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Sufficiency Conditions for Weak Local Minima in Multidimensional Optimal Control Problems with Mixed Control-State Restrictions

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In [13] a new sufficiency criterion for strong local minimality in multidimensional non-convex control problems with pure state constraint was developed. In this paper we use a similar method to obtain sufficient conditions for weak local minimality in multidimensional control problems with mixed state-control restrictions. The result is obtained by applying duality theory for control problems of KLÖTZLER [11] as well as first and second order optimality conditions for optimization problems described by C^1-functions having a locally Lipschitzian gradient mapping. The main theorem contains the result of ZEIDAN [17] for one-dimensional problems without state restrictions.

Key words: *Sufficient optimality conditions, multidimensional control problems, parametric optimization*

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1. Introduction

Let Ω be a domain in $\mathbb{R}^m$, $m > 1$, with piecewise smooth boundary. Then we study the following basic problem (P) of multidimensional optimal control theory:

$$\text{Minimize} \quad J(x,u) = \int_\Omega r(t, x(t), u(t))\, dt \tag{1}$$

subject to all pairs (x, u) of state functions $x \in D^{1,n}(\bar{\Omega})$ and control functions $u \in D^{0,r}(\bar{\Omega})$ such that the *state equations*

$$x_{t_\alpha}(t) = g_\alpha(t, x(t), u(t)) \quad \text{a. e. on } \Omega \quad (\alpha = 1, \ldots, m), \tag{2}$$

the *mixed state-control restrictions*

$$(x(t), u(t)) \in Y(t) \quad \text{a. e. on } \bar{\Omega}, \tag*{(3)$_1$}$$

with

$$Y(t) = \{(\xi, v) \in \mathbb{R}^n \times \mathbb{R}^r \mid f_i(t, \xi, v) \geq 0\, (i = 1, \ldots, l)\} \tag*{(3)$_2$}$$

and the *boundary condition*

$$x(s) = \varphi(s) \qquad \text{on } \partial\Omega \tag{4}$$

are fulfilled. Here $\partial\Omega$ is the boundary and $\bar{\Omega}$ is the closure of Ω. The functions r, g_α and f_i $(\alpha = 1, \ldots, m;\ i = 1, \ldots, l)$ are assumed to be continuous with respect to all arguments. $D^{0,r}(\bar{\Omega})$ is the space of all continuous vector functions on $\bar{\Omega}^j$ $(j = 1, \ldots, \nu)$, where $[\Omega^1, \ldots, \Omega^\nu]$ is a finite decomposition of Ω into domains Ω^j with piecewise smooth boundary, and $D^{1,n}(\bar{\Omega})$ is the space of all continuous vector functions on $\bar{\Omega}$ having continuous first partial derivatives in $\bar{\Omega}^j$ $(j = 1, \ldots, \nu)$. A pair $(x, u) \in D^{1,n}(\bar{\Omega}) \times D^{0,r}(\bar{\Omega})$ satisfying (2) - (4) is called *admissible* to (P) and the set of all admissible pairs is denoted by Z.

2. Construction of a dual problem to (P)

In a very general sense we call a problem

(D) maximize $L(S)$ subject to $S \in \mathbf{S}$

a dual problem to

(P) minimize $J(x,u)$ subject to $(x,u) \in Z$,

if the weak duality relation

$$L(S) \leq J(x,u) \tag{5}$$

is fulfilled for all $S \in \mathbf{S}$ and $(x,u) \in Z$. If for $S \in \mathbf{S}$ and $(x,u) \in Z$ the equation

$$L(S) = J(x,u) \tag{6}$$

holds, then the pair (x,u) is a global minimum of the problem (P).

There are many different conceptions of duality in literature, see, e. g., [3, 6, 7]. In contrast to the earlier concepts the duality construction of Klötzler [11], introduced in 1979, works without any convexity assumptions.

Using the Pontrjagin function H of (P) given by

$$H(t,\xi,v,\eta) = -r(t,\xi,v) + \sum_{\alpha=1}^{m} \eta^{\alpha\,\mathsf{T}} g_\alpha(t,\xi,v) \tag{7}$$

a dual problem (D) to (P) can be formulated in the following way [1]:

$$\text{maximize} \quad L(S) = \int_{\partial\Omega} S(s,\varphi(s))^{\mathsf{T}} n(s)\, do(s) \tag{8}$$

subject to $S \in \mathbf{S}$.

