

Axially Symmetric Flow with Finite Cavities I

F. BROCK

The existence of an axially symmetric flow of an incompressible inviscid fluid moving around an obstacle is proved. The flow is either in a cylindrical pipe or an unbounded region and the cavity may be finite. The proof is based on a variational approach for a non-continuous functional. Essentially is the assumption that the obstacle is starlike with respect to some point on the axis of symmetry. This allows to apply the technique of symmetrization on the stream function.

Key words: *Non-continuous functionals, symmetrizations, axially symmetric flows*

AMS subject classification: 35J85

1. Introduction

A three-dimensional axially symmetric flow of an incompressible inviscid fluid is moving around an obstacle. In its shadow occurs a cavity bounded by a free surface, on which the modulus of the velocity is constant. We consider two cases : the flow is either in an unbounded region or in a cylindrical pipe. Many examples of axially symmetric free surface flows have been studied in the literature. The existence proofs are mostly based on a minimum principle for a functional depending on the stream function of the flow. The considered functional is non-continuous but lower semicontinuous on a convex set which is sufficient for the existence of a minimum. The theory of such functionals was developed by Alt and Caffarelli in [1] and then applied on some free boundary problems, see [2,3,5,6]. We also follow this way in our paper. The existence of an axially symmetric infinite cavity is proved in [5]. The authors investigate the flow around an obstacle which is an y -graph, i.e., any straight line which is parallel to the axis of symmetry $\{y = 0\}$ can intersect it in at most one point or along a segment . The free streamline then has the same property. Our geometrical assumption is :

The obstacle is *starlike* with respect to some point on the axis of symmetry.

Then we can ensure the existence of flows of required type the free streamlines of which are also starlike. It appears that there are obstacles for which the cavity is finite, i.e., the free streamlines end on the axis of symmetry. We note that two-dimensional analogues of our flows are given by Serrin [9] (uniform flow in an unbounded region) and Hilbig [8] (flow in a monotonously narrowing channel) . In these papers the existence proof works with the hodograph mapping using the fact that the stream function is harmonic and with the Leray-Schauder fixed point theorem.

F. Brock: Universität Leipzig, Inst. Math., Augustuspl. 10, D(Ost) - 7027 Leipzig

ISSN 0232 - 2064 © 1993 Heldermann Verlag Berlin

2. Definition of two axisymmetric cavity problems

We denote by $X = (x, y)$ points in \mathbb{R}^2 and by $B_R(X)$ the interior of the ball with centre X and radius R , and we set $B_R = B_R(O)$, where O is the origin $(0, 0)$. We infer a continuous curve N (the obstacle), satisfying the following conditions $(0 < a, b < +\infty, 0 < \alpha < 1)$:

$$\begin{aligned} N : X = X_0(t) = (x_0(t), y_0(t)) \text{ for } 0 \leq t \leq a ; \\ X_0(0) = (-b, 0) ; y_0(t) > 0 \text{ for } 0 < t \leq a ; \\ X_0 \text{ is piecewise of class } C^{1+\alpha} ; \nabla X_0(t \pm 0) \neq 0 \text{ for } 0 \leq t \leq a . \end{aligned} \tag{1}$$

$$\begin{aligned} N \text{ is starlike with respect to the origin , i.e., a straight line} \\ \text{through the origin intersects the curve } N \text{ in at most one point} \\ \text{or touches her along a segment .} \end{aligned} \tag{2}$$

We set $A = X_0(a) = (x_A, y_A)$, $S = (\{x < -b\} \cap \{y = 0\}) \cup N \cup \overline{AO} \cup (\{x > 0\} \cap \{y = 0\})$ and $H = \{y = h\}$, where $h > \max\{y_0(t) : 0 \leq t \leq a\}$. By Ω we denote the domain between the curves S and H . The straight line through O and A intersects H in a point A' . D denotes the part of Ω lying to the left of $\overline{OA'}$ (see Figure 1).

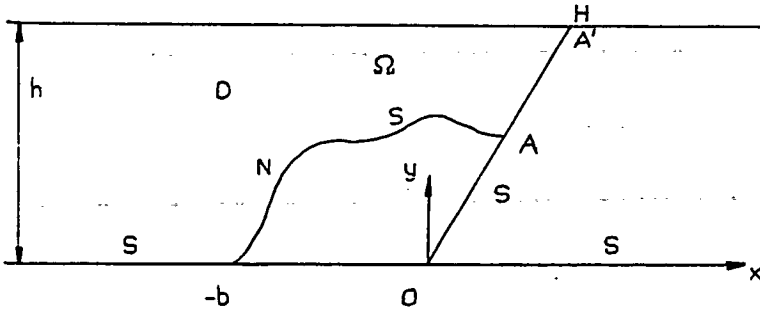


Figure 1

Problem 2.1 (Axially symmetric cavity problem in a pipe) : Find a function u , a number $\lambda > 0$ and a curve Γ , such that the following conditions are satisfied $(0 \leq d \leq +\infty)$:

$$\begin{aligned} (B1) \quad \Gamma : X = X_1(t) = (x_1(t), y_1(t)) \text{ for } 0 \leq t \leq d , X_1(0) = X_0(a) , \\ X_1 \text{ is of class } C^1 , \nabla X_1(t \pm 0) \neq 0 \text{ for } 0 \leq t \leq d ; \\ u \in C^{0,1}(\Omega) \text{ and } u = 0 \text{ on } S ; \\ u = 1 \text{ on } H \text{ and } 0 \leq u \leq 1 \text{ in } \Omega ; \end{aligned} \tag{3}$$

$$\begin{aligned} u \in C^2(G) \cap C(\overline{G}) , \text{ where } G = \{u > 0\} \text{ and } \partial\{u > 0\} \cap \Omega = \Gamma , \\ Lu := u_{xx} + u_{yy} - \frac{u_y}{y} = 0 \text{ in } G , \end{aligned} \tag{4}$$

$$\frac{1}{y} \frac{\partial u}{\partial \nu} = \lambda \text{ on } \Gamma , \nu - \text{ inner normal .} \tag{5}$$

In the triple (u, λ, Γ) we call u the stream function, λ the cavity speed and Γ the free stream line. In view of (4) the vector $(\frac{u_x}{v}, -\frac{u_x}{v})$ gives the velocity of an axially symmetric inviscid incompressible flow where $\{y = 0\}$ is the axis of symmetry.

