

Weighted Spaces of Pseudocontinuable Functions and Approximations by Rational Functions with Prescribed Poles

V. E. KATSNELSON

The main result of this paper provides an essential intermediate step for the proof of the author's theorem on left and right Blaschke-Potapov products which yields more insight into the structure of J -inner functions. The goal of this paper is to prepare an appropriate backward Schur-Potapov algorithm by solving a weighted approximation problem.

Key words: *Spectral synthesis, Blaschke-Potapov products, J -contractive analytic matrix-valued functions, Arov-singular J -inner matrix-valued functions, rational approximation.*

AMS subjects classification: 30D50, 41A20

Let $\bar{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, $\mathbb{D}_+ := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{D}_- := \{z \in \bar{\mathbb{C}} : 1 < |z| \leq \infty\}$, $\mathbb{T} = \{t \in \mathbb{C} : |t| = 1\}$, and let m be the normalized Lebesgue measure on \mathbb{T} . Suppose that $w : \mathbb{T} \rightarrow [0, \infty]$ is an m -integrable function on \mathbb{T} which satisfies $\int_{\mathbb{T}} \ln[w(t)]m(dt) > -\infty$. Let L_w^2 be the set of all square integrable functions with respect to the weight w with norm

$$\|f\|^2 := \int_{\mathbb{T}} |f(t)|^2 w(t)m(dt).$$

Further assume that $(z_k)_{k \in \mathbb{N}}$ is some sequence of points from \mathbb{D}_+ which satisfies the Blaschke condition $\sum_{k=1}^{\infty} [1 - |z_k|] < \infty$. For $n \in \mathbb{N}$ there will be considered the closed linear hull (in L_w^2) $R_n := \overline{\bigvee_{k=n}^{\infty} \frac{1}{i-z_k}}$ of the functions $\frac{1}{i-z_k}$, $k \geq n$. Clearly, then $R_1 \supseteq R_2 \supseteq R_3 \supseteq \dots$. It turns out that $\bigcap_{n \in \mathbb{N}} R_n \subseteq PCH_w^2$ where PCH_w^2 denotes the set of all functions $f : \mathbb{D}_+ \cup \mathbb{D}_- \rightarrow \mathbb{C}$ with the following properties:

- (i) f is holomorphic in $\mathbb{D}_+ \cup \mathbb{D}_-$.
- (ii) The restrictions of f onto \mathbb{D}_+ and \mathbb{D}_- belong to the corresponding Smirnov classes in those domains.
- (iii) $\lim_{r \rightarrow -1+0} f(rt) = \lim_{r \rightarrow -1+0} f(rt) (= f(t))$ m -a.e. on \mathbb{T} .
- (iv) $\int_{\mathbb{T}} |f(t)|^2 w(t)m(dt) < \infty$.

The main result in this paper is that, for each weight function w with the above stated properties, a Blaschke sequence $(z_k)_{k \in \mathbb{N}}$ can be constructed such that $\bigcap_{n \in \mathbb{N}} R_n = PCH_w^2$ is satisfied.

V. E. Katsnelson: The Weizmann Inst. Sci., Dep. Theor. Math., Rehovot 76100, Israel, and The Kharkov State University, Dep. Math., Kharkov 310077, Ukraine

1 Motivation of the Problem : Left Blaschke-Potapov Products are not Necessarily Right Blaschke-Potapov Products. An Arov-singular World.

This paper is concerned with particular aspects of rational approximation. Our main problem has been arisen in the process of investigating some questions related with factorization of analytic J -contractive matrix functions. A reader who is not interested in the origin of the weighted rational approximation problem considered in this paper can omit Section 1 and start to read this paper beginning with Section 2. The following question stands in the background of the considerations: Let W be a meromorphic $m \times m$ matrix function in the open unit disc which is J -contractive, i.e.

$$J - W^*(z) J W(z) \geq 0, \quad z \in \mathbb{D}_+ := \{u \in \mathbb{C} : |u| < 1\}. \quad (1.1)$$

Here J is a matrix satisfying the conditions

$$J = J^*, \quad J^2 = I, \quad (1.2)$$

where I is the $m \times m$ unit matrix. (The matrix J is called a *signature matrix*).

Assume that W is a left Blaschke-Potapov product, i.e.,

$$W(z) = \prod_{k=1}^{\infty} B(z; z_k, P_k^{(l)}) \quad (1.3)$$

where $B(z; z_k, P_k^{(l)})$ is a Blaschke-Potapov factor with pole at z_k where $|z_k| \neq 1$. Splitting off from this matrix function W Blaschke-Potapov factors on the opposite side, i.e. on the right side, we obtain a representation

$$W(z) = E^{(r)}(z) \prod_{k=1}^{\infty} B(z; z_k, P_k^{(r)}), \quad (1.4)$$

where $B(z; z_k, P_k^{(r)})$ are some Blaschke-Potapov factors (with the same poles as the factors in (1.1) but in general with different residues in these poles). After successive splitting-off from this matrix function on the right side all possible Blaschke-Potapov factors (which correspond to all poles of W and W^{-1} in the unit disc) we obtain some matrix function $E^{(r)}$ which is J -contractive in the unit disc, i.e.,

$$J - (E^r(z))^* J \cdot E^{(r)}(z) \geq 0 \quad (z \in \mathbb{D}_+) \quad (1.5)$$

and for which both $E^{(r)}$ and $[E^r]^{-1}$ are holomorphic in the unit disc. It is a well-known fact that such a matrix function can be represented as a multiplicative integral. In the scalar case (i.e. if W is a complex-valued function) and even in the definite matrix-valued case (if $J = I$ or $J = -I$ where I is the unit matrix) the function $E^{(r)}$ obtained after the mentioned splitting-off procedure turns out to be constant. Hence, in the definite case a left Blaschke-Potapov product is at the same time a right Blaschke-Potapov product.

However, in the indefinite case, i.e. the signature matrix J is not definite, it can happen that the matrix function $E^{(r)}$ is not constant. Hence, there are matrix functions which are a left Blaschke-Potapov product but not a right Blaschke-Potapov product. Furthermore, it proves to be possible to give a function-theoretical description of the class of all such functions $E^{(r)}$, which occur in the right multiplicative representations of left Blaschke-Potapov products. In order to give an exact formulation of this result we need the following definition:

Definition 1.1.: Let J be a fixed signature $m \times m$ matrix. An $m \times m$ matrix-valued function, A which is defined in the unit disc \mathbb{D}_+ and has a non-identically vanishing determinant is called Arov-singular if the following conditions are satisfied:

- (i) Both matrix functions A and A^{-1} are holomorphic in \mathbb{D}_+ .
- (ii) The matrix function A is J -contractive in \mathbb{D}_+ :

$$J - A^*(z) J A(z) \geq 0 \quad (\forall z \in \mathbb{D}_+),$$

and its boundary function $A(t) = \lim_{\varrho \rightarrow 1-0} A(\varrho t)$ (which exists for m -almost every $t \in \mathbb{T}$) is J -unitary for m -almost every $t \in \mathbb{T}$:

$$J - A^*(t) J A(t) = 0 \quad (m - \text{a.e.})$$

- (iii) The family $\{\ln^+(||A(\varrho t)|| + ||(A(\varrho t)^{-1})||)\}_{0 \leq \varrho < 1}$ defined for $t \in \mathbb{T}$ is uniformly integrable with respect to the Lebesgue measure m (ϱ is a parameter indexing the family). In other words, both matrix-valued functions A and A^{-1} belong to the Smirnov class N_* . (Details concerning the Smirnov class N_* of complex-valued functions are given below in Section 3).

We want to recall that a matrix-valued function A , which is meromorphic in the unit disc, is called a J -inner function if it satisfies only the condition (ii) of the previous definition.

In the definite case (if $J = I$ or if $J = -I$) any Arov-singular matrix function is necessarily constant. In the indefinite case there always exist non-constant Arov-singular matrix functions.

A more detailed analysis of function-theoretical properties of Blaschke-Potapov products yields the following result: the factor $E^{(r)}$ in the right multiplicative decomposition (1.3) of a function W representable as a left Blaschke-Potapov product (1.3) is an Arov-singular matrix-valued function.

A much more delicate result is the converse statement:

Main Theorem on left and right Blaschke-Potapov products. *Let E be an arbitrary Arov-singular matrix-valued function. Then there exists an infinite convergent right Blaschke-Potapov product*

$$B^{(r)}(z) := \prod_{1 \leq k < \infty} B(z; z_k, P_k^m) \tag{1.6}$$

(whose poles are located in the interior \mathbb{D}_+ of the unit circle \mathbb{T}) such that the matrix-function W defined by

$$W(z) := E \cdot B^{(r)} \tag{1.7}$$

is representable as a left Blaschke-Potapov product (1.3).

What concerns a detailed treatment of this circle of questions we refer the reader to [19]-[21].

The core of proving the just formulated theorem lies in reducing it to weighted approximation by vector-valued rational functions with prescribed poles. The tools for realizing such a reduction are Nevanlinna-Pick interpolation and reproducing kernel Hilbert spaces of analytic functions.

The main goal in this paper is a detailed investigation of the announced weighted approximation problem. As well the problem as the method to solve it are interesting themselves. Taking into account the interests of the large group of analysts, which are far away from studying vector-valued functions, we confine our investigations in this paper to the case of complex function, i.e. the so-called scalar case. This case arises naturally in the study of the links between left and right Blaschke-Potapov products of size 2×2 .

In this section, we will not carry out an exact derivation of the reduction of the problem concerned with left and right Blaschke-Potapov products to a problem of weighted rational approximation. Even we will not give an exact formulation of the approximation problem in this section. The main aim of this section is to motivate this problem and the choice of Hilbert spaces of analytic functions related with them. These spaces will be introduced in Sections 4 and 5. The exact formulation of the approximation problem and its full solution will be presented in Sections 6 and 7. Now let us consider for the 2×2 case the announced reduction to a scalar problem of weighted rational approximation. Without loss of generality, we can choose the particular 2×2 signature matrix

$$J_0 := \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \quad (1.8)$$

(Every indefinite 2×2 signature matrix proves to be unitarily equivalent to J_0 .) Denote $\mathcal{C}(\mathbb{D}_+)$ the Carathéodory class in the unit disc, i.e. the set of all holomorphic functions $P : \mathbb{D}_+ \rightarrow \mathbb{C}$ having nonnegative real part:

$$P(z) + \overline{P(z)} \geq 0 \quad (z \in \mathbb{D}_+). \quad (1.9)$$

Now we will explain what Nevanlinna-Pick interpolation in the class $\mathcal{C}(\mathbb{D}_+)$ means. Given is a non-empty index set A and a family $(\{z_\alpha, p_\alpha\})_{\alpha \in A}$ of ordered pairs of numbers $z_\alpha \in \mathbb{D}_+$ and $p_\alpha \in \mathbb{C}$ with $p_\alpha + \overline{p_\alpha} \geq 0$. Furthermore, we suppose that $z_\alpha \neq z_{\alpha'}$ for all $\alpha, \alpha' \in A$ with $\alpha \neq \alpha'$. Then the problem is to describe all functions P belonging to $\mathcal{C}(\mathbb{D}_+)$ which satisfy

$$P(z_\alpha) = p_\alpha, \quad \alpha \in A. \quad (1.10)$$

The Nevanlinna-Pick problem with interpolation data $(\{z_\alpha, p_\alpha\})_{\alpha \in A}$ will be denoted in the sequel by $NP((\{z_\alpha, p_\alpha\})_{\alpha \in A})$. Every function $P \in \mathcal{C}(\mathbb{D}_+)$ which satisfies the interpolation conditions (1.10) for all $\alpha \in A$ is called a solution of $NP((\{z_\alpha, p_\alpha\})_{\alpha \in A})$. Depending on the concrete data of interpolation there are three possibilities: The problem $NP((\{z_\alpha, p_\alpha\})_{\alpha \in A})$ can have no solutions, a unique solution or an infinite set of solutions. In the case that $NP((\{z_\alpha, p_\alpha\})_{\alpha \in A})$ has infinitely many solutions the whole solution set can be parametrized via a fractional linear transformation:

$$P = (r_{11} \sigma + r_{12}) (r_{21} \sigma + r_{22})^{-1} \quad (1.11)$$

where $R = (r_{jk})_{j,k=1}^2$ is a 2×2 matrix-valued function which is meromorphic in \mathbb{D}_+ and which is only built from the interpolation data $(\{z_\alpha, p_\alpha\})_{\alpha \in A}$. Formula (1.11) realizes a bijective correspondence between the extended Carathéodory class (which consists of the class $\mathcal{C}(\mathbb{D}_+)$ supplemented by the constant function in \mathbb{D}_+ with value ∞) and the set of all solutions P of $NP(\{z_\alpha, p_\alpha\}_{\alpha \in A})$. The matrix generating the fractional linear transformation (1.11), which yields the set of all solutions of $NP(\{z_\alpha, p_\alpha\}_{\alpha \in A})$, is called a resolvent matrix of this problem and will be denoted by $R(z; \{z_\alpha, p_\alpha\}_{\alpha \in A})$.

If $NP(\{z_\alpha, p_\alpha\}_{\alpha \in A})$ has an infinite set of solutions, then the family $(z_\alpha)_{\alpha \in A}$ forms necessarily a discrete subset of \mathbb{D}_+ (and, hence, is finite or countable) and satisfies necessarily the Blaschke condition.

The resolvent matrix of a nonuniquely solvable Nevanlinna-Pick problem can be chosen as a J_0 -inner function in the unit disc the poles of which are located in the interpolation knots. Such a resolvent matrix is essentially unique (i.e. up to right multiplication with a constant J_0 -unitary matrix).

The strategy in proving the Main Theorem on left and right Blaschke-Potapov products is as follows: A left Blaschke-Potapov product W , for which the factor $E^{(r)}$ in its right multiplicative decomposition (1.6)–(1.7) coincides with the given Arov-singular J_0 -inner function E , will be constructed as a resolvent matrix of an appropriate Nevanlinna-Pick problem in the class $\mathcal{C}(\mathbb{D}_+)$, i.e.

$$W(z) := R(z; \{z_k, p_k\}_{k \in \mathbb{N}}) . \tag{1.12}$$

(Observe that in the scalar case TUMARKIN [44] studied various questions of approximating singular inner functions by sequences of Blaschke products.) It is important for us that the index set is not only a countable set but even an ordered set. This means we will essentially use the natural order in \mathbb{N} . Clearly, the sequence $(z_k)_{k \in \mathbb{N}}$ of interpolation nodes must fulfill the Blaschke condition

$$\sum_{k \in \mathbb{N}} [1 - |z_k|] < \infty . \tag{1.13}$$

This is one of the necessary conditions of nonunique solvability of $NP(\{z_k, p_k\}_{k \in \mathbb{N}})$. Our construction of the interpolation data $(\{z_k, p_k\})_{k \in A}$ will be done in such a way that the nonunique solvability of $NP(\{z_k, p_k\}_{k \in \mathbb{N}})$ and, hence, the existence of a resolvent matrix $R(z; \{z_k, p_k\}_{k \in \mathbb{N}})$ will be guaranteed automatically. This is the so-called priming function method. This method works as follows. Let $(p_k)_{k \in A}$ be an at most countable sequence of pairwise different points from \mathbb{D}_+ and let P_{pr} an arbitrary function from $\mathcal{C}(\mathbb{D}_+)$. For $k \in A$, we set

$$p_k := P_{pr}(z_k) . \tag{1.14}$$

We will consider the set $(\{z_k, p_k\})_{k \in A}$ as interpolation data of a Nevanlinna-Pick problem. This problem $NP(\{z_k, p_k\}_{k \in A})$ has clearly solutions since by construction P_{pr} is a solution. The following nonuniqueness criterion is important for us. We set

$$w_{pr}(t) := \operatorname{Re} \left[\lim_{r \rightarrow 1-0} P_{pr}(t) \right] , \quad t \in \mathbb{T} . \tag{1.15}$$

Then the function w_{pr} is defined m -a.e. on \mathbb{T} , has nonnegative values and is integrable with respect to m .

Lemma on priming functions. *Let $(z_k)_{k \in A}$ be an at most countable sequence of pairwise different points from \mathbb{D}_+ which satisfies the Blaschke condition, and suppose that the priming function $P_{pr} \in \mathcal{C}(\mathbb{D}_+)$ has a convergent logarithmic integral, i.e.*

$$\int_{\mathbf{T}} \ln w_{pr} dm > -\infty, \tag{1.16}$$

where w_{pr} is defined in (1.15). Then the problem $NP((z_k, p_k)_{k \in A})$ with $(p_k)_{k \in A}$ defined via (1.14) has an infinite set of solutions.

