Zeitschrift für Analysis
und ihre Anwendungen
Vol. 11 (1992) 4, 539 - 552

A Stochastic Nonlinear Evolution Equation

W. GRECKSCH

A stochastic evolution equation for processes with values in two orthogonal subspaces of a Hilbert space is considered. Such types of equations arise in the study of quasistatic processes of elastic viscoplastic materials with random disturbances. Using the theory of Hilbert-space valued Ito equations an existence and uniqueness theorem is proved. Finally a time discrete approximation is discussed.

Key words: *Stochastic partial differential equations, difference equations*

AMS subject classification: 60 H 15, 60 H 30

1. Introduction

Abstract first order stochastic evolution equations are very important for applications in physics, statistics, engineering and neurophysiology. Navier-Stokes equations with random exterior forces [15, 18], the continuous limit of a random walk in random velocity fields [7], stochastoc partial differential equations related to non-linear filtering [14], diffusion-reaction equations with random interior disturbances [12] and stochastic differential equations for neuronal behaviour [9] are examples for such stochastic evolution equations.

Here we study random nonlinear evolution equations which arise in the study of quasi-static processes of elastic viscoplastic materials with random disturbances (for the deterministic case see [4, 8]). Let us start with an example. We discuss a special random constitutive equation of quasistatic processes of elastic viscoplastic materials. Let G be a bounded domain in $\mathbb{R}^d$ $(d = 1,2,3)$ with a smooth boundary Γ, let Γ_1 be an open subset of Γ, $\Gamma_2 = \Gamma \setminus \overline{\Gamma}_1$, and $T > 0$. Real independent Wiener processes $(w_i(t), F_t)$ $(i = 1,2)$ are defined on a complete probability space (Ω, F, P). We consider the following problem:

(P) Find a random (adapted) displacement function $u: [0, T] \times G \to \mathbb{R}^d$ and stress function σ: $[0, T] \times G \to S$ (being the set of second order symmetric tensors on $\mathbb{R}^d$) such that

$\operatorname{div}_x \sigma(t,x) = 0$ (in $[0,T) \times G$, P-a.s.)

$\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla^T u)$ (in $(0,T) \times G$, P-a.s.; strain tensor of small deformations)

$d\varepsilon(u(t,x)) = \varepsilon(\dot{u}(t,x))dt + \gamma w_2(dt)$ with $\gamma \in S$ (random disturbances in Ito's sense)

$d\varepsilon(u(t,x)) = A d\sigma(t,x) + B_1(\sigma(t,x))dt + B_2(\sigma(t,x))w_1(dt)$ (constitutive equation in Ito's sense A - forth order tensor, B_1 constitutive function, $B_2(\sigma(t,x))w_1(dt)$ describes rand. dist. in Ito's sense)

$u(t,x)|_{(0,T)\times\Gamma_1} = 0$ and $\sigma(t,x)\nu(x)|_{(0,T)\times\Gamma_2} = 0$ (P-a.s. ν is the exterior unit normal at Γ)

$u(0,x) = 0$ and $\sigma(0,x) = 0.$ (in G, P-a.s.)

We will define Problem (P) in a generalized sense for which we introduce the following spaces:

$$K = \left\{\tau = (\tau_{ij})_{i,j=1}^{d}:\ \tau_{ij} \in L^2(G),\ \tau_{ij} = \tau_{ji}\right\} \qquad \boldsymbol{K} = \left\{v = (v_i)_{i=1}^{d}:\ v_i \in L^2(G)\right\}$$

$$V = \left\{\tau \in K:\ \frac{\partial \tau_{ij}}{\partial x_j} \in L^2(G),\ \operatorname{div}\tau = 0,\ \tau\nu = 0 \text{ on } \Gamma_2\right\} \qquad \boldsymbol{V} = \left\{v \in \boldsymbol{K}:\ \frac{\partial v_i}{\partial x_j} \in L^2(G),\ v = 0 \text{ on } \Gamma_1\right\}.$$

The scalar products on K and $\boldsymbol{K}$ are defined by the formulas $(\sigma,\tau)_K = \sum_{i,j}\int_G \sigma_{ij}\tau_{ij}\,dx$ and $(u,v)_{\boldsymbol{K}} = \sum_i \int_G u_i v_i\,dx$, respectively. We note that (see [4])

$$\left(\tfrac{1}{2}(\nabla v + \nabla^{\mathsf{T}} v), \tau\right)_K + (v, \operatorname{div}\tau)_{\boldsymbol{K}} = 0 \quad \forall\, v \in \boldsymbol{V} \text{ and } \tau \in K \text{ with } \operatorname{div}\tau \in \boldsymbol{K} \text{ and } \tau \circ \nu = 0 \text{ on } \Gamma_2. \tag{1.1}$$

Consequently, $(\varepsilon(v),\tau)_K = 0$ (for all $\tau \in V$) by the definition of K. A scalar product on V is induced by $(\cdot,\cdot)_K$. By $A(\sigma,\tau)$ we denote the multilinear form $\sum_{i,j,k,l}\int_G A_{i,j,k,l}\sigma_{kl}\tau_{ij}\,dx$, where $A = (A_{i,j,k,l})$ is the forth order tensor from (P).

So we can define Problem P in the following way:

(P') Find adapted V-valued processes $(u(t))_{t\in[0,T]}$ and $(\sigma(t))_{t\in[0,T]}$ such that

$$\begin{aligned}(\varepsilon(u(t,\cdot)),\tau(\cdot))_V &= (\varepsilon(u(0,\cdot)),\tau(\cdot))_V + (A\sigma(t,\cdot),\tau(\cdot))_V \\ &\quad + \int_0^t (B_1(\sigma(s,\cdot)),\tau(\cdot))_V\,ds + \int_0^t (B_2(\sigma(s,\cdot)),\tau(\cdot))_V\,w_1(ds)\end{aligned} \tag{1.2}$$

for all $\tau \in V$ and $t \in [0,T]$, P-a.s. Because equation (1.2) holds for all $\tau \in V$, the operator equation

$$\varepsilon(u(t,x))(t) = \varepsilon(u(0,x)) + A\sigma(t,x) + \int_0^t B_1(\sigma(s))\,ds + \int_0^t B_2(\sigma(s))\,w_1(ds) \tag{1.3}$$

is fulfilled for all $t \in [0,T]$, P-a.s., where the integrals with respect to w_1 are Ito integrals.

Problem (P) in sense of (1.2) is a system of random partial differential equations with homogeneous initial and boundary conditions. The case of non-homogeneous initial and boundary conditions can be obtained by an easy transformation, which is analogous to the deterministic case [16].