Here n denotes the exterior normal unit vector to $\partial\Omega$ and $\mathbf{S}$ is the set of all vector functions $S = (S^1, \ldots, S^m)^{\mathsf{T}} \in C^{1,m}(\Omega \times \mathbb{R}^n)$ fulfilling the Hamilton-Jacobi differential inequality

$$\operatorname{div}_t S(t,\xi) + H(t,\xi,v,\operatorname{grad}_\xi S(t,\xi)) \leq 0 \quad \text{on } W \tag{9$_1$}$$

with

$$W = \{(t,\xi,v) \mid t \in \bar{\Omega}, \quad (\xi,v) \in Y(t)\}. \tag{9$_2$}$$

Assertion 1. *Between the problems (D) and (P) the weak duality relation*

$$L(S) \leq J(x,u)$$

holds for all $S \in \mathbf{S}$ and $(x,u) \in Z$.

Proof. This relation follows immediately from (7) with $y(t) = \operatorname{grad}_\xi S(t,x(t))$ and Gauss' theorem:

$$\begin{aligned} J(x,u) &= \int_\Omega -H(t,x(t),u(t),\operatorname{grad}_\xi S(t,x(t)))\,dt \\ &\quad + \sum_{\alpha=1}^{m} \int_\Omega \operatorname{grad}_\xi S^\alpha(t,x(t))^{\mathsf{T}} g_\alpha(t,x(t),u(t))\,dt \\ &= -\int_\Omega \{H(t,x(t),u(t),\operatorname{grad}_\xi S(t,x(t))) + \operatorname{div}_t S(t,x(t))\}\,dt \\ &\quad + \int_{\partial\Omega} S(s,\varphi(s))^{\mathsf{T}} n(s)\,do(s) \geq L(S). \end{aligned} \tag{10}$$

Assertion 2 (Generalized Maximum Principle). *An admissible pair* (x^*, u^*) *is a global minimizer of (P) if there exists a vector function* $S^* \in \mathbf{S}$ *satisfying the Hamilton–Jacobi equation*

$$\text{(HJ)} \qquad \operatorname{div}_t S^*(t, x^*(t)) + H(t, x^*(t), u^*(t), \operatorname{grad}_\xi S^*(t, x^*(t))) = 0 \qquad \text{on } \bar{\Omega}.$$

Proof. The condition (HJ) effects that especially the equation $J(x^*, u^*) = L(S^*)$ holds in (10) for $(x^*, u^*) \in Z$ and $S^* \in \mathbf{S}$. Thus (x^*, u^*) is a global minimizer of (P) ∎

Remark. The existence of an S^* in Assertion 2 is hypothetical and, in general, it is very difficult to find a suitable S^*. Nevertheless, it was done for some interesting problems, see, e. g., [1, 2].

For that reason it is also helpfull to give sufficient criterions for local minima. In [13] we proved such a criterion for *strong* local minimality. The present paper can be regarded as the second part of those investigations which gives a sufficient optimality condition for *weak* local minimality.

Definition 1. An admissible pair (x^0, u^0), with $u^0 \in C^{0,r}(\bar{\Omega})$, is a *weak local minimum* of (P) if there exists an $\epsilon > 0$ such that (x^0, u^0) minimizes $J(x, u)$ over all admissible pairs $(x, u) \in Z$ with $\|x - x^0\|_{C^{0,n}} < \epsilon$ and $\|u - u^0\|_{C^{0,r}} < \epsilon$.

In a similar way as in Assertion 2 we now can deduce from the Generalized Maximum Principle

Assertion 3. *An admissible pair* $(x^0, u^0) \in Z$, *with* $u^0 \in C^{0,r}(\bar{\Omega})$, *is a weak local minimum of (P), if there exists an* $\epsilon > 0$ *and an* $S \in C^1(\bar{\Omega} \times \mathbb{R}^n)$ *such that the condition (HJ) and the Hamilton-Jacobi inequality*

$$\operatorname{div}_t S(t, \xi) + H(t, \xi, v, \operatorname{grad}_\xi S(t, \xi)) \le 0 \tag{11$_1$}$$

on

$$W_\epsilon = \{(t, \xi, v) \mid t \in \bar{\Omega},\ (\xi, v) \in Y_\epsilon(t)\} \tag{11$_2$}$$

with

$$Y_\epsilon(t) = Y(t) \cap K_\epsilon(x^0(t), u^0(t)) \tag{11$_3$}$$

and

$$K_\epsilon(x^0(t), u^0(t)) = \{(\xi, v) \mid \|((\xi - x^0(t))^\top, (v - u^0(t))^\top)^\top\| \le \epsilon\} \tag{11$_4$}$$

($\|\cdot\|$ *denotes the Euklidean norm in* $\mathbb{R}^n \times \mathbb{R}^r$) *is fulfilled.*