Problem 2.2 (Axially symmetric cavity problem) : Find a function u , a number $\lambda > 0$ and a curve Γ , such that the conditions (B1) of Problem 2.1 except (3) are satisfied in a domain Ω_∞ (instead of Ω), lying above S (see Figure 2). Here D_∞ denotes the domain to the left of the straight line \overline{OA} . Instead of the condition (3) we claim

$$0 \leq u(x, y) \leq \frac{y^2}{2} \text{ in } G. \tag{6}$$

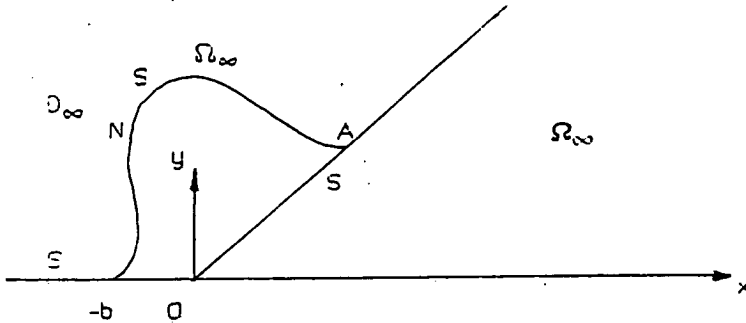


Figure 2

First we discuss some details in our problems.

Because of the homogeneity of the operator L we could demand instead of the conditions (3) and (6), that $u = Q$ on H and

$$0 \leq u(x, y) \leq Q \frac{y^2}{2} \text{ in } G$$

with a prescribed number $Q > 0$, respectively. If then (u, λ, Γ) is a solution of the earlier problem we get a solution $(\bar{u}, \bar{\lambda}, \bar{\Gamma})$ for the modified problem by setting $\bar{\Gamma} = \Gamma$, $\bar{\lambda} = Q\lambda$ and $\bar{u} = Qu$. We can heighten without further ado the demand of the smoothness of Γ because the free streamline is always analytic (see [5]).

The main result of this paper is that the above stated problems have solutions. More precisely we prove the following two theorems.

Theorem 2.3: *The axially symmetric cavity problem in a pipe has a solution (u, λ, Γ) with the following properties :*

$$(B2) \quad \partial\{u > 0\} \cap B_\epsilon(A) \text{ is of class } C^1 \text{ for small } \epsilon > 0; \tag{7}$$

$\partial\{u > 0\} \setminus H$ is starlike with respect to the origin ;

Γ is local analytic and has a representation

$$\Gamma : r = R(\phi) \text{ for } \phi \in (0, \phi_A),$$

$$\text{where } R \in C^\infty(0, \phi_A) \text{ and } \lim_{\phi \rightarrow 0} R(\phi) = d \in (0, +\infty),$$

$$\phi_A = \arctan \frac{y_A}{x_A}, \quad (r, \phi) - \text{polar coordinates to } (x, y).$$

In the case $d = +\infty$ (infinite cavity) Γ has for great $x > 0$ a representation $y = f(x)$ with $\lim_{x \rightarrow +\infty} f(x) = \sqrt{h^2 - \frac{\lambda}{k^2}}$ and $\lambda \geq \frac{\lambda}{k^2}$. In the case $d < +\infty$ (finite cavity) we have $\lambda < \frac{\lambda}{k^2}$.

Theorem 2.4: The axially symmetric cavity problem has a solution (u, λ, Γ) , which suffices the properties (B2) and $\lambda = 1$ in the case $d = +\infty$, and $\lambda < 1$ in the case $d < +\infty$.

Remark 2.5: In a further paper (see [4]) we study the behavior of the free boundaries near the axis of symmetry $\{y = 0\}$ and at infinity. There are the following main results:

1) In the case of a finite cavity we have smooth fit at the endpoint of Γ on $\{y = 0\}$, i.e., $\lim_{\phi \rightarrow 0} R'(\phi) = +\infty$. The proof is lengthy because the uniform ellipticity of the operator L in (4) is violated on the axis $\{y = 0\}$.

2) In our solutions the tangents on the free streamlines are asymptotically horizontal (i.e., parallel to the x -axis) near infinity.

We now outline the contents of the paper. In Section 3 we introduce a variational functional with one real parameter λ . The theory of such functionals was developed by Alt, Caffarelli and Friedman (see [1-3,5]). Here we use a version, in which the integral is extended over the unbounded domain Ω . This makes it more suitable for the numerical aspect and simplifies some proofs. To secure the convergence of the integral it requires additional terms in the integrand. However these terms do not influence on the variations of the functional. As in the above mentioned papers the functionals have absolute minima u in a suitable function class. In the open domain $\{u > 0\}$ the equation (4) is valid and on the local analytic free boundary $\partial\{u > 0\}$ the condition (5) is satisfied. In Section 4 we prove that a suitable "starlike" increasing rearrangement of any function decreases our functionals. This shows that the free boundary is starlike. The Lipschitz continuity of absolute minima is shown in Section 5. In Section 6 we prove that $\partial\{u > 0\}$ cannot oscillate touching the fixed boundary. In Section 7 we show that for a certain value λ^* of the parameter λ the absolute minimum of the functional gives the solution of the cavity problem in the pipe. Here we use the uniqueness of absolute minima and their monotonicity in λ . The second property means that for two parameters $\lambda_1 < \lambda_2$ the corresponding minima suffice the inequality $u_1 > u_2$. To get a solution of the Problem 2.2 we choose a convergent sequence of solutions of the cavity problem in the pipe where the corresponding sequence of diameters of the pipe tends to infinity. This finishes the proof of the Theorems 2.3 and 2.4. Finally we show that for certain obstacles (for instance if the detaching point A lies near the origin) the cavities in the solutions are finite.

