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Numerical Solutions for Some Free Boundary Value Problems Occuring in Planar Fluid Dynamics

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In the paper [1], we consider the solvability of some free boundary value problems occuring in planar fluid dynamics. The object of the present paper is to present numerical methods for solving free boundary problems. It was shown in [1] that such free boundary problems may be transformed into a mixed boundary value problem for a linear or nonlinear elliptic complex equations, which in turn, might be reformulated into a conformal mapping or quasiconformal mapping from a general domain onto some canonical domain. In this paper we direct our discussion to the numerical solution of mixed boundary value problems for generalized Beltrami equations.

Key words: *Numerical solutions, free boundary problems*

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§1 Numerical solutions of mixed boundary value problems

Let D be a $(N+1)$-connected domain with the boundary $\Gamma = \bigcup_{j=0}^{N} \Gamma_j$, where the sets $\Gamma_j = \{z: |z - z_j| = \gamma_j\}$ $(j = 1,\dots,N)$ are situated inside $\Gamma_0 = \{z: |z| = 1\}$ and $z = 0 \in D$. We consider, in D, the nonlinear elliptic complex equation

$$w_{\bar z} = F(z, w, w_z), \quad F = Q_1 w_z + Q_2 \overline{w}_{\bar z} + A_1 w + A_2 \overline{w} + A_3, \tag{1.1}$$

in which $Q_j = Q_j(z, w, w_z)$ $(j = 1,2)$ and $A_j = A_j(z, w)$ $(j = 1,2,3)$. Suppose that the complex equation (1.1) satisfies condition

(C) 1. The functions $Q_j(z, w, U)$ $(j = 1,2)$ and $A_j(z, w)$ $(j = 1,2,3)$ are measurable in $z \in D$ for all continuous functions w and all measurable functions U in D and satisfy the bounds

$$L_p[A_j(z, w(z)), \overline{D}] \le k_0 < \infty \quad (j = 1,2,3)$$

where p $(2 < p < \infty)$ and k_0 $(0 \le k_0 < \infty)$ are real constants.

2. The complex equation (1.1) satisfies the uniform ellipticity condition

$$|F(z, w, U_1) - F(z, w, U_2)| \le q_0 |U_1 - U_2| \text{ for a.e. } z \in D \text{ and } w, U_1, U_2 \in \mathbb{C}, \tag{1.2}$$

where q_0 $(0 \le q_0 < 1)$ is a constant.

3. The functions $Q_j(z, w, U)$ $(j = 1,2)$ and $A_j(z, w)$ $(j = 1,2,3)$ are continuous for $w \in \mathbb{C}$ for

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a.e. point $z \in D$, and $F(z,w,U)$ satisfies the condition

$$|F(z,w_1,U) - F(z,w_2,U)| \le R(z)|w_1 - w_2| \text{ for } w_1, w_2, U \in \mathbb{C}, \tag{1.3}$$

where $L_p(R(z),\bar{D}) \le k_0$.

4. The function

$$G(w,V) = [w_{\bar{z}} - F(z,w,w_z)][\overline{V_z} - \overline{F_w V} - \overline{F_{\bar{w}} \bar{V}} - \overline{F_{w_z} V_z} - \overline{F_{\bar{w}_{\bar{z}}} V_{\bar{z}}}]$$

satisfies the condition

$$G(w_1, w_1 - w_2) - G(w_2, w_1 - w_2) = |w_{1\bar{z}} - F(z,w_1,w_{1z}) - (w_{2\bar{z}} - F(z,w_2,w_{2z}))|^2. \tag{1.4}$$

In particular, if the complex equation (1.1) is linear, i.e. $Q_j = Q_j(z)$ $(j = 1,2)$ and $A_j = A_j(z)$ $(j = 1,2,3)$, then the conditions (1.2) - (1.4) obviously hold.