Proof. The proof follows from the fact that $(x^0, u^0) \in Z$ is a weak local minimum of (P) if there exists an $\epsilon > 0$ such that (x^0, u^0) is a global minimum of (P_ϵ). Here (P_ϵ) is defined in the same way as (P), where only Y(t) is replaced by $Y_\epsilon(t)$ ∎

3. On strongly stable local maximizers of parametric optimization problems

As in [13] we will use now results concerning strong stable local maximizers of parametric optimization problems to prove the main result. Therefore we study the general parametric problem

$$\text{(P(t))} \qquad \text{maximize } f_0(t, \xi, v) \qquad \text{with respect to } (\xi, v) \in Y_\epsilon(t).$$

Let $(x^0, u^0) \in Z$, with $u^0 \in C^{0,r}(\bar{\Omega})$, be given. Then our aim is to develop sufficient conditions for the existence of an $\epsilon > 0$ (independent on $t \in \bar{\Omega}$) such that, for $t \in \bar{\Omega}$,

$$f_0(t, x^0(t), u^0(t)) \ge f_0(t, \xi, v) \quad \text{for all } (\xi, v) \in Y_\epsilon(t). \tag{12}$$

The relation (12) means not only that $(x^0(t), u^0(t))$ is a local maximizer of (P(t)) for all $t \in \bar{\Omega}$ but also the existence of a uniform radius $\epsilon > 0$ such that $(x^0(t), u^0(t))$ is a global maximizer with respect to $Y_\epsilon(t)$. In a natural way this problem (P(t)) appears if we are looking for an admissible S^* for which the Hamilton–Jacobi equation (HJ) is already satisfied.

3. 1. Linear statements of S in the dual problem

Now we choose the following statement S in (D):

$$S^\alpha(t,\xi) = a^\alpha(t) + p^\alpha(t)^\top(\xi - x^0(t)) \tag{13}$$

with $p^\alpha \in C^{1,n}(\bar\Omega)$ and $a^\alpha \in C^1(\bar\Omega)$ $(\alpha = 1,\dots,m)$. Then the function f_0 appearing in (P(t)) has the special form

$$f_0(t,\xi,v) = \sum_{\alpha=1}^{m}\{a^\alpha_{t_\alpha}(t) + p^\alpha_{t_\alpha}(t)^\top(\xi - x^0(t)) - p^\alpha(t)^\top x^0_{t_\alpha}(t)\} + H(t,\xi,v,p(t)).$$

In accordance with the basic assumptions, the function H and therefore the function f_0 is continuous with respect to all its arguments. In addition to these assumptions we require the following condition:

(V) For an $\epsilon > 0$ let $f_0(t,\cdot,\cdot)$ and $f_i(t,\cdot,\cdot)\,(i = 1,\dots,l)$ be proper concave functions on $K_\epsilon(x^0(t),u^0(t))$ for all $t \in \bar\Omega$.

According to [16] $f_i(t,\cdot,\cdot)\,(i = 0,\dots,l)$ are Lipschitz functions on $\bar K_\epsilon(x^0(t),u^0(t))$ with respect to the second and third argument and, by the Rademacher theorem [14], they are almost everywhere on $K_\epsilon(x^0(t),u^0(t))$ totally differentiable. Functions having such properties possess a non–empty supergradient [16]

$$\begin{aligned}&\partial_{\xi,v} f_i(t,x^0(t),u^0(t))\\ &\quad = \{(\xi^*,v^*) \in \mathbb{R}^{n+r} \mid f_i(t,\xi',v') - f_i(t,x^0(t),u^0(t))\\ &\qquad \le \langle\xi^*,\xi' - x^0(t)\rangle + \langle v^*,v' - u^0(t)\rangle \forall\xi' \in \mathbb{R}^n, \forall v' \in \mathbb{R}^r\}\end{aligned}$$

which in this case coincides with the supergradient in the sense of CLARKE [4]

$$\begin{aligned}&\partial^c_{\xi,v} f_i(t,x^0(t),u^0(t))\\ &\quad = \operatorname{conv}\{M \mid \exists\{h_j\}_{j=1}^\infty \text{ with}(x^0(t) + h^1_j, u^0(t) + h^2_j) \in E_t f_i(t,\cdot,\cdot)\\ &\qquad \operatorname{grad}_{\xi,v} f(t,x^0(t) + h^1_j, u^0(t) + h^2_j) \to M \text{ for } j \to \infty\}.\end{aligned} \tag{14}$$

(Here $E_t f_i(t,\cdot,\cdot)$ is the set of all $(\xi,v) \in \bar K_\epsilon(x^0(t),u^0(t))$, for which $f_i(t,\cdot,\cdot)$ is continuously differentiable and conv denotes the convex hull.) Thereby the problem (P(t)) is a concave maximization problem. If we assume that in $(x^0(t),u^0(t))$ the following **L***inear* **I***ndependent* **C***onstraint* **Q***ualification*