If a solution of (1.3) exists, then the equation $(\varepsilon(u(t,\cdot)),\sigma(t,\cdot))_V = 0$ (for all $t \in [0,T]$, P-a.s.) follows from (1.1), that is the processes $(\varepsilon(u(t,\cdot)))_{t\in[0,T]}$ and $(\sigma(t,\cdot))_{t\in[0,T]}$ have values which are orthogonal in V. Thus, the following abstract random evolution equation is motivated:

Let H be a real separable Hilbert space, with scalar product $(\cdot,\cdot)_H$, and let $H_1, H_2 \subset H$ be two orthogonal (closed) subspaces with $H = H_1 \dot{+} H_2$. We introduce a linear bounded operator $A\colon H \to H$ and continuous operarors $B_1\colon [0,T] \times H_1 \times H_2 \to H$. By $(w(t), F_t)$ we denote a Wiener process with values in a separable Hilbert space K_1 and with a kernel operator Q as covariance operator. Let L_Q be the Hilbert space of linear operators $B\colon Q^{1/2}K_1 \to H$ so that $BQ^{1/2}$ is a Hilbert-Schmidt operator and let $\|B\|_Q$ be the Hilbert-Schmidt norm of $BQ^{1/2}$. Furthermore we introduce an operator $B_2\colon [0,T] \times H_1 \times H_2 \to L_Q$. We are looking for adapted continuous H_1- and H_2-valued processes $(X(t))_{t\in[0,T]}$ and $(Y(t))_{t\in[0,T]}$ with

$$dY(t) = A\,dX(t) + B_1(t,X(t),Y(t))\,dt + B_2(t,X(t),Y(t))\,w(dt),\quad X(0) = X_0,\ Y(0) = Y_0 \tag{1.4}$$

in the Ito sense

$$Y(t) - Y_0 = AX(t) - AX_0 + \int_0^t B_1(s, X(s), Y(s))\,ds + \int_0^t B_2(s, X(s), Y(s))w(ds) \qquad (1.5)$$

for all $t \in [0, T]$, P-a.s., where X_0 and Y_0 are given F_0-measurable H_1- and H_2-valued elements with $\mathrm{E}\|X_0\|_H^2 < \infty$ and $\mathrm{E}\|Y_0\|_H^2 < \infty$, and where the stochastic integral in (1.5) is defined as H-valued Ito integral.

Obviously, Problem (P') is an example for the abstract problem (1.3). If we choose $B_1(t, X(t), Y(t)) = -\lambda(Y(t) - S(t, X(t)))$ $(\lambda > 0)$ and if $B_2(t, X(t), Y(t))$ doesn't depend on $X(t)$ and $Y(t)$, so we have the viscoelastic case with additive random disturbances (for the deterministic case see, e.g., [4]).

In Chapter 2 of this paper we prove an existence and uniqueness theorem of a Hilbert-space valued Ito equation. In Chapter 3 we use this result to prove the existence and uniqueness of a solutuion of (1.5). Chapter 3 contains some a-priori estimations, too. In Chapter 4 we discuss a time discretization of (1.5).

2. A Hilbert-space valued Ito equation

We consider progressive measurable operators

$$C_1: \Omega \times [0, T] \times H \to H \qquad \text{and} \qquad C_2: \Omega \times [0, T] \times H \to L_Q$$

for which there are constants $\varkappa_1, \varkappa_2 > 0$ such that

$$\begin{aligned} &\|C_1(t, x)\|_H^2 \le \varkappa_1(1 + \|x\|^2) \quad (P\text{-a.s.}) \\ &\mathrm{E}\left\|\int_0^t C_2(s, a(s))w(ds)\right\|_H^2 \le \varkappa_1\left(1 + \mathrm{E}\int_0^t \|a(s)\|_H^2\,ds\right) \end{aligned} \qquad (2.1)$$

and

$$\begin{aligned} &\|C_1(t, x) - C_1(t, y)\|_H \le \varkappa_2\|x - y\|_H \quad (P\text{-a.s.}) \\ &\mathrm{E}\left\|\int_0^t \big(C_2(s, a(s)) - C_2(s, b(s))\big)w(ds)\right\|_H^2 \le \varkappa_2^2\,\mathrm{E}\int_0^t \|a(s) - b(s)\|_H^2\,ds \end{aligned} \qquad (2.2)$$

for all $t \in [0, T]$ and $x, y \in H$ and for all progressive measurable H-valued processes

$$(a(t))_{t\in[0,T]} \text{ with } \mathrm{E}\int_0^T \|a(s)\|_H^2\,ds < \infty \quad \text{and} \quad (b(t))_{t\in[0,T]} \text{ with } \mathrm{E}\int_0^T \|b(s)\|_H^2\,ds < \infty.$$

Let X_0 be a F_0-measurable H-valued element with $\mathrm{E}\|X_0\|_H^2 < \infty$. We introduce the Ito equation

$$dX(t) = C_1(t, X(t))\,dt + C_2(t, X(t))\,w(dt), \quad X(0) = X_0 \qquad (2.3)$$

in the sense

$$X(t) = X_0 + \int_0^t C_1(s, X(s))\, ds + \int_0^t C_2(s, X(s)) w(ds). \tag{2.4}$$

Remark 2.1: Sufficient conditions for the validity of the second inequalities in (2.1) and (2.2) are $\|C_2(t,x)\|_Q \le \varkappa_1(1 + \|x\|_H^2)$ and $\|C_2(t,x) - C_2(t,y)\|_Q < \varkappa_2 \|x - y\|_H$, respectively. Then these inequalities follow from the property $E\|\int_0^t C_2(s, a(s)) w(ds)\|_H^2 = E\int_0^t \|C_2(s, a(s))\|_Q^2 ds$ of the Ito integral.

Remark 2.2: There are many other papers dealing with abstract Hilbert-space valued Ito equations. For example, Ito equations with monotonous opeartors and Ito equations with an infinitesimal operator of a semigroup are discussed in [6,13,14] and in [3,9]. Random partial differential equations are examples for such equations. We will see that the above strong assumptions are suitable for the discussion of (1.3).

Remark 2.3: An equation of the type (2.3) in the finite-dimensional case is discussed in [11], under the assumptions that the first inequalities in (2.1) and (2.2) and the inequalities of Remark 2.1 are satisfied.

We can prove by the same technique as in the proof of [6: Lemma 3.2.1.3] the following

Teorem 2.4: *There is a unique (with probability 1) H-valued continuous adapted process* $(X(t))_{t\in[0,T]}$ *such that* (2.4) *holds for all* $t \in [0,T]$ *with probability* 1 *and* $E\sup_t \|X(t)\|_H^2 < \infty$.

We obtain by the Schwarz and Doob inequalities that the solution of the equation (2.3) depends on the initial conditions in the following sense.

Theorem 2.5: *Let* $(X_1(t))$ *and* $(X_2(t))$ be solutions of equation (2.4) with the initial conditions $X_1(0) = X_{01}$ *and* $X_2(0) = X_{02}$. *Then*

$$E\sup_t \|X_1(t) - X_2(t)\|_H^2 \le 2E\|X_{01} - X_{02}\|_H^2 \left(\exp\{4\varkappa_2^2(1+T)t\} - 1\right).$$

3. Existence and uniqueness theorem

Now we prove an existence and uniqueness theorem for (1.4). For this we make the following assumptions:

(A1) There is a constant $\gamma > 0$ such that $(Ax, x) \ge \gamma \|x\|_H^2$ for all $x \in H$.

(A2) There is $(Ax, y)_H = (Ay, x)_H$ for all $(x, y) \in H \times H$.