3. A variational problem with a parameter

We take $K = \{v \in H_{loc}^{1,2}(\Omega) \mid v = 0 \text{ on } S, v = 1 \text{ on } H, 0 \leq v \leq 1 \text{ in } \Omega\}$, $e = (0,1)$ and $\lambda > 0$. For $v \in K$ we define a functional J_λ by setting

$$\begin{aligned}
 J_\lambda(v) = & \iint_D y \left| \frac{\nabla v}{y} - \frac{2e}{h^2} \right|^2 dx dy \tag{8} \\
 & + \begin{cases} \iint_{\Omega \setminus D} y \left| \frac{\nabla v}{y} - \lambda I(\{v > 0\})e \right|^2 dx dy & \text{if } \lambda \geq \frac{2}{h^2} \\ \iint_{\Omega \setminus D} y \left[\left| \frac{\nabla v}{y} - \frac{2}{h^2} I(\{v > 0\})e \right|^2 \right. \\ \left. + \left(\frac{4}{h^4} - \lambda^2 \right) I(\Omega \setminus \{v > 0\}) \right] dx dy & \text{if } \lambda < \frac{2}{h^2} \end{cases}
 \end{aligned}$$

in the case that the integrals in (8) converge and otherwise $J_\lambda(v) = +\infty$. Here I denotes the characteristic function. We look for absolute minima u_λ of J_λ .

Theorem 3.1: *There is a function $u_\lambda \in K$ such that $J_\lambda(u_\lambda) = \min_{v \in K} J_\lambda(v)$.*

Proof: We set

$$A_0 = \begin{cases} \sqrt{1 - \frac{2}{h^2 \lambda}} (x_A, y_A) & \text{if } \lambda \geq \frac{2}{h^2} \\ (0, 0) & \text{if } \lambda < \frac{2}{h^2} \end{cases}$$

and

$$g_0 = \begin{cases} \{y = y_A \sqrt{1 - \frac{2}{h^2 \lambda}}\} \cap \{x > x_A \sqrt{1 - \frac{2}{h^2 \lambda}}\} & \text{if } \lambda \geq \frac{2}{h^2} \\ \{y = 0\} \cap \{x > 0\} & \text{if } \lambda < \frac{2}{h^2}. \end{cases}$$

Then let G_0 be the domain between the curves H and $(\{y = 0\} \cap \{x < -b\}) \cup N \cup \overline{AA_0} \cup g_0$ and v_0 the solution of the boundary value problem

$$\begin{aligned}
 v_0 & \in C^2(G_0) \cap C(\overline{G_0}), \quad 0 < v_0 < 1 \text{ in } G_0; \\
 v_0 & = 0 \text{ on } \partial G_0 \setminus H, \quad v_0 = 1 \text{ on } H; \\
 Lv_0 & = \Delta v_0 - \frac{(v_0)_y}{y} = 0 \text{ in } G_0, \quad v_0 = 0 \text{ in } \Omega \setminus G_0.
 \end{aligned}$$

Obviously v_0 is in K and we have $J_\lambda(v_0) < +\infty$. The existence of u_λ then follows as in [1] ■

The next definition will be useful for us in the following.

Definition 3.2: We call $u \in K$ a local minimum to J_λ , if there exists a number $\epsilon > 0$, such that for any $v \in K$ with

$$\iint_\Omega \left\{ \frac{1}{y} |\nabla(v - u)|^2 + y |I(\{v > 0\} \setminus D) - I(\{u > 0\} \setminus D)| \right\} dx dy < \epsilon \tag{9}$$

there follows $J_\lambda(u) \leq J_\lambda(v)$.

Local minima already have some of the properties which we require to the solution of the cavity problem in the pipe.

Theorem 3.3: Let u be a local minimum to J_λ . Then it follows

$$\begin{aligned}
 (B3) \quad & Lu \geq 0 \text{ in } \Omega \text{ in the sense of distributions, } D \subset \{u > 0\}, \\
 & \Gamma := \partial\{u > 0\} \cap \Omega \text{ is local analytic, } Lu = 0 \text{ in } \{u > 0\}, \\
 & \frac{1}{y} \frac{\partial u}{\partial \nu} = \lambda \text{ on } \Gamma, \text{ where } \nu \text{ is the inner normal to } \{u > 0\}, \\
 & \partial\{u > 0\} \cap \{y > \epsilon\} \text{ is of class } C^1 \text{ for any } \epsilon > 0, \\
 & \lim_{x \rightarrow -\infty} \nabla u(x, y) = (0, \frac{2}{h^2}) \text{ for all } y \in (0, h). \quad (10)
 \end{aligned}$$

Proof: We take $R > 0$, set $K_R = \{v \in K \mid v(X) = u(X) \text{ for } |X| > R\}$ and, for a $v \in K_R$,

$$J_{\lambda, R}(v) = \iint_{\Omega \cap B_R} \left(\frac{|\nabla v|^2}{y} + yI(\{v > 0\} \setminus D) \right) dx dy. \quad (11)$$

