{\varphi},\underline{f},\underline{K},\Omega).$$

The proof of Theorem 16 is done ∎

7. Proof of the solubility of non–linear temperature–coupled Bingham problems

For problem (1),(2) we define the space

$$\mathcal{X} = \left\{ \mathbf{u}\in\mathcal{H}^{1,q}(\Omega)\times W^{2,t}(\Omega): \begin{array}{ll} \mathbf{u}=(\underline{u},p,\vartheta) & \text{and } \operatorname{div}\underline{u}=0 \text{ and } \underline{u}|_{R_1}=0 \text{ and} \\ u_n|_{R_2}=0 & \text{and } u_t|_{R_3}=0 \text{ and } \vartheta|_{\Gamma_D}=0 \end{array} \right\}$$

for $2 < q < \infty$ and $1 < t < \infty$ and an operator $\mathbf{B} : \mathcal{X}\times\mathcal{X} \to \underline{W}^{-1,q}\times L_t$ by $(\mathbf{v}=(\underline{v},\pi,\theta))$

$$\begin{aligned}(\mathbf{B}(\mathbf{u},\mathbf{v}),(\underline{w},\eta)) = {} & a(\theta,\underline{u},\underline{w}) + b_1(\underline{v},\underline{v},\underline{w}) + k(\theta,\underline{w}) + (\Phi'(\underline{u}),\underline{w}) \\ & + e(\theta,\vartheta,\eta) + b_2(\underline{u},\vartheta,\eta)\end{aligned} \tag{29}$$

with

$$\begin{array}{llllll} a(\theta,\underline{u},\underline{w}) & = & \int_\Omega \mu(\theta)\mathbf{D}_{ij}(\underline{u})\mathbf{D}_{ij}(\underline{w})\,dx, & e(\theta,\vartheta,\eta) & = & \int_\Omega \mu(\theta)D_i\vartheta D_i\eta\,dx, \\ b_1(\underline{v},\underline{v},\underline{w}) & = & \int_\Omega k v_i(D_i v_j)w_j\,dx, & b_2(\underline{u},\vartheta,\eta) & = & \int_\Omega u_i(D_i\vartheta)\eta\,dx, \\ (\Phi'(\underline{u}),\underline{w}) & = & \int_\Omega \tau\frac{\mathbf{D}_{ij}(\underline{u})}{\mathbf{D}_{II}(\underline{u})}\mathbf{D}_{ij}(\underline{w})\,dx, & k(\theta,\underline{w}) & = & \int_\Omega K_j\theta w_j\,dx, \\ (\underline{w},\eta)\in \underline{W}^{1,q'}(\Omega)\times L_{t'}(\Omega) & & (1/t+1/t'=1,\ 1/q+1/q'=1). & & & \end{array}$$

In analogy to the approximation of $\mathbf{A}$ by $\mathbf{A}_\epsilon$ (cf. Section 4) we define now a sequence $\{\mathbf{B}_\epsilon\}_{\epsilon>0}$, substituting

$$(\Phi'(\underline{u}),\underline{w}) \longleftrightarrow (\Phi'_\epsilon(\underline{u}),\underline{w}) \quad\text{with}\quad (\Phi'_\epsilon(\underline{u}),\underline{w}) = \int_\Omega \tau\frac{\mathbf{D}_{ij}(\underline{u})}{\mathbf{D}_{II}^{1-\epsilon}(\underline{u})}\mathbf{D}_{ij}(\underline{w})\,dx$$

in (29). It is easily seen that $\mathbf{B}_\epsilon$ converges uniformly to $\mathbf{B}$ on bounded sets of $\mathcal{X}\times\mathcal{X}$ if ϵ tends to zero (cf. the same property for $\mathbf{A}$ and $\mathbf{A}_\epsilon$).