(A3) There is a constant $r' > 0$ such that

$$\|B_1(t,x,y)\|_H^2 + \|B_2(t,x,y)\|_Q^2 \le r'(1 + \|x\|_H^2 + \|y\|_H^2) \text{ for all } (t,x,y) \in [0,T] \times H_1 \times H_2.$$

(A4) There is a constant $r > 0$ such that

$$\|B_1(t,x_1,y_1) - B_1(t,x_2,y_2)\|_H + \|B_2(t,x_1,y_1) - B_2(t,x_2,y_2)\|_Q$$

$$\leq r\left(\|x_1 - x_2\|_H + \|y_1 - y_2\|_H\right) \text{ for all } (t, x_1, y_1), (t, x_2, y_2) \in [0, T] \times H_1 \times H_2.$$

The inverse operator A^{-1} is linear and bounded. There exists a number $\tilde{\gamma} > 0$ such that $\|A^{-1}y\|_H \geq \tilde{\gamma}\|y\|_H$ for all $y \in H$. The last inequality follows from the remark to [10: Theorem 2/(2.V)]. Let $y \in H$ be arbitrary. We conclude that, for all $x \in H$ with $Ax = y$,

$$(A^{-1}y, y)_H = (x, Ax)_H \geq \gamma\|x\|_H^2 = \gamma\|A^{-1}y\|_H^2 \geq \gamma\tilde{\gamma}^2\|y\|_H^2.$$

Let us introduce the Hilbert space $Z = H_1 \times H_2$ which is isomorphic to $H = H_1 \oplus H_2$. We define over Z the scalar product

$$(z_1, z_2)_Z = (Ax_1, x_2)_H + (A^{-1}y_1, y_2)_H \quad \text{for all } z_i = (x_i, y_i) \in Z \qquad (i = 1, 2).$$

Then $(Z, (\cdot,\cdot)_Z)$ is a real separable Hilbert space. We define an operator E_1: $[0, T] \times Z \to Z$ by the scalar product equation

$$(E_1(t, z_1), z_2)_Z = -(B_1(t, x_1, y_1), x_2)_H + (A^{-1}B_1(t, x_1, y_1), y_2)_H \tag{3.1}$$

and a Z-valued Ito integral $\int_0^t E_2(s, z)w(ds)$ by the scalar product equation

$$\left(\int_0^t E_2(s, z_1)w(ds), z_2\right)_Z = -\left(\int_0^t B_2(s, x_1, y_1)w(ds), x_2\right)_H + \left(A^{-1}\int_0^t B_2(s, x_1, y_1)w(ds), y_2\right)_H \tag{3.2}$$

for all $z_i = (x_i, y_i) \in Z$ $(i = 1, 2)$ and $z \in Z$.

Theorem 3.1: *Adapted processes $(X(t))_{t\in[0,T]}$ and $(Y(t))_{t\in[0,T]}$ with values in H_1 and H_2,* respectively, *are solutions of* (1.5) *if and only if the Z-valued process $(Z(t))_{t\in[0,T]}$ with $Z(t) = (X(t), Y(t))$ is a solution of the equation*

$$Z(t) = Z_0 + \int_0^t E_1(s, Z(s))\, ds + \int_0^t E_2(s, Z(s))w(ds) \tag{3.3}$$

in the sense of the scalar product equation

$$(Z(t), z)_Z = (Z_0, z)_Z + \int_0^t (E_1(s, Z(s)), z)_Z\, ds + \left(\int_0^t E_2(s, Z(s))w(ds), z\right)_Z \tag{3.4}$$

for all $z = (x, y) \in Z$, *where* $Z_0 = (X_0, Y_0)$.

Proof: *Step* 1. Let $(X(t))_{t\in[0,T]}$ and $(Y(t))_{t\in[0,T]}$ be a solution of (1.5). Then, for $x \in H_1$,

$$\left(AX(t) - AX_0, x\right)_H = -\int_0^t \left(B_1(s, X(s), Y(s)), x\right)_H ds - \left(\int_0^t B_2(s, X(s), Y(s))w(ds), x\right)_H. \tag{3.5}$$

If we apply A^{-1} to both sides of (1.5), then for all $y \in H_2$ we obtain

$$\left(A^{-1}(Y(t) - Y_0), y\right)_H = \int_0^t \left(A^{-1}B_1(s, X(s), Y(s)), y\right)ds + \left(A^{-1}\int_0^t B_2(s, X(s), Y(s))w(ds), y\right)_H. \tag{3.6}$$

If we sum up the equations (3.5) and (3.6), then the equation (3.4) follows from the definition of $(\cdot,\cdot)_Z$ and from the formulas (3.1) and (3.2).

Step 2. Let $(Z(t))_{t\in[0,T]} = (X(t),Y(t))_{t\in[0,T]}$ be a solution of (3.4). Then

$$(AX(t),x)_H = (AX_0,x)_H - \int_0^t \big(B_1(s,X(s),Y(s)),x\big)_H\,ds - \Big(\int_0^t B_2(s,X(s),Y(s))w(ds),x\Big)_H$$

follows from (3.4) for $z = (x,0)$ by the definition of $(\cdot,\cdot)_Z$ and the formulas (3.1) and (3.2). Obviously $(Y_0,x)_H = 0$ and $(Y(t),x)_H = 0$ hold for $x \in H_1$. Then we have $(R(t),x)_H = 0$ with

$$R(t) = A(X(t)-X_0) + (Y(t)-Y_0) + \int_0^t B_1(s,X(s),Y(s))\,ds + \int_0^t B_2(s,X(s),Y(s))w(ds).$$

Consequently the process $(R(t))_{t\in[0,T]}$ has values in H_2 and

$$0 = (X(t),R(t))_H = (A^{-1}AX(t),R(t))_H \quad \text{and} \quad 0 = (X_0,R(t))_H = (A^{-1}AX_0,R(t))_H.$$

Then for $y = R(t)$ we obtain from (3.6)

$$0 = \Big(A^{-1}\Big[A(X(t)-X_0) + (Y(t)-Y_0) + \int_0^t B_1(s,X(s),Y(s))\,ds + \int_0^t B_2(s,X(s),Y(s))w(ds)\Big],R(t)\Big)_H.$$

The term $[\dots]$ equals $R(t)$. Since A^{-1} is a positive operator, we get $R(t) = 0$, that is, the equation (1.5) holds ∎

The last theorem shows that we can consider equation (1.5) as a Hilbert-space valued Ito equation of type (2.3). We introduce the following positive constants:

$$D_1 = 2r'r_0 \text{ and } D_2 = 2rr_0, \text{ with } r_0 = \max\{1/\gamma,\ \|A^{-1}\|^2/\gamma\tilde{\gamma}^2\}\max\{1,\ 1/\gamma,\ 1/\gamma\tilde{\gamma}^2\}.$$

Lemma 3.2: *For all $t\in[0,T]$ and $x,x_1,x_2 \in H_1$ as well as $y,y_1,y_2 \in H_2$, and all progressive measurable H_1-valued processes $(a(t))$ and $(a_i(t))$ $(i = 1,2)$ as well as for progressive measurable H_2-valued processes $(\alpha(t))$ and $(\alpha_i(t))$ $(i = 1,2)$ the norm squares of which are integrable over $\Omega\times[0,T]$ the following inequalities hold:*

$$\|E_1(t,x,y)\|_Z^2 \le D_1(1+\|(x,y)\|_Z^2) \tag{3.7}$$

$$\mathrm{E}\Big\|\int_0^t E_2(s,a(s),\alpha(s))w(ds)\Big\|_Z^2 \le D_1\Big(1+\mathrm{E}\int_0^t\|(a(s),\alpha(s))\|_Z^2\,ds\Big) \tag{3.8}$$

$$\|E_1(t,x_1,y_1)-E_1(t,x_2,y_2)\|_Z \le D_2\|(x_1,y_1)-(x_2,y_2)\|_Z^2 \tag{3.9}$$

$$\mathrm{E}\Big\|\int_0^t\big(E_2(s,a_1(s),\alpha_1(s))-E_2(s,a_2(s),\alpha_2(s))\big)w(ds)\Big\|_Z^2 \le D_2\,\mathrm{E}\int_0^t\|(a_1(s),\alpha_1(s))-(a_2(s),\alpha_2(s))\|_Q^2\,ds. \tag{3.10}$$