(LICQ) For each $t \in \bar\Omega$ the vectors $\operatorname{grad}_{\xi,v} f_i(t,x^0(t),u^0(t))$, $\quad i \in I^0(t) = \{i \in \{1,\dots,l\} \mid f_i(t,x^0(t),u^0(t)) = 0\}$ are linearly independent

is satisfied, then the necessary conditions

$$0 \in \partial^c_{\xi,v} f_0(t,x^0(t),u^0(t)) + \sum_{i=1}^{l}\lambda_i(t)\partial^c_{\xi,v} f_i(t,x^0(t),u^0(t)) \tag{15}$$

and

$$\lambda_i(t) \ge 0, \quad \lambda_i(t) f_i(t,x^0(t),u^0(t)) = 0 \qquad (i = 1,\dots,l) \tag{16}$$

are sufficient for optimality.

We remark that the function $f_0(t,\cdot,\cdot)$ fulfils the assumption (V) for an $\epsilon > 0$ if the Pontrjagin function $H(t,\cdot,\cdot,p(t))$ satisfies the assumption (V) for all $t \in \bar\Omega$. Then it holds

Theorem 1. *Let (x^0, u^0), with $u^0 \in C^{0,r}(\bar{\Omega})$, be an admissible process to (P). Assume that there exists an $\epsilon > 0$ such that the condition (V) is fulfilled for the corresponding Pontrjagin function $H(t,\cdot,\cdot,p(t))$ and $f_i(t,\cdot,\cdot)(i = 1,\ldots,l)$ and that the condition (LICQ) is satisfied in (x^0, u^0). If there are vector functions $p^\alpha \in C^1(\bar{\Omega})$ $(\alpha = 1,\ldots,m)$ and multiplier functions λ_i $(i = 1,\ldots,l)$ with*

$$-\sum_{\alpha=1}^{m} p^\alpha_{t_\alpha}(t) \in \partial^c_\xi\{H(t,x^0(t),u^0(t),p(t)) + \sum_{i=1}^{l} \lambda_i(t) f_i(t,x^0(t),u^0(t))\}, \tag{17}$$

$$\lambda_i(t) \geq 0, \quad \lambda_i(t) f_i(t,x^0(t),u^0(t)) = 0 \qquad (i = 1,\ldots,l), \tag{18}$$

$$0 \in \partial^c_v\{H(t,x^0(t),u^0(t),p(t)) + \sum_{i=1}^{l} \lambda_i(t) f_i(t,x^0(t),u^0(t))\}, \tag{19}$$

for $t \in \bar{\Omega}$, then (x^0, u^0) provides a weak local minimum of (P).

Proof. We show that the conditions of Assertion 3 are satisfied. By the special choice of a^α,

$$\sum_{\alpha=1}^{m} a^\alpha_{t_\alpha}(t) = \sum_{\alpha=1}^{m} p^\alpha(t) x^0_{t_\alpha}(t) - H(t,x^0(t),u^0(t),p(t)),$$

we ensure that the Hamilton-Jacobi equation (HJ) is fulfilled on $\bar{\Omega}$. Furthermore we choose p^α $(\alpha = 1,\ldots,m)$ in such a way that $(x^0(t), u^0(t))$ is a global maximizer of (P(t)) on $Y_\epsilon(t)$ for all $t \in \bar{\Omega}$. Following (15) and (16) the conditions

$$0 \in \partial^c_{\xi,v}\Big\{\sum_{\alpha=1}^{m} p^\alpha_{t_\alpha}(t)^\top(\xi - x^0(t)) + H(t,\xi,v,p(t)) + \sum_{i=1}^{l} \lambda_i(t) f_i(t,\xi,v)\Big\}_{|\xi=x^0(t),v=u^0(t)}$$

that means relation (17) and relation (19) of the theorem and the complementary condition (18) are sufficient for maximality, see [4]. Now it follows that S in (12) fulfils the Hamilton-Jacobi inquality (11), i. e. $S \in \mathbf{S}_\epsilon$, and Assertion 3 holds. This completes the proof ∎

3.2. Quadratic statements for S in the dual problem

By nonlinear statements we shall overcome the a priori concavity assumptions of Theorem 1. Therefore, we now use the statement

$$S^\alpha(t,\xi) = a^\alpha(t) + p^\alpha(t)^\top(\xi - x^0(t)) + \frac{1}{2}(\xi - x^0(t))^\top Q^\alpha(t)(\xi - x^0(t)) \tag{20}$$

with $a^\alpha \in C^1(\bar{\Omega}), p^\alpha \in C^{1,n}(\bar{\Omega})$, and a symmetric matrix function Q^α with components $Q^\alpha_{i,j} \in C^1(\bar{\Omega})\,(\alpha = 1,\ldots,m;\ i,j = 1,\ldots,n)$ in the dual problem.