Problem A: The so-called *mixed boundary value problem* for (1.1) is to find a bounded solution $w \in D^*$ satisfying the boundary condition

$$\mathrm{Re}[\overline{\lambda(t)}w(t)] = r(t) + h(t) \ (t \in \Gamma^*), \tag{1.5}$$

in which $|\lambda(t)| = 1$, λ and r posses discontinuities of the first kind at the points $t_1, \ldots, t_m$, $D^* = \bar{D} \backslash \{t_1, \ldots, t_m\}$ and $\Gamma^* = \Gamma \backslash \{t_1, \ldots, t_m\}$ consists of the arcs $\Gamma^1, \ldots, \Gamma^n$ with end points from the set $\{t_1, \ldots, t_m\}$, where the $t_1, \ldots, t_m$ are arranged in with a positive orientation. If λ is continuous on Γ_j $(0 \le j \le N)$, then we add an arbitrarily chosen point $t^* \in \Gamma_j$ to the set $\{t_1, \ldots, t_m\}$ and relabel this set as $\{t_1, \ldots, t_m\}$. Moreover λ and r satisfy the conditions

$$\lambda(t) = e^{i\varphi_j}, \ \varphi_j = \beta_j \pi, \ 0 \le \beta_j < 2$$
$$r(t) = r_j(t), \ C_\alpha[r_j(t), \Gamma^j] \le l < \infty, \ j = 1, \ldots, n$$

where $1/2 < \alpha < 1$ and β_j, l are nonnegative constants. Besides,

$$h(t) = \begin{cases} h_j, t \in \Gamma_j, \ j = 1,\ldots,N \\ 0, t \in \Gamma_j, \ j = 0 \end{cases} \quad \text{for } \varkappa = 0$$
$$\qquad\quad \ \ h_j, t \in \Gamma_j, \ j = 0,1,\ldots,N \quad \text{for } \varkappa = -1,$$

where h_j $(j = 0,1,\ldots,N)$ are unknown constants to be determined appropriately, and $\varkappa = 0$ or $\varkappa = -1$ is the index of Problem A (see [1, 2]). If $\varkappa = 0$, we may require that the solution w satisfies the point condition

$$\mathrm{Im}[\overline{\lambda(a_j)}w(a_j)] = b_j, \ j \in \{j\} = 1, \tag{1.6}$$

in where $a_1 (\neq t_k) \in \Gamma_0$, b_1 is a real constant with the condition $|b_1| \le l$. The mixed boundary value problem in [1] can be reduced to the Problem A above.

By $H^1(D)$ we denote the Sobolev space $H^1(D) = W_2^1(D) = \{w | w, w_{\bar{z}}, w_z \in L_2(D)$, and denote by **Problem B** the extremal problem for the functional $I = I(u)$, i.e. we find a bounded function $w \in H^1(D) \cap C(D^*)$, such that

$$I(w) = \min_{u \in H^1(D) \cap C(D^*)} I(u),$$

where

$$I(u) = \iint_D |u_{\bar z} - F(z,u,u_z)|^2 d\sigma_z + \int_\Gamma |\mathrm{Re}\,\overline{\lambda(t)}u(t) - r(t) - h(t)|^2 ds + \sum_{j\in\{j\}} (\mathrm{Im}\,\overline{\lambda(a_j)}u(a_j) - b_j)^2. \tag{1.7}$$

It is easy to see that the unknown constants $h_j = \mathrm{Re}\,\overline{\lambda(a_j)}u(a_j) - r(a_j)$, in which a_j ($\neq t_k$) is a point on Γ_j ($j = 0,1,2,\dots,N$), which we sometimes denote as $h(t) = h(t,u^*)$. In the formula (1.7), the double integral is over $D_\varepsilon = D\setminus\sum_{j=1}^m\{|z - t_j| < \varepsilon\}$, where ε is an arbitrary small positive number.

Theorem 1.1: *The function w is a solution of Problem A if and only if it is a solution of Problem B.*

Proof: Substitute the solution w of Problem A for u in (1.7). Obviously $I(w) = 0$, which shows that w is a solution of Problem B.

Conversely, let w be a solution of Problem B. Then it is clear that w satisfies the first order complex equation (1.1) for almost every point $z \in D$, the boundary condition (1.5) on Γ^* and the point condition (1.6). If we can prove that $w \in W^1_{p_0}(D_\varepsilon)$, $2 < p_0 < p$, then w is also a solution of Problem A. According to [3: Theorem 2.1] and [4: Chap. 4/Theorem 1.3], w can be expressed as $w(z) = \Phi[\zeta(z)]e^{\varphi(z)} + \Psi(z)$, where $\zeta, \varphi, \Psi \in W^1_{p_0}(D)$, $w \in W^1_{p_0}[\zeta(D)]$ and Φ is analytic in $\zeta(D)$. It can be shown that $w \in W^1_{p_0}(D_\varepsilon)$ ∎