Proof: We choose $z = (x, y)$ and $\eta = (\eta_1, \eta_2) \in Z$. Then, by the definition of $(\cdot,\cdot)_Z$,(A3), the equality $(u+v)^2 \le 2u^2 + 2v^2$ $(u, v \in \mathbb{R})$ and the properties of the operators A and A^{-1} we get

$$\begin{aligned}\|E_1(t,x,y)\|_Z^2 &\le \sup\left\{(E_1(t,x,y),\eta)_Z^2 : \|\eta\|_Z = 1\right\}\\ &= \sup\left\{\left[(-B_1(t,x,y),\eta_1)_H + (A^{-1}B_1(t,x,y),\eta_2)_H\right]^2 : \|\eta\|_Z = 1\right\}\\ &\le \sup\left\{\|\eta_1\|_H^2 + \|\eta_2\|_H^2\|A^{-1}\|^2 : \|\eta\|_Z = 1\right\} r'(1 + \|x\|_H^2 + \|y\|_H^2)\\ &\le \max\left\{1/\gamma, \|A^{-1}\|^2/\gamma\tilde{\gamma}^2\right\} r' \max\left\{1, 1/\gamma, 1/\gamma\tilde{\gamma}^2\right\}(1 + \|(x,y)\|_Z^2),\end{aligned}$$

that is, inequality (3.7) holds.

In a similar way, with the second moment of the norm of an Ito integral and (A3) we get

$$\begin{aligned}&\mathrm{E}\left\|\int_0^t E_2(s,a(s),\alpha(s))w(ds)\right\|_Z^2\\ &= \sup\left\{-\left(\int_0^t B_2(s,a(s),\alpha(s))w(ds),\eta_1\right)_H + \left(A^{-1}\int_0^t B_2(s,a(s),\alpha(s))w(ds),\eta_2\right)_H^2 : \|\eta\|_Z = 1\right\}\\ &\le 2\max\left\{1/\gamma, \|A^{-1}\|^2/\gamma\tilde{\gamma}^2\right\}\int_0^t \|B_2(s,a(s),\alpha(s))\|_Q^2\, ds\\ &\le 2\max\left\{1/\gamma, \|A^{-1}\|^2/\gamma\tilde{\gamma}^2\right\} r' \max\left\{1, 1/\gamma, 1/\gamma\tilde{\gamma}^2\right\}\left(1 + \mathrm{E}\int_0^t \|(a(s),\alpha(s))\|_Z^2\, ds\right),\end{aligned}$$

that is inequality (3.8) holds. The inequalities (3.9) and (3.10) are proved analogously ∎

Theorem 3.3: *There is a unique (with probability 1) adapted Z-valued solution process $(X(t), Y(t))_{t\in[0,T]}$ of (1.5) with continuous paths and $\mathrm{E}\sup_{t\in[0,T]}\|(X(t), Y(t))\|_Z^2 < \infty$.*

Proof: Lemma 3.2 shows that the assumptions (2.1) and (2.2) are satisfied for $C_1 = E_1$, $C_2 = E_2$ and $H = Z$. Then the existence of a unique (with probability 1) solution process $(X(t), Y(t))$ of (3.3) with the above properties follows from Theorem 2.4. The process $(X(t), Y(t))$ is the solution of (1.5), too. This statement follows from Theorem 3.1 ∎

Corollary 3.4: *Let $(X_i(t), Y_i(t))$ be the solutions of (1.4) with the initial conditions (X_{0i}, Y_{0i}) $(i = 1, 2)$. Then*

$$\begin{aligned}&\mathrm{E}\sup_{t\in[0,T]}\left(\|X_1(t) - X_2(t)\|_H^2 + \|Y_1(t) - Y_2(t)\|_H^2\right)\\ &\quad\le \frac{\exp\{4D_2^2(1+T)T\} - 1}{\min\{\gamma, \gamma\tilde{\gamma}^2\}}\left(\|A\|\,\mathrm{E}\|X_{01} - X_{02}\|_H^2 + \|A^{-1}\|\,\mathrm{E}\|Y_{01} - Y_{02}\|_H^2\right).\end{aligned}$$

Proof: We obtain from Theorem 2.5 for the solution process $(X(t), Y(t))_{t\in[0,T]}$ of (3.3)

$$\mathrm{E}\sup_{t\in[0,T]}\|(X_1(t), Y_1(t)) - (X_2(t), Y_2(t))\|_H^2 \le \left(\exp\{4D_2^2(1+T)T\} - 1\right)\mathrm{E}\|(X_{01}, Y_{01}) - (X_{02}, Y_{02})\|_Z^2.$$

The statement follows from the two inequalities

$$\|(X_1(t),Y_1(t)) - (X_2(t),Y_2(t))\|_Z^2 \geq \gamma\|X_1(t) - X_2(t)\|_H^2 + \gamma\tilde{\gamma}^2\|Y_1(t) - Y_2(t)\|_H^2$$

$$\|(X_{01},Y_{01}) - (X_{02},Y_{02})\|_Z^2 \leq \|A\|\,\|X_{01} - X_{02}\|_H^2 + \|A^{-1}\|\,\|Y_{01} - Y_{02}\|_H^2 \quad \blacksquare$$

Now we consider an a priori estimation for the special case of viscoelastic materials with additive disturbances. Therefore, we assume (λ, $p > 0$ given)

$$B_1(t,x,y) = \lambda(y - S(t,x)) \text{ and } B_2(t,x,y) = B_2(t) \quad \forall (t,x,y) \in [0,T] \times H_1 \times H_2$$

and

$$(S(t,x_1) - S(t,x_2), x_1 - x_2) \geq p\|x_1 - x_2\|_H^2 \quad \forall (t,x_1),(t,x_2) \in [0,T] \times H_1. \tag{3.11}$$

Lemma 3.5: *Let* $(X_i(t), Y_i(t))$ *be the solutions of* (1.4) *with the initial conditions* (X_{0i},Y_{0i}) $(i = 1,2)$. *Then*

$$\mathrm{E}\|X_1(t) - X_2(t)\|_H^2 \leq \frac{\|A\|}{\gamma}\mathrm{E}\|X_{01} - X_{02}\|_H^2 \exp\left\{-\frac{2p\lambda}{\gamma}t\right\} \tag{3.12}$$

$$\begin{aligned}\mathrm{E}\|Y_1(t) - Y_2(t)\|_H^2 \leq \exp\{-\lambda t\}\Big[&\|A^{-1}\|\,\mathrm{E}\|Y_0\|_H^2 \\ &+ \lambda\|A\|\,\|A^{-1}\|\frac{\mathrm{E}\|X_0\|_H^2}{2p+\gamma}\left(1 - \exp\left\{-\lambda\left(\frac{2p}{\gamma}+1\right)t\right\}\right)\Big].\end{aligned} \tag{3.13}$$