We note that the integrand in the functional $J_{\lambda, R}$ is the same as in [5:p.98]. Then we can define a local minimum to $J_{\lambda, R}$ as in Definition 3.2 with the set K replaced by K_R , the functional J_λ by $J_{\lambda, R}$ and in formula (9) the domain of integration restricted on $\Omega \cap B_R$. Since the difference $J_{\lambda, R}(v) - J_\lambda(v)$ does not depend on v , we can conclude that u is also a local minimum to $J_{\lambda, R}$. Now we can prove as in [1-3], that the conclusions (B3) except (10) hold in the region $\Omega \cap B_R$, and, since $R > 0$ was arbitrary, in the whole domain Ω . Finally the property (10) follows from the convergence of the integrals in J_λ ■

4. Starlike rearrangement

Let (r, ϕ) be polar coordinates to (x, y) and $\phi_A = \arctan \frac{y_A}{x_A}$. In view of the assumption (1) $0 < \phi_A < \pi$. We set $z = \frac{r^2}{3}$. Then let $G = \{(z, \phi) \mid 0 < \phi < \pi, z_0(\phi) < z < z_1(\phi)\}$ be the domain which corresponds with Ω in the (z, ϕ) -plane. The functions $z_0 = z_0(\phi)$, $\phi_A < \phi < \pi$, and $z_1 = z_1(\phi)$, $0 < \phi < \pi$, are representations of N and H , respectively, and we have $z_0(\phi) = 0$ for $0 < \phi < \phi_A$. We define for a function $u \in K$:

$$U(z, \phi) = \begin{cases} \frac{u(x(z, \phi), y(z, \phi))}{(3z)^{-2/3}} & \text{if } (z, \phi) \in G \\ \frac{u(x(z_1(\phi), \phi), y(z_1(\phi), \phi))}{(3z_1(\phi))^{-2/3}} & \text{if } z \geq z_1(\phi) \\ 0 & \text{if } 0 \leq z \leq z_0(\phi). \end{cases}$$

Then we denote with U^* the monotonous increasing rearrangement (or Steiner symmetrization) of U in the variable z (see [6: pp.293-296]). Finally we call the function u^* defined by $u^*(x, y) = (x^2 + y^2)U^*(z(x, y), \phi(x, y))$ a starlike rearrangement of u . It follows that $u^* \in K$ and $\frac{\partial}{\partial x} u^*(x, \phi), y(z, \phi) \geq 0$ in G . We can prove that the above defined rearrangement decreases the functional $J_\lambda(u^*) \leq J_\lambda(u)$, where the equality is valid only in the case $u = u^*$ a.e. Then there follows that absolute minima are already starlike and we get the following

Theorem 4.1: Let u_λ be an absolute minimum to J_λ . Then the free boundary $\partial\{u_{\lambda>0}\} \cap \Omega$ is starlike with respect to the origin. Moreover it has a representation

$$r = R(\phi) \text{ for } \phi \in (0, \phi_A), \text{ where } R \in C^\infty(0, \phi_A). \tag{12}$$

Proof: The assertion follows from the fact that the free boundary cannot contain straight line pieces because of its analyticity ■

Now we prove the inequality $J_\lambda(u^*) \leq J_\lambda(u)$. First let us assume that u is a C^1 -function. Then as in [7: p.567ff.] it follows, that to each $c \in (0, 1)$ and $\phi \in (0, \pi)$ there exist numbers z_n , $1 \leq n \leq 2k + 1$, $k \in \mathbb{N}$, such that $0 \leq z_0(\phi) \leq z_1 \leq z_2 \leq \dots \leq z_{2k+1} \leq z_1(\phi)$ and

$$U(z, \phi) \begin{cases} \geq c & \text{if } z_{2n+1} \leq z \leq z_{2n+3}, \quad 0 \leq n \leq k-1, \\ \leq c & \text{if } z_{2n} \leq z \leq z_{2n+1}, \quad 1 \leq n \leq k, \end{cases}$$

$$U^*(Z, \phi) = c \text{ with } Z = z_1 - z_2 + z_3 - \dots + z_{2k+1}$$

(see [6]). To the equations $U(z_n, \phi) = c$ and $U^*(z_n, \phi) = c$, $1 \leq n \leq 2k + 1$, we can define inverse functions $z_n = z_n(U, \phi)$, $1 \leq n \leq 2k + 1$, and $Z = Z(U^*, \phi)$. Then

$$(-1)^{n-1} \frac{\partial z_n}{\partial U} \geq 0 \text{ for } 1 \leq n \leq 2k + 1, \tag{13}$$

and

$$\frac{\partial Z}{\partial U^*} \geq 0. \tag{14}$$

Expressing $J_\lambda(u)$ in terms of the function U we get now

$$J_\lambda(u) = \int \int_{Gr\{\phi>\phi_A\}} \left\{ (9z^2U_z^2 + 12zU_zU + 4U^2 + U_\phi^2) \frac{1}{\sin \phi} - \frac{4}{h^2} (2U \sin \phi + U_\phi \cos \phi + 3zU_z \sin \phi) + \frac{4 \sin \phi}{h^4} \right\} dzd\phi$$

$$+ \begin{cases} \int \int_{Gr\{\phi<\phi_A\}} \left\{ (9z^2U_z^2 + 12zU_zU + 4U^2 + U_\phi^2) \frac{1}{\sin \phi} - 2\lambda(2U \sin \phi + U_\phi \cos \phi + 3zU_z \sin \phi) + \lambda^2 \sin \phi I(\{U > 0\}) \right\} dzd\phi & \text{if } \lambda \geq \frac{2}{h^2} \\ \int \int_{Gr\{\phi<\phi_A\}} \left\{ (9z^2U_z^2 + 12zU_zU + 4U^2 + U_\phi^2) \frac{1}{\sin \phi} - \frac{4}{h^2} (2U \sin \phi + U_\phi \cos \phi + 3zU_z \sin \phi) + \sin \phi \left(\frac{4}{h^4} - \lambda^2 + \lambda^2 I(\{U > 0\}) \right) \right\} dzd\phi & \text{if } \lambda < \frac{2}{h^2}. \end{cases}$$

Naturally we can replace in this representation the functions u and U by u^* and U^* , respectively.