We use a regular triangular net to subdivide the domain D, such that the diameter of each triangular unit Δ_j is not greater than a positive constant h. Denote the vertices by $z_1,\dots,z_n$ so that $a_j \in \{z_1,\dots,z_n\}$ ($j = 0,1,2,\dots,N$). Moreover, we insist that $t_k \notin \{z_1,\dots,z_n\}$ ($k = 1,2,\dots,m$). We introduce the space E of complex linear splines,

$$E = \{v \mid v \text{ is linear in } z,\bar z \text{ on each } \Delta_j \text{ and continuous at } z_j \in D\}.$$

It is not difficult to see that $E \subset H^1(D)\cap C(D^*)$. We use $\varphi_k \in E$, where $\varphi_k(z_j) = 1$ for $j = k$ and $\varphi_k(z_j) = 0$ for $j \neq k$ ($j,k = 1,\dots,n$). A basis for the space E consists of the function set $\{\varphi_1,\dots,\varphi_n, i\varphi_2,\dots,i\varphi_n\}$, where i is the imaginary unit.

Theorem 1.2: *Suppose that $R \in L^\infty(\bar D)$ in Condition C. Let w be an exact solution of Problem B and $\hat w \in E$ be a linear interpolation function of w. Then Problem B has a solution.*

Proof: We first prove $I(\hat w) \le M_1 h^2$, where M_1 is constant. In fact, we have that

$$\begin{aligned} I(\hat w) &= \iint_D |\hat w_{\bar z} - F(z,\hat w,\hat w_z)|^2 d\sigma_z \\ &\quad + \int_\Gamma |\mathrm{Re}\,\overline{\lambda(t)}\,\hat w(t) - r(t) - h(t)|^2 ds + \sum_{j\in\{j\}} (\mathrm{Im}\,\overline{\lambda(a_j)}\,\hat w(a_j) - b_j)^2 \\ &= \iint_D |\hat w_{\bar z} - F(z,\hat w,\hat w_z)|^2 d\sigma_z + \int_\Gamma |\mathrm{Re}\,\overline{\lambda(t)}(\hat w(t) - w(t))|^2 ds. \end{aligned}$$

Noting that $w \in H^1(D)\cap C(D^*)$, it may be extended to a function $w^* \in H^1(\mathbb{R}^2)\cap C(D^*)$, i.e. $w^*(z) = w(z)$, for $z \in D$, and $\|w^*\|_{H^1(\mathbb{R}^2)} \le M_2\|w\|_{H^1(D)}$ where M_2 is a constant. From the trace theorem, it follows that

$$\int_\Gamma |\mathrm{Re}\,\overline{\lambda(t)}(\hat{w}(t)-w(t))|^2 ds \le M_3\|\hat{w}-w\|^2_{H^1(D)} \le M_3\|\hat{w}-w^*\|^2_{H^1(D)}$$
$$\le M_3 h^2\|w^*\|^2_{H^1(D)} \le M_3 h^2 \|w\|^2_{H^1(D)} \le M_4 h^2,$$

where M_3, M_4 are constants. Moreover,

$$\iint_D |\hat{w}_{\bar{z}} - F(z,\hat{w},\hat{w}_z)|^2 d\sigma_z$$
$$= \iint_D |(\hat{w}-w)_{\bar{z}} - (F(z,\hat{w},\hat{w}_z) - F(z,w,w_z))|^2 d\sigma_z$$
$$\le \iint_D |(\hat{w}-w)_{\bar{z}} - (F(z,\hat{w},\hat{w}_z) - F(z,\hat{w},w_z)) - (F(z,\hat{w},w_z) - F(z,w,w_z))|^2 d\sigma_z$$
$$\le \iint_D ((\hat{w}-w)_{\bar{z}} + q_0(w_z - \hat{w}_{\bar{z}}) + |R(z)|\,|\hat{w}-w|^2)\, d\sigma_z$$
$$\le M_5\|\hat{w}-w\|^2_{H^1(D)} \le M_6 h^2,$$

in which M_5, M_6 are constants.