Proof: We introduce $X_0 = X_{01} - X_{02}$, $Y_0 = Y_{01} - Y_{02}$, $X(t) = X_1(t) - X_2(t)$, $Y(t) = Y_1(t) - Y_2(t)$. So we get from (1.4)

$$dX(t) = A^{-1}dY(t) - \left[B_1(t,X_1(t),Y_1(t)) - B_1(t,X_2(t),Y_2(t))\right]dt \tag{3.14}$$

$$d(AX(t)) = dY(t) - \left[B_1(t,X_1(t),Y_1(t)) - B_1(t,X_2(t),Y_2(t))\right]dt. \tag{3.15}$$

Then $d(AX(t),X(t))_H = (AX(t),dX(t))_H + (X(t), d(AX(t)))_H$ follows from the Ito formula. Consequently, $(X(t),Y(t))_H = 0$ and $(Ax,A^{-1}y)_H = (x,y)_H$ yield $d(AX(t),X(t))_H = -2\lambda(X(t), S(t,X_1(t)) - S(t,X_2(t)))_H dt$, and we obtain by (3.11) and Assumption A4

$$\mathrm{E}\|X(t)\|_H^2 \leq \frac{1}{\gamma}\mathrm{E}(AX_0,X_0)_H - \frac{2p\lambda}{\gamma}\int_0^t \mathrm{E}\|X(s)\|_H^2\,ds.$$

Hence the function $\varphi(t) = \mathrm{E}\|X(t)\|_H^2$ fulfils the inequality $\dot{\varphi}(t) \leq (2p\lambda/\gamma)\varphi(t)$ with $\varphi(0) = \gamma^{-1} \times \mathrm{E}(AX_0,X_0)_H$. The function $\varphi(t) = \gamma^{-1}\mathrm{E}(AX_0,X_0)_H \exp\{-2p\lambda t/\gamma\}$ solves the initial value problem $\dot{\varphi}(t) = (2p\lambda/\gamma)\varphi(t)$ with $\varphi(0) = \gamma^{-1}\mathrm{E}(AX_0,X_0)_H$. Then the inequality

$$\mathrm{E}\|X(t)\|_H^2 \leq \frac{\|A\|}{\gamma}\mathrm{E}\|X_0\|_H^2 \exp\left\{-\frac{2p\lambda}{\gamma}t\right\}$$

holds. We get from (3.14) by $(X(t),Y(t))_H = 0$ the relation

$$(Y(t), A^{-1}dY(t))_H = -\lambda(A^{-1}Y(t), Y(t))_H dt + \lambda(A^{-1}[S(t, X_1(t)) - S(t, X_2(t))], Y(t))_H dt$$

and from the Ito formula we get

$$\begin{aligned} E(Y(t), A^{-1}Y(t))_H &= E(Y_0, A^{-1}Y_0)_H - 2\lambda E\int_0^t (A^{-1}Y(s), Y(s))_H ds \\ &\quad + 2\lambda E\int_0^t \left(A^{-1}[S(s, X_1(s)) - S(s, X_2(s))], Y(s)\right)_H ds \\ &= E(Y_0, A^{-1}Y_0)_H - 2\lambda E\int_0^t (A^{-1}Y(s), Y(s))_H ds \\ &\quad + 2\lambda E\int_0^t \left(A^{-1/2}[S(s, X_1(s)) - S(s, X_2(s))], A^{-1/2}Y(s)\right)_H ds. \end{aligned}$$

Then

$$\begin{aligned} E(Y(t), A^{-1}Y(t))_H &\le E(Y_0, A^{-1}Y_0)_H - 2\lambda E\int_0^t (A^{-1}Y(s), Y(s))_H ds \\ &\quad + \lambda \|A^{-1}\| r^2 E\int_0^t \|X(s)\|_H^2 ds + \lambda E\int_0^t (A^{-1}Y(s), Y(s))_H ds \end{aligned}$$

follows from Assumption A4, the Schwartz inequality, the properties of the operator A and the inequality $2ab \le a^2 + b^2$ $(a, b \in \mathbb{R})$. Hence the function $\psi(t) = E(Y(t), A^{-1}Y(t))_H$ fulfils the inequality $\dot{\psi}(t) \le \lambda\psi(t) + \|A^{-1}\| r^2 E\|X(t)\|_H^2$ with $\psi(0) = E(Y_0, A^{-1}Y_0)_H$. Consequently, we have

$$\begin{aligned} \psi(t) &= \exp\{-\lambda t\}\left[\|A^{-1}\| E\|Y_0\|_H^2 + \lambda\|A^{-1}\| r^2 E\int_0^t \|X(s)\|_H^2 \exp\{-\lambda s\}\, ds\right] \\ &\le \exp\{-\lambda t\}\left[\|A^{-1}\| E\|Y_0\|_H^2 + \lambda\|A\|\,\|A^{-1}\| \frac{E\|X_0\|_H^2}{2p+\gamma}\left(1 - \exp\left\{-\lambda\left(\frac{2p}{\gamma} + 1\right)t\right\}\right)\right] \blacksquare \end{aligned}$$

4. Approximation by difference equations

We will discuss here approximations for equation (1.4). In Chapter 3 we had seen that we can consider this equation as a Hilbert-cpace valued Ito equation of type (2.3). Therefore let us start with some known approximations for equation (2.3). Let $(h_n) \subset H$ be an orthonormal base. Let $(X^1(t), \dots, X^n(t))$ be adapted real processes with $E(X^i(t))^2 < \infty$ $(i = 1, \dots, n;\ t \in [0, T])$ which are the solutions of

$$dX_n(t) = \sum_{i=1}^{n}\left(C_1(t, X_n(t)), h_i\right)_H h_i + \sum_{i=1}^{n}\left(C_2(t, X_n(t))w(dt), h_i\right)_H h_i \tag{4.1}$$

where $X_n(t) = \sum_{i=1}^{n} X^i(t) h_i$ and $X_n(0) = \sum_{i=1}^{n}(X_0, h_i)_H h_i$. Formula (4.1) is a Galerkin approximation of (2.3) and we obtain by the properties of the Fourier coefficients and the Ito integral, by (2.2) and Gronwall's lemma the following

Theorem 4.1: *The limit* $\lim_{n\to\infty} \mathrm{E}\|X_n(t) - X(t)\|_H^2$ *equals* 0 *for all* $t \in [0,T]$.

We introduce a partition $(t_{iN})_{i=0}^N$ of $[0,t]$ with $t \in [0,T]$, $t_{iN} = iK_N$ $(i = 1,\dots,N)$, $t_{NN} = t$, $K_N > 0$ and $K_N \to 0$ for $N \to \infty$. Let us consider the difference equations

$$\begin{aligned}\overline{X}^n_{i+1N} &= \overline{X}^n_{iN} + \sum_{j=1}^{n}\left(C_1(t_{iN}, \overline{X}^n_{iN}), h_j\right)_H h_j K_N \\ &\quad + \sum_{j=1}^{n}\left(C_2(t_{iN}, \overline{X}^n_{iN})(w(t_{i+1N}) - w(t_{iN})), h_j\right)_H h_j\end{aligned} \tag{4.2}$$

with $i = 0,\dots,N-1$ and $\overline{X}^n_{oN} = X_n(0)$.