After a change of the variable z with u and U^* , respectively, there follows

$$J_\lambda(u) = \int_{\phi_A}^{\pi} \int_0^1 \sum \left\{ \frac{1}{\sin \phi} \left(\frac{9z_n^2}{|(z_n)U|} + \frac{(z_n)_\phi^2}{|(z_n)U|} \right) + (-1)^{n-1} \left(\frac{1}{\sin \phi} (12U z_n + 4U^2(z_n)U) \right) \right.$$

$$\begin{aligned}
 & -\frac{4}{h^2} \left[\sin \phi (2U(z_n)U + 3z_n) + \cos \phi (z_n)\phi \right] + \frac{4}{h^2} \sin \phi (z_n)U \Big\} dU d\phi \\
 & + \left\{ \int_0^{\phi_A} \int_0^1 \sum \left\{ \frac{1}{\sin \phi} \left(\frac{9z_n^2}{|(z_n)U|} + \frac{(z_n)\phi^2}{|(z_n)U|} \right) \right. \right. \\
 & \quad + (-1)^{n-1} \left(\frac{1}{\sin \phi} (12U z_n + 4U^2(z_n)U) \right. \\
 & \quad \left. \left. - 2\lambda \left[\sin \phi (2U(z_n)U + 3z_n) + \cos \phi (z_n)\phi \right] \right. \right. \\
 & \quad \left. \left. + \lambda^2 \sin \phi (z_n)U \right\} dU d\phi \right. \quad \text{if } \lambda \geq \frac{2}{h^2} \\
 & + \left\{ \int_0^{\phi_A} \int_0^1 \left[\sum \left\{ \frac{1}{\sin \phi} \left(\frac{9z_n^2}{|(z_n)U|} + \frac{(z_n)\phi^2}{|(z_n)U|} \right) \right. \right. \right. \\
 & \quad + (-1)^{n-1} \left(\frac{1}{\sin \phi} (12U z_n + 4U^2(z_n)U) \right. \\
 & \quad \left. \left. - \frac{4}{h^2} \left[\sin \phi (2U(z_n)U + 3z_n) + \cos \phi (z_n)\phi \right] + \lambda^2 \sin \phi (z_n)U \right) \right] \right. \\
 & \quad \left. + (z_1(\phi) - z_0(\phi)) \sin \phi \left(\frac{4}{h^4} - \lambda^2 \right) \right] dU d\phi \quad \text{if } \lambda < \frac{2}{h^2} ,
 \end{aligned}$$

and

$$\begin{aligned}
 \bar{j}_\lambda(u^*) &= \int_{\phi_A}^\pi \int_0^1 \left\{ \frac{1}{\sin \phi} \frac{9(\sum (-1)^{n-1} z_n)^2}{\sum |(z_n)U|} + \frac{(\sum (-1)^{n-1} (z_n)\phi)^2}{\sum |(z_n)U|} \right. \\
 & \quad + \sum (-1)^{n-1} \left(\frac{1}{\sin \phi} (12U z_n + 4U^2(z_n)U) \right. \quad (15) \\
 & \quad \left. \left. - \frac{4}{h^2} \left[\sin \phi (2U(z_n)U + 3z_n) + \cos \phi (z_n)\phi \right] + \frac{4}{h^4} \sin \phi (z_n)U \right\} dU d\phi \right. \\
 & + \left\{ \int_0^{\phi_A} \int_0^1 \left\{ \frac{1}{\sin \phi} \frac{9(\sum (-1)^{n-1} z_n)^2}{\sum |(z_n)U|} + \frac{(\sum (-1)^{n-1} (z_n)\phi)^2}{\sum |(z_n)U|} \right. \right. \\
 & \quad + \sum (-1)^{n-1} \left(\frac{1}{\sin \phi} (12U z_n + 4U^2(z_n)U) \right. \\
 & \quad \left. \left. - 2\lambda \left[\sin \phi (2U(z_n)U + 3z_n) + \cos \phi (z_n)\phi \right] \right. \right. \\
 & \quad \left. \left. + \lambda^2 \sin \phi (z_n)U \right\} dU d\phi \right. \quad \text{if } \lambda \geq \frac{2}{h^2} \\
 & + \left\{ \int_0^{\phi_A} \int_0^1 \left\{ \frac{1}{\sin \phi} \frac{9(\sum (-1)^{n-1} z_n)^2}{\sum |(z_n)U|} + \frac{(\sum (-1)^{n-1} (z_n)\phi)^2}{\sum |(z_n)U|} \right. \right. \\
 & \quad + \sum (-1)^{n-1} \left(\frac{1}{\sin \phi} (12U z_n + 4U^2(z_n)U) \right. \\
 & \quad \left. \left. - \frac{4}{h^2} \left[\sin \phi (2U(z_n)U + 3z_n) + \cos \phi (z_n)\phi \right] + \lambda^2 \sin \phi (z_n)U \right) \right. \\
 & \quad \left. + \left(\frac{4}{h^4} - \lambda^2 \right) (z_1(\phi) - z_0(\phi)) \sin \phi \right\} dU d\phi \quad \text{if } \lambda < \frac{2}{h^2} .
 \end{aligned}$$

(Here and in the following the summation is over n from 1 to $2k + 1$.) By (13) and the Schwarz inequality we obtain

$$\left(\sum (-1)^{n-1} (z_n)_\phi\right)^2 \leq \left(\sum \frac{(z_n)_\phi^2}{|(z_n)U|}\right) \left(\sum (-1)^{n-1} (z_n)U\right)$$

and

$$\left(\sum (-1)^{n-1} z_n\right)^2 \leq \left(\sum |(z_n)U|\right) \left(\sum \frac{z_n^2}{|(z_n)U|}\right)$$

and the inequality $J_\lambda(u^*) \leq J_\lambda(u)$ follows from the representations (15). Now let u be any function in K . Then the assertion of the theorem follows by approximation .