As in the section before, we study the parametric optimization problem (P(t)) and give sufficient conditions for the fact that $(x^0(t), u^0(t))$ is a global maximizer of (P(t)) for a given $\epsilon > 0$. With the quadratic statement of S, the objective functional f_0 in (P(t)) has the form

$$\begin{aligned} f_0(t,\xi,v) = \sum_{\alpha=1}^{m} \Big\{ & a^\alpha_{t_\alpha}(t) + p^\alpha_{t_\alpha}(t)^\top(\xi - x^0(t)) \\ & - p^\alpha(t)^\top x^0_{t_\alpha}(t) - x^0_{t_\alpha}(t)^\top Q^\alpha(t)(\xi - x^0(t)) \\ & + \frac{1}{2}(\xi - x^0(t))^\top Q^\alpha_{t_\alpha}(t)(\xi - x^0(t))\Big\} \\ & + H(t,\xi,v,p(t) + Q(t)(\xi - x^0(t))). \end{aligned} \tag{21}$$

We are mainly interested in second-order conditions which require further assumptions on the differentiability of the appearing functions. Therefore, with

$$N(t,\epsilon,\delta) := \{(\xi,v,\eta) \in \mathbb{R}^{n+r+nm} \,|(\xi,v) \in \bar{K}_\epsilon(x^0(t),u^0(t)), \\ \|\eta - (p^1(t)^\top,\ldots,p^m(t)^\top)^\top\| \le \delta\}$$

and

$$N(\epsilon,\delta) := \{(t,\xi,v,\eta) \in \mathbb{R}^{m+n+r+nm} \mid (\xi,v,\eta) \in N(t,\epsilon,\delta),\, t \in \bar{\Omega}\}$$

for a given $\epsilon, \delta > 0$ we additionally assume:

(W)

1. $H(t,\cdot,\cdot,\cdot) \in C^1(N(t,\epsilon,\delta))$ for all $t \in \bar{\Omega}$,
2. $H(\cdot,\cdot,\cdot,\cdot)$, $\mathrm{grad}_{\xi,v,\eta}H(\cdot,\cdot,\cdot,\cdot) \in C(N(\epsilon,\delta))$,
3. $\mathrm{grad}_{\xi,v,\eta}H(t,\cdot,\cdot,\cdot)$ be locally Lipschitzian on $N(t,\epsilon,\delta)$ for all $t \in \bar{\Omega}$,
4. the mapping $(t,\xi,v,\eta) \mapsto \partial^c_{\xi,v,\eta}(\mathrm{grad}_{\xi,v,\eta}H(t,\xi,v,\eta))$ be closed and locally bounded on $N(\epsilon,\delta)$,
5. $f_k \in C^2(N_\epsilon)\,(k=1,\ldots,l)$, $N_\epsilon := \{(t,\xi,v) \mid t \in \bar{\Omega}, (\xi,v) \in K_\epsilon(x^0(t),u^0(t))\}$.

If the function H satisfies the assumptions (1) - (4) of (W), then f_0 in (21) is almost everywhere twice differentiable with respect to (ξ,v) in a neighbourhood of $(x^0(t),u^0(t))$. In the following, we use the generalized Hessian $\partial^2_{\xi,v}f_0$ in the sense of HIRIART-URRUTY et al. [9]:

$$\begin{aligned}\partial^2_{\xi,v}f_0(t,x^0(t),u^0(t)) \\ = \mathrm{conv}\{M(t) \mid \exists\{h^1_j,h^2_j\}_{j=1}^\infty \text{ with } \{h^1_j,h^2_j\} \to 0 \text{ for } j\to\infty, \\ (x^0(t)+h^1_j, u^0(t)+h^2_j) \in E_t(\mathrm{grad}_{\xi,v}f_0(t,\cdot,\cdot)), \\ \mathrm{d}^2_{\xi,v}f_0(t,x^0(t)+h^1_j,u^0(t)+h^2_j) \to M(t) \text{ for } j\to\infty\}\end{aligned} \tag{22}$$