Then $\lim_{N\to\infty}\mathrm{E}\|\overline{X}^n_{NN} - X_n(t)\|_H^2 = 0$ follows from a stochastic version of the Euler approximation for the finite-dimensional case [11]. We have together with Theorem 4.1 the following

Theorem 4.2: *The limit* $\lim_{n\to\infty}\lim_{N\to\infty}\mathrm{E}\|\overline{X}^n_{NN} - X(t)\|_H^2$ *equals* 0 *for all* $t \in [0,T]$.

Now we consider the difference equations

$$\overline{X}_{i+1N} = \overline{X}_{iN} + C_1(t_{iN}, \overline{X}_{iN})K_N + C_2(t_{iN}, \overline{X}_{iN})(w(t_{i+1N}) - w(t_{iN}))$$

with $i = 0,\dots,N-1$ and $\overline{X}_{oN} = X_o$.

Corollary 4.3: *The limit* $\lim_{n\to\infty}\mathrm{E}\|\overline{X}^n_{iN} - \overline{X}_{iN}\|_H^2$ *equals* 0 *for all* $i = 0,\dots,N$ *and* $N \in \mathbb{N}$.

We can prove this assertion by induction using the following

Lemma 4.4: *Let* Φ *be a Lipschitz continuous function from H into H. Then*

$$\lim_{n\to\infty}\left\|\sum_{i=1}^{n}(\Phi(a_n), h_i)_H h_i - \Phi(a)\right\|_H = 0 \quad \forall \text{ sequences } (a_n) \subset H \text{ with } \lim_{n\to\infty}\|a_n - a\|_H = 0.$$

Now we discuss two types of approximations for (1.4). For this we consider partitions $(t_{iN})_{i=0}^N$ of $[0,t]$ with $N = 2^q$ $(q \in \mathbb{N})$, $K_N = 1/N$, $t_{iN} = iK_N t$ $(i = 1,\dots,N)$. We introduce the following recursive system $(j = 0,\dots,N-1;\ N \in \mathbb{N})$:

$$\begin{aligned}&\overline{X}_{oN} = X_o, \quad \overline{Y}_{oN} = Y_o, \quad \overline{X}_{j+1N} \in H_1, \quad \overline{Y}_{j+1N} \in H_2 \\ &\overline{Y}_{j+1N} - A\overline{X}_{j+1N} = \overline{Y}_{jN} - A\overline{X}_{jN} + K_n B_1(t_{jN}, \overline{X}_{jN}, \overline{Y}_{jN}) \\ &\qquad + B_2(t_{jN}, \overline{X}_{jN}, \overline{Y}_{jN})(w(t_{j+1N}) - w(t_{jN})).\end{aligned} \tag{4.3}$$

We obtain by the definition of Z, E_1, E_2 that problem (4.3) is equivalent to the following one:

$$\begin{aligned}&\overline{Z}_{oN} = Z_o, \ \overline{Z}_{j+1N} \in Z \\ &\overline{Z}_{j+1N} = \overline{Z}_{jN} + K_n E_1(t_{jN}, \overline{Z}_{jN}) + E_2(t_{jN}, \overline{Z}_{jN})(w(t_{j+1N}) - w(t_{jN})).\end{aligned} \tag{4.4}$$

($j = 0, \dots, N-1$; $N \in \mathbb{N}$). We obtain with Corollary 4.3 and Theorem 4.2 the following

Lemma 4.5: *The limit* $\lim_{N\to\infty} \mathrm{E}\|\overline{Z}_{NN} - Z(t)\|_Z^2$ *equals* 0 *for all* $t \in [0,T]$.

Furthermore we get by the definition of Z the following

Corollary 4.6: *The limits* $\lim_{N\to\infty} \mathrm{E}\|\overline{X}_{NN} - X(t)\|_H^2$ *and* $\lim_{N\to\infty} \mathrm{E}\|\overline{Y}_{NN} - Y(t)\|_H^2$ *equals* 0 *for all* $t \in [0,T]$.

Now we consider a different approximation of (1.4). Let H_{1h} $(h > 0)$ be a closed subspace of H_1 with the following property: Let $x \in H_1$ be chosen arbitrary. Then there exist elements $h_h \in H_{1h}$ with $\lim_{h\to 0}\|x_h - x\|_H = 0$. We choose H_{1h}- and H-valued elements X_{ho} and Y_{ho}, respectively, with $\lim_{h\to 0} \mathrm{E}\|(X_{ho}, Y_{ho}) - (X_o, Y_o)\|_Z^2 = 0$. We define the following recursive system ($j = 0, \dots, N-1$; $N \in \mathbb{N}$; $x \in H_{1h}$):

$$
\begin{aligned}
&\widetilde{X}_{oN}^h = X_{ho}, \quad \widetilde{Y}_{oN}^h = Y_{ho},\\
&(A\widetilde{X}_{j+1N}^h, x)_H = (A\widetilde{X}_{jN}^h, x)_H - K_n\big(B_1(t_{jN}, \widetilde{X}_{jN}^h, \widetilde{Y}_{jN}^h), x\big)_H\\
&\qquad\qquad - \big(B_2(t_{jN}, \widetilde{X}_{jN}^h, \widetilde{Y}_{jN}^h)(w(t_{j+1N}) - w(t_{jN})), x\big)_H \qquad (4.5)\\
&\widetilde{Y}_{j+1N}^h = \widetilde{Y}_{jN}^h + A(\widetilde{X}_{j+1N}^h - \widetilde{X}_{jN}^h) + K_N B_1(t_{jN}, \widetilde{X}_{jN}^h, \widetilde{Y}_{jN}^h)\\
&\qquad\qquad + B_2(t_{jN}, \widetilde{X}_{jN}^h, \widetilde{Y}_{jN}^h)(w(t_{j+1N}) - w(t_{jN})).
\end{aligned}
$$

We want to show that (4.5) gives an approximation of (1.4). Summing up (4.5) from $j = 0$ to $j = m$ we obtain for all $x \in H_{1h}$

$$(A\widetilde{X}_{m+1N}^h, x)_H = f_m^h(x) \qquad (4.6)$$

with

$$
\begin{aligned}
f_m^h(x) = (A\widetilde{X}_{oN}^h, x)_H &- \sum_{j=0}^m K_N\big(B_1(t_{jN}, \widetilde{X}_{jN}^h, \widetilde{Y}_{jN}^h), x\big)_H\\
&- \sum_{j=0}^m \big(B_2(t_{jN}, \widetilde{X}_{jN}^h, \widetilde{Y}_{jN}^h)(w(t_{j+1N}) - w(t_{jN})), x\big)_H.
\end{aligned}
$$