Now let

$$E = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.17}$$

be an J_0 -inner matrix function which is holomorphic in \mathbb{D}_+ (but not necessarily Arov-singular). Outgoing from this E we define

$$P_{pr} := [a \cdot (+1) + b] \cdot [c \cdot (+1) + d]^{-1}. \tag{1.18}$$

From the J_0 -contractivity of E in \mathbb{D}_+ it follows that the function P_{pr} defined in (1.18) belongs to $\mathcal{C}(\mathbb{D}_+)$. Moreover, the m -a.e. J_0 -unitarity of the boundary values of E on \mathbf{T} implies that the function $w_{pr} := \operatorname{Re} P_{pr}$ satisfies

$$w_{pr}(t) = \frac{1}{|d(t) + c(t)|^2} \quad m\text{-a.e. on } \mathbf{T}. \tag{1.19}$$

Clearly,

$$\int_{\mathbf{T}} w_{pr} dm \leq \operatorname{Re} [P_{pr}(0)] < \infty. \tag{1.20}$$

Furthermore,

$$\begin{aligned} \int_{\mathbf{T}} \ln \frac{1}{w_{pr}} dm &= 2 \int_{\mathbf{T}} [\ln |d(t) + c(t)|] m(dt) \\ &\leq 2 \ln 2 + 2 \int_{\mathbf{T}} [\ln \|E(t)\|] m(dt). \end{aligned}$$

Since every J -contractive matrix function E satisfies

$$\int_{\mathbf{T}} [\ln \|E(t)\|] m(dt) < \infty$$

the function P_{pr} defined in (1.18) fulfills (1.16). Thus, our Lemma on priming functions implies the following result:

Theorem on generating Nevanlinna-Pick problems with infinitely many solutions by holomorphic J_0 -inner matrix-valued functions: *Let E be a J_0 -inner function which is holomorphic in \mathbb{D}_+ , and let E be partitioned as in (1.17). Suppose that $(z_k)_{k \in A}$ is an at most countable sequence of pairwise different points of \mathbb{D}_+ which satisfies*

the Blaschke condition. Assume that the function P_{pr} is defined from E via (1.18) (i.e. via a transformation of type (1.11) with parameter $\sigma \equiv 1$) and let the sequence $(p_k)_{k \in A}$ defined from P_{pr} via (1.16). Then the interpolation problem $NP((z_k, p_k)_{k \in A})$ has an infinite set of solutions.

Our Main Theorem on left and right Blaschke-Potapov products will be proved if we will succeed in constructing a matrix-valued function W having the following two properties:

- (α) W is a left Blaschke-Potapov product only formed from Blaschke-Potapov elementary factors with poles in \mathbb{D}_+ .
- (β) After successive-right hand side splitting off from W all the Blaschke-Potapov elementary factors which correspond to the set of all poles of W in \mathbb{D} we obtain the originally given Arov-singular function E .

We will construct such a matrix function W as a resolvent matrix of some Nevanlinna-Pick problem, i.e. via (1.12). The sequence $(z_k)_{k \in \mathbb{N}}$ of interpolation knots will fulfill the Blaschke condition (1.13). The most difficult part of our construction consists of the concrete choice of the sequence $(z_k)_{k \in \mathbb{N}}$. Just this choice leads us to the approximation problem which is the central one in this paper. After this choice is realized the interpolating values $(p_k)_{k \in \mathbb{N}}$ will be constructed via the priming function method where the concrete priming function is built from the originally given Arov-singular J_0 -inner function (1.17) via (1.18). The above theorem ensures that $NP((z_k, p_k)_{k \in \mathbb{N}})$ has infinitely many solutions and, consequently, there exists a resolvent matrix $R(z; ((z_k, p_k)_{k \in \mathbb{N}}))$.

Theorem on multiplicative structure of resolvent matrices (V.P. Potapov):
The resolvent matrix of a Nevanlinna-Pick problem can be chosen as a left Blaschke-Potapov product which is only built of Blaschke-Potapov elementary factors with poles in the interpolation knots (and, consequently, in \mathbb{D}_+).

This theorem was proved by V.P. Potapov at the end of the sixties. Its formulation was firstly published in the joint paper I.V. KOVALISHINA / V.P. POTAPOV [27] which appeared in 1974. A detailed proof is contained in the paper I.V. KOVALISHINA [26] from 1983, where she unfortunately forgot to mention V.P. Potapov's contribution.

This theorem of V.P. Potapov implies the following:

If we proceed in the above-mentioned way outgoing from a Nevanlinna-Pick problem then the corresponding resolvent matrix W can always be chosen as a left Blaschke-Potapov product. This means that (α) is automatically satisfied. The method which V.P. Potapov used to prove this theorem is essentially a matricial variant of the classical stepwise algorithm of I. SCHUR [37] or, more precisely, of its modification due to R. NEVANLINNA [31]. It will turn out that the right multiplicative decomposition (1.6–1.7) of a matrix-valued function of type (1.12) can also be obtained by a stepwise procedure. This stepwise procedure can be conceived in some sense as an Schur type algorithm in the reverse direction.

Both types of Schur algorithms are based on the following result:

Theorem on resolvent matrices for Nevanlinna-Pick problems with one interpolation node. *If a Nevanlinna-Pick problem in the class $\mathcal{C}(\mathbb{D}_+)$ with only one interpolation node in \mathbb{D}_+ has infinitely many solutions, then some Blaschke-Potapov elementary factor with pole in this interpolation node can be taken as corresponding resolvent matrix.*

Just this theorem of V.P. Potapov was the starting point for his further investigations on the theory of classical interpolation problems. V.P. Potapov used this result to construct a matricial generalization of the classical Schur algorithm which leads to a left multiplicative decomposition

$$R(z; ([z_k, p_k])_{k \in \mathbb{N}}) = \prod_{k=1}^{\infty} B(z; z_k, P_k^{(k)}) \quad (1.21)$$

of an appropriately chosen resolvent matrix for a Nevanlinna-Pick problem $NP((z_k, p_k)_{k \in \mathbb{N}})$ which has infinitely many solutions, i.e. an interpolation problem with interpolation conditions

$$P(z_k) = p_k, \quad k \in \mathbb{N}. \quad (1.22)$$

Our stepwise procedure leads us to the right multiplicative decomposition of the same resolvent matrix.

In the first step of our algorithm we consider the problem $NP((z_k, p_k)_{k \geq 2})$ which is a truncation of $NP((z_k, p_k)_{k \geq 1})$. The solutions of this truncated problem satisfy all conditions (1.22) of the original problem with exception of $k = 1$. If the original problem has infinitely many solutions then, clearly, this is also true for all of its truncations. Hence, $NP((z_k, p_k)_{k \geq 2})$ has an infinite set of solutions which can be parametrized via a fractional linear transformation of type (1.11) the generating matrix of which is the resolvent matrix $R(z; ([z_k, p_k]_{k \geq 2}))$. The free parameter σ varies here over $\mathcal{C}(\mathbb{D}_+) \cup \{\infty\}$. If we want to extract from the solutions of the truncated problem the solutions of the original we have to take into account the removed interpolation condition $P(z_1) = p_1$. Doing this we obtain an interpolation condition

$$\sigma(z_1) = p_1^{(0)} \quad (1.23)$$

where the interpolation value $p_1^{(0)}$ can be obtained from the interpolation data $([z_k, p_k]_{k \geq 1})$ by some recalculation. Thus, the parameter σ must necessarily be a solution of the interpolation problem $NP([z_1, p_1^{(0)}])$ with exactly one interpolation node, namely z_1 . In view of V.P. Potapov's theorem the resolvent matrix for such a problem can be chosen as Blaschke-Potapov-elementary factor:

$$R(z; [z_1, p_1^{(0)}]) = B(z; z_1, P_1^{(r)}) \quad (1.24)$$

Since composition of fractional linear transformations can be expressed by forming the product of their generating matrices we get the identity

$$R(z; ([z_k, p_k]_{k \geq 1})) = R(z; ([z_k, p_k]_{k \geq 2})) \cdot B(z; z_1, P_1^{(r)}) \quad (1.25)$$

For arbitrary $m \in \mathbb{N}$ we obtain analogously

$$R(z; ([z_k, p_k]_{k \geq m})) = R(z; ([z_k, p_k]_{k \geq m+1})) \cdot B(z; z_m, P_m^{(r)}) \quad (1.26)$$

where $B(z; z_m, P_m^{(r)})$ is a Blaschke-Potapov elementary factor with pole at z_m and arises as resolvent matrix of some interpolation problem $NP(\{z_m, p_m^{(0)}\})$. Let n be a fixed positive integer. Then writing successively the identities (1.26) for $m = 1, 2, \dots, n$ we get

$$W(z) = R\left(z; \{z_k, p_k\}_{k \geq n}\right) \cdot \prod_{k=1}^n B\left(z; z_k, P_k^{(r)}\right), \tag{1.27}$$

where W is a matrix function of type (1.12). In identity (1.27) we can carry out the limit process $n \rightarrow \infty$. As usually in such contexts the convergence of the infinite right Blaschke-Potapov product is based on V.P. Potapov's famous "Theorem on the Modulus". The limit process leads us to equation (1.4) where $B^{(r)}$ is a right Blaschke-Potapov product of type (1.6) and where

$$E^{(r)}(z) = \lim_{n \rightarrow \infty} R\left(z; \{z_k, p_k\}_{k \geq n}\right). \tag{1.28}$$

It only remains to verify that after realizing an appropriate choice of the sequence $(z_k)_{k \in \mathbb{N}}$ the matrix function $E^{(r)}$ where

$$E^{(r)} = \begin{pmatrix} a^{(r)} & b^{(r)} \\ c^{(r)} & d^{(r)} \end{pmatrix} \tag{1.29}$$

coincides with the originally given Arov-singular matrix-valued function E . The key for that lies in following:

Theorem on generating Arov-singular J_0 -inner functions via nested sequences of Nevanlinna-Pick problems. *Let E be a given Arov-singular J_0 -inner function. Then there exists a Nevanlinna-Pick problem $NP(\{z_k, p_k\}_{k \in \mathbb{N}})$ for the class $C(\mathbb{D}_+)$ with infinitely many solutions such that the function E can be obtained as limit of a sequence of appropriately chosen resolvent matrices of the 'nested' sequence of truncations of this problem:*

$$E(z) = \lim_{n \rightarrow \infty} R\left(z; \{z_k, p_k\}_{k \geq n}\right), \quad z \in \mathbb{D}_+. \tag{1.30}$$

It is exactly the proof of the just formulated theorem which requires considerations linked with rational weighted approximation. The bridge between Nevanlinna-Pick interpolation and rational weighted approximation is built by the theory of reproducing kernels. A detailed treatment of the theory of classical interpolations problems including Nevanlinna-Pick interpolation which is based on reproducing kernels is given in H. DYM's monograph [12] which was influenced by the pioneering work of L. DE BRANGES [7] and L. DE BRANGES / J. ROVNYAK [8].

Let $(z_\alpha)_{\alpha \in A}$ be a sequence of pairwise different points from \mathbb{D}_+ which satisfies the Blaschke condition. Further let w be a weight function which is m -a.e. defined on \mathbb{T} and fulfills the conditions (4.4) and (4.5) given below. There are two objects which are associated with the sequence $(z_\alpha)_{\alpha \in A}$ and the weight function w , namely first a reproducing kernel Hilbert space (RKHS) and second some Nevanlinna-Pick problem for the class $C(\mathbb{D}_+)$ which has infinitely many solutions and, consequently, an appropriate resolvent matrix. Denote $R((z_\alpha)_{\alpha \in A})$ the set of all rational functions the poles of which are contained in $(z_\alpha)_{\alpha \in A}$ whereas $R_w((z_\alpha)_{\alpha \in A})$ stands for the closure of $R((z_\alpha)_{\alpha \in A})$ with respect

to the Hilbert space norm (5.8). The elements of the space $R_w((z_\alpha)_{\alpha \in A})$ are naturally determined as functions meromorphic in the (nonconnected) open set $\mathbb{D}_+ \cup \mathbb{D}_-$ and not only as functions defined m -a.e. on \mathbb{T} . The functions belonging to $R_w((z_\alpha)_{\alpha \in A})$ have a lot of interesting function-theoretical properties which will be described below. At the moment it is most important for us that the evaluation in a fixed point $z \in ((\mathbb{D}_+ \cup \mathbb{D}_-) \setminus \{(z_\alpha)_{\alpha \in A}\})$ turns out to be a continuous linear functional on this space. In other words, the space $R_w((z_\alpha)_{\alpha \in A})$ is a RKHS. Denote $K(t, z; R_w((z_\alpha)_{\alpha \in A}))$ the corresponding reproducing kernel, i.e. for each $f \in R_w((z_\alpha)_{\alpha \in A})$ and each $z \in ((\mathbb{D}_+ \cup \mathbb{D}_-) \setminus \{(z_\alpha)_{\alpha \in A}\})$ we have

$$f(z) = \int_{\mathbb{T}} f(t) \cdot \overline{K(t, z; R_w((z_\alpha)_{\alpha \in A}))} w(t) m(dt). \quad (1.31)$$

On the other side, outgoing from the weight function w we can define the priming function P_{pr} via

$$P_{pr}(z) := \frac{1}{2} \int_{\mathbb{T}} \frac{t+z}{t-z} w(z) m(dt), \quad z \in \mathbb{D}_+. \quad (1.32)$$

After that on the basis of P_{pr} and $(z_\alpha)_{\alpha \in A}$ we can define

$$p_\alpha := P_{pr}(z_\alpha), \quad \alpha \in A.$$

Thus, the sequence $(z_\alpha)_{\alpha \in A}$ and the weight function w generate a Nevanlinna-Pick problem. In view of the Priming function lemma this problem $NP((z_\alpha, p_\alpha)_{\alpha \in A})$ has infinitely many solutions. Let

$$R(z; ((z_\alpha, p_\alpha)_{\alpha \in A})) = \begin{pmatrix} a(z; ((z_\alpha, p_\alpha)_{\alpha \in A})) & b(z; ((z_\alpha, p_\alpha)_{\alpha \in A})) \\ c(z; ((z_\alpha, p_\alpha)_{\alpha \in A})) & d(z; ((z_\alpha, p_\alpha)_{\alpha \in A})) \end{pmatrix} \quad (1.33)$$

be an appropriate resolvent matrix of $NP((z_\alpha, p_\alpha)_{\alpha \in A})$. The following Reproducing Kernel Formula is of principal importance for us:

$$\begin{aligned} & K(t, z; R_w((z_\alpha)_{\alpha \in A})) \\ &= \frac{c(t; ((z_\alpha, p_\alpha)_{\alpha \in A})) \cdot \overline{d(z; ((z_\alpha, p_\alpha)_{\alpha \in A}))} + d(t; ((z_\alpha, p_\alpha)_{\alpha \in A})) \cdot \overline{c(z; ((z_\alpha, p_\alpha)_{\alpha \in A}))}}{1 - t\bar{z}}. \end{aligned} \quad (1.34)$$

Now we are able to perform the conclusive step. Let E be an 2×2 Arov-singular J_0 -inner function as in (1.17). Let us assume that the associated priming function P_{pr} of type (1.18) admits an integral representation of type (1.32), i.e. the measure realizing the Riesz-Herglotz representation of $P_{pr} \in \mathcal{C}(\mathbb{D}_+)$ is absolutely continuous with respect to m . The general case can be reduced to this case by replacing E by UE where U is an appropriate constant J_0 -unitary matrix. We will not explicitly carry out this procedure here since our aim is to elucidate the main ideas but not the details. As already mentioned the condition $J - E^*(t)JE(t) = 0$ m -a.e. implies

$$w(t) = \frac{1}{|c(t) + d(t)|^2} \quad m\text{-a.e. on } \mathbb{T} \quad (1.35)$$

Let us fix a function w of type (1.35). Suppose that $(z_k)_{k \in \mathbb{N}}$ is some sequence from \mathbb{D}_+ which satisfies the Blaschke condition. We will consider the following decreasing chain of subspaces:

$$R_w((z_k)_{k \geq 1}) \supseteq R_w((z_k)_{k \geq 2}) \supseteq R_w((z_k)_{k \geq 3}) \supseteq \dots \quad (1.36)$$

Since all subspaces of this chain were constructed from the same weight function w all embeddings in (1.36) are isometric ones. Because all the subspaces forming the chain (1.36) are RKHS spaces the same is true for their intersection $\bigcap_{n \in \mathbb{N}} R_w((z_k)_{k \geq n})$. From elementary properties of RKHS it follows

$$K \left(t, z; \bigcap_{n \in \mathbb{N}} R_w \left((z_k)_{k \geq n} \right) \right) = \lim_{n \rightarrow \infty} K \left(t, z; R_w \left((z_k)_{k \geq n} \right) \right). \tag{1.37}$$