By $\mathbf{B}_\mathbf{v}$ and $\mathbf{B}_{\epsilon,\mathbf{v}}$ we denote the operators resulting from $\mathbf{B}$ and $\mathbf{B}_\epsilon$ by fixing the second argument. The properties which we have proved for $\mathbf{A}_\epsilon$ and Lemma 13 ensure that $\{\mathbf{B}^{-1}_{\epsilon,\mathbf{v}}(\mathbf{y})\}_{\epsilon>0,\mathbf{v}\in\mathcal{G}}$ is bounded for each $\mathbf{y}=(\underline{f},g,c,\underline{\varphi})\in\mathcal{Y}$ with

$$\begin{aligned}\mathcal{Y} = \underline{W}^{-1,q}(\Omega)\times L_t(\Omega) & \times \textstyle\prod_{j=1}^N \left[W^{2-m_{P_j}-1/t,t}(\Gamma_j)\right] \\ & \times \textstyle\prod_{j=1}^N \left[W^{1-1/q,q}(\Gamma_j)\times W^{1-m_{S_j}-1/q,q}(\Gamma_j)\right]\end{aligned}$$

and each bounded set $\mathcal{G}\subset\mathcal{X}$. Obviously $\mathbf{B}_\mathbf{v}$ and $\mathbf{B}_{\epsilon,\mathbf{v}}$ are operators from $\mathcal{X}$ to $\mathcal{Y}$.

Because the operators $\mathbf{B}_{\epsilon,\mathbf{v}}$ are well defined not only for $\mathbf{v} \in \mathcal{X}$ but also for $\mathbf{v} \in \underline{L}_s(\Omega) \times W^{-1,2}(\Omega) \times L_\infty(\Omega)$ for every $s > 2$ and $\mathcal{X}$ is compactly imbedded in the last space, we have

Lemma 19: *Let $\mathcal{G} \subset \mathcal{X}$ be a bounded open set. The operator-valued operators $\mathcal{B}_\epsilon : \mathbf{v} \longmapsto \mathbf{B}_{\epsilon,\mathbf{v}}$ are completely continuous with respect to the uniform convergence on $\mathcal{G}$, i.e., every weak Cauchy sequence $\{\mathbf{v}_n\} \subset \mathcal{G}$ will be transformed into a sequence $\{\mathbf{B}_{\epsilon,\mathbf{v}_n}\}$, which, with respect to the norm of uniform convergence on $\mathcal{G}$, is a strong Cauchy sequence.*

Proof: With $(\underline{w}, \eta) \in \underline{W}^{1,q'} \times L_{t'}$ we estimate the difference

$$\begin{aligned} &|(\mathbf{B}_\epsilon(\mathbf{u}_1, \mathbf{v}_1) - \mathbf{B}_\epsilon(\mathbf{u}_2, \mathbf{v}_2), (\underline{w}, \eta))| \\ \le\ & |a(\theta_1, \underline{u}_1, \underline{w}) - a(\theta_2, \underline{u}_1, \underline{w})| + |a(\theta_2, \underline{u}_1 - \underline{u}_2, \underline{w})| + |(\Phi'_\epsilon(\underline{u}_1) - \Phi'_\epsilon(\underline{u}_2), \underline{w})| \\ & + |b_1(\underline{v}_1, \underline{v}_1, \underline{w}) - b_1(\underline{v}_2, \underline{v}_2, \underline{w})| + |k(\theta_1 - \theta_2, \underline{w})| + |e(\theta_2, \vartheta_1 - \vartheta_2, \eta)| \\ & + |e(\theta_1, \vartheta_1, \eta) - e(\theta_2, \vartheta_1, \eta)| + |b_2(\underline{u}_1, \vartheta_1, \eta) - b_2(\underline{u}_2, \vartheta_2, \eta)|. \end{aligned} \tag{30}$$

We restrict the estimation to the first four summands of the right-hand side of the last inequality; the other one may be managed in a similar way.