Taking the scalar product of (4.3) with $u \in H_1$ and summing up from $j = 0$ to $j = m$ we get

$$(A\widetilde{X}_{m+1N}, u)_H = g_m(u), \qquad (4.7)$$

where

$$
\begin{aligned}
g_m(u) = (A\widetilde{X}_{oN}, u)_H &- K_N\sum_{j=0}^m \big(B_1(t_{jN}, \overline{X}_{jN}, \overline{Y}_{jN}), u\big)_H\\
&-\sum_{j=0}^m \big(B_2(t_{jN}, \overline{X}_{jN}, \overline{Y}_{jN})(w(t_{j+1N}) - w(t_{jN})), u\big)_H.
\end{aligned}
$$

The existence and uniqueness of $F_{(m+1)K_N}$-measurable solutions $\tilde{X}^h_{m+1N}$ with $\mathrm{E}\|\tilde{X}^h_{m+1N}\|^2_H < \infty$ and $\bar{X}_{m+1N}$ with $\mathrm{E}\|\bar{X}_{m+1N}\|^2_H < \infty$ follow from the abstract Browder-Minty theorem about monotonous operators (see [19: Theorem 25.1]). We obtain by (A1), (4.6) and (4.7)

$$\begin{aligned}\gamma\|\tilde{X}^h_{m+1N} - x\|^2_H &\le \big(A(\tilde{X}^h_{m+1N} - x), \tilde{X}^h_{m+1N} - x\big)_H \\ &= f^h_m(\tilde{X}^h_{m+1N} - x) - (Ax, \tilde{X}^h_{m+1N} - x)_H + (A\tilde{X}^h_{m+1N}, \tilde{X}^h_{m+1N} - x)_H - g_m(\tilde{X}^h_{m+1N} - x) \\ &= \big(A(\tilde{X}^h_{m+1N} - x), \tilde{X}^h_{m+1N} - x\big)_H + f^h_m(\tilde{X}^h_{m+1N} - x) - g_m(\tilde{X}^h_{m+1N} - x) \\ &\le \|A\|\|\bar{X}_{m+1N} - x\|_H\|\tilde{X}^h_{m+1N} - x\|_H + (f^h_m - g_m)(\tilde{X}^h_{m+1N} - x).\end{aligned}$$

Then, using

$$\|\tilde{X}^h_{m+1N} - \bar{X}_{m+1N}\|_H \le \|\tilde{X}^h_{m+1N} - x\|_H + \|\bar{X}_{m+1N} - x\|_H$$

$$|(f^h_m - g_m)(\tilde{X}^h_{m+1N} - x)| \le \|f^h_m - g_m\|\|\tilde{X}^h_{m+1N} - x\|_H$$

we get $\|\tilde{X}^h_{m+1N} - \bar{X}_{m+1N}\|_H \le (\gamma^{-1}\|A\| + 1)\|\bar{X}_{m+1N} - x\|_H + \|f^h_m - g_m\|_H$ and, consequently,

$$\mathrm{E}\|\tilde{X}^h_{m+1N} - \bar{X}_{m+1N}\|^2_H \le 2(\gamma^{-1}\|A\| + 1)^2\mathrm{E}\|\bar{X}_{m+1N} - x\|^2_H + 2\mathrm{E}\|f^h_m - g_m\|^2_H. \tag{4.8}$$

The inequality

$$\begin{aligned}\mathrm{E}\|f^h_m - g_m\|^2_H &\le (1 + (1+m)K_N)\mathrm{E}\|A(\tilde{X}^h_{0N} - \bar{X}_{0N})\|^2_H \\ &\quad + K_N\sum_{j=0}^m \mathrm{E}\big\|B_1(t_{jN}, \tilde{X}_{jN}, \tilde{Y}_{jN}) - B_1(t_{jN}, \bar{X}_{jN}, \bar{Y}_{jN})\big\|^2_H \\ &\quad + 2^{m+1}K_N^2\sum_{j=0}^m \mathrm{E}\big\|B_1(t_{jN}, \tilde{X}_{jN}, \tilde{Y}_{jN}) - B_1(t_{jN}, \bar{X}_{jN}, \bar{Y}_{jN})\big\|^2_H \\ &\quad + \sum_{j=0}^m \mathrm{E}\big\|\big(B_2(t_{jN}, \tilde{X}_{jN}, \tilde{Y}_{jN}) - B_2(t_{jN}, \bar{X}_{jN}, \bar{Y}_{jN})\big)(w(t_{j+1N}) - w(t_{jN}))\big\|^2_H\end{aligned}$$

follows from the inequalities $2ab \le a^2 + b^2$ $(a, b \in \mathbb{R})$ and $\big(\sum_{j=0}^m a_j\big)^2 \le 2^{m+1}\sum_{j=0}^m a_j^2$ $(a_0, \dots, a_m \in \mathbb{R})$ and from the independence of the increments of the Wiener process. Then, using (A4) and

$$\begin{aligned}&\mathrm{E}\big\|\big(B_2(t_{jN}, \tilde{X}_{jN}, \tilde{Y}_{jN}) - B_2(t_{jN}, \bar{X}_{jN}, \bar{Y}_{jN})\big)(w(t_{j+1N}) - w(t_{jN}))\big\|^2_H \\ &= K_N\mathrm{E}\big\|\big(B_2(t_{jN}, \tilde{X}_{jN}, \tilde{Y}_{jN}) - B_2(t_{jN}, \bar{X}_{jN}, \bar{Y}_{jN})\big)\big\|^2_Q,\end{aligned}$$

we have

$$\begin{aligned}\mathrm{E}\|f^h_m - g_m\|^2_H &\le (1 + (1+m)K_N)\mathrm{E}\|A(\tilde{X}^h_{0N} - \bar{X}_{0N})\|^2_H \\ &\quad + K_N(2 + 2^{m+1}K_N)r^2\sum_{j=0}^m\big(\mathrm{E}\|\tilde{X}^h_{jN} - \bar{X}_{jN}\|^2_H + \mathrm{E}\|\tilde{Y}^h_{jN} - \bar{Y}_{jN}\|^2_H\big).\end{aligned}$$

Therefore we have proved that

$$\begin{aligned}\mathrm{E}\|\tilde{X}^h_{m+1N} - \bar{X}_{m+1N}\|^2_H &\le 2(\gamma^{-1}\|A\|+1)^2\mathrm{E}\|\bar{X}_{m+1N} - x\|^2_H \\ &+ 2(1+(1+m)K_N)\|A\|^2\mathrm{E}\|\tilde{X}^h_{0N} - \bar{X}_{0N}\|^2_H \\ &+ 2K_N(2+2^{m+1}K_N)r^2\sum_{j=0}^{m}\left(\mathrm{E}\|\tilde{X}^h_{jN} - \bar{X}_{jN}\|^2_H + \mathrm{E}\|\tilde{Y}^h_{jN} - \bar{Y}_{jN}\|^2_H\right).\end{aligned} \tag{4.9}$$

In a similar way we obtain

$$\begin{aligned}&\mathrm{E}\|\tilde{Y}^h_{m+1N} - \bar{Y}_{m+1N}\|^2_H \\ &\le 2^6\Big[\mathrm{E}\|\tilde{Y}^h_{0N} - \bar{Y}_{0N}\|^2_H + \|A\|^2\mathrm{E}\|\tilde{X}^h_{0N} - \bar{X}_{0N}\|^2_H + \|A\|^2\mathrm{E}\|\tilde{X}^h_{m+1N} - \bar{X}^2_{m+1N}\|^2_H \\ &+ 2K_N(2+2^{m+1}K_N)r^2\sum_{j=0}^{m}\mathrm{E}\left(\|\tilde{X}^h_{jN} - \bar{X}_{jN}\|^2_H + \|\tilde{Y}^h_{jN} - \bar{Y}_{jN}\|^2_H\right).\end{aligned} \tag{4.10}$$