In view of (1.34), (1.28), (1.29) and (1.37) we obtain then

$$K \left(t, z; \bigcap_{n \in \mathbb{N}} R_w \left((z_k)_{k \geq n} \right) \right) = \frac{c^{(r)}(t) \cdot \overline{d^{(r)}(z)} + d^{(r)}(t) \cdot \overline{c^{(r)}(z)}}{1 - t\bar{z}} \tag{1.38}$$

where $c^{(r)}$ and $d^{(r)}$ are the elements of the second row of the matrix function $E^{(r)}$ occurring in the right multiplicative decomposition (1.4) of a matrix function of type (1.12). The space $\bigcap_{n \in \mathbb{N}} R_w((z_k)_{k \geq n})$ consists of all functions which have the following properties. They are holomorphic in (the nonconnected) open set $\mathbb{ID}_+ \cup \mathbb{ID}_-$, are pseudocontinuable (see Section 2), belong to the Smirnov class (see Section 3) and are square integrable with respect to the weight w . Moreover, it turns out that

$$\bigcap_{n \in \mathbb{N}} R_w \left((z_k)_{k \geq n} \right) \subseteq PCH_w^2, \tag{1.39}$$

where the Hilbert space PCH_w^2 will be introduced in Section 4. The embedding in (1.39) proves to be an isometric one. It is important for us that PCH_w^2 is a RKHS such that if the weight function w has the form (1.35), the reproducing kernel $K(t, z; PCH_w^2)$ of this kernel is given by

$$K(t, z; PCH_w^2) = \frac{c(t) \cdot \overline{d(z)} + d(t) \cdot \overline{c(z)}}{1 - t\bar{z}}. \tag{1.40}$$

If, for a given weight function w of type (1.35), we will be able to choose a Blaschke sequence $(z_k)_{k \in \mathbb{N}}$ such that the subspaces occurring in (1.39) coincide, i.e.

$$\bigcap_{n \in \mathbb{N}} R_w \left((z_k)_{k \geq n} \right) = PCH_w^2 \tag{1.41}$$

then we will ensure that the reproducing kernels standing at the left-hand sides of the equations (1.38) and (1.40) coincide. This means that, for all $z \in \mathbb{ID}_+ \cup \mathbb{ID}_-$ and all $\xi \in \mathbb{ID}_+ \cup \mathbb{ID}_-$, the identity

$$c(\xi) \overline{d(z)} + d(\xi) \overline{c(z)} \equiv c^{(r)}(\xi) \cdot \overline{d^{(r)}(z)} + d^{(r)}(\xi) \overline{c^{(r)}(z)} \tag{1.42}$$

holds true. It is the main result in this paper that it will be shown that the choice of a Blaschke sequence $(z_k)_{k \in \mathbb{N}}$ guaranteeing (1.41) is always possible. This concerns even nonnegative m -integrable weight functions w with convergent logarithmic integral (see (4.3)-(4.5)) below) and not only weights w of type (1.35). This assertion which has purely function-theoretical character will be formulated in terms of coincidence of subspaces in Section 6 and, moreover, as a problem of rational weighted approximation in Section 7.

Clearly, from (1.42) it will not follow that the matrix function $E^{(r)}$ and E coincide. However, with the aid of a little more complicate and longer consideration one can also verify that for all $\xi, z \in \mathbb{ID}_+ \cup \mathbb{ID}_-$ the identity

$$a(\xi) \overline{b(z)} + b(\xi) \overline{a(z)} = a^{(r)}(\xi) \overline{b^{(r)}(z)} + b^{(r)}(\xi) \cdot \overline{a^{(r)}(z)} \tag{1.43}$$

holds true. Using (1.42) and (1.43) some computations provide

$$E(\xi) J_0 [E(z)]^* = E^{(r)}(\xi) J_0 \cdot [E^{(r)}(z)]^* \quad (1.44)$$

for all $\xi, z \in \mathbb{D}_+ \cup \mathbb{D}_-$. Choosing an appropriate normalization from (1.44) we obtain $E = E^{(r)}$.

2 The Class of Pseudocontinuable Functions

Let us start this section with introducing some notation. Let \mathbb{C} be the complex plane, whereas $\overline{\mathbb{C}}$ stands for the extended complex plane $\mathbb{C} \cup \{\infty\}$. Further, let \mathbb{T} be the unit circle, i.e. $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and denote $\mathbb{D}_+ := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{D}_- := \{z \in \overline{\mathbb{C}} : |z| > 1\}$ the interior and exterior of the unit circle, respectively. Hence, $\overline{\mathbb{C}} \setminus \mathbb{T} = \mathbb{D}_+ \cup \mathbb{D}_-$. Moreover, $m(dt)$ stands for the normalized linear Lebesgue-Borel measure on \mathbb{T} . (In particular, $m(\mathbb{T}) = 1$.)

If G is some subset of \mathbb{C} , then $G^\#$ denotes that subset of $\overline{\mathbb{C}}$ which lies symmetric to G with respect to the unit circle, i.e.

$$G^\# := \left\{ z \in \overline{\mathbb{C}} : \frac{1}{\bar{z}} \in G \right\} \quad (2.1)$$

If $f : G \rightarrow \mathbb{C}$, then $f^\#$ is that complex-valued function which is defined on $G^\#$ by the rule

$$f^\#(z) := \overline{f\left(\frac{1}{\bar{z}}\right)}, \quad z \in G^\#. \quad (2.2)$$

If the function f is holomorphic (resp. meromorphic) in G , then the 'symmetric' function $f^\#$ is holomorphic (resp. meromorphic) in $G^\#$. Now we introduce some distinguished classes of functions. We will be concerned with functions which are meromorphic (in particular, holomorphic) in one of the following three domains: the interior \mathbb{D}_+ of the unit disc, the exterior \mathbb{D}_- of the unit disc or the (non-connected) set $\mathbb{D}_+ \cup \mathbb{D}_- (= \overline{\mathbb{C}} \setminus \mathbb{T})$.

All considered classes will be subclasses of the Nevanlinna class of all meromorphic functions of bounded type in the corresponding open set.

We will give some definitions.

The class $NM(\mathbb{D}_+)$ consists of all functions f , which are meromorphic in \mathbb{D}_+ and satisfy the following two conditions

$$\overline{\lim}_{r \rightarrow 1-0} \int_{\mathbb{T}} \ln^+ |f(rt)| m(dt) < \infty, \quad (2.3)$$

$$\sum_k [1 - |\zeta_k(f)|] < \infty \quad (2.4)$$

where the sum is taken over the set $\{\zeta_k(f)\}$ of all poles of the function f (regarding their multiplicities).

In a symmetric way, the class $NM(\mathbb{D}_-)$ consists of all functions f , which are meromorphic in \mathbb{D}_- and fulfill the conditions

$$\overline{\lim}_{r \rightarrow 1+0} \int_{\mathbb{T}} \ln^+ |f(rt)| m(dt) < \infty \quad (2.5)$$

$$\sum_k [1 - |\zeta_k(f)|^{-1}] < \infty \tag{2.6}$$

where the sum is taken over the set $\{\zeta_k(f)\}$ of all poles of the function f (regarding their multiplicities).

The class $N(\mathbb{D}_+)$ is the subclass of all those functions $f \in NM(\mathbb{D}_+)$ which are holomorphic in \mathbb{D}_+ . In other words, the class $N(\mathbb{D}_+)$ consists of all functions f which are holomorphic in \mathbb{D}_+ and satisfy condition (2.3), which for holomorphic functions f is equivalent to the condition

$$\sup_{0 \leq r < 1} \int_{\mathbb{T}} \ln^+ |f(rt)| m(dt) < \infty. \tag{2.7}$$

Analogously, the class $N(\mathbb{D}_-)$ is the subclass of all functions $f \in NM(\mathbb{D}_-)$ which are holomorphic in \mathbb{D}_- . Thus, the class $N(\mathbb{D}_-)$ consists of all functions f which are holomorphic in \mathbb{D}_- and fulfill condition (2.5), which for holomorphic functions f is equivalent to

$$\sup_{1 < r \leq \infty} \int_{\mathbb{T}} \ln^+ |f(rt)| m(dt) < \infty. \tag{2.8}$$

Obviously,

$$f \in NM(\mathbb{D}_+) \iff f^\# \in NM(\mathbb{D}_-)$$

and

$$f \in N(\mathbb{D}_+) \iff f^\# \in N(\mathbb{D}_-).$$

The class $N(\mathbb{D}_+)$ of all functions of bounded type which are holomorphic in \mathbb{D}_+ was introduced in the paper [29] by the brothers F. and R. NEVANLINNA, whereas the class $NM(\mathbb{D}_+)$ of all functions of bounded type which are meromorphic in \mathbb{D}_+ was considered first by R. NEVANLINNA in [30]. In this paper, we will consider functions f which are meromorphic in the non-connected open set $\mathbb{D}_+ \cup \mathbb{D}_- (= \mathbb{C} \setminus \mathbb{T})$ and which are pseudocontinuable. The property of pseudocontinuity is formulated in terms of boundary values.

It is well-known that each function f belonging to the meromorphic Nevanlinna class $NM(\mathbb{D}_+)$ has boundary values m -a.e. on \mathbb{T} , i.e., for m -almost all $t \in \mathbb{T}$, there exists the radial limit

$$f_+(t) := \lim_{r \rightarrow 1-0} f(rt). \tag{2.9}$$

Analogously, if f belongs to the meromorphic Nevanlinna class $NM(\mathbb{D}_-)$, then, for m -almost all $t \in \mathbb{T}$, there exists the radial limit

$$f_-(t) := \lim_{r \rightarrow 1+0} f(rt). \tag{2.10}$$

If g is a function defined on a set K and L is a subset of K then $Rstr.Lg$ stands for the restriction of g onto L .

Definition 2.1.: A function f which is meromorphic in the (non-connected) open set $\mathbb{D}_+ \cup \mathbb{D}_- (= \mathbb{C} \setminus \mathbb{T})$ is called *pseudocontinuable*, if $Rstr.\mathbb{D}_+ f$ and $Rstr.\mathbb{D}_- f$ belong to the classes $NM(\mathbb{D}_+)$ and $NM(\mathbb{D}_-)$, respectively, and if, additionally, the equality

$$f_+(t) = f_-(t) \tag{2.11}$$

holds for m -almost all $t \in \mathbb{T}$. Here f_+ and f_- are the radial limits of f on \mathbb{T} from the interior and from the exterior of the unit disc, respectively.

Definition 2.2.: $PCNM$ is the class of all functions which are meromorphic in $\mathbb{D}_+ \cup \mathbb{D}_-$ and pseudocontinuable.

Definition 2.3.: PCN is the subclass of all those functions belonging to $PCNM$ which are holomorphic in $\mathbb{D}_+ \cup \mathbb{D}_-$.

Remark 2.1.: A function f belonging to $PCNM$ is originally defined in $\mathbb{D}_+ \cup \mathbb{D}_-$ but, in view of Definition 2.1, for such a function the interior and exterior radial limits $f_+(t)$ and $f_-(t)$ coincide m -a.e. For this reason, a function $f \in PCNM$ can be extended to such points $t \in \mathbb{T}$ where the radial limits $f_+(t)$ and $f_-(t)$ defined in (2.9) and (2.10), respectively, exist and satisfy (2.11). For such points $t \in \mathbb{T}$, we define

$$f(t) := f_+(t) (= f_-(t)). \quad (2.12)$$

If we extend a function $f \in PCNM$ in this way, the extended function will be defined everywhere in the extended complex plane $\overline{\mathbb{C}}$ with exception of some subset of the unit circle having linear Lebesgue-Borel measure zero. (In particular, this subset can also be empty.)

Remark 2.2.: If $f \in PCNM$ is such that for all $z \in \mathbb{D}_+$ we have $f(z) = 0$, then, obviously, $f_+(t) = 0$ for all $t \in \mathbb{T}$. Because of (2.11) then $f_-(t) = 0$ for m -almost all $t \in \mathbb{T}$. According to the unicity theorem for the boundary values, then it also holds $f(z) = 0$ for all $z \in \mathbb{D}_-$. Consequently, a pseudocontinuable function is completely determined by its restriction onto the connected component \mathbb{D}_+ of its domain.

The class of pseudocontinuable functions was introduced in G.Ts. TUMARKIN's paper [45] in connection with the problem of describing the class of functions which can be approximated by rational functions with prescribed poles. In the sequel this class has been occurred in the paper [10] by R.G. DOUGLAS, H.S. SHAPIRO and A.L. SHIELDS on the description of cyclic vectors and invariant subspaces of the backward shift operator, in the paper [35] by M. ROSENBLUM and J. ROVNYAK on factorization of operator-valued functions which are non-negative on the unit circle and, finally, in the work of D.Z. AROV [1],[2] and R.G. DOUGLAS and J.W. HELTON [9] on Darlington synthesis.

3 The Smirnov Class N_*

The classes $PCNM$ (resp. PCN) of pseudocontinuable meromorphic (resp. holomorphic) functions are too large for our aims and we will be actually concerned with functions from the smaller subclasses $PCNM_*$ and PCN_* . These classes are subclasses of $PCNM$ and PCN and outgoing from the smaller Smirnov classes NM_* and N_* they will be defined in the same way as this was done above with $PCNM$ and PCN as subclasses of NM and N .

Definition 3.1.: A function $f : \mathbb{D}_+ \rightarrow \mathbb{C}$ belongs to the Smirnov class $N_*(\mathbb{D}_+)$ if f is holomorphic in \mathbb{D}_+ and if the family $(\ln^+ |f(rt)|)_{0 \leq r < 1}$ is uniformly integrable on \mathbb{T} with respect to the normalized Lebesgue-Borel measure m .

A function $f : \mathbb{D}_- \rightarrow \mathbb{C}$ belongs to the *Smirnov class* $N_*(\mathbb{D}_-)$ if f is holomorphic in \mathbb{D}_- and if the family $(\ln^+ |f(rt)|)_{1 < r \leq \infty}$ is uniformly integrable on \mathbb{T} with respect to m . Uniform integrability means the following: For each $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for every Borel subset e of the unit circle \mathbb{T} which satisfies

$$m(e) < \delta \tag{3.1}$$

and for all values of the parameter r indexing the family the inequality

$$\int_e \ln^+ |f(rt)| m(dt) < \varepsilon \tag{3.2}$$

is satisfied. (In particular, $\delta(\varepsilon)$ is independent on r .)

We have not assumed a priori in Definition 3.1 that the function f has to belong to the Nevanlinna class $N(\mathbb{D}_+)$ (resp. $N(\mathbb{D}_-)$) but the given Definition 3.1 easily implies that

$$N_*(\mathbb{D}_+) \subseteq N(\mathbb{D}_+), \quad N_*(\mathbb{D}_-) \subseteq N(\mathbb{D}_-).$$

We will be touched not only with the holomorphic Smirnov classes $N_*(\mathbb{D}_+)$ and $N_*(\mathbb{D}_-)$ but also with the meromorphic classes $NM_*(\mathbb{D}_+)$ and $NM_*(\mathbb{D}_-)$.

Definition 3.2.: A meromorphic function f in \mathbb{D}_+ (resp. \mathbb{D}_-) is said to belong to the *meromorphic Smirnov class* $NM_*(\mathbb{D}_+)$ (resp. $NM_*(\mathbb{D}_-)$) if the following two conditions are satisfied:

- (i) f belongs to $NM(\mathbb{D}_+)$ (resp. $NM(\mathbb{D}_-)$).
- (ii) The family $(\ln^+ |f(rt)|)_r$ is uniformly integrable with respect to the Lebesgue measure on \mathbb{T} . If we speak about $NM_*(\mathbb{D}_+)$, we take $r \in [\frac{1}{2}, 1)$, whereas in the case $NM_*(\mathbb{D}_-)$ the parameter r runs over $(1, 2]$.

Obviously,

$$f \in N_*(\mathbb{D}_+) \quad \Leftrightarrow \quad f^\# \in N_*(\mathbb{D}_-)$$

and

$$f \in NM_*(\mathbb{D}_+) \quad \Leftrightarrow \quad f^\# \in NM_*(\mathbb{D}_+).$$

The Smirnov classes are additive and multiplicative, e.g. if $f_1, f_2 \in N_*(\mathbb{D}_+)$, then $f_1 + f_2 \in N_*(\mathbb{D}_+)$, $f_1 \cdot f_2 \in N_*(\mathbb{D}_+)$.