Secondly, using reductio ad absurdum, we can prove that there exists a positive constant $M_7(h)$, such that

$$\iint_D |u_{\bar{z}} - F(z,u,u_z) + F(z,0,0)|^2 d\sigma_z$$
$$+ \sum_{j=0}^{N}\int_{\Gamma_j} |\mathrm{Re}\,\overline{\lambda(t)}u(t) - \mathrm{Re}\,\overline{\lambda(a_j)}u(a_j)|^2 ds + \sum_{j\in\{j\}} \mathrm{Im}\,\overline{\lambda(a_j)}u(a_j)$$
$$= I_0(u) \ge M_7(h)\|u\|^2_{H^1(D)\cap C(D^*)}$$

where $\|u\|_{H^1(D)\cap C(D^*)} = \|u\|^2_{H^1(D)} + C(u,D^*)$, and if $\varkappa = 0$, $\mathrm{Re}\,\overline{\lambda(a_0)}u(a_0)$ is represented by 0. Obviously,

$$\min_{u\in E} I_0(u) = \min_{u\in E,\, I(u)\le M_4 h^2} I_0(u). \tag{1.8}$$

It remains to be shown that (1.8) possesses a minimum. As a matter of fact,

$$I_0(u) = \iint_D |u_{\bar{z}} - F(z,u,u_z) + F(z,0,0)|^2 d\sigma$$
$$+ \sum_{j=0}^{N}\int_{\Gamma_j} \left|(\mathrm{Re}\,\overline{\lambda(t)}u(t) - r(t) - \mathrm{Re}\,\overline{\lambda(a_j)}u(a_j) + r(a_j)) + r(t) - r(a_j)\right| ds$$
$$+ \sum_{j\in\{j\}} \left((\mathrm{Im}\,\overline{\lambda(a_j)}u(a_j) - b_j) + b_j\right)^2$$
$$\le 2\left(M_4 h^2 + \iint_D |F(z,0,0)|^2 d\sigma_z + \sum_{j=0}^{N}\int_{\Gamma_j} |r(t) - r(a_j)|^2 ds + \sum_{j\in\{j\}} |b_j|^2\right) = M_8(h).$$

Consequently, $\max_{z\in D^*}|u| \le M_9(h) < \infty$. We consider next $u = \sum_{i=1}^{n} c_i q_i + \sum_{i=n+1}^{2n} c_i(iq_i)$; we notice that $\max_{1\le i\le 2n}|c_i| \le M_9(h)$. Hence there exists a $w_h \in E$, such that $I(w_h) = \min_{u\in E} I(u)$∎

If $u = w_h$, the variation of $I(u)$ is equal to zero, i.e.

$$\delta I(w_h) = 2\,\mathrm{Re}\iint_D (w_{h\bar z} - F(z,w_h,w_{hz}))(\bar u_z - F_{w_h}u - F_{\overline{w}_h}\bar u - F_{w_{hz}}u_z - F_{\overline{w}_{h\bar z}}\bar u_{\bar z})\,d\sigma_z$$

$$+\sum_{j=0}^{N}\int_{\Gamma_j}(\mathrm{Re}\,\overline{\lambda(t)}w_h(t) - r(t) - \mathrm{Re}\,\overline{\lambda(a_j)}w_h(a_j) + r(a_j))(\mathrm{Re}\,\overline{\lambda(t)}u(t) - \mathrm{Re}\,\overline{\lambda(a_j)}u(a_j))ds$$

$$+\sum_{j\in\{j\}}(\mathrm{Im}\,\overline{\lambda(a_j)}w_h(a_j) - b_j)\,\mathrm{Im}\,\overline{\lambda(a_j)}u(a_j)$$

$$= a(w_h,u) = 0.$$

Variational Problem C: Find a complex function $w_h \in E$, such that $a(w_h,u) = 0$ for all $u \in E$.