(i) The first term may be estimated using Hölder's inequality, that means

$$\begin{aligned} &|a(\theta_1, \underline{u}_1, \underline{w}) - a(\theta_2, \underline{u}_1, \underline{w})| \\ &\quad = \left|\int_\Omega [\mu(\theta_1) - \mu(\theta_2)] \mathbf{D}_{ij}(\underline{u}_1) \mathbf{D}_{ij}(\underline{w})\, dx\right| \\ &\quad \le C\, \|\mu(\theta_1) - \mu(\theta_2)|\, L_\infty\|\, \|\mathbf{D}_{II}(\underline{u}_1)|\, L_q\|\, \|\mathbf{D}_{II}(\underline{w})|\, L_{q'}\| \\ &\quad \le C(\operatorname{Lip}\mu)\, \|\theta_1 - \theta_2|\, L_\infty\|\, \left\|\underline{u}_1|\, \underline{W}^{1,q}\right\| \left\|\underline{w}|\, \underline{W}^{1,q'}\right\|. \end{aligned}$$

(ii) For the second term we get

$$\begin{aligned} &|a(\theta_2, \underline{u}_1 - \underline{u}_2, \underline{w})| \\ &\quad = \left|\int_\Omega \mu(\theta_2) \mathbf{D}_{ij}(\underline{u}_1 - \underline{u}_2) \mathbf{D}_{ij}(\underline{w})\, dx\right| \\ &\quad \le C(\operatorname{Lip}\mu)\, \|\theta_2|\, L_\infty\|\, \|\mathbf{D}_{II}(\underline{u}_1 - \underline{u}_2)|\, L_q\|\, \|\mathbf{D}_{II}(\underline{w})|\, L_{q'}\| \\ &\quad \le C(\operatorname{Lip}\mu)\, \|\theta_2|\, L_\infty\|\, \left\|\underline{u}_1 - \underline{u}_2|\, \underline{W}^{1,q}\right\| \left\|\underline{w}|\, \underline{W}^{1,q'}\right\|. \end{aligned}$$

(iii) Because of the boundedness of the function $f(\lambda) = (\lambda^{2\epsilon} + 1 - 2\lambda^\epsilon \cos\gamma)/[\lambda^2 + 1 - 2\lambda\cos\gamma]^\epsilon$ for $\lambda \in \mathbb{R}_+$, for any elements $\mathbf{a}, \mathbf{b}$ of a Banach space E we get the inequality

$$\left\| \frac{\mathbf{a}}{\|\mathbf{a}|E\|^{1-\epsilon}} - \frac{\mathbf{b}}{\|\mathbf{b}|E\|^{1-\epsilon}} \right| E \Bigg\| \le C\|\mathbf{a} - \mathbf{b}|E\|^\epsilon.$$

And therefore for the third term of (30) we get the estimate

$$\begin{aligned} &|(\Phi'_\epsilon(\underline{u}_1) - \Phi'_\epsilon(\underline{u}_2), \underline{w})| \\ &= \left|\int_\Omega \tau \left[\frac{\mathbf{D}_{ij}(\underline{u}_1)}{\mathbf{D}_{II}(\underline{u}_1)^{1-\epsilon}} - \frac{\mathbf{D}_{ij}(\underline{u}_2)}{\mathbf{D}_{II}(\underline{u}_2)^{1-\epsilon}}\right] \mathbf{D}_{ij}(\underline{w})\, dx\right| \\ &\le \int_\Omega \tau \left\{\sum_{i,j=1}^2 \left[\frac{\mathbf{D}_{ij}(\underline{u}_1)}{\mathbf{D}_{II}(\underline{u}_1)^{1-\epsilon}} - \frac{\mathbf{D}_{ij}(\underline{u}_2)}{\mathbf{D}_{II}(\underline{u}_2)^{1-\epsilon}}\right]^2\right\}^{1/2} \left\{\sum_{i,j=1}^2 \mathbf{D}_{ij}^2(\underline{w})\right\}^{1/2} dx \\ &= \int_\Omega \tau \mathbf{D}_{II} \left[\frac{\mathbf{D}_{ij}(\underline{u}_1)}{\mathbf{D}_{II}(\underline{u}_1)^{1-\epsilon}} - \frac{\mathbf{D}_{ij}(\underline{u}_2)}{\mathbf{D}_{II}(\underline{u}_2)^{1-\epsilon}}\right] \mathbf{D}_{II}(\underline{w})\, dx \end{aligned}$$

$$\leq C_T \int_\Omega [\mathrm{D}_{II}(\underline{u}_1 - \underline{v}_2)]^\varepsilon \mathrm{D}_{II}(\underline{w})\, dx$$
$$\leq C_T |\Omega|^{(1-\varepsilon)/q} \left\|\underline{u}_1 - \underline{u}_2 | \underline{W}^{1,q}\right\|^\varepsilon \left\|\underline{w} | \underline{W}^{1,q'}\right\|.$$