The fact whether a function f belongs to some distinguished subclass of NM can be characterized with the aid of the Riesz-Nevanlinna-Smirnov factorization. Assume that f belongs to $NM(\mathbb{D}_+)$ and that $f \neq 0$. Then it is well-known that f can be represented in the form

$$f(z) = C \cdot \frac{B_1(z)}{B_2(z)} \cdot \exp \left\{ - \int_{\mathbb{T}} \frac{t+z}{t-z} \delta_s(dt) \right\} \cdot \exp \left\{ \int_{\mathbb{T}} \frac{t+z}{t-z} \ln |f(t)| m(dt) \right\} \quad (z \in \mathbb{D}_+), \tag{3.3}$$

where C is some unimodular constant, B_1 and B_2 are Blaschke products which are built from the zeros and poles of f in \mathbb{D}_+ , respectively, and δ_s is some signed Borel measure on \mathbb{T} which is mutually singular with respect to the Lebesgue-Borel measure m . (If a function f belongs to $NM(\mathbb{D}_+)$, then the sets of their zeros and poles satisfy necessarily the Blaschke condition so that the Blaschke products B_1 and B_2 are well-defined and the function $|\ln |f(t)||$ is integrable on \mathbb{T} with respect to m .) Obviously, the Blaschke products B_1 and B_2 and also the singular measure δ_s are uniquely determined by f . Of course as well both Blaschke products as the singular measure can be missing in (3.3). Let f be a function belonging to $NM(\mathbb{D}_+)$ with multiplicative representation (3.3). Clearly,

$$f \in N(\mathbb{D}_+) \quad \Leftrightarrow \quad B_2 \equiv 1. \quad (3.4)$$

The following assertions are little bit less obvious, but nevertheless well-known:

$$f \in NM_*(\mathbb{D}_+) \quad \Leftrightarrow \quad \delta_s \geq 0 \quad (3.5)$$

and

$$f \in N_*(\mathbb{D}_+) \quad \Leftrightarrow \quad (B_2 \equiv 1 \quad \text{and} \quad \delta_s \geq 0). \quad (3.6)$$

Suppose $p \in (0, \infty)$.

Definition 3.3.: A function $f : \mathbb{D}_+ \rightarrow \mathbb{C}$ is said to belong to the *Hardy class* $H^p(\mathbb{D}_+)$ if f is holomorphic in \mathbb{D}_+ and if

$$\sup_{r \in [0,1)} \int_{\mathbb{T}} |f(rt)|^p m(dt) < \infty. \quad (3.7)$$

We say that a function $f : \mathbb{D}_- \rightarrow \mathbb{C}$ belongs to the *Hardy class* $H^p(\mathbb{D}_-)$ if f is holomorphic in \mathbb{D}_- and if

$$\sup_{r \in (1,\infty]} \int_{\mathbb{T}} |f(rt)|^p m(dt) < \infty. \quad (3.8)$$

A function $f : \mathbb{D}_+ \rightarrow \mathbb{C}$ is referred to as belonging to the *Hardy class* $H^\infty(\mathbb{D}_+)$ if f is holomorphic and bounded in \mathbb{D}_+ , i.e. if

$$\sup_{z \in \mathbb{D}_+} |f(z)| < \infty. \quad (3.9)$$

Finally, a function $f : \mathbb{D}_- \rightarrow \mathbb{C}$ is referred to as belonging to the *Hardy class* $H^\infty(\mathbb{D}_-)$ if f is holomorphic and bounded in \mathbb{D}_- , i.e. if

$$\sup_{z \in \mathbb{D}_-} |f(z)| < \infty. \quad (3.10)$$

Obviously,

$$f \in H^p(\mathbb{D}_+) \quad \Leftrightarrow \quad f^\# \in H^p(\mathbb{D}_-).$$

The following fact due to V.I. Smirnov is of principal importance for us.

Maximum principle of V.I.Smirnov: *Suppose that the function f belongs to $N_*(\mathbb{D}_+)$ and suppose that for its boundary values $f(t) := \lim_{r \rightarrow 1-0} f(rt)$ and for some $p \in (0, \infty)$ the condition*

$$\int_{\mathbb{T}} |f(t)|^p m(dt) < \infty \tag{3.11}$$

is satisfied. Then $f \in H^p(\mathbb{D}_+)$. If

$$\text{ess sup}_{t \in \mathbb{T}} |f(t)| < \infty, \tag{3.12}$$

then $f \in H^\infty(\mathbb{D}_+)$.

The class $N_*(\mathbb{D})$ was introduced by V.I. SMIRNOV in [40]. There is also contained that theorem which we have called "maximum principle of V.I. Smirnov". (Smirnov himself had chosen the symbol D for the class which we have denoted by N_* . For this class one can also often find the symbols N_+ or N^+ .) To our knowledge, the meromorphic Smirnov class (which we have denoted by NM_*) was not considered before. What concerns basic facts on the classes of Nevanlinna, Smirnov and Hardy we refer the reader to the monographs P.L. DUREN [11], J.B. GARNETT [14] and P. KOOSIS [25]. In particular, the maximum principle of V.I. Smirnov occurs as Theorem 2.11 in [11]. A selection of papers by V.I. Smirnov on complex analysis including comments on further progress is contained in the work of N.K. NIKOLSKII and V.P. KHAVIN [33]. The monograph by W. RUDIN [36] deals with classes of functions of several variables in the polydisc which can be conceived as natural analogues of the classes of Nevanlinna, Smirnov and Hardy. The results presented there have also sense for functions of one variable. In [36] there is also a well-written representation of special questions concerning functions of one variable (see Chapter III). Several aspects of the theory of Hardy spaces are also contained in the monograph by K. HOFFMAN [18].

4 The Smirnov Class of Pseudocontinuable Functions. Weighted Spaces PCH_w^2 of Pseudocontinuable Holomorphic Functions

Let us turn our attention to the function class PCN_* .

Definition 4.1.: By the class PCN_* we mean the class of all holomorphic pseudocontinuable functions $f : \mathbb{D}_+ \cup \mathbb{D}_- \rightarrow \mathbb{C}$ for which $Rstr_{\mathbb{D}_+} f \in N_*(\mathbb{D}_+)$ and $Rstr_{\mathbb{D}_-} f \in N_*(\mathbb{D}_-)$.

The class PCN_* is invariant with respect to changing to the symmetric function: $f \in PCN_*$ if and only if $f^\# \in PCN_*$.

Remark 4.1.: If $f \in PCN_*$ and if the boundary values $f(t)$ of this function on the unit circle are integrable with respect to m , i.e.

$$\int_{\mathbb{T}} |f(t)| m(dt) < \infty, \tag{4.1}$$

then in view of the maximum principle of V.I. Smirnov applied to $Rstr_{\mathbb{D}_+} f$ we get that this function belongs to $H^1(\mathbb{D}_+)$. Analogously, by virtue of the corresponding version of the maximum principle of V.I. Smirnov for the exterior of the unit circle we obtain that $Rstr_{\mathbb{D}_-} f$ belongs to $H^1(\mathbb{D}_-)$.

Hence, there are two functions defined in the interior and exterior of the unit circle \mathbb{T} , respectively, which belong to the corresponding Hardy classes $H^1(\mathbb{D}_+)$ and $H^1(\mathbb{D}_-)$ and for which the boundary values coincide m -a.e. on \mathbb{T} . Consequently, a variant of Painlevé's Theorem on removing singularities implies that these functions can be analytically continued through \mathbb{T} into each other (see KOOSIS [25, Theorem III.E.2]). For this reason, the original function f is holomorphic in the extended complex plane and has to be constant.

However, there exist non-constant functions $f \in PCN_*$ such that for all $p \in (0, 1)$ the condition

$$\int_{\mathbb{T}} |f(t)|^p m(dt) < \infty \quad (4.2)$$

is satisfied. For example, all rational functions, the poles of which are located at the unit circle and have the order one, have this property.

Suppose that w is a Borel-measurable function on \mathbb{T} which satisfies the following three conditions:

$$w(t) \geq 0, \quad m - \text{a.e. on } \mathbb{T}, \quad (4.3)$$

$$\int_{\mathbb{T}} w(t) m(dt) < \infty \quad (4.4)$$

and

$$\int_{\mathbb{T}} \ln[w(t)] m(dt) > -\infty. \quad (4.5)$$

We will call such a function a *weight function* or, for short, a *weight*.

Of course assumption (4.5) implies that the function w is m -a.e. on \mathbb{T} positive:

$$w(t) > 0 \quad m - \text{a.e. on } \mathbb{T}. \quad (4.6)$$

Definition 4.2.: Let w be a weight function. Then by the class PCH_w^2 we mean the set of all functions $f \in PCN_*$ which satisfy the condition

$$\int_{\mathbb{T}} |f(t)|^2 w(t) m(dt) < \infty. \quad (4.7)$$

Remark 4.2.: If the weight w satisfies

$$\int_{\mathbb{T}} [w(t)]^{-1} m(dt) < \infty, \quad (4.8)$$

then (4.7) implies (4.1). In view of Remark 4.1, this means that f is constant. Hence, the space PCH_w^2 can be non-trivial in the sense that it does not contain constant functions alone only in the case

$$\int_{\mathbb{T}} [w(t)]^{-1} m(dt) = \infty. \quad (4.9)$$

By virtue of a theorem due to G. SZEGÖ [42], a weight function satisfying (4.3)–(4.5) admits a factorization. Namely, define

$$\Phi_+(z) := \exp \left\{ \frac{1}{2} \int_{\mathbb{T}} \frac{t+z}{t-z} \ln[w(t)] m(dt) \right\}, \quad z \in \mathbb{D}_+ \tag{4.10}$$

and

$$\Phi_-(z) := \exp \left\{ \frac{1}{2} \int_{\mathbb{T}} \frac{z+t}{z-t} \ln[w(t)] m(dt) \right\}, \quad z \in \mathbb{D}_-. \tag{4.11}$$

Then, obviously, $\Phi_- = (\Phi_+)^{\#}$. The functions Φ_+ and Φ_- satisfy

$$\Phi_+ \in H^2(\mathbb{D}_+), \quad \Phi_- \in H^2(\mathbb{D}_-) \tag{4.12}$$

and are outer, i.e., we have

$$\frac{1}{\Phi_+} \in N_*(\mathbb{D}_+), \quad \frac{1}{\Phi_-} \in N_*(\mathbb{D}_-). \tag{4.13}$$

Moreover, the boundary values of these functions fulfill the factorization identities

$$|\Phi_+(t)|^2 = w(t), \quad |\Phi_-(t)|^2 = w(t) \quad m - \text{a.e. on } \mathbb{T}. \tag{4.14}$$

Clearly, we say that two functions $f_1 \in PCH_w^2$ and $f_2 \in PCH_w^2$ coincide, if $f_1(z) = f_2(z)$ for all $z \in \mathbb{D}_+ \cup \mathbb{D}_-$. The space PCH_w^2 equipped with the natural linear operations is a complex vector space. The null element of this vector space is the null function in $\mathbb{D}_+ \cup \mathbb{D}_-$.

Lemma 4.1.: *The expression*

$$\|f\| := \left\{ \int_{\mathbb{T}} |f(t)|^2 w(t) m(dt) \right\}^{1/2} \tag{4.15}$$

is a norm in the complex vector space PCH_w^2 .

Proof: The homogeneity property ($\|\lambda f\| = |\lambda| \|f\|$ for all $\lambda \in \mathbb{C}$ and all $f \in PCH_w^2$) and the triangle inequality ($\|f_1 + f_2\| \leq \|f_1\| + \|f_2\|$ for all $f_1, f_2 \in PCH_w^2$) are obviously satisfied. Assume now that $f \in PCH_w^2$ satisfies $\|f\| = 0$. In view of (4.15), then it follows $|f(t)|^2 w(t) = 0$ $m - \text{a.e. on } \mathbb{T}$. By virtue of (4.6) then $f(t) = 0$ $m - \text{a.e. on } \mathbb{T}$. The unicity of the boundary values implies then that $f(z) = 0$ for all $z \in \mathbb{D}_+ \cup \mathbb{D}_-$. ■

Theorem 4.1.: *The normed space PCH_w^2 is complete.*

Proof: Suppose that $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in PCH_w^2 , i.e.

$$\int_{\mathbb{T}} |f_n(t) - f_l(t)|^2 w(t) m(dt) \rightarrow 0 \quad (l, n \rightarrow \infty). \tag{4.16}$$

Using the first factorization identity in (4.14), we can rewrite (4.16) in the form

$$\int_{\mathbf{T}} |f_n(t) - f_l(t)|^2 |\Phi_+(t)|^2 m(dt) \rightarrow 0 \quad (l, n \rightarrow \infty). \quad (4.17)$$

For $n \in \mathbb{N}$, we introduce the function $h_{n,+} : \mathbb{D}_+ \rightarrow \mathbb{C}$ defined by

$$h_{n,+}(z) := f_n(z) \cdot \Phi_+(z), \quad z \in \mathbb{D}_+. \quad (4.18)$$

Because of $Rstr_{\mathbb{D}_+} f_n \in N_*(\mathbb{D}_+)$ (compare the definition of PCH_w^2) and $\Phi_+ \in N_*(\mathbb{D}_+)$, then $h_{n,+} \in N_*(\mathbb{D}_+)$, $n \in \mathbb{N}$. In view of $f_n \in PCH_w^2$, we have

$$\int_{\mathbf{T}} |f_n(t)|^2 w(t) m(dt) < \infty.$$

This means

$$\int_{\mathbf{T}} |h_{n,+}(t)|^2 m(dt) < \infty. \quad (4.19)$$

By virtue of the maximum principle of V.I. Smirnov applied to the function $h_{n,+} \in N_*(\mathbb{D}_+)$, from (4.19) we obtain $h_{n,+} \in H^2(\mathbb{D}_+)$, $n \in \mathbb{N}$. The limit relation (4.17) can be written in the form

$$\int_{\mathbf{T}} |h_{n,+}(t) - h_{l,+}(t)|^2 m(dt) \rightarrow 0 \quad (l, n \rightarrow \infty). \quad (4.20)$$

This condition means that $(h_{n,+})_{n \in \mathbb{N}}$ is a Cauchy sequence in the Hardy space $H^2(\mathbb{D}_+)$. As $H^2(\mathbb{D}_+)$ (equipped with the standard norm) is complete there exists a function h_+ belonging to $H^2(\mathbb{D}_+)$ such that

$$\int_{\mathbf{T}} |h_{n,+}(t) - h_+(t)|^2 m(dt) \rightarrow 0 \quad (n \rightarrow \infty). \quad (4.21)$$

We define

$$f_+(z) := h_+(z) \cdot [\Phi_+(z)]^{-1}, \quad z \in \mathbb{D}_+. \quad (4.22)$$

In view of $h_+ \in H^2(\mathbb{D}_+)$, $\Phi_+^{-1} \in N_*(\mathbb{D}_+)$ and (4.18), it follows

$$f_+ \in N_*(\mathbb{D}_+). \quad (4.23)$$

The boundary values $f_+(t)$ of the function f_+ are given by

$$f_+(t) = h_+(t) \cdot [\Phi_+(t)]^{-1}. \quad (4.24)$$

In view of this relation and the first one in (4.14), the condition

$$\int_{\mathbf{T}} |h_+(t)|^2 m(dt) < \infty$$

takes the form

$$\int_{\mathbf{T}} |f_+(t)|^2 w(t) m(dt) < \infty. \quad (4.25)$$

Clearly, from (4.18) it follows

$$f_{n,+}(t) = h_{n,+}(t) \cdot [\Phi_+(t)]^{-1}, \quad t \in \mathbb{T}, \quad (4.26)$$

where

$$f_{n,+}(t) := \lim_{r \rightarrow 1-0} f_n(rt), \quad t \in \mathbb{T}. \quad (4.27)$$

From (4.21), (4.24), (4.26) and the first relation in (4.14) we infer

$$\int_{\mathbb{T}} |f_{n,+}(t) - f_+(t)|^2 w(t) m(dt) \rightarrow 0 \quad (n \rightarrow \infty). \quad (4.28)$$

Analogously, using the second of the factorization identities (4.14) we rewrite (4.16) in the form

$$\int_{\mathbb{T}} |f_n(t) - f_l(t)|^2 \cdot |\Phi_-(t)|^2 m(dt) \rightarrow 0 \quad (l, n \rightarrow \infty). \quad (4.29)$$

We introduce the function $h_{n,-} : \mathbb{D}_- \rightarrow \mathbb{C}$ defined by

$$h_{n,-}(z) := f_n(z) \cdot \Phi_-(z), \quad z \in \mathbb{D}_-. \quad (4.30)$$

In a similar manner, as we have derived the corresponding facts for the sequence $(h_{n,+})_{n \in \mathbb{N}}$ we obtain that $h_{n,-} \in H^2(\mathbb{D}_-)$ and that $(h_{n,-})_{n \in \mathbb{N}}$ is a Cauchy sequence in $H^2(\mathbb{D}_-)$. As $H^2(\mathbb{D}_-)$ is complete, $(h_{n,-})_{n \in \mathbb{N}}$ converges to some function $h_- \in H^2(\mathbb{D}_-)$, i.e.