Theorem 1.3: *A necessary and sufficient condition that w_h is a solution of Problem B, is that it is a solution of Problem C.*

Proof: The necessity of the hypothesis is obvious. Now suppose w_h is a solution of Problem C. By Theorem 1.1 and by the necessity put of this theorem, w_h is a solution of Problem B. If the solution of Problem C is unique, then sufficiency is proved. Suppose to the contrary that w_{1h}, w_{2h} are two solutions of Problem C. Since $a(w_{1h},u) = 0$, $a(w_{2h},u) = 0$, for all $u \in E$, it follows that $a(w_{1h},u) - a(w_{2h},u) = 0$. Setting $w_h^* = w_{1h} - w_{2h}$ and using Condition C, it can be seen that

$$\iint_D |w_{1h\bar z} - F(z,w_{1h},w_{1hz}) - (w_{2h\bar z} - F(z,w_{2h},w_{2hz}))|^2 d\sigma_z$$

$$+\sum_{j=0}^{N}\int_{\Gamma_j}|\mathrm{Re}\,\overline{\lambda(t)}(w_{1h}(t) - w_{2h}(t)) - \mathrm{Re}\,\overline{\lambda(a_j)}(w_{1h}(a_j) - w_{2h}(a_j))|^2 ds$$

$$+\sum_{j\in\{j\}}|\mathrm{Im}\,\overline{\lambda(a_j)}(w_{1h}(a_j) - w_{2h}(a_j))|^2 = 0,$$

and w_h^* is a bounded solution of the following boundary value problem:

$$w_{h\bar z}^* = Q(z)w_{hz}^* + A(z)w_h^* \qquad (|Q(z)| \le q_0 < 1,\ A \in L_p(\overline{D}))$$

$$\mathrm{Re}(\overline{\lambda(t)}w_h^*(t)) = \mathrm{Re}(\overline{\lambda(a_j)}w_h(a_j)) \qquad (a_j \in \Gamma_j,\ 0 \le j \le N)$$

$$\mathrm{Im}(\overline{\lambda(a_j)}(w_h^*(a_j)) = 0 \qquad (j = 1,\ \text{for}\ \varkappa = 0).$$

According to the uniqueness theorem, we conclude $w_h^* = 0$. i.e. $w_{1h} = w_{2h}$∎

Similar arguments can be used to obtain error estimates.

§ 2 Numerical method for some quasiconformal mappings

A quasiconformal mapping may be shown to be a homeomorphic solution of an elliptic complex equation of first order

$$w_{\bar z} = F(z, w, w_z), \quad F = Q_1 w_z + Q_2 \overline{w}_{\bar z} = Q w_z \tag{2.1}$$

defined in the domain D, where $Q := Q_1(z) w_z + Q_2(z)\overline{w}_{\bar z}/w_z$. We assume that (2.1) satisfies Condition C stated as §1. In [4: Chap. 3], we proved that there exists a unique solution w of the complex equation (2.1), and that this solution quasiconformally maps the domain D onto some canonical domains. The conformal mappings and quasiconformal mappings may be reduced to **Boundary Value Problem Q** for the complex equation (2.1), i.e. we improve the condition

$$\mathrm{Re}(\overline{\lambda(t)} w(t)) = r(t) + h(t), \quad t \in \Gamma, \tag{2.2}$$

where

$$\lambda(t) = \begin{cases} i, & t \in \Gamma_j,\ j = 1,\dots,N \\ 1, & t \in \Gamma_0' \\ i, & t \in \Gamma_0'' \end{cases} (\Gamma_0 = \Gamma_0' \cup \Gamma_0'',\ \Gamma_0' \cap \Gamma_0'' = \Phi), \quad r(t) = \begin{cases} r(t), & t \in \Gamma\backslash\Gamma_0,\ r \in C_\alpha(\Gamma\backslash\Gamma_0) \\ r_1(t), & t \in \Gamma_0',\ r_1 \in C_\alpha(\Gamma_0') \\ r_2(t), & t \in \Gamma_0'',\ r_2 \in C_\alpha(\Gamma_0'') \end{cases} \quad (0 < \alpha < 1)$$

$$h(t) = \begin{cases} h_j, & t \in \Gamma_j,\ j = 1,\dots,N \\ h_0, & t \in \Gamma_0, \text{ if } \varkappa = 0, \end{cases}$$

in which the h_j $(j = 1,\dots,N)$ are unknown constants to be determined, Γ_0' and Γ_0'' are some subarcs of Γ_0. Similarly to §1, the indexes of Problem Q are $\varkappa = 0$ or $\varkappa = -1$. If $\varkappa = 0$, we take $h_0 = 0$. In some cases, the solutions w of Problem Q are not bounded. Using the method of [2: Chap. 3/Theorem 6.1], Problem Q can be transformed into another boundary value problem. Applying the method mentioned and the representation theorem therein for solutions of the complex equation (2.1), we can be proved that the solution of Problem Q under certain conditions is a homeomorphism from D onto the desired canonical domain.