($d^2_{\xi,v}$ denotes the usual Hessian). With these assumptions we get

Assertion 4. *Let the assumptions (W) and (LICQ) be fulfilled in $(x^0(t),u^0(t))$ for $t \in \bar{\Omega}$. Then, the following conditions are sufficient for optimality of a continuous pair $(x^0(t),u^0(t))$ to the problem (P(t)):*
There exists a multiplier function $\lambda_i : \bar{\Omega} \to \mathbb{R}$ $(i=1,\ldots,l)$ such that $(x^0(t),u^0(t),\lambda(t))$ is a stationary point of (P(t)), i. e.

$$\mathrm{grad}_{\xi,v}f_0(t,x^0(t),u^0(t)) + \sum_{i=1}^{l}\lambda_i(t)\mathrm{grad}_{\xi,v}(x^0(t),u^0(t)) = 0, \tag{23}$$

$$\lambda_i(t)f_i(t,x^0(t),u^0(t)) = 0, \quad \lambda_i(t) \ge 0 \quad (i=1,\ldots,l) \tag{24}$$

and each matrix function $M(t)$, with

$$M(t) \in \partial^2_{\xi,v}f_0(t,x^0(t),u^0(t)) + \sum_{i\in I^+(t)}\lambda_i(t)\mathrm{d}^2_{\xi,v}f_i(t,x^0(t),u^0(t)),$$

is negative definite on

$$R^+(t) := \{h \in \mathbb{R}^{n+r} \mid h^\top \mathrm{grad}_{\xi,v}f_i(t,x^0(t),u^0(t)) = 0, \quad i \in I^+(t)\},$$
$$I^+(t) = \{i \in \{1,\ldots,l\} \mid \lambda_i(t) > 0\},$$

i. e.

$$h^\top M(t)h < 0 \quad \text{on } R^+(t)\setminus\{0\}, \quad t \in \bar{\Omega}. \tag{25}$$

Proof. Firstly we remark that the conditions (23) - (25) are a natural generalization of the classical second-order conditions for strict local maximality of an $(x^0(t),u^0(t))$ with respect to the objective functional f_0 and the constraints $f_i(t,\xi,v) \geq 0,\ t \in \bar{\Omega}$ [10]. For that reason, there exists an $\epsilon(t) > 0$ such that there holds

$$f_0(t,\xi,v) \leq f_0(t,x^0(t),u^0(t)) \quad \text{for all } (\xi,v) \in Y(t) \cap K_{\epsilon(t)}(x^0(t),u^0(t)).$$

It remains to prove that

$$\inf_{t\in\bar{\Omega}} \epsilon(t) = \epsilon_0 > 0 \tag{26}$$

exists. This was done in [13, Assertion 3]. The proof is based on the property of strict stability in the sense of KOJIMA [12] which is an immediate consequence of the validity of the condition (LICQ) ∎

Assertion 5. *Let us consider the parametric optimization problem (P(t)), for which the condition (W) is fulfilled. If (x^0,u^0) is a continuous optimal solution for which the condition (LICQ) is satisfied on $\bar{\Omega}$, then the multipier λ is unique and depends continuously on $t \in \bar{\Omega}$.*

Proof. This is a standard result of nonlinear parametric optimization, see e. g. Theorem 2. 3 in [15] ∎

After these preparatory assertions, we can formulate the main result of this paper.

Theorem 2. *Let (x^0,u^0), with $u^0 \in C^{0,r}(\bar{\Omega})$, be an admissible process to (P). Assume that there exists an $\epsilon > 0$ such that the condition (W) is fulfilled for the corresponding Pontrjagin function $H(t,\cdot,\cdot,p(t))$ and for $f_i(t,\cdot,\cdot)$ and that the condition (LICQ) is satisfied in (x^0,u^0). If there are vector functions $p^\alpha \in C^1(\bar{\Omega})$, multiplier functions λ_i and matrix functions $Q^\alpha \in C^{1,nn}(\bar{\Omega})$ $(\alpha = 1,\ldots,m;\ i = 1,\ldots,l)$ such that, for $t \in \bar{\Omega}$,*

$$-\sum_{\alpha=1}^{m} p^\alpha_{t_\alpha}(t) = \operatorname{grad}_\xi \Big\{ H(t,x^0(t),u^0(t),p(t)) + \sum_{i=1}^{l} \lambda_i(t) f_i(t,x^0(t),u^0(t)) \Big\}, \tag{27}$$

$$\lambda_i(t) \geq 0, \quad \lambda_i(t) f_i(t,x^0(t),u^0(t)) = 0 \quad (i = 1,\ldots,l), \tag{28}$$

$$0 = \operatorname{grad}_v \left\{ H(t,x^0(t),u^0(t),p(t)) + \sum_{i=1}^{l} \lambda_i(t) f_i(t,x^0(t),u^0(t)) \right\} \tag{29}$$