$$\int_{\mathbb{T}} |h_{n,-}(t) - h_-(t)|^2 m(dt) \rightarrow 0 \quad (n \rightarrow \infty). \quad (4.31)$$

We define

$$f_-(z) := h_-(z) \cdot [\Phi_-(z)]^{-1}, \quad z \in \mathbb{D}_-. \quad (4.32)$$

In view of $h_- \in H^2(\mathbb{D}_-)$, $\Phi_-^{-1} \in N_*(\mathbb{D}_-)$ and (4.18), we get

$$f_- \in N_*(\mathbb{D}_-). \quad (4.33)$$

The boundary values $f_-(t)$ of the function f_- are given by

$$f_-(t) = h_-(t) \cdot [\Phi_-(t)]^{-1}. \quad (4.34)$$

In a similar manner as we have verified (4.28), we obtain

$$\int_{\mathbb{T}} |f_{n,-}(t) - f_-(t)|^2 w(t) m(dt) \rightarrow 0 \quad (n \rightarrow \infty) \quad (4.35)$$

where

$$f_{n,-}(t) := \lim_{r \rightarrow 1+0} f_n(rt). \quad (4.36)$$

We define $f : \mathbb{D}_+ \cup \mathbb{D}_- \rightarrow \mathbb{C}$ by

$$f(z) := \begin{cases} f_+(z), & z \in \mathbb{D}_+ \\ f_-(z), & z \in \mathbb{D}_- \end{cases} \quad (4.37)$$

where f_+ and f_- are given in (4.22) and (4.32), respectively. We have already obtained (compare (4.23) and (4.33)) that $Rstr_{\mathbb{D}_+} f \in N_*(\mathbb{D}_+)$ and $Rstr_{\mathbb{D}_-} f \in N_*(\mathbb{D}_-)$. Now we show that the function f is pseudocontinuable, i.e. $f_+(t) = f_-(t)$ m -a.e. on \mathbb{T} . The functions f_n forming the original Cauchy sequence are pseudocontinuable as elements of the space PCH_w^2 (note that $PCH_w^2 \subseteq PCN_*$). In other words, we have

$$f_{n,+}(t) = f_{n,-}(t) (= f_n(t)) \quad m - \text{a.e.} \quad (4.38)$$

Taking into account this relation the limit relations (4.28) and (4.35) have the form

$$\int_{\mathbb{T}} |f_n(t) - f_+(t)|^2 w(t) m(dt) \rightarrow 0 \quad (n \rightarrow \infty) \quad (4.39)$$

and

$$\int_{\mathbb{T}} |f_n(t) - f_-(t)|^2 w(t) m(dt) \rightarrow 0 \quad (n \rightarrow \infty). \quad (4.40)$$

From here it follows

$$\int_{\mathbb{T}} |f_+(t) - f_-(t)|^2 w(t) m(dt) = 0. \quad (4.41)$$

Using (4.6) from this we get $f_+(t) = f_-(t) (= f(t))$ m -a.e. on \mathbb{T} .

Hence, f is pseudocontinuable and in view of (4.23) and (4.33) we obtain $f \in PCN_*$. Condition (4.25), i.e.

$$\int_{\mathbb{T}} |f(t)|^2 w(t) m(dt) < \infty \quad (4.42)$$

means now that $f \in PCH_w^2$. The conditions (4.39) and (4.40) provide

$$\int_{\mathbb{T}} |f_n(t) - f(t)|^2 w(t) m(dt) \rightarrow 0 \quad (n \rightarrow \infty) \quad (4.43)$$

or, in other words, $\|f_n - f\| \rightarrow 0$ for $n \rightarrow \infty$. Hence, the space PCH_w^2 is complete ■

5 Weighted Spaces $PCH_w^2(S)$ of Pseudocontinuable Meromorphic Functions with Prescribed Set of Poles

Our aim is to indicate the possibility of approximating the elements of the space PCH_w^2 by some system of rational functions the poles of which belong to some prescribed set. At the beginning we will introduce the corresponding spaces of meromorphic functions. Let S be some discrete subset of $\mathbb{D}_+ \cup \mathbb{D}_-$:

$$S \subseteq \mathbb{D}_+ \cup \mathbb{D}_-. \quad (5.1)$$

We set

$$S_+ := S \cap \mathbb{D}_+, \quad S_- := S \cap \mathbb{D}_-. \quad (5.2)$$

Definition 5.1.: We say that S satisfies a *Blaschke condition* (in $\mathbb{D}_+ \cup \mathbb{D}_-$) if

$$\sum_{z_k \in S_+} [1 - |z_k|] < \infty \tag{5.3}$$

and

$$\sum_{z_k \in S_-} [1 - |z_k|^{-1}] < \infty . \tag{5.4}$$

(The points of the set S can occur with a finite multiplicity. The sums are taken with respect to this multiplicity.)

Definition 5.2.: By the *space* $PCNM(S)$ we mean the set of all functions f which are meromorphic in $\mathbb{D}_+ \cup \mathbb{D}_-$ and pseudocontinuable and the poles of which are assumed to be located in the discrete subset S of $\mathbb{D}_+ \cup \mathbb{D}_-$ which satisfies the Blaschke condition. (The order of each pole of f does not exceed the multiplicity of the corresponding point of S .)

Definition 5.3.: The *space* $PCNM_*(S)$ consists of all functions $f \in PCNM(S)$ which are contained in $PCNM_*$, i.e.

$$PCNM_*(S) := PCNM(S) \cap PCNM_*$$

Suppose now that w is a given weight function on \mathbb{T} which satisfies (4.3)-(4.5) and suppose further that S is a discrete subset of $\mathbb{D}_+ \cup \mathbb{D}_-$ which fulfills the Blaschke condition.

Definition 5.4.: The *space* $PCH_w^2(S)$ is the set of all functions $f \in PCNM_*(S)$ which satisfy

$$\int_{\mathbb{T}} |f(t)|^2 w(t) m(dt) < \infty . \tag{5.5}$$

In other words, the space $PCH_w^2(S)$ consists of all functions f which are meromorphic in $\mathbb{D}_+ \cup \mathbb{D}_-$ and which have the following properties:

(i) The restrictions of f onto \mathbb{D}_+ and \mathbb{D}_- satisfy

$$Rstr_{\mathbb{D}_+} f \in NM_*(\mathbb{D}_+), \quad Rstr_{\mathbb{D}_-} f \in NM_*(\mathbb{D}_-). \tag{5.6}$$

(ii) The function f is pseudocontinuable:

$$f_+(t) = f_-(t) (= f(t)) \quad m - \text{a.e. on } \mathbb{T} . \tag{5.7}$$

(Note that f_+ and f_- are defined in (2.9) and (2.10), respectively.)

(iii) The poles of f belong to S . (Here it is assumed that the order of each pole of f does not exceed the multiplicity of the corresponding point of S .)

(iv) The function f satisfies condition (5.5).

The set $PCH_w^2(S)$ equipped with the natural linear operations is a complex linear vector space and

$$\|f\| := \left[\int_{\mathbb{T}} |f(t)|^2 w(t) m(dt) \right]^{1/2} \quad (5.8)$$

turns out to be a norm in it.

Theorem 5.1.: *The normed space $PCH_w^2(S)$ is complete.*

Proof: This theorem will be proved in a similar manner as Theorem 4.1. (It should be mentioned that Theorem 4.1 can be conceived as that particular special case of Theorem 5.1 which corresponds to $S = \emptyset$.) Let Φ_+ and Φ_- be the outer functions defined in (4.10) and (4.11) which realize the factorizations (4.14) of the weight function w . Further, let B_+ and B_- be the Blaschke products built on the basis of S_+ and S_- , i.e.

$$B_+(z) := \prod_{z_k \in S_+} \frac{z_k - z}{1 - \bar{z}_k z} \cdot \frac{|z_k|}{z_k}, \quad z \in \mathbb{D}_+ \cup \mathbb{D}_-, \quad (5.9)$$

and

$$B_-(z) := \prod_{z_k \in S_-} \frac{z_k - z}{1 - \bar{z}_k z} \cdot \frac{|z_k|}{z_k}, \quad z \in \mathbb{D}_+ \cup \mathbb{D}_-. \quad (5.10)$$

The functions B_+ and B_- are pseudocontinuable. More precisely, B_+ and B_- belong to $PCNM_*$, B_+ is holomorphic in \mathbb{D}_+ whereas B_- is holomorphic in \mathbb{D}_- and their boundary values $B_+(t)$ and $B_-(t)$ satisfy

$$|B_+(t)| = 1, \quad |B_-(t)| = 1 \quad m - \text{a.e. on } \mathbb{T}. \quad (5.11)$$

Suppose now that $(f_n)_{n \in \mathbb{N}}$ is some Cauchy sequence belonging to $PCH_w^2(S)$, i.e.

$$\int_{\mathbb{T}} |f_n(t) - f_l(t)|^2 w(t) m(dt) \rightarrow 0 \quad (l, n \rightarrow \infty). \quad (5.12)$$

For $n \in \mathbb{N}$ we set

$$h_{n,+}(z) := f_n(z) \cdot B_+(z) \cdot \Phi_+(z), \quad z \in \mathbb{D}_+ \quad (5.13)$$

and

$$h_{n,-}(z) := f_n(z) \cdot B_-(z) \cdot \Phi_-(z), \quad z \in \mathbb{D}_-. \quad (5.14)$$

The zeros of the Blaschke products B_+ and B_- "compensate" possible poles of $h_{n,+}$ in \mathbb{D}_+ and $h_{n,-}$ in \mathbb{D}_- . Hence, $h_{n,+}$ is holomorphic in \mathbb{D}_+ whereas $h_{n,-}$ is holomorphic in \mathbb{D}_- . Considerations analogously to those used in proving Theorem 4.1 provide that $h_{n,+} \in H^2(\mathbb{D}_+)$ and $h_{n,-} \in H^2(\mathbb{D}_-)$, $n \in \mathbb{N}$. From (4.14) and (4.15) we infer that

$$\int_{\mathbb{T}} |h_{n,+}(t) - h_{l,+}(t)|^2 m(dt) \rightarrow 0 \quad (l, n \rightarrow \infty) \quad (5.15)$$

and

$$\int_{\mathbb{T}} |h_{n,-}(t) - h_{l,-}(t)|^2 m(dt) \rightarrow 0 \quad (l, n \rightarrow \infty). \quad (5.16)$$

Condition (5.15) expresses that $(h_{n,+})_{n \in \mathbb{N}}$ is a Cauchy sequence in $H^2(\mathbb{D}_+)$, whereas (5.16) says that $(h_{n,-})_{n \in \mathbb{N}}$ is a Cauchy sequence in $H^2(\mathbb{D}_-)$. Because of the completeness of the spaces $H^2(\mathbb{D}_+)$ and $H^2(\mathbb{D}_-)$, there exist functions $h_+ \in H^2(\mathbb{D}_+)$ and $h_- \in H^2(\mathbb{D}_-)$ satisfying

$$\int_{\mathbb{T}} |h_{n,+}(t) - h_+(t)|^2 m(dt) \rightarrow 0 \quad (n \rightarrow \infty) \tag{5.17}$$

and

$$\int_{\mathbb{T}} |h_{n,-}(t) - h_-(t)|^2 m(dt) \rightarrow 0 \quad (n \rightarrow \infty) \tag{5.18}$$

where, of course, $h_+(t)$ and $h_-(t)$ are the boundary values of h_+ and h_- , respectively. We set

$$f_+(z) := h_+(z) \cdot [B_+(z)]^{-1} \cdot [\Phi_+(z)]^{-1}, \quad z \in \mathbb{D}_+ \setminus S_+ \tag{5.19}$$

$$f_-(z) := h_-(z) \cdot [B_-(z)]^{-1} \cdot [\Phi_-(z)]^{-1}, \quad z \in \mathbb{D}_- \setminus S_- \tag{5.20}$$

and

$$f(z) := \begin{cases} f_+(z), & z \in \mathbb{D}_+ \setminus S_+ \\ f_-(z), & z \in \mathbb{D}_- \setminus S_- \end{cases} \tag{5.21}$$

Since h_+ and Φ_+^{-1} are holomorphic in \mathbb{D}_+ , whereas h_- and Φ_-^{-1} are holomorphic in \mathbb{D}_- , and the zeros of the Blaschke products B_+ and B_- coincide with S_+ and S_- (regarding their multiplicities), respectively, the function f is meromorphic in the open set $\mathbb{D}_+ \cup \mathbb{D}_-$, and the set of poles of f is contained in S . Since the functions Φ_+^{-1}, Φ_-^{-1} belong to the corresponding holomorphic Smirnov classes, whereas the functions B_+^{-1}, B_-^{-1} are members of the corresponding meromorphic Smirnov classes, we obtain

$$Rstr_{\mathbb{D}_+} f \in NM_*(\mathbb{D}_+), \quad Rstr_{\mathbb{D}_-} f \in NM_*(\mathbb{D}_-). \tag{5.22}$$

From (5.17), (5.19), (4.14) and (5.11) it follows

$$\int_{\mathbb{T}} |f_n(t) - f_+(t)|^2 w(t) m(dt) \rightarrow 0 \quad (n \rightarrow \infty), \tag{5.23}$$

whereas (5.18), (5.20), (4.14) and (5.11) imply that

$$\int_{\mathbb{T}} |f_n(t) - f_-(t)|^2 w(t) m(dt) \rightarrow 0 \quad (n \rightarrow \infty). \tag{5.24}$$

From (5.23) and (5.24) we infer that

$$\int_{\mathbb{T}} |f_+(t) - f_-(t)|^2 w(t) m(dt) = 0 \tag{5.25}$$

and, consequently,

$$f_+(t) = f_-(t) \quad m - \text{a.e. on } \mathbb{T}. \tag{5.26}$$

Thus, f is pseudocontinuable, which together with (5.22) means that $f \in PCNM_*(S)$. Combining

$$|f_+(t)|^2 \cdot |\Phi_+(t)|^2 = |h_+(t)|^2, \quad |f_-(t)|^2 \cdot |\Phi_-(t)|^2 = |h_-(t)|^2$$

(5.19), (5.20) and (4.14), we obtain

$$|f(t)|^2 w(t) = |h_+(t)|^2 = |h_-(t)|^2$$

and, hence, f satisfies condition (5.5). Each of the conditions (5.17) and (5.18) implies

$$\int_{\mathbb{T}} |f_n(t) - f(t)|^2 w(dt) \rightarrow 0 \quad (n \rightarrow \infty) \quad (5.27)$$

or $\|f_n - f\| \rightarrow 0, n \rightarrow \infty$. Thus, the space $PCH_w^2(S)$ is complete ■

Lemma 5.1.: *The normed space $PCH_w^2(S)$ is separable. In particular, PCH_w^2 is separable.*

Proof: We continue to use the notations of the proof of Theorem 5.1. As it was shown in the proof of Theorem 5.1 the mapping $f \rightarrow fB_+\Phi_+$ realizes an isometric embedding of the space $PCH_w^2(S)$ into the separable space $H^2(\mathbb{D}_+)$ ■

6 The Closure of the Set of Rational Functions with Poles in a Prescribed Set. Formulation of the Main Theorem on Impossibility of Spectral Synthesis

Let S be a discrete subset of the open subset $\mathbb{D}_+ \cup \mathbb{D}_-$. (To each point of S a positive integer is assigned - its multiplicity.) It is assumed that S satisfies the Blaschke conditions (5.3) and (5.4).

The symbol $R(S)$ stands for the set of all rational functions the poles of which are contained in S . (We suppose that the order of each such pole does not exceed the multiplicity of the corresponding point in S .) If all points in S are simple (i.e. have multiplicity one), then each function $r \in R(S)$ has the form

$$r(z) = \xi_0 + \sum_{z_k \in S} \frac{\xi_k}{z - z_k}$$

where all complex numbers ξ_k with exception of a finite number are zero. The set $R(S)$ is a complex linear vector space.

Suppose now that w is a weight function on \mathbb{T} satisfying conditions (4.3)-(4.5). Every rational function is clearly pseudocontinuable and its restrictions belong to the corresponding meromorphic Smirnov classes $NM_*(\mathbb{D}_+)$ and $NM_*(\mathbb{D}_-)$. Since $S \cap \mathbb{T} = \emptyset$ every rational function $r \in R(S)$ is bounded on \mathbb{T} and, therefore,

$$\int_{\mathbb{T}} |r(t)|^2 w(t) m(dt) < \infty.$$

Hence, the set $R(S)$ can be conceived (and will be conceived) as (non-closed) subspace of $PCH_w^2(S)$:

$$R(S) \subseteq PCH_w^2(S). \quad (6.1)$$

Definition 6.1.: The space $R_w^2(S)$ is the closure of $R(S)$ in $PCH_w^2(S)$ (with respect to the above introduced norm in $PCH_w^2(S)$).