5. Lipschitz continuity of minima

In this section we show that the absolute minima to the functionals J_λ are Lipschitz continuous . Alt, Caffarelli and Friedman [3] derived an analogue result for an axially symmetric jet problem . We follow their proof. Differences only come from the fact that in our problems the free boundary can touch the axis $\{y = 0\}$. We write u for (absolute) minima to J_λ . With the next two lemmata we give estimates for u from below and from above near the free boundary $\partial\{u > 0\} \cap \Omega$. For proofs see [3 : Lemmata 2.2 and 2.4 , respectively] .

Lemma 5.1: *There is a constant $C > 0$, independent of λ , such that for any ball $B_r(X_0) \subset \Omega$ with $X_0 = (x_0, y_0)$ and $r \leq \frac{y_0}{2}$ the inequality $\frac{1}{2\pi r^2} \int_{\partial B_r(X_0)} u \geq C\lambda y_0$ implies $u > 0$ in $B_r(X_0)$.*

Lemma 5.2: *There is a constant $c > 0$, independent of λ , such that for any ball $B_r(X_0) \subset \Omega$ with $X_0 = (x_0, y_0) \in \Omega \setminus \bar{D}$ and $r \leq \frac{y_0}{2}$ the inequality $\frac{1}{2\pi r^2} \int_{\partial B_r(X_0)} u \leq c\lambda y_0$ implies $u = 0$ in $B_{r/8}(X_0) \cap (\Omega \setminus D)$. (Here we set $u = 0$ in $B_r(X_0) \setminus \bar{\Omega}$.)*

The next lemma is an inner estimate of $|\nabla u|$. It is a variant of [5 : Lemma 8.1].

Lemma 5.3: *Let X_0 be a free boundary point in Ω and let G be a compact subset of a ball $B_r(x_0) \subset \Omega \cap \{y \geq c_0\}$, $c_0 > 0$. Then $|\nabla u(X)| \leq C$ in G , where C is a constant depending only on r, c_0, λ and G .*

We now formulate the main result of this chapter .

Theorem 5.4: *Absolute minima are Lipschitz continuous in Ω and at all boundary points where $\partial\Omega$ is of class $C^{1+\alpha}$, $0 < \alpha < 1$.*

Proof: Because of Lemma 5.3 we get $u \in C^{0,1}(\Omega)$. The function u is also Lipschitz continuous at all points of the obstacle N , where N is of class $C^{1+\alpha}$ since we have $Lu = 0$ in D (see (B3)). Now we prove the Lipschitz continuity near the segment \overline{AO} . Let $X_0 = (x_0, y_0)$ be an inner point of \overline{AO} . Then we choose $r_0 = \frac{1}{2} \min\{y_0; \text{dist}(A, X_0)\} > 0$ and define in $B_{r_0}(X_0) \cap (\Omega \setminus \bar{D})$ a function v with $Lv = 0$ such that $v = 1$ on $\partial B_{r_0}(X_0) \cap (\Omega \setminus \bar{D})$ and $v = 0$ on $\overline{AO} \cap B_{r_0}(X_0)$. Since we have $v \geq u$ on $\partial(B_{r_0}(X_0) \cap (\Omega \setminus \bar{D}))$, and $Lu \geq 0$ in Ω we get $v \geq u$ in $B_{r_0}(X_0) \cap (\Omega \setminus \bar{D})$. By elliptic estimates we can show that there is a number $C > 0$

such that, for all $X' \in B_{\frac{r_0}{2}}(X_0) \cap (\Omega \setminus \overline{D})$, we have $v(X') \leq C |X_0 - X'|$. We set $r = \text{dist}(X, \overline{AO})$ for any $X \in B_{\frac{r_0}{2}}(X_0) \cap (\Omega \setminus \overline{D})$ and get in the case $B_r(X) \subset \{u > 0\}$

$$|\nabla u(X)| \leq \frac{c_1}{2\pi r^2} \int_{\partial B_r(X)} u \leq \frac{c_2}{2\pi r^2} \int_{\partial B_r(X)} v \leq \frac{c_3}{r_0}, \tag{16}$$

where the constants c_1 , c_2 and c_3 do not depend on X . The first inequality in (16) we derive if we represent the scaled function $w(\overline{X}) = u(X + r\overline{X})$, $|\overline{X}| < 1$, by the formula

$$w(\overline{X}) = \int_{|Y|=1} \frac{\partial G(\overline{X}, Y)}{\partial \nu} w(Y) ds_Y, \nu - \text{inner normal}, G - \text{Greens function},$$

and then apply ∇ at the point $\overline{X} = 0$. In the case $B_r(X) \not\subset \{u > 0\}$ there follows $\text{dist}(X, \overline{AO}) > \text{dist}(X, \partial\{u > 0\} \cap (\Omega \setminus \overline{D}))$, and we can argue as in the proof of [6 : Theorem 3.2, p.278].