(iv) Finally the fourth of the terms of inequality (30) may be estimated in the following way:

$$\begin{aligned}
&|b_1(\underline{v}_1, \underline{v}_1, \underline{w}) - b_1(\underline{v}_2, \underline{v}_2, \underline{w}|\\
&= \left|\int_\Omega [v_{1i} D_i v_{1j} - v_{2i} D_i v_{2j}] w_j\, dx\right|\\
&\leq \left|\int_\Omega (v_{1i} - v_{2i})(D_i v_{1j}) w_j\, dx\right| + \left|\int_\Omega v_{2i}[D_i(v_{1j} - v_{2j})] w_j\, dx\right|\\
&\leq \|v_{1i} - v_{2i} | L_2\| \, \|D_i v_{1j} | L_q\| \, \|w_j | L_{\tilde{t}}\|\\
&\quad + \|v_{2i} | W^{1,q}\| \, \|v_{1j} - v_{2j} | L_q\| \, \|w_j | W^{1,q'}\|\\
&\leq C \left\|\underline{v}_1 - \underline{v}_2 | \underline{L}_q\right\| \left[\left\|\underline{v}_1 | \underline{W}^{1,q}\right\| + \left\|\underline{v}_2 | \underline{W}^{1,q}\right\|\right] \left\|\underline{w} | \underline{W}^{1,q'}\right\|
\end{aligned}$$

There we have used the imbeddings $W^{1,q'} \hookrightarrow L_{\tilde{t}}$ for $\frac{1}{\tilde{t}} = \frac{1}{q'} - \frac{1}{2}$ and $W^{1,q}(\Omega) \hookrightarrow L_\infty$.

Summing up the estimates (i)–(iv) and the analogous ones for the other terms of (30) we get

$$\begin{aligned}
&|(\mathbf{B}_\varepsilon(\mathbf{u}_1, \mathbf{v}_1) - \mathbf{B}_\varepsilon(\mathbf{u}_2, \mathbf{v}_2), (\underline{w}, \eta))|\\
&\quad \leq C(\operatorname{Lip}\mu, \operatorname{Lip}\kappa) \left\|(\underline{w}, \eta) | \underline{W}^{1,q'} \times L_{t'}\right\|\\
&\qquad \times \left[\left\|\underline{u}_1 | \underline{W}^{1,q}\right\| + \left\|\underline{u}_2 | \underline{W}^{1,q}\right\| + \left\|\underline{v}_1 | \underline{W}^{1,q}\right\| + \left\|\underline{v}_2 | \underline{W}^{1,q}\right\|\right]\\
&\qquad \times \left[\left\|\underline{u}_1 - \underline{u}_2 | \underline{W}^{1,q}\right\| + \left\|\underline{u}_1 - \underline{u}_2 | \underline{W}^{1,q}\right\|^\varepsilon\right.\\
&\qquad\quad \left. + \left\|\underline{v}_1 - \underline{v}_2 | \underline{L}_2\right\| + \|\vartheta_1 - \vartheta_2 | W^{1,s} \cap L_\infty\|\right].
\end{aligned}$$

That means that the conclusion is now proved. ∎

The last assertion results obviously in the following

Corollary 20: *Each operator of the family $\mathbf{B}_\varepsilon : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{Y}$ fulfils the properties of a mapping with restricted representation by F.E. Browder [2, Definition 12.6.] if $\mathcal{G}$ is a bounded and open subset of the space $\mathcal{X}$.*

The trace $\mathbf{B}(\mathbf{u}, \mathbf{u})$ of $\mathbf{B}$ is the operator of problem (1), (2). It is denoted by $\mathbf{S}$. We now prove