Since the space $PCH_w^2(S)$ is complete, then its closed subspace $R_w^2(S)$ is also complete. By definition, we have

$$R_w^2(S) \subseteq PCH_w^2(S). \tag{6.2}$$

In particular, the restrictions onto ID_+ and ID_- of each function $r \in R_w^2(S)$ belong to the corresponding meromorphic Smirnov classes:

$$r \in R_w^2(S) \Rightarrow Rstr_{ID_+} r \in NM_*(ID_+), Rstr_{ID_-} r \in NM_*(ID_-). \tag{6.3}$$

Observe that the subspace $R_w^2(S)$ not necessarily coincides with the whole space $PCH_w^2(S)$ or, in other words, the inclusion in (6.3) can be strict.

Suppose now that $(z_k)_{k \in \mathbb{N}}$ is a given sequence of points in $ID_+ \cup ID_-$ satisfying the Blaschke condition

$$\sum_{|z_k| < 1} [1 - |z_k|] < \infty, \quad \sum_{|z_k| > 1} [1 - |z_k|^{-1}] < \infty. \tag{6.4}$$

For $n \in \mathbb{N}$ we set

$$S_n := \bigcup_{k=n}^{\infty} \{z_k\}. \tag{6.5}$$

In a natural way, a multiplicity is assigned to each point of S_n . The Blaschke condition provides that this multiplicity is finite. Clearly, we have

$$S_1 \supseteq S_2 \supseteq S_3 \supseteq \dots \tag{6.6}$$

and

$$\bigcap_{n \in \mathbb{N}} S_n = \emptyset. \tag{6.7}$$

From (6.6) it follows

$$R(S_1) \supseteq R(S_2) \supseteq R(S_3) \supseteq \dots, \tag{6.8}$$

whereas (6.7) implies

$$\bigcap_{n \in \mathbb{N}} R(S_n) = C \tag{6.9}$$

where C stands for the (one-dimensional) complex vector space containing all complex-valued constant functions. Clearly, from (6.8) it follows

$$R_w^2(S_1) \supseteq R_w^2(S_2) \supseteq R_w^2(S_3) \supseteq \dots, \tag{6.10}$$

but it will turn out that it is possible that

$$\bigcap_{n \in \mathbb{N}} R_w^2(S_n) \neq C. \tag{6.11}$$

In view of property (6.2) we infer that, for each $k \in \mathbb{N}$, the space $\bigcap_{n \in \mathbb{N}} R_w^2(S_n)$ is a subspace of $PCH_w^2(S_k)$. In particular, all possible poles of a function r belonging to $\bigcap_{n \in \mathbb{N}} R_w^2(S_n)$ are contained in $\bigcap_{n \in \mathbb{N}} S_n$. Consequently, in view of (6.7) every function r

which belongs to $\bigcap_{n \in \mathbb{N}} R_w^2(S_n)$ is holomorphic in $\mathbb{ID}_+ \cup \mathbb{ID}_-$. Thus, we have proved the following result.

Theorem 6.1.: *The space $\bigcap_{n \in \mathbb{N}} R_w^2(S_n)$ is a subspace of PCH_w^2 :*

$$\bigcap_{n \in \mathbb{N}} R_w^2(S_n) \subseteq PCH_w^2. \quad (6.12)$$

The inclusion in (6.12) can be strict or, in other words, it is possible that not each function belonging to PCH_w^2 is contained in $\bigcap_{n \in \mathbb{N}} R_w^2(S_n)$.

Now we formulate our main result in this paper.

Main Theorem: *Let w be a weight function on \mathbb{T} satisfying the conditions (4.3)-(4.5). Then there exists a sequence (z_k) of points in $\mathbb{ID}_+ \cup \mathbb{ID}_-$ which satisfies the Blaschke condition (6.4) such that*

$$\bigcap_{n \in \mathbb{N}} R_w^2(S_n) = PCH_w^2. \quad (6.13)$$

Remark 6.1.: From the spectral point of view the assertion $\bigcap_{n \in \mathbb{N}} R_w^2(S_n) \neq C$ has the character of an assertion on impossibility of spectral synthesis. The notion of spectral synthesis in the context of harmonic analysis (or in the context of shift operators or operators which commute with them and act in some spaces of functions defined on the real axis or even on some group) goes back to papers of A. BEURLING [5] and [6], L. SCHWARTZ [38], R. GODEMENT [15]. This notion was prepared by preceding work of N. Wiener in abstract harmonic analysis. We particularly emphasize the paper [38] of L. SCHWARTZ where a translation invariant operator with discrete spectrum was considered. (In A. BEURLING [5], [6] and R. GODEMENT [15] the case of an arbitrary spectrum was studied.) For a detailed discussion of questions associated with spectral synthesis in the context of harmonic analysis we refer the reader to chapter X of the monograph E. HEWITT / K. ROSS [17] (compare also chapter V of the survey paper V.P. KHAVIN [22]). Starting with the papers H.L. HAMBURGER [16] and J. WERMER [46] related problems were even studied for operators acting in an abstract Hilbert space (and not only in spaces of functions defined on groups). Hereby it is usually assumed that the operator under consideration has a complete system of eigenvectors, whereas its spectrum is to be assumed as discrete (i.e. every eigenvalue is an isolated point of spectrum). It was studied the question whether the restriction of an operator to an invariant subspace has a complete system of eigenvectors. Such problems are called spectral synthesis problems for operators. In this framework H.L. HAMBURGER's paper [16] is of principal importance. It contains an example of a compact operator in Hilbert space which has a complete system of eigenvectors and at the same time a nontrivial Volterra part. In the sequel problems of spectral synthesis of operators were considered by N.K. NIKOLSKII [32] and A.S. MARKUS [28].

Our Main Theorem can be conceived as some result on impossibility of spectral synthesis for operators which are near to unitary ones. It characterizes exactly to what extent this spectral synthesis is impossible. However, in this paper we will not discuss the spectral interpretation of our Main Theorem.

7 Fundamental Approximation Lemma. Proof of Our Main Theorem

The formulation of our Main Theorem given in Section 6 a little bit disguises its approximation character. Now we present a reformulation which emphasizes its approximation character.

Main Theorem (Approximation Version): *Let w be a weight function on \mathbb{T} which satisfies (4.3) and (4.4). Then there exists a sequence $(z_k)_{k \in \mathbb{N}}$ of points taken from $\mathbb{D}_+ \cup \mathbb{D}_-$ which satisfies the Blaschke condition and which has the approximation property that, for each function $f \in PCH_w^2$, for each $\varepsilon \in (0, \infty)$ and for each $n \in \mathbb{N}$, there exists a rational function $r \in R(S_n)$ with*

$$\int_{\mathbb{T}} |f(t) - r(t)|^2 w(t) m(dt) < \varepsilon \tag{7.1}$$

holds true.

Observe, that in the approximation formulation of our Main Theorem we do not suppose that the Szegő condition (4.5) is satisfied for the weight function w .

We emphasize once more that the same sequence $(z_k)_{k \in \mathbb{N}}$ enables us to 'serve' all functions f from the space PCH_w^2 and even not the sequence itself but each of its truncated sequences $(z_k)_{k \geq n}$. A large portion of difficulties concerning the proof of our Main Theorem is contained in the following result.

Fundamental Approximation Lemma: *Let w be a weight function on \mathbb{T} which satisfies (4.3) and (4.4). Suppose that f is an arbitrary function from PCH_w^2 and that ε is an arbitrary positive number. Then there exists a rational function r with poles (z_k) located in $\mathbb{D}_+ \cup \mathbb{D}_-$ such that the two inequalities*

$$\int_{\mathbb{T}} |f(t) - r(t)|^2 w(t) m(dt) < \varepsilon^2 \tag{7.2}$$

and

$$\sum_{|z_k| < 1} [1 - |z_k|] + \sum_{|z_k| > 1} [1 - |z_k|^{-1}] < \varepsilon \tag{7.3}$$

are satisfied.

The proof of this lemma is not very long but requires deep facts, namely Frostman's theorem from the theory of value distribution of holomorphic functions (more precisely, its generalization due to W. Rudin) and a result of D.Z. Arov on approximation of a pseudocontinuable function which is bounded on \mathbb{T} . This theorem of D.Z. Arov relies on two deep facts itself, namely the possibility of constructing Darlington realizations for each holomorphic matrix-valued function in \mathbb{D}_+ which is contractive and pseudocontinuable (this was also proved by D.Z. AROV in [2]; see also [4, Ch. I]) and the theorem due to V.P. POTAPOV [34] on the multiplicative representation of a matrix-valued function which is holomorphic and contractive in the open unit disc. Thus, the following proof of our Main Theorem, which was formulated in a 'purely scalar context' (at least it is

not obviously connected with matrix-valued functions), uses deep facts of the theory of analytic matrix-valued functions.

We prove the Fundamental Approximation Lemma later. First we show how one can derive our Main Theorem from it.

Proof of the Main Theorem (on the basis of Fundamental Approximation Lemma): From Lemma 5.1 we know that the space PCH_w^2 is separable. Denote $(h_k)_{k \in \mathbb{N}}$ a sequence the elements of which form a countable dense subset of elements of this space. Now we construct a sequence $(f_k)_{k \in \mathbb{N}}$ of elements of PCH_w^2 such that every function h_s is contained infinitely many times in this sequence. For example, outgoing from the sequence

$$(h_k)_{k \in \mathbb{N}} \quad (7.4)$$

we form the sequence

$$\underbrace{h_1}, \underbrace{h_1, h_2}, \underbrace{h_1, h_2, h_3}, \underbrace{h_1, h_2, h_3, h_4}, \underbrace{h_1, h_2, h_3, h_4, h_5}, \dots \quad (7.5)$$

and preserving the indicated order we rewrite its elements as

$$f_1, f_2, f_3, f_4, f_5, \dots$$

Suppose that $(\varepsilon_p)_{p \in \mathbb{N}}$ is a sequence of positive real numbers such that

$$\sum_{p=1}^{\infty} \varepsilon_p < \infty. \quad (7.6)$$

For every index $p \in \mathbb{N}$ we apply the Fundamental Approximation Lemma to the function $f := f_p$ and the positive real number $\varepsilon := \varepsilon_p$. By virtue of this lemma there exists a rational function r_p , the set of poles

$$z_1^{(p)}, z_2^{(p)}, \dots, z_{n_p}^{(p)} \quad (7.7)$$

of which does not intersect the unit circle, such that the inequalities

$$\int_{\mathbb{T}} |f_p(t) - r_p(t)|^2 w(t) m(dt) < \varepsilon_p^2 \quad (7.8)$$

and

$$\sum_{z_k^{(p)} \in \mathbb{D}_+} [1 - |z_k^{(p)}|] + \sum_{z_k^{(p)} \in \mathbb{D}_-} [1 - |z_k^{(p)}|^{-1}] < \varepsilon_p \quad (7.9)$$

are fulfilled.

Now we form the sequence

$$\underbrace{z_1^{(1)}, z_2^{(1)}, \dots, z_{n_1}^{(1)}}_{}, \underbrace{z_1^{(2)}, z_2^{(2)}, \dots, z_{n_2}^{(2)}}_{}, \underbrace{z_1^{(3)}, z_2^{(3)}, \dots, z_{n_3}^{(3)}}_{}, \dots \quad (7.10)$$

consisting of all poles of all functions $r_p, p \in \mathbb{N}$. Preserving here the indicated order we rewrite its elements as

$$z_1, z_2, z_3, \dots \quad (7.11)$$

The sequence $(z_k)_{k \in \mathbb{N}}$ will turn out to be such a sequence which has all properties asserted in the Approximation Version of our Main Theorem.

From (7.6), (7.9) and from the principle of constructing the sequence $(z_k)_{k \in \mathbb{N}}$ we infer that $(z_k)_{k \in \mathbb{N}}$ satisfies the Blaschke condition (6.4).

Now suppose that f is an arbitrary element from the countable dense subset given in (7.4) which was constructed at the beginning of the proof. Since this f occurs infinitely many times in the sequence $(f_p)_{p \in \mathbb{N}}$, there exists some subsequence $(r_{p_k})_{k \in \mathbb{N}}$ of $(r_p)_{p \in \mathbb{N}}$ such that

$$\int_{\mathbb{T}} |f(t) - r_{p_k}(t)|^2 w(t) m(dt) \rightarrow 0 \quad (k \rightarrow \infty). \tag{7.12}$$

Hereby, for each positive integer n , all poles of the functions r_{p_k} are located in the truncated sequence $S_n := (z_l)_{l \geq n}$ if the index k is sufficiently high. Consequently, if f belongs to the chosen dense subset and if $\varepsilon \in (0, \infty)$ and $n \in \mathbb{N}$ are given, then for sufficiently high indexes k the function r_{p_k} can be taken to play the role of the function $r \in R(S_n)$ in (7.1). Thus, the assertion of our Main Theorem (in its approximation version) is verified for each f from a given dense subset.

Obvious considerations enable us now to conclude the assertion for arbitrary functions $f \in PCH_w^2$ (which do not necessarily belong to the chosen dense subset) ■

Proof of the Fundamental Approximation Lemma: We divide the proof into two steps. In the first step we approximate an arbitrary function $f \in PCH_w^2$ (which is necessarily holomorphic in $\mathbb{D}_+ \cup \mathbb{D}_-$) by a function which is meromorphic in $\mathbb{D}_+ \cup \mathbb{D}_-$, pseudocontinuable, bounded on \mathbb{T} and the poles of which are very near to the unit circle. In the second step we approximate this pseudocontinuable function, which is bounded on \mathbb{T} by rational functions.

Step 1. Let f be an arbitrary function from PCH_w^2 . We introduce a family $(f_\alpha)_{\alpha \in (0, \infty)}$ of functions. Namely, for $\alpha \in (0, \infty)$ we define $f_\alpha : \mathbb{D}_+ \cup \mathbb{D}_- \rightarrow \mathbb{C}$ by

$$f_\alpha(z) := \frac{f(z)}{1 + \alpha f(z) f^\#(z)}, \quad z \in \mathbb{D}_+ \cup \mathbb{D}_-. \tag{7.13}$$

Recall that the function $f^\#$ was defined in (2.2). By virtue of $f \in PCH_w^2 \subseteq PCNM$, we have $f_\alpha \in PCNM$. Moreover,

$$f(t) f^\#(t) = |f(t)|^2 \geq 0, \quad t \in \mathbb{T}. \tag{7.14}$$

From (7.13) and (7.14) it follows

$$|f_\alpha(t)| = \frac{|f(t)|}{1 + \alpha |f(t)|^2}, \quad t \in \mathbb{T}. \tag{7.15}$$

On the one hand, this implies

$$|f_\alpha(t)| \leq \frac{1}{2\sqrt{\alpha}}, \quad t \in \mathbb{T}, \alpha > 0, \tag{7.16}$$

i.e. each function f_α of the family (7.13) is bounded on \mathbb{T} by a constant (depending on α). On the other hand, from (7.15) we infer

$$|f_\alpha(t)| \leq |f(t)|, \quad t \in \mathbb{T}, \alpha \in (0, \infty), \tag{7.17}$$

i.e. the family $(f_\alpha)_{\alpha \in (0, \infty)}$ has a common majorant. Since clearly $f_\alpha(t) \rightarrow f(t)$, $\alpha \rightarrow 0+0$, and since f satisfies (4.7), then Lebesgue's dominated convergence theorem yields

$$\int_{\mathbb{T}} |f(t) - f_\alpha(t)|^2 w(t) m(dt) \rightarrow 0, \quad (\alpha \rightarrow 0+0). \quad (7.18)$$

For $\alpha \in (0, \infty)$, set

$$d_\alpha(z) := 1 + \alpha f(z) f^\#(z), \quad z \in \mathbb{D}_+ \cup \mathbb{D}_-. \quad (7.19)$$

The function d_α is 'symmetric', i.e. $d_\alpha = d_\alpha^\#$. In view of $f \in PCH_w^2 \subseteq PCN_*$, for each $\alpha \in (0, \infty)$, the function d_α belongs to the Smirnov class PCN_* . In particular, for each $\alpha \in (0, \infty)$, the restriction $Rstr_{\mathbb{D}_+} d_\alpha$ satisfies

$$Rstr_{\mathbb{D}_+} d_\alpha \in N_*(\mathbb{D}_+). \quad (7.20)$$

The zero set of d_α is symmetric with respect to \mathbb{T} . Applying Jensen's inequality to $Rstr_{\mathbb{D}_+} d_\alpha$, we obtain

$$\sum_{z_k(\alpha) \in \mathbb{D}_+} \ln \frac{1}{|z_k(\alpha)|} \leq \int_{\mathbb{T}} \ln |d_\alpha(t)| m(dt) - \ln |d_\alpha(0)|. \quad (7.21)$$