and each matrix $m(t)$,

$$m(t) \in \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix} + \sum_{i\in I^+(t)} \lambda_i(t) d^2_{\xi,v} f_i(t,x^0(t),u^0(t)), \qquad t \in \bar{\Omega} \tag{30}$$

with

$$\begin{aligned}
\alpha(t) = \sum_{\alpha=1}^{m} Q^\alpha_{t_\alpha}(t) &+ \partial^2_{\xi,\xi} H(t,x^0(t),u^0(t),p(t)) + \sum_{\alpha=1}^{m} Q^\alpha(t) \partial^2_{y^\alpha,\xi} H(t,x^0(t),u^0(t),p(t)) \\
&+ \sum_{\alpha=1}^{m} \partial^2_{\xi,y^\alpha} H(t,x^0(t),u^0(t),p(t)) Q^\alpha(t) \\
&+ \sum_{\alpha,\beta=1}^{m} Q^\alpha(t) \partial^2_{y^\alpha,y^\beta} H(t,x^0(t),u^0(t),p(t)) Q^\beta(t)
\end{aligned}$$

$$\beta(t) = \partial^2_{\xi,v} H(t,x^0(t),u^0(t),p(t)) + \sum_{\alpha=1}^{m} Q^\alpha(t) \partial^2_{y^\alpha,v} H(t,x^0(t),u^0(t),p(t)),$$

$$\gamma(t) = \partial^2_{v,\xi} H(t,x^0(t),u^0(t),p(t)) + \sum_{\alpha=1}^{m} \partial^2_{v,y^\alpha} H(t,x^0(t),u^0(t),p(t)) Q^\alpha(t),$$

$$\delta(t) = \partial^2_{v,v} H(t,x^0(t),u^0(t),p(t))$$

is negative definite on $R^+(t)\setminus\{0\}$ *, i. e.*

$$h^\top m(t)h < 0 \quad \text{on } R^+(t)\setminus\{0\}. \tag{31}$$

Then (x^0, u^0) *provides a weak local minimum of (P).*

Proof. The idea of the proof is also to apply Assertion 4 with the quadratic statement (20) for S in (P(t)). Indeed, we can choose $a^\alpha\,(\alpha = 1,\ldots,m)$ in such way that the condition (HJ) in Assertion 3 is satisfied on $\bar{\Omega}$, namely

$$\sum_{\alpha=1}^{m} a^\alpha_{t_\alpha}(t) = \sum_{\alpha=1}^{m} p^\alpha(t)x^0_{t_\alpha}(t) - H(t, x^0(t), u^0(t), p(t)).$$

Now we shall prove that,for an $\epsilon > 0$, S satisfies the assumptions of Assertion 3. Obviously, this is true if $(x^0(t), u^0(t))$ maximizes $f_0(t,\cdot,\cdot)$ on $Y_\epsilon(t)$ for all $t \in \bar{\Omega}$. We want use Assertion 4. Therefore, we choose $p^\alpha\ (\alpha = 1,\ldots,m)$ and $\lambda_j\ (j = 1,\ldots,l)$ in such way that $(x^0(t), u^0(t), \lambda(t))$ is a stationary point of (P(t)) for all $t \in \bar{\Omega}$. According to (22) and (23), this is true if the canonical equations (27) and (29) as well as the complementary conditions (28) of the theorem are satisfied. Further on, we verify condition (25) of Assertion 4. From [5] the inclusion

$$\partial^2_{\xi,v} f_0(t, x^0(t), u^0(t)) \subseteq \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix} \tag{32}$$

holds. If now for each $m(t)$ fulfilling (30) it follows

$$h^\top m(t)h < 0 \quad \text{on } R^+(t)\setminus\{0\},$$

then, according to (32),for each $n(t)$ with

$$n(t) \subseteq \partial^2_{\xi,v} f_0(t, x^0(t), u^0(t)) + \sum_{i\in I^+(t)} \lambda_i(t) \mathrm{d}^2_{\xi,v} f_i(t, x^0(t), u^0(t))$$

it holds

$$h^\top n(t)h < 0 \quad \text{on } R^+(t)\setminus\{0\}.$$

Thus we finally conclude condition (25). Taking the regularity assumption (W) into account Assertion 4 can be applied to our situation and the proof is complete ∎

Remark. The matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ appearing in (31) is symmetrically because of the definition of the generalized Hessian. Therefore it make sense to reformulate condition (31) in special cases. Preparatorily we use the following

Assertion 6. *Let be given a symmetric block matrix* $\begin{pmatrix} A & B \\ B^\top & D \end{pmatrix} =: K$ *, where* A *is an* $n \times n$ *matrix and* D *is an* $m \times m$ *matrix. Then* K *is negative definite on* $\mathbb{R}^{nm}$ *if and only if* D *and* $A - BD^{-1}B^\top$ *are negative definite on* $\mathbb{R}^{nm}$.