Next we prove the Lipschitz continuity near $\{y = 0\}$. Let u be a minimum to J_λ . Since $Lu \geq 0$ in Ω we conclude

$$u(x, y) \leq \frac{y^2}{h^2} \text{ in } \Omega. \tag{17}$$

We set $X_0 = (x_0, 0)$, $x_0 > 0$, and choose $r_0 = \min\{h, \text{dist}(X_0, \overline{AO})\}$ and $X \in B_{\frac{r_0}{2}}(X_0) \cap \Omega$. First we consider the case $B_{\frac{r_0}{2}}(X) \subset \{u > 0\}$. Then we define $\overline{u}(\overline{X}) = \frac{1}{r} u(X + r\overline{X})$, where $r = \frac{r_0}{2}$, $\overline{X} = (\overline{x}, \overline{y})$, and get $\overline{u} \leq \overline{C}$, $\overline{u}_{\overline{x}\overline{x}} + \overline{u}_{\overline{y}\overline{y}} - \frac{1}{2\overline{y}} \overline{u}_{\overline{y}} = 0$ in B_1 , where the number \overline{C} does not depend on y . It follows $|\nabla u(X)| = r |\nabla \overline{u}(0)| \leq C'y$, with the constant C' independent of X . On the other hand if $B_{\frac{r_0}{2}}(X) \not\subset \{u > 0\}$ we can conclude by Lemma 5.1 that $\frac{4}{\pi r} \int_{\partial B_{\frac{r_0}{2}}(X)} u \leq \lambda C y$ with the constant C independent of y . Since $\nabla u = 0$ a.e. in $\{u > 0\}$, we can assume $u(X) > 0$. Let $B_{r_1}(X)$ be the greatest ball around X , which is contained in $\{u > 0\}$. Then in view of Lemma 5.1 we also have $\frac{1}{2\pi r_1^2} \int_{\partial B_{r_1}(X)} u \leq \lambda C y$. It follows

$$|\nabla u(X)| \leq \frac{c_1}{2\pi r_1^2} \int_{\partial B_{r_1}(X)} u \leq \lambda c_1 C y, \tag{18}$$

where c_1 is the constant from (16).

We still have to study the minima u near the straight line H . To this it suffices to prove the inequality

$$u \geq w \text{ in } \Omega, \text{ where } w(x, y) = \begin{cases} \frac{y^2 - k^2}{h^2 - k^2} & \text{if } y \geq k, \\ 0 & \text{if } 0 < y < k, \end{cases} \tag{19}$$

with the number $k = \max\{\sqrt{\max\{0; h^2 - \frac{2}{\lambda}\}}; \max\{y_0(t) \mid 0 \leq t \leq a\}\}$, after which the Lipschitz continuity near H follows by well-known elliptic estimates.

Now the idea is to prove $J_\lambda(u_0) \leq J_\lambda(u)$, where $u_0(x, y) = \max\{u(x, y); w(x, y)\}$. We derive by an easy calculation

$$J_\lambda(u) - J_\lambda(u_0) = \int_{\Omega} \left(\frac{1}{y} \nabla(u - u_0) \nabla(u + u_0) + \lambda^2 y [I(\{u > 0\}) - I(\{u_0 > 0\})] \right)$$

$$\begin{aligned}
 &= \int_{\Omega} \frac{1}{y} \nabla \max\{w - u; 0\} \nabla(u + w) - \lambda^2 \int_{\Omega \setminus \bar{D}} y I(\{u > 0\} \cap \{w > 0\}) \\
 &= \int_{\Omega} \frac{1}{y} [|\nabla \max\{w - u; 0\}|^2 - 2 \nabla w \nabla \max\{w - u; 0\}] \\
 &\quad - \lambda^2 \int_{\Omega \setminus \bar{D}} y I(\{u = 0\} \cap \{w > 0\}) \\
 &\geq \int_{\Omega \cap \{w > u > 0\}} \frac{1}{y} |\nabla(w - u)|^2 + \int_{(\Omega \setminus \bar{D}) \cap \{w > 0\} \cap \{u > 0\}} \left(\frac{|\nabla w|^2}{y} - \lambda^2 y \right) \\
 &\geq \int_{\Omega \cap \{w > u > 0\}} \frac{1}{y} |\nabla(w - u)|^2 \geq 0.
 \end{aligned}$$

It follows that $J_{\lambda}(u_0) \leq J_{\lambda}(u)$, where the equality is possible only in the case that $\nabla w = \nabla u$ a.e. in $\Omega \cap \{w > u > 0\}$. Since u minimizes J_{λ} , we get $w = u$ a.e. in $\{w > u > 0\} \cap \Omega$, from which (19) follows. This finishes the proof of the theorem ■

6. Non-oscillation of the free boundary

In this chapter u again is a minimum to J_{λ} . Let be $G = \{(r, \phi) \mid r_1 \leq r \leq r_2, \phi_1 \leq \phi \leq \phi_2\}$, ($0 < r_1 < r_2, 0 \leq \phi_1 < \phi_2$), with $G \subset (\Omega \setminus \bar{D})$, where (r, ϕ) are polar coordinates to (x, y) , and let the domain G contain two disjoint arcs $\gamma_k, (k = 1, 2)$, of the free boundary $\partial\{u > 0\} \cap \Omega$ satisfying

$$\begin{aligned}
 &\gamma_k : r = r_k(t) \text{ and } \phi = \phi_k(t) \text{ for all } 0 < t < T, \\
 &r_1(0) = r_2(0) = r_1, r_1(T) = r_2(T) = r_2 \text{ and } r_1 < r_k(t) < r_2 \text{ for all } 0 < t < T, \\
 &\phi_1(0) < \phi_2(0) \text{ and } \phi_1(T) < \phi_2(T).
 \end{aligned}$$

Further let G' denote the subdomain of G lying between the arcs γ_1 and γ_2 . Finally let be $u > 0$ in a G' -neighbourhood of $\gamma_1 \cup \gamma_2$.