(The version of Jensen's inequality where the integral on the right-hand side of (7.21) is taken over the boundary of the unit circle is not generally true for all functions belonging to the holomorphic Nevanlinna class $N(\mathbb{D})$, but it is true for all functions which are members of the holomorphic Smirnov class $N_*(\mathbb{D})$. This fact should be well-known to specialists dealing with boundary properties of analytic functions. However, we were not able to found any reference to it. For the convenience of the reader, we will give a detailed proof of (7.21), but to avoid an interruption of the basic line of realizing step I we will defer this to the end of step I.) From (7.19) it follows that $\ln |d_\alpha| \rightarrow 0$ for $\alpha \rightarrow 0+0$. In view of

$$\int_{\mathbb{T}} \ln[1 + |f(t)|^2] m(dt) < \infty$$

and

$$0 \leq \ln |d_\alpha(t)| \leq \ln[1 + |f(t)|^2], \quad t \in \mathbb{T}, \alpha \in (0, 1],$$

Lebesgue's dominated convergence theorem provides

$$\int_{\mathbb{T}} \ln |d_\alpha(t)| m(dt) \rightarrow 0, \quad (\alpha \rightarrow 0+0).$$

Now from (7.21) we infer

$$\lim_{\alpha \rightarrow 0+0} \left(\sum_{z_k(\alpha) \in \mathbb{D}_+} \ln \frac{1}{|z_k(\alpha)|} \right) = 0. \quad (7.22)$$

Suppose that ε is an arbitrarily given positive real number. By virtue of (7.18) and (7.22) it follows that there exists a $\delta = \delta(\varepsilon, f) > 0$ such that, for each $\alpha \in (0, \delta)$, the two inequalities

$$\int_{\mathbb{T}} |f(t) - f_\alpha(t)|^2 w(t) m(dt) < \varepsilon^2 \quad (7.23)$$

and

$$\sum_{z_k(\alpha) \in \mathbb{D}_+} [1 - |z_k(\alpha)|] + \sum_{z_k(\alpha) \in \mathbb{D}_-} [1 - |z_k(\alpha)|^{-1}] < \varepsilon \tag{7.24}$$

are satisfied. The sums in (7.24) are taken over all poles of f_α which are located in $\mathbb{D}_+ \cup \mathbb{D}_-$. (Since f is holomorphic in $\mathbb{D}_+ \cup \mathbb{D}_-$ the set of poles of f_α coincides with the set of zeros of d_α .) Now we show that the function f_α belongs to the meromorphic Smirnov class $PCNM_*$ of pseudocontinuable functions or, more precisely, that for every $\delta > 0$, one can choose a number $\alpha \in (0, \delta)$ such that the function f_α belongs to $PCNM_*$. We will rely on a theorem of Frostman or, more precisely, on its generalization given by W. RUDIN (see Section 3.6 of the monograph [36] and the references there). The function f_α can be represented as quotient

$$f_\alpha = \frac{f}{\alpha \left(\frac{1}{\alpha} + g\right)}. \tag{7.25}$$

where

$$g = f \cdot f^\# . \tag{7.26}$$

The function f belongs to the holomorphic Smirnov class PCN_* . Thus, we have to ensure that the function $\left(\frac{1}{\alpha} + g\right)^{-1}$ belongs to the meromorphic Smirnov class for sufficiently small positive real numbers α . Since by definition (7.26) the function g is symmetric, i.e. $g = g^\#$, it suffices to ensure that

$$Rstr_{\mathbb{D}_+} \left(\frac{1}{\alpha} + g\right)^{-1} \in NM_*(\mathbb{D}_+) . \tag{7.27}$$

This condition is guaranteed if in the multiplicative representation (3.3) of the function $\frac{1}{\alpha} + g \in N_*(\mathbb{D}_+)$ the factor containing the singular measure δ_s is missing or, equivalently, the limit relation

$$\lim_{r \rightarrow 1-0} \int_{\mathbb{T}} \ln \left| \frac{1}{\alpha} + g(rt) \right| m(dt) = \int_{\mathbb{T}} \ln \left| \frac{1}{\alpha} + g(t) \right| m(dt) \tag{7.28}$$

holds.

In Section 3.6 of W. RUDIN's work [36] the following theorem is proved. *Let g be a function which belongs to the holomorphic Smirnov class $N_*(\mathbb{D}_+)$ and let K be an arbitrary compact subset of the complex plane having positive logarithmic capacity. Then there exists a $\lambda \in K$ such that the function $\lambda + g$ satisfies*

$$\lim_{r \rightarrow 1-0} \int_{\mathbb{T}} \ln |\lambda + g(rt)| m(dt) = \int_{\mathbb{T}} \ln |\lambda + g(t)| m(dt) . \tag{7.29}$$

Since every interval of the real axis has positive logarithmic capacity, in each interval there exists a number λ such that (7.29) is fulfilled. Letting $\lambda := \frac{1}{\alpha}$ we get that the set of all real λ satisfying the limit relation (7.28) is dense everywhere in \mathbb{R} . In particular, such a number λ can be found in every interval $(0, \delta)$, $\delta = \delta(\varepsilon, f)$. (W. RUDIN [36] gives the following definition: A compact subset K of the complex plane is said to have *positive logarithmic capacity* if there exists a positive Borel measure $\rho \neq 0$ which is concentrated on K such that its logarithmic potential $U_\rho : \mathbb{C} \rightarrow \mathbb{R}$ given by $U_\rho(z) := \int_K \log |z - \zeta| \rho(d\zeta)$

is continuous at each point $z \in \mathbb{C}$. Every interval of the real axis has positive logarithmic capacity because one can choose as such a measure ρ , e.g., the restriction of the one-dimensional Lebesgue measure onto the Borelian subsets of this interval. One can easily check by straightforward computations that the logarithmic potential of this measure is continuous in the whole complex plane \mathbb{C} .)

Frostman's original result we referred to is not related to arbitrary functions of the Smirnov class but only to inner functions. It is proved in O. FROSTMAN's remarkable paper [13] which also contains many important applications of potential theory in complex function theory.

Hence, in the first step it was verified that, for an arbitrarily chosen $f \in PCH_{\omega}^2$ and an arbitrarily given $\varepsilon > 0$, there exists a function $a \in PCNM_*$ such that the following conditions are satisfied:

$$\int_{\mathbf{T}} |f(t) - a(t)|^2 w(t) m(dt) < \varepsilon^2, \quad (7.30)$$

$$\sup_{t \in \mathbf{T}} |a(t)| < \infty, \quad (7.31)$$

$$\sum_{z_k \in \mathbb{D}_+} [1 - |z_k|] + \sum_{z_k \in \mathbb{D}_-} [1 - |z_k|^{-1}] < \varepsilon. \quad (7.32)$$

The sums in (7.32) are taken over all poles of a contained in \mathbb{D}_+ and \mathbb{D}_- , respectively. Furthermore, the set of poles of this function lies symmetric with respect to the unit circle.

According to the Frostman-Rudin theorem as a particular choice for such a function we can take a function f_{α} of the form (7.13) with specially chosen sufficiently small $\alpha > 0$.

Comments: We explain inequality (7.21). Suppose that d is a member of the Smirnov class $N_*(\mathbb{D}_+)$ having the zero set (z_k) . Moreover, assume that $d(0) \neq 0$. Jensen's formula applied to the function d in the disc $|z| \leq r$ with $r \in [0, 1)$ provides

$$\sum_{|z_k| < r} \ln \frac{r}{|z_k|} = \int_{\mathbf{T}} \ln |d(rt)| m(dt) - \ln |d(0)|.$$

From this it follows, for $r \in [0, 1)$,

$$\sum_{|z_k| < r} \ln \frac{r}{|z_k|} \leq \sup_{\rho \in [0, 1)} \int_{\mathbf{T}} \ln |d(\rho t)| m(dt) - \ln |d(0)|.$$

Now on the left-hand side we carry out the limit process $r \rightarrow 1 - 0$ and obtain

$$\sum_{|z_k| < 1} \ln \frac{1}{|z_k|} \leq \sup_{\rho \in [0, 1)} \int_{\mathbf{T}} \ln |d(\rho t)| m(dt) - \ln |d(0)|.$$

The last inequality is not only true for functions d of the Smirnov class $N_*(\mathbb{D}_+)$ but also for functions of the larger Nevanlinna class $N(\mathbb{D}_+)$. However, for each function d belonging to the Smirnov class, one can estimate the supremum on the right-hand side of the last inequality from below by the logarithmic integral over the boundary \mathbf{T} , namely

$$\int_{\mathbf{T}} \ln |d(\rho t)| m(dt) \leq \int_{\mathbf{T}} |d(t)| m(dt), \quad \rho \in [0, 1).$$

This inequality can be verified, e.g., by use of the Riesz-Nevanlinna-Smirnov factorization $d = c \cdot B_d S_d E_d$ of the function d where B_d is a Blaschke product, S_d is a singular function and E_d is an outer function. We have $|B_d(z)| \leq 1$ for $z \in \mathbb{D}_+$, whereas $|B_d(t)| = 1$ m -a.e. on \mathbb{T} . Because of $d \in N_*(\mathbb{D}_+)$ the singular measure in the representation of S_d is nonnegative. Thus, $|S_d(z)| \leq 1$ for $z \in \mathbb{D}_+$ whereas $|S_d(t)| = 1$ m -a.e. on \mathbb{T} . Consequently, $|d(z)| \leq |E_d(z)|$ for $z \in \mathbb{D}_+$ and $|d(t)| = |E_d(t)|$ m -a.e. on \mathbb{T} . From the canonical representation of an outer function it follows immediately

$$\int_{\mathbb{T}} \ln |E_d(t)| \, m(dt) = \ln |E_d(0)| = \int_{\mathbb{T}} \ln |E_d(\rho t)| \, m(dt), \quad \rho \in [0, 1).$$

Step 2. Let ε be an arbitrary positive real number. Suppose that a is a pseudocontinuable meromorphic function of the Smirnov class $PCNM_*$, i.e. $a \in PCNM_*$, which is bounded on \mathbb{T} , i.e. (7.31) is fulfilled. Denote B_+ the Blaschke product constructed from the poles (regarding its multiplicities) of a which are located in \mathbb{D}_+ . We consider the function S given by

$$S(z) := a(z) \cdot B_+(z), \quad z \in \mathbb{D}_+ \cup \mathbb{D}_-. \tag{7.33}$$

Since every Blaschke product is pseudocontinuable, the function S is also pseudocontinuable. The Blaschke product B_+ was constructed in such a way that S is holomorphic in \mathbb{D}_+ . Since $|B(t)| = 1$, $t \in \mathbb{T}$ and since in view of (7.31) the function a is bounded on \mathbb{T} , the function S is also bounded on \mathbb{T} : $\sup_{t \in \mathbb{T}} |S(t)| < \infty$. Since a belongs to the Smirnov class $N_*(\mathbb{D}_+)$ we get by virtue of the maximum principle of V.I. Smirnov then

$$\sup_{z \in \mathbb{D}_+} |S(z)| < \infty, \quad \text{i.e.} \quad Rstr_{\mathbb{D}_+} S \in H^\infty(\mathbb{D}_+). \tag{7.34}$$

Here the set of poles of S is the union of two subsets, namely the set of poles of a which are located in \mathbb{D}_- and the set which is symmetric to the set of poles of a which lie in \mathbb{D}_+ . (If the set of poles of a would be symmetric, then the set of poles of S would be the set of poles of a in \mathbb{D}_- and each pole would have the double multiplicity.)

Analogously, if B_- denotes the Blaschke product built on the basis of the zeros of a in \mathbb{D}_- , then the function $a \cdot B_- = S \cdot B_+^{-1} B_-$ belongs to the Hardy class $H^\infty(\mathbb{D}_-)$.

Now we use the following approximation theorem due to D.Z. AROV [3] (see also [4, Section 3.5]):

Let S be a pseudocontinuable function such that

$$Rstr_{\mathbb{D}_+} S \in H^\infty(\mathbb{D}_+). \tag{7.35}$$

Suppose that there exists Blaschke product B such that

$$Rstr_{\mathbb{D}_-} S \cdot B^{-1} \in H^\infty(\mathbb{D}_-). \tag{7.36}$$

Then there exists a sequence $(\rho_n)_{n \in \mathbb{N}}$ of rational functions with the following properties:

- (i) *For $n \in \mathbb{N}$, the poles of ρ_n are located in the set of poles of S (and, consequently, in \mathbb{D}_-).*
- (ii) *The sequence $(\rho_n)_{n \in \mathbb{N}}$ is uniformly bounded on \mathbb{T} , i.e.*

$$\sup_{n \in \mathbb{N}} \max_{t \in \mathbb{T}} |\rho_n(t)| \leq \sup_{t \in \mathbb{T}} |s(t)|. \tag{7.37}$$

(iii) One has the limit relation

$$\lim_{n \rightarrow \infty} \rho_n(t) = S(t) \quad m - a.e. \quad (7.38)$$

We will comment condition (7.36) after finishing Step 2.

We are going to apply the theorem of D.Z. Arov to functions S of type (7.33). (For the Blaschke product, which occurs in Arov's theorem, one has to take $B := B_+ B_-^{-1}$.) Given S let $(\rho_n)_{n \in \mathbb{N}}$ be a sequence of rational functions the existence of which is ensured by Arov's theorem. The sequence

$$\varphi_n := \rho_n \cdot B_+^{-1} \quad (7.39)$$

is uniformly bounded, satisfies the inequality

$$\sup_{n \in \mathbb{N}} \left(\sup_{t \in \mathbb{T}} |\varphi_n(t)| \right) \leq \sup_{t \in \mathbb{T}} |a(t)|, \quad (7.40)$$

and converges almost everywhere to the function a . Since the Blaschke product B_+ is not rational the functions φ_n are not rational. Denote $B_{n,+}$ the n -th partial Blaschke product generated by the 'full' Blaschke product B_+ . Each of the functions $B_{n,+}$ is rational. Moreover, it is well-known that the sequence of partial products of an arbitrary Blaschke product converges to it in Lebesgue measure m on \mathbb{T} and even with respect to square mean norm convergence,

$$\int_{\mathbb{T}} |B_{n,+}(t) - B_+(t)|^2 m(dt) \rightarrow 0 \quad (n \rightarrow \infty)$$

(see, e.g., Chapter 5 in HOFFMAN [18]). Hence, there is a subsequence $(B_{k_n,+})_{n \in \mathbb{N}}$ of partial Blaschke products which converges to B_+ m -a.e. on \mathbb{T} . The sequence (r_n) ,

$$r_n := \rho_n B_{k_n,+}^{-1} \quad (7.41)$$

consists only of rational functions. Moreover, it is uniformly bounded on \mathbb{T} , i.e.

$$\sup_{n \in \mathbb{N}} \max_{t \in \mathbb{T}} |r_n(t)| \leq \sup_{t \in \mathbb{T}} |a(t)| \quad (7.42)$$

and converges to the function a m -a.e. on \mathbb{T} . From this and (4.4) it follows

$$\int_{\mathbb{T}} |a(t) - r_n(t)|^2 w(t) m(dt) \rightarrow 0 \quad (n \rightarrow \infty). \quad (7.43)$$

It is readily checked that the set of poles of all these functions r_n is contained in the set of poles of a in $\mathbb{ID}_+ \cup \mathbb{ID}_-$.

Hence, in Step 2 the following fact was established: Let a be a meromorphic pseudo-continuable function belonging to the Smirnov class $PCNM_*$ which is bounded on the unit circle (i.e. (7.31) is fulfilled) and let ε be an arbitrary positive real number. Then there exists a rational function r the set of poles of which is contained in the set of poles of this function a such that the inequality

$$\int_{\mathbb{T}} |a(t) - r(t)|^2 w(t) m(dt) < \varepsilon^2 \quad (7.44)$$

is satisfied. The role of such a function r can be played by each of the functions r_n when n is sufficiently large. Step 2 is realized.