Next, we construct a homeomorphic solution of (2.1) that quasiconformally maps D onto a strip lying between $\mathrm{Im}\, w = 0$, $\mathrm{Im}\, w = 1$ and having N level rectilinear slits deleted from in $0 < \mathrm{Im}\, w < 1$. In this case, the boundary condition (2.2) of Problem Q may be put in the following form:

$$\mathrm{Re}(\overline{\lambda(t)} w(t)) = r(t) + h(t) := \begin{cases} 0, & -\pi < \vartheta < 0 \\ 1, & 0 < \vartheta < \pi \\ h_j, & t \in \Gamma_j,\ j = 1,\dots,N \end{cases} (t = e^{i\vartheta} \in \Gamma_0) \; (w(\mathrm{i}) = \mathrm{i}\pi), \quad \overline{\lambda(t)} = -\mathrm{i}, \quad t \in \Gamma.$$

We choose an analytic function $H(z) = 1/\pi(\ln\frac{1+z}{1-z} + \mathrm{i}\pi/2)$ such that $H(0) = \mathrm{i}/2$, which conformally maps $|z| < 1$ onto the strip $0 < \mathrm{Im}\, H < 1$. Setting $w = \mathrm{i}W + H$, Problem Q for (1.1) can be reduced to the following boundary value **Problem R**:

$$W_{\bar z} = G(z, W_z), \quad G = Q_1(z) W_z - Q_2(z)\overline{W}_{\bar z} - \mathrm{i}A(z), \quad A = Q_1 H^1 + Q_2 \overline{H}^1 \tag{2.3}$$

$$\mathrm{Re}\, W(t) = R(t) = \begin{cases} 0, & t \in \Gamma_0 \\ -\mathrm{Im}\, H(t) + h(t), & t \in \Gamma\backslash(\Gamma\backslash\{0\}) \end{cases} \quad (W(\mathrm{i}) = 0).$$

We assume that $A \in L_p(\overline{D})$, $p > 2$.

Theorem 2.1: *The function w is a quasiconformal mapping for Problem Q if and only if W* $= -\mathrm{i}(w - H)$ *is a bounded solution of Problem R.*

Proof: It is clear that if w is a quasiconformal mapping, then $W = -\mathrm{i}(w - H)$ is a bounded solution of Problem R. Conversly, if W is a bounded solution of Problem R, then $w = \mathrm{i}W + h$ can be expressed as $w(z) = \Phi(\zeta(z))$, where ζ is a homeomorphism which quasiconformally maps D onto a bounded circular domain Ω and, $\Phi(\zeta)$ ia an analytic function in the domain Ω, satisfying the boundary condition $\mathrm{Re}(\overline{\lambda(z(\zeta)}\Phi(\zeta)) = r(z(\zeta)), \zeta \in \Gamma = \partial\Omega$, where $z(\zeta)$ is the inverse function of $\zeta(z)$. Using the method of [2: Chap. 3, § 3] we can prove that $\Phi(\zeta)$ conformally maps the domain $\zeta(D)$ onto the strip domain with the boundary $\mathrm{Im}w = 0$, $\mathrm{Im}w = 1$ and N level rectilinear slits lying between $0 < \mathrm{Im}w < 1$ ∎

In order to find the numerical solution of Problem R for the complex equation (2.3), we introduce the extremal **Problem S** for $I(U)$:

$$I(W) = \min_{u \in H^1(D) \cap C(\overline{D})} \left\{ \iint_D |U_{\bar{z}} - G(z, U_z)|^2 d\sigma_z + \int_\Gamma |\mathrm{Re}\, W(t) - R(t)|^2 ds + |W(\mathrm{i})|^2 \right\}$$

and the variational **Problem T**:

$$\partial I(W) = 2\,\mathrm{Re} \iint_D |W_{\bar{z}} - G(z, W_z)|\,|\overline{U}_z - \overline{G(z, U_z)}|\, d\sigma_z$$

$$+ 2\int_\Gamma |\mathrm{Re}\, W(z) - R(z,W)|\,|\mathrm{Re}\, U(z) - R(z,U)|\, ds + 2\,\mathrm{Re}\, W(\mathrm{i})\overline{U(\mathrm{i})},$$

where $R(z,W) = 0$ for $t \in \Gamma_0$ and $R(z,W) = -\mathrm{Im}\, H(t) + \mathrm{Re}\, W(a_j) + \mathrm{Im}\, H(a_j)$, $j = 1,\dots,N$. Using the finite element method, we can construct a numerical solution W_h of Problem T and Problem S. The function W_h is also a numerical solution of Problem R. Therefore $w_h = \mathrm{i}W_h + H$ is a numerical solution of Problem Q.