Lemma 21: *The set $\mathbf{S}(\bar{\mathcal{G}})$ is closed for each bounded set $\mathcal{G} \subset \mathcal{X}$.*

Proof: Assume $\{\mathbf{y}_n\}_{n \in \mathbb{N}} \subset \mathbf{S}(\bar{\mathcal{G}})$ with $\mathbf{y}_n \longrightarrow \mathbf{y} \in \mathcal{Y}$. For every n there exists an element $\mathbf{x}_n \in \bar{\mathcal{G}}$ with $\mathbf{S}(\mathbf{x}_n) = \mathbf{y}_n$. Because of the a priori estimate for solutions of problem (1),(2) the sequence $\{\mathbf{x}_n\}$ is bounded and consequently weakly compact. Then $\mathbf{x}_n$ converges to $\mathbf{x} \in \mathcal{X}$ weakly with respect to the norm in $\mathcal{X}$ and strongly with respect to the norm in $\underline{L}_s(\Omega) \times W^{-1,2}(\Omega) \times L_\infty(\Omega)$ for $s > 2$. Using the properties already proved for the operators $\mathbf{A}_\varepsilon$ and $\mathbf{B}_\varepsilon$ we get

$$\begin{aligned}
\|\mathbf{S}(\mathbf{x}_n) - \mathbf{S}(\mathbf{x})\| &\leq \|\mathbf{S}(\mathbf{x}_n) - \mathbf{B}(\mathbf{x}, \mathbf{x}_n)\| + \|\mathbf{B}(\mathbf{x}, \mathbf{x}_n) - \mathbf{B}_\varepsilon(\mathbf{x}, \mathbf{x}_n)\|\\
&\quad + \|\mathbf{B}_\varepsilon(\mathbf{x}, \mathbf{x}_n) - \mathbf{B}_\varepsilon(\mathbf{x}, \mathbf{x})\| + \|\mathbf{B}_\varepsilon(\mathbf{x}, \mathbf{x}) - \mathbf{S}(\mathbf{x})\| \quad \leq \quad \delta
\end{aligned}$$

if $n \geq n_o(\delta)$ and $\varepsilon \leq \varepsilon_o(\delta)$. The uniqueness of limes in $\mathcal{Y}$ shows that $\mathbf{S}(\mathbf{x}) = \mathbf{y}$ ∎

We define a homotopy $\mathbf{S}_\gamma$ by $\mathbf{S}_\gamma(\mathbf{u}) = \mathbf{B}_{\mathbf{u}}^{(\gamma)}(\mathbf{u}) = \mathbf{B}(\mathbf{u}, \gamma \cdot \mathbf{u})$ and choose the set $\mathcal{G} = \{\mathbf{u} \in \mathcal{X} : \|\mathbf{u}|\mathcal{X}\| < 2C_a\}$, where C_a is the constant for which we have proved the a priori estimate in Theorem 16. Above we have proved the existence, uniqueness and regularity of weak solutions for the energy equation with fixed velocity and for the Bingham problem with fixed temperature. In the case $\gamma = 0$ the operator $\mathbf{S}_\gamma$ defines an uncoupled problem and we conclude therefore the unique solubility for the equation $\mathbf{S}_0(\mathbf{u}) = \mathbf{y}$ and the regularity of its solution for every $\mathbf{y} \in \mathcal{Y}$.

The properties just proved for the couple $(\mathbf{S}, \mathbf{S}_\gamma, \mathbf{B}, \mathbf{B}_\epsilon)$ of operators show that the assumptions of [2, Theorems 12.5, 12.6 and 12.7] are fulfilled and consequently we get

Theorem 22: *Let* $\mathbf{f} = (\underline{f}, g, c, \varphi) \in \mathcal{Y}$ *and assume that Assumption I is fulfilled and that the summing exponents q and t comply with the assumptions of Proposition 14 and Theorem 9. Then there exists a solution* $\mathbf{u} \in \mathcal{X}$ *for problem (1),(2) with right-hand side* $\mathbf{f}$.

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