Comments: We explain condition (7.30) in D.Z. Arov's theorem. Without loss of generality one can assume that $\sup_{|z|<1} |s(z)| \leq 1$. In [2] D.Z. AROV has proved that a pseudocontinuable function s can be considered as (1,2)-element of some holomorphic matrix-valued function

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$$

which is contractive, i.e.

$$I - S^*(z)S(z) \geq 0, \quad z \in \mathbb{D}_+,$$

and which has unitary boundary values m - a.e. on \mathbb{T} , i.e.

$$S^*(t)S(t) = I \quad m - \text{a.e. on } \mathbb{T}.$$

(Such matrix functions are called *inner*.) In AROV [2] the representation of a given contractive holomorphic function s as 12-element of some inner matrix function is called a *Darlington realization* of s . As an inner function, S is pseudocontinuable. The construction of such an inner function S with prescribed element $s_{12} = s$ is not unique. D.Z. AROV proved in [2] that the completion of a given block s_{12} to a 'full' inner function S can be realized in such a way that this 'completed' function S does not contain new singularities (in comparison with the singularities of s) in \mathbb{D}_- . In particular, if $Rstr_{\mathbb{D}_-} s$ belongs to the meromorphic Smirnov class $NM_*(\mathbb{D}_-)$, then it is possible to construct a such completion S of s which satisfies $Rstr_{\mathbb{D}_-} S \in NM_*(\mathbb{D}_-)$. However, an inner function S the pseudocontinuation of which to \mathbb{D}_- belongs to the meromorphic Smirnov class $NM_*(\mathbb{D}_-)$ must be a Blaschke-Potapov product. The functions of the sequence $(\rho_n)_{n \in \mathbb{N}}$ given in (7.38) can be represented as 12-elements of partial Blaschke-Potapov products. For a pseudocontinuable function s which is bounded on \mathbb{T} condition (7.36) is equivalent to $Rstr_{\mathbb{D}_-} s \in NM^*(\mathbb{D}_-)$.

Continuation of the Proof of the Fundamental Approximation Lemma: The Fundamental Approximation Lemma follows by combining the results obtained in Steps 1 and 2. Indeed, given $\epsilon > 0$ and a function $f \in PCH_w^2$ we construct in Step 1 a function $a \in PCNM_*$ which satisfies conditions (7.30) and (7.32) whereas in Step 2 we construct for this function a , which was just obtained, a rational function r which satisfies (7.44) and whose set of poles is contained in the set of poles of a . In particular, condition (7.32) is satisfied for a .

From (7.30) and (7.44) it follows

$$\int_{\mathbb{T}} |f(t) - r(t)|^2 w(t) m(dt) \leq 4\epsilon^2. \tag{7.45}$$

The proof of the Fundamental Approximation Lemma is complete and this means that our Main Theorem is also proved ■

Remark 7.1.: By a slight modification of the proof of the Fundamental Approximation Lemma one can show that, given f and $\epsilon > 0$ one can construct a rational function r which satisfies (7.2) and (7.3), whose poles are only located in \mathbb{D}_- and which are simple.

Remark 7.2.: The sequence $S = (z_k)_{k \in \mathbb{N}}$ which occurs in our Main Theorem is clearly not unique. As one can see from the proof of this theorem one can move the points of this sequence a little bit. Only the asymptotic behaviour of these points plays a role but not the points themselves. Here we will not give a formal definition of what we mean by 'asymptotic behaviour'. Nevertheless, it is possible to indicate some general aspects on the structure of such a sequence of points. We call a point $t_0 \in \mathbb{T}$ singular with respect to the weight function w if for every open arc γ ($\gamma \subseteq \mathbb{T}$) containing t_0 the condition

$$\int_{\gamma} [w(t)]^{-1} m(dt) = \infty \quad (7.46)$$

is satisfied. Denote by $\text{sing } w$ the set of all points of \mathbb{T} which are singular with respect to w . Then $\text{sing } w$ turns out to be a closed subset of the unit circle. A function $f \in PCH_w^2$ proves to be 'well-adapted' with its values on each arc of the unit circle which belongs to the complement of $\text{sing } w$ which means that f is holomorphic at each such arc. (Compare the considerations in Remark 4.2) Hence, each function $f \in PCH_w^2$ is holomorphic in the open subset $\overline{\mathbb{C}} \setminus \text{sing } w$. A sharper analysis of the proof of our Main Theorem shows that the sequence $S = (z_k)_{k \in \mathbb{N}}$ can be chosen such that the set of its accumulation points coincides with $\text{sing } w$.

Remark 7.3.: The considered summable weight function w satisfied the conditions (4.5) and (4.9). The condition (4.5) ensures the completeness of the PCH_w^2 , whereas the condition (4.9) ensures the non-triviality of this space. The conditions (4.5) and (4.9) occur in the theory of (in wide sense) stationary random processes with discrete time. In the theory of random processes these conditions were introduced by A.N. Kolmogorov, [23] and [24]. (A good exposition of the theory of stationary random processes (in wide sense) can be found, e.g., in [39], Chapter 6.) Suppose, that w is a spectral density of a stationary random process with discrete time. The condition (4.5) means, that the considered process is a regular one. In particular, a perfect prediction (extrapolation) of the 'future' of this process in terms of its 'past' is impossible. The condition (4.9) means, that a perfect interpolation of an omitted value of this process from the rest of its values is possible.

Remark 7.4.: Results which are analogous to the Main Theorem and the Fundamental Approximation Lemma can be obtained not only for weighted L^2 -metrics but also for weighted L^p -metrics and even for many other metrics. One can give an axiomatic description of the class of all metrics for which the proofs of our basic results go through.

Remark 7.5.: In his paper [45] (see also [43]) G.Ts. TUMARKIN considered classes of functions f on the unit circle which can be approximated by sequences (r_n) of rational functions in weighted L^p -metrics, i.e.

$$\int_{\mathbb{T}} |f(t) - r_n(t)|^p w(t) m(dt) \rightarrow 0 \quad (n \rightarrow \infty).$$

Here it is assumed that the poles of these rational functions belong to a given scheme of numbers and, moreover, that certain Blaschke type conditions are satisfied 'uniformly with respect to the rows of this scheme'. G.Ts. Tumarkin has proved that each such function f is the boundary value on \mathbb{T} of some pseudocontinuable function. In particular,

his results contain our results obtained in Section 5 in those parts in which it is asserted that the functions from $R_w^2(S)$ are pseudocontinuable and belong to $PCNM$, (if the set S satisfies the Blaschke condition). However, a full description of the class of functions which admit an approximation of the considered type was obtained by G.Ts. Tumarkin only for the case of the weight $w \equiv 1$. The problem of describing this class in the case of a general weight function with convergent logarithmic integral was posed as an open problem in [45]. A full solution of Tumarkin's problem can be given in an analogous way as we have proved the Fundamental Approximation Lemma.

Acknowledgement. This paper was written during the authors's stay at Leipzig University in 1991 which was sponsored by the Deutsche Forschungsgemeinschaft. The author thanks the Leipzig Schur Analysis Group for its warm hospitality. In particular, the author expresses his gratitude to Bernd Kirstein who carefully read an earlier draft of this manuscript, gave some useful comments which improved its presentation and even translated it into English.

Moreover, the author thanks the administration of Kharkov State University for giving him the possibility to work a whole year in Leipzig.

References

- [1] AROV, D.Z.: *On Darlington's method for the investigation of dissipative systems*. Soviet Phys. Dokl. 16 (1971/72), 954 - 956.
- [2] AROV, D.Z.: *Darlington realization of matrix-valued functions* (in Russian). Izvestija Akad. Nauk SSSR, Ser. Mat. 37 (1973), 1299 - 1331. Engl. transl. in: Math. USSR Izvestija 7 (1973), 1295 - 1326.
- [3] AROV, D.Z.: *Approximating characteristic of functions of class BII* (in Russian). Funktsional'nyi analiz i ego prilozheniya, 12 (1978) 2, 70 - 71. Engl. transl. in: Funct. Anal. Appl. 12 (1978), 133 - 134.
- [4] AROV, D.Z.: *Linear stationary passive systems with loss* (in Russian). Thesis. Odessa 1985.
- [5] BEURLING, A.: *Un théorème sur les fonctions bornées et uniformément continues sur l'axe réel*. Acta Math. 77 (1945), 127 - 136.
- [6] BEURLING, A.: *On the spectral synthesis of bounded functions*. Acta Math. 81 (1949), 225 - 239.
- [7] BRANGES, L. DE: *Hilbert spaces of entire functions*. Englewood Cliffs, N.-J.: Prentice Hall 1968.
- [8] BRANGES, L. DE, and J. ROVNYAK: *Canonical models in quantum scattering theory*. In: Perturbation Theory and its Applications in Quantum Mechanics (ed.: C. Wilcox). New York: Wiley 1966, pp. 295 - 392.
- [9] DOUGLAS, R.G., and J.W. HELTON: *Inner dilations of analytical matrix functions and Darlington synthesis*. Acta Sci. Math. (Szeged) 34 (1973), 61 - 67.
- [10] DOUGLAS, R.G., SHAPIRO, H.S., and A.L. SHIELDS: *Cyclic vectors and invariant subspaces for the backward shift operator*. Ann. Inst. Fourier (Grenoble) 20 (1970), 37 - 76.
- [11] DUREN, P.L.: *Theory of H^p Spaces*. New York: Academic Press 1970.
- [12] DYM, H.: *J Contractive Matrix Functions, Reproducing Kernel Hilbert Spaces and Interpolation* (CBMS Lecture Notes: Vol. 71). Providence, R.I.: Amer. Math. Soc. 1989.
- [13] FROSTMAN, O.: *Potential d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions*. Meddel. Lunds Univ. Mat. Sem. 3 (1935), 1 - 118.

- [14] GARNETT, J.B.: *Bounded Analytic Functions*. New York: Academic Press 1981.
- [15] GODEMENT, R.: *Théorèmes taubériens et théorie spectrale*. Ann. Sci. École Norm. Super. 64 (1947), 119 - 138.
- [16] HAMBURGER, H.L.: *Über die Zerlegung des Hilbertschen Raumes durch vollstetige lineare Transformationen*. Math. Nachr. 4 (1951), 56 - 69.
- [17] HEWITT, E., and K. ROSS: *Abstract Harmonic Analysis*, Vol II. Berlin-Heidelberg-New York: Springer-Verlag 1970.
- [18] HOFFMAN, K.: *Banach Spaces of Analytic Functions*. Englewood Cliffs, N.J.: Prentice-Hall 1962.
- [19] KATSNELSON, V.E.: *A left Blaschke-Potapov product is not necessarily a right Blaschke-Potapov product* (in Russian). Dokl. Akad. Nauk Ukrainian SSR (Series A) 10 (1989), 15 - 17.
- [20] KATSNELSON, V.E.: *Left and right Blaschke-Potapov products and Arov-singular matrix-valued functions*. Int. Equ. Oper. Theory 13 (1990), 836 - 848.
- [21] KATSNELSON, V.E.: *Left and right Blaschke-Potapov products and Arov-singular J -inner functions*. To appear.
- [22] KHAVIN, V.P.: *Methods and Structure of Commutative Harmonic Analysis*. In: Commutative Harmonic Analysis I. Encyclopedia of Mathematical Sciences: Vol. 15 (eds.: Khavin and N.K. Nikolskii). Berlin-Heidelberg-New York-Tokyo: Springer-Verlag 1991, pp. 1 - 111.
- [23] KOLMOGOROV, A.N.: *Stationary sequences in Hilbert space* (in Russian). Bull. Moskov. Univ. 2 (1941), 1 - 40. Reprinted in: A.N. Kolmogorov: Selected Papers - Probability and Statistics (in Russian). Moscow: Nauka 1986, pp. 215 - 255.
- [24] KOLMOGOROV, A.N.: *Interpolation and extrapolation of stationary random sequences* (in Russian). Izvestija Akad. Nauk USSR, Ser. Mat. 5 (1941), 3 - 14. Reprinted in: A.N. Kolmogorov: Selected Papers - Probability and Statistics (in Russian). Moscow: Nauka 1986, pp. 255 - 263.
- [25] KOOSIS, P.: *Introduction to H_p Spaces*. Cambridge: Cambridge University Press 1980.
- [26] KOVALISHINA, I.V.: *Analytic theory of a class of interpolation problems* (in Russian). Izvestija Akad. Nauk SSSR, Ser. Mat. 47 (1983), 455 - 497. Engl. transl. in: Math. USSR Izvestija 22 (1984), 419 - 463.
- [27] KOVALISHINA, I.V., and V.P. POTAPOV: *An indefinite metric in the Nevanlinna-Pick problem* (in Russian). Dokl. Akad. Nauk Armyanskoi SSR. 59 (1974), 17 - 22. Engl. transl. in: Amer. Math. Soc. Transl. (Ser. 2) 138 (1988), 15 - 19.
- [28] MARKUS, A.S.: *The problem of spectral synthesis for operators with point spectrum* (in Russian). Izvestija Akad. Nauk SSSR, Ser. Mat. 34 (1970), 662 - 688. Engl. transl. in: Math. USSR Izvestija 4 (1970), 669 - 696.
- [29] NEVANLINNA, F., and R. NEVANLINNA: *Über die Eigenschaften analytischer Funktionen in der Umgebung einer singulären Stelle oder Linie*. Acta Soc. Sci. Fenn. 50 (1922), Nr.5 (46 Seiten).
- [30] NEVANLINNA, R.: *Über eine Klasse meromorpher Funktionen*. Math. Ann. 92 (1924), 145 - 154.
- [31] NEVANLINNA, R.: *Über beschränkte analytische Funktionen*. Ann. Acad. Sci. Fenn. A 32 (1929), 1 - 75. Reprinted in: Ausgewählte Arbeiten zu den Ursprüngen der Schur-Analysis (Teubner-Archiv zur Mathematik: Vol. 16; eds.: B. Fritzsche and B. Kirstein). Stuttgart-Leipzig: B.G. Teubner 1991, pp. 97-171.
- [32] NIKOLSKII, N.K.: *Complete extensions of Volterra operators* (in Russian). Izvestija Akad. Nauk SSSR, Ser. Mat. 33 (1969), 1349 - 1355. Engl. transl. : Math. USSR Izvestija 3 (1969), 1271 - 1276.
- [33] NIKOLSKII, N.K., and V.P. KHAVIN: *The results of V.I. Smirnov in complex analysis and their further development* (in Russian). In: V.I. Smirnov: Selected Papers - Complex Analysis. Mathematical Theory of Diffraction (in Russian). Leningrad: Leningrad University Press 1988, pp. 111 - 145.

- [34] POTAPOV, V.P.: *The multiplicative structure of J -contractive matrix functions* (in Russian). Trudy Moskov. Mat. Obšč. 4 (1955), 125 - 236. Engl. transl. in: Amer. Math. Soc. Transl. (Ser. 2) 15 (1960), 131 - 243.
- [35] ROSENBLUM, M., and J. ROVNYAK: *The factorization problem for nonnegative operator-valued functions*. Bull. Amer. Math. Soc. 77 (1971), 287 - 318.
- [36] RUDIN, W.: *Function Theory in Polydiscs*. New York: Benjamin 1969.
- [37] SCHUR, I.: *Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind I*. J. reine angew. Math. 147 (1917), 205 - 232. Reprinted in: *Ausgewählte Arbeiten zu den Ursprüngen der Schur-Analyse* (Teubner-Archiv zur Mathematik: Vol. 16; eds.: B. Fritzsche and B. Kirstein). Stuttgart-Leipzig: B.G. Teubner 1991, pp. 22-49.
- [38] SCHWARTZ, L.: *Théorie générale des fonctions moyenne-périodiques*. Ann. Math. 48 (1947), 857 - 929.
- [39] SHIRYAEV, A.N.: *Probability*. Berlin-Heidelberg-New York: Springer-Verlag 1984.
- [40] SMIRNOV, V.I.: *Sur les formules de Cauchy et de Green et quelques problèmes qui s'y rattachent*. Izv. Akad. Nauk SSSR, Ser. Mat. 3 (1932), 338 - 372. Russian transl. in: V.I. Smirnov: *Selected Papers - Complex Analysis*. Mathematical Theory of Diffraction. Leningrad: Leningrad University Press 1988, pp. 82 - 111.
- [41] SMIRNOV, V.I.: *Selected Papers - Complex Analysis. Mathematical Theory of Diffraction* (in Russian). Leningrad: Leningrad University Press 1988.
- [42] SZEGÖ, G.: *Über die Randwerte einer analytischen Funktion*. Math. Ann. 84 (1921), 232 - 244.
- [43] TUMARKIN, G.Ts.: *Boundary properties of sequences of analytic functions* (in Russian). Thesis. Leningrad 1961.
- [44] TUMARKIN, G.Ts.: *Convergent sequences of Blaschke products* (in Russian). Siberian Math. J. 5 (1964), 201 - 233.
- [45] TUMARKIN, G.Ts.: *Description of a class of functions which admit an approximation by rational functions with prescribed poles* (in Russian). Izv. Akad. Nauk Armjanskoi SSR, Ser. Mat. 1 (1966), 89 - 105.
- [46] WERMER, J.: *On invariant subspaces of normal operators*. Proc. Amer. Math. Soc. 3 (1952), 270 - 277.

Received 16.3.1992, in revised version 6.11.1992