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K. STREHMEL (ed.): **Numerical Treatment of Differential Equations.** Proceedings of the Fifth Seminar "NUMDIFF-5" held in Halle 1989 (Teubner-Texte zur Mathematik: Vol. 121). Stuttgart - Leipzig: B.G. Teubner Verlagsges. 1991; 372 pp.

With the present Volume 121 of the series "*Teubner-Texte zur Mathematik*", Teubner-Verlag continues the publication of proceedings of the by now traditional NUMDIFF workshops which have been held at the Martin Luther University of Halle-Wittenberg since 1981. As can be estimated from both the list of speakers, and the papers presented, NUMDIFF-5 has found considerable resonance among the European mathematical community. The central goal of the workshops, to support the exchange of the latest scientific developments in the broad field of numerical analysis for ordinary differential equations including its relationships to other mathematical disciplines and to applications in modelling real-world processes, is reflected by the papers selected for the proceedings by the editor (and chairman of the NUMDIFF seminars) K. Strehmel.

The first and main part of the proceedings is devoted to the analysis of numerical methods for ordinary differential equations. Initial value problems for algebro-differential equations are discussed in the contributions by Arnold, Griepentrog, Hanke, März, Niepage, Reich, and Roche, who cover almost all aspects of this topic of current research interest. The survey article by Hairer and the paper by Strehmel/Weiner focus on the relation between algebro-differential equations and certain classes of singularly perturbed stiff systems of ordinary differential equations. Efficient solvers for stiff problems are dealt with by Auzinger/Frank/Kirlinger (extrapolation schemes), Kaps/Ostermann und Scholz (methods of Rosenbrock type). Houwen addresses parallelization aspects in the implementation of block Runge-Kutta methods, and Hout/Spijker contribute some results on algorithms for delay equations. The papers on numerical methods for boundary value problems for ordinary differential equations are relatively isolated from each other, and cover very different methods and problem classes: shooting algorithms for parameter dependent nonlinear two-point boundary value problems (Hermann/Ullmann) and algebro-differential equations (Lamour), the analysis of the τ-method (Pfeifer/Roos), iterative techniques using a formulation as integral equation (Quinney/Croft), and issues of grid selection in finite difference discretizations (Schmitt/Schild).

The second part of the proceedings is concentrating on the application of numerical techniques for ordinary differential equations to the discretization of partial differential equations. The most typical such problem class, the method of lines for evolution problems, is discussed in the papers by Farago, Grigorief, Hundsdorfer, Mann, Oliveira and Verwer/Hundsdorfer/Sommeijer. A convergence concept for weak solutions of certain flow problems including entropy conditions is introduced by Ansorge. Some further contributions deal with finite element and finite difference methods for singularly perturbed convection-diffusion problems (Groen/Veldhuizen, Stoyan, Tobiska).

The concluding third part contains applications of differential equations and of corresponding solution methods to the numerical modelling of real-world problems arising in different fields of natural sciences. Models involving ordinary differential equations are discussed by Bohl (transport of information in cells), Denk/Rentrop (simulation of electric circuits by the software package SPICE), Führer/Leimkuhler, Rösch/Kretzschmar (mechanical motion with restrictions), and Großmann/Juggi (heart physiology). The numerical determination of invariant tori for reaction-diffusion equations is investigated in a paper by Holodniok/Kubicek/Marek, thematically close to this topic is the note by Veldhuizen from Part 1 on the Josephson equation. Particular problems of fluid modelling (Abia/Sanz-Serna), models for chemical kinetics (Erhardt/Klusacek), solitons (Ortiz) and electrical field problems (Lucht/Radke) are treated using partial differential equations.

Concluding remark after this short survey of the papers included into the proceedings of NUMDIFF-5: NUMDIFF-6 will be held in Halle in September 1992!

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