Proof. It follows from the fact that a symmetric block–matrix

$$\begin{pmatrix} A & B \\ B^\top & D \end{pmatrix}$$

is negative definite if and only if D and the Schur complement $A - BD^{-1}B^{\top}$ of this matrix are negative definite, see [8].

Remark. Considering the interesting special case of problem (P) where the mixed state–control restrictions are missing and the Pontrjagin function H belongs to C^2 with respect to all its arguments. Then condition (31) of Theorem 2 can be replaced by a *classical* second order condition: Let the matrices

$$G(t) := a(t) - b(t)d^{-1}(t)b(t) \text{ and } d(t) \tag{31'}$$

be negative definite on $\mathbb{R}^n \setminus \{0\}$ or $\mathbb{R}^m \setminus \{0\}$, respectively, where

$$\begin{aligned} a(t) = \sum_{\alpha=1}^{m} Q^{\alpha}_{t_\alpha}(t) + \mathrm{d}^2_{\xi,\xi} H(t, x^0(t), u^0(t), p(t)) \\ + \sum_{\alpha=1}^{m} Q^{\alpha}(t)\mathrm{d}^2_{y^\alpha,\xi} H(t, x^0(t), u^0(t), p(t)) \\ + \sum_{\alpha=1}^{m} \mathrm{d}^2_{\xi,y^\alpha} H(t, x^0(t), u^0(t), p(t))Q^{\alpha}(t) \\ + \sum_{\alpha,\beta=1}^{m} Q^{\alpha}(t)\mathrm{d}^2_{y^\alpha,y^\beta} H(t, x^0(t), u^0(t), p(t))Q^{\beta}(t) \end{aligned}$$

$$b(t) = \mathrm{d}^2_{\xi,v} H(t, x^0(t), u^0(t), p(t)) + \sum_{\alpha=1}^{m} Q^{\alpha}(t)\mathrm{d}^2_{y^\alpha,v} H(t, x^0(t), u^0(t), p(t)),$$

and

$$d(t) = \partial^2_{v,v} H(t, x^0(t), u^0(t), p(t)).$$

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Book reviews

M. BEALS, R. B. MELROSE and J. RAUCH (eds.): **Microlocal Analysis and Nonlinear Waves** (The IMA Volumes in Mathematics and its Applications: Vol. 30). Berlin - Heidelberg - New York: Springer-Verlag 1991; XIII + 199 pp., 18 fig.

The volume is based on the proceedings of a workshop organized by the editors. This meeting was an integral of the 1988 - 1989 IMA program on "Nonlinar Waves". The 14 articles included in the volume offer a useful overview of recent trends in the current research in this field. The majority of these papers is devoted to the propagation and interaction of nonlinear hyperbolic waves. There are used two kinds of substantial ideas: Results on the propagation of suitably strong singularities of suitably smooth solutions are proved by microlocal arguments developed in the study of the corresponding linear problems and, in addition, by a simultaneous analysis of interactions. Since weaker singularities in solutions of nonlinear problems show nonlinear effects different methods are else needed. Results of this type are crucially based on commutator relations satisfied by vector fields from a Lie Algebra and the underlying differential operators. These commutator methods are closely related to similar ideas used in the analysis of the long-time behaviour of solutions to nonlinear hyperbolic equations. Each of the contributions is well thought out, but we can only sketch the content, by grouping them for convenience.

A. Sá Barreto describes interactions of conormal waves for semilinear wave equations. He uses spaces of distributions associated with the geometry (conormal disteibutions). A more general approach to conormality, cusps and nonlinear interaction in semilinear hyperbolic problems is presented in a paper of R. B. Melrose. The discussion shows how microlocalization and blow-up techniques are mixed in the framework of C^∞-varieties. M. Beals also deals with conormal regularity of nonlinear waves associated with a cusp in solutions to partial differential equations of the same type, but in a different way. Here, the commutator techniques of Bony and extra regularity arguments yield the results. Moreover, the methods are adapted to conormal nonlinear Tricomi problems. Commutator relations and microlocal energy estimates enable J. Y. Chemin to study the evolution of a punctual singularity in an Eulerian flow. The propagation of stronger singularities of solutions from Sobolev spaces $H^s_{loc}(\Omega)$, $s <$ dim $\Omega/2$, to semilinear wave equations is analyzed by Liu Linqi. His paper contains the trick to reduce the loss of smoothness of microlocal products of the solutions. M. Williams considers