We introduce

$$a_j = \mathrm{E}\|\tilde{X}^h_{jN} - \bar{X}_{jN}\|^2_H, \quad b_j = \mathrm{E}\|\tilde{Y}^h_{jN} - \bar{Y}_{jN}\|^2_H \quad \text{and} \quad d_m = \sum_{j=0}^{m}(a_j + b_j)$$

$$C = 2^6\max\left\{2(\gamma^{-1}\|A\|+1)^2,\ 2(1+2T)\|A\|^2,\ 2(1+2T)\right\}.$$

From (4.9) and (4.10) we get

$$a_{m+1} \le C(\mathrm{E}\|\bar{X}_{m+1N} - x\|^2_H + K_N r^2 d_m + d_0) \text{ and } b_{m+1} \le C(a_{m+1} + K_N r^2 d_m + d_0), \tag{4.11}$$

and from (4.11) we get

$$d_{m+1} \le d_m(1 + CK_N r^2) + C(d_0 + \mathrm{E}\|\bar{X}_{m+1N} - x\|^2_H).$$

Recursively we see that, for $m = 0, \dots, N-1$,

$$d_{m+1} \le \frac{1}{K_N r^2}(d_0 + \mathrm{E}\|\bar{X}_{m+1N} - x\|^2_H)(\exp(Cr^2T) - 1) + d_0\exp(Cr^2T)$$

and we obtain by (4.11)

$$\begin{aligned}a_{m+1} + b_{m+1} &\le C(1+C)\mathrm{E}\|\bar{X}_{m+1N} - x\|^2_H \\ &+ C(2+C)\Big[(d_0 + \mathrm{E}\|\bar{X}_{m+1N} - x\|^2_H)(\exp(Cr^2T) - 1) + K_N r^2 d_0\exp(Cr^2T)\Big].\end{aligned}$$

So we have proved the following

Theorem 4.7: *There are constants $D > 0$ and $C > 0$ such that*

$$\begin{aligned}&\mathrm{E}\|\tilde{X}^h_{m+1N} - \bar{X}_{m+1N}\|^2_H + \mathrm{E}\|\tilde{Y}^h_{m+1N} - \bar{Y}_{m+1N}\|^2_H \le D\exp(Cr^2T)\mathrm{E}\|\bar{X}_{m+1N} - x\|^2_H \\ &+ \left(\mathrm{E}\|\tilde{X}^h_{0N} - \bar{X}_{0N}\|^2_H + \mathrm{E}\|\tilde{Y}^h_{0N} - \bar{Y}_{0N}\|^2_H\right)(1 + K_N r^2)(\exp(Cr^2T) - 1)\end{aligned}$$

Remark 4.8: With the help of Corollary 4.6 we see that problem (4.3) defines an approximation of (1.4). Problem (4.6) is an elliptical one. We consider an example for H_{1h}. Let $(\varphi_n) \subset H_1$ be a complete orthonormal system and let $H_{1\,1/n}$ $(n \in \mathbb{N})$ be finite-dimensional subspaces which are generated by $\varphi_1, \dots, \varphi_n$. Then (4.5) is the Galerkin approximation of an elliptical equation.

REFERENCES

[1] BUCAN, G. P.: *Stochastic semigroups* (in Russian). Kiev: Naukova Dumka 1977.

[2] CIARL ET, P.: *The Finite Element Method for Elliptic Problems*. Amsterdam - New York - Oxford: North-Holland Publ. Comp. 1978.

[3] CURTAIN, R. and A. J. PRITCHARD: *Infinite-Dimensional Linear Systems Theory* (Lect. Notes Control Inform. Sci.: Vol. 8). Berlin-Heidelberg-New York: Springer-Verlag 1978.

[4] DUVANT, G. and J. L. LIONS: *Inequalities in Mechanics and Physics*. Berlin-Heidelberg-New York: Springer-Verlag 1970.

[5] GICHMAN, I. I. and A. V. SKOROCHOD: *Controlled Stochastic Processes* (in Russioan). Kiev: Naukova Dumka 1977.

[6] GRECKSCH, W.: *Stochastische Evolutionsgleichungen und deren Steuerung* (Teubner-Texte zur Mathematik: Vol. 98). Leipzig: B. G. Teubner Verlagsges. 1987.

[7] HONKONEN, J.: *Renormalisation of random walks memory*. Nordisk Inst. Teor. Atomfys. Nordita Preprint **26 S** (1989), 1 - 24.

[8] IONESCU, I. R.: *Error estimates of a numerical method for a nonlinear evolution equation*. Anal. Univ. Bucuresti - Mat.-Inf. **37** (1988)2, 64 - 74.

[9] KALLIANPUR, G. and V. PEREZ-ABREU: *Stochastic evolution equations driven by nuclear-space-valued martingales*. Appl. Math. Optim. **17** (1988), 237 - 272.

[10] KANTOROWITSCH, L. W. and G. P. AKILOW: *Funktionalanalysis in normierten Räumen*. Berlin: Akademie-Verlag 1964.

[11] KLOEDEN, P. E. and E. PLATEN: *A survey of numerical mehtods for stochastic differential equations*. Report-Reihe Inst. Dyn. Systeme Univ. Bremen **204** (1989), 1 - 3.

[12] KOTELENEZ, P.: *Gaussian approximation to the nonlinear reaction-diffusion equation*. Report-Reihe Inst. Dyn. Systeme Univ. Bremen **146** (1986), 1 - 57.

[13] KRYLOV, N. V. and B. L. ROZOVSKIJ: *On stochastic evolution equations* (in Russian). In: Itogi nauki i techniki 14. Moskva: VINITI 1979, 71 - 146.

[14] OCONE, D.: *Probability distributions of solutions to some stochastic partial differential equations*. Lect. Notes Math. **1236** (1987), 184 - 199.

[15] SCHMALFUSZ, B.: *Invariant attracting sets of nonlinear stochastic differential equations*. Math. Research **54** (1989), 217 - 228.

[16] SOFENEA, M.: *On existence and uniqueness of the solution of a dynamical elastic-visco-plastic problem*. Anal. Univ. Bucuresti - Mat. - Inf. **37** (1988)1, 53 - 59

[17] TUDOR, C.: *On stochastic evolution equations driven by continuous semimartingales*. Stochastics **23** (1988), 179 - 195.

[18] VISHIK, M. I. and A. O. FUSIKOV: *Mathematical Problems of Statistical Hydromechanics*. Dortrecht: Kluwer Acad. Publ. 1988.

[19] ZEIDLER, E.: *Vorlesungen über nichtlineare Funktionalanalysis*. Vol. II: *Monotone Operatoren* (Teubner-Texte zur Mathematik: Vol. 9). Leipzig: B. G. Teubner Verlagsges. 1978.

Received 03.09.1990; in revised form 10.03.1992

Prof. Dr. Wilfried Grecksch
FB Mathematik und Informatik der Techn. Hochschule
Geusaer Str.
D (*Ost*) - 4200 Merseburg