Theorem 6.1: Under the above assumptions $|r_1 - r_2| \leq C|\phi_1 - \phi_2|^{\frac{1}{2}}$, where the constant C depends only on λ and the number $\text{dist}(G', A)$.

Proof: We define a function v in the domain G by $v(x, y) = \sqrt{x^2 + y^2} - r_1$. Applying Greens formula on u and v we get

$$\int_{\partial(G' \cup \{u > 0\})} \frac{1}{y} \left(\frac{\partial u}{\partial \nu} v - \frac{\partial v}{\partial \nu} u \right) ds = 0, \quad (\nu - \text{inner normal to } G' \cup \{u > 0\}).$$

From this, the condition (17), and since we have $v = 0$ for $r = r_1$ and $u = 0, \frac{1}{y} \frac{\partial u}{\partial \nu} = \lambda$ on $\partial(G' \setminus \{u > 0\}) \setminus (\{r = r_1\} \cup \{r = r_2\})$ we can estimate

$$\lambda \int_{\Gamma \cap \Omega} v ds \leq c \int_{\{r=r_1\} \cup \{r=r_2\}} \left| \frac{\partial v}{\partial \nu} \right| ds + \int_{\{r=r_2\}} \left| \frac{v}{y} \frac{\partial u}{\partial \nu} \right| ds, \tag{20}$$

where the constant $c > 0$ only depends on h . The first term on the right-hand side equals $c(\phi_2 - \phi_1)(r_1 + r_2)$. Because of Theorem 5.4 and the inequalities (16) and (18) there is a constant C^* , which depends only on the number $\text{dist}(G', A)$, such that $|\nabla u(x, y)| \leq C^*y$ in G' . Therefore the second term on the right-hand side of (20) is less or equal to $(r_2 - r_1)^2$. The left-hand side of (20) is greater or equal to $\lambda(r_2 - r_1)^2$. We conclude

$$\lambda(r_2 - r_1)^2 \leq c(\phi_2 - \phi_1)(r_1 + r_2) + (r_2 - r_1)(\phi_2 - \phi_1)C^*,$$

and the assertion follows ■

Now an easy conclusion is the following

Theorem 6.2 (Continuous fit of the free boundary): *The free boundary $\partial\{u > 0\} \cap (\Omega \setminus \bar{D})$ has only one limit point A_λ on the straight line through A and O and at most one limit point E_λ on $\{y = 0\}$.*

7. Existence of the solutions

First we have to prove the uniqueness of the (absolute) minima.

Theorem 7.1: *The (absolute) minima to the functionals J_λ are unique.*

Proof: Let be u_0 and u_1 two absolute minima to J_λ . Because of Theorem 4.1 they are starlike and we can infer the corresponding $U = U_k(z, \phi)$ with their inverse functions $Z = Z_k(U, \phi)$, ($k = 0, 1$), as in the proof of Theorem 4.1. Then we define a set of admissible functions u_ϵ , $0 \leq \epsilon \leq 1$, and the corresponding U_ϵ , by their inverse functions Z_ϵ , setting $Z_\epsilon = (1 - \epsilon)Z_0 + \epsilon Z_1$. The idea is to prove that $u_\epsilon \in K$ and $J_\lambda(u_\epsilon)$ is convex in $\epsilon \in (0, 1)$. We denote with $J_{\lambda, R}(u_\epsilon)$ the integral from formula (8) restricted on the finite domain $\Omega \cap B_R$, $R > 0$. After the change of the variables used in Section 3 ($u = u_\epsilon \longleftrightarrow Z = Z_\epsilon$) the integrand in $J_{\lambda, R}(u_\epsilon)$ consists of terms which are linear in ϵ beyond $\frac{Z_\epsilon^2}{Z_U}$ and $\frac{Z_\epsilon^2}{Z_V}$. But an easy calculation shows that

$$\frac{\partial^2}{\partial \epsilon^2} \left(\frac{Z_\epsilon^2}{Z_U} \right) \geq 0 \text{ and } \frac{\partial^2}{\partial \epsilon^2} \left(\frac{Z_\epsilon^2}{Z_V} \right) \geq 0$$

a.e. in the domain of integration. Thus $J_{\lambda, R}(u_\epsilon)$ is convex in $\epsilon \in (0, 1)$. Since $u_0, u_1 \in K$ and J_λ is non-negative, taking $R \rightarrow +\infty$ it follows that $u_\epsilon \in K$, $\epsilon \in (0, 1)$. Therefore $J_\lambda(u_{\frac{1}{2}}) \leq J_\lambda(u_0) = J_\lambda(u_1)$. But from the proof it is easy to see that $J_\lambda(u_{\frac{1}{2}}) = J_\lambda(u_0)$ only if $u = u_0$ a.e. in Ω . This proves the uniqueness of the minimum to the functional J_λ ■

To prove the continuity of the mapping $\lambda \rightarrow u_\lambda$, we need the following monotonicity property

Theorem 7.2: *Let be $0 < \lambda_1 < \lambda_2$. If then u_1 and u_2 are the corresponding absolute minima to J_{λ_1} and J_{λ_2} , respectively, it follows $\{u_1 > 0\} \supset \{u_2 > 0\}$.*

Proof: We set $v_1 = \max\{u_1; u_2\}$ and $v_2 = \min\{u_1; u_2\}$. Both v_1 and v_2 are admissible and we have $\{v_1 > 0\} \supseteq \{u_k > 0\} \supseteq \{v_2 > 0\}$ for $k = 1, 2$. We denote with J_1 and J_2

the functionals J_{λ_1} and J_{λ_2} , respectively. Then we get $J_1(u_1) + J_1(u_2) \geq J_1(v_1) + J_2(v_2)$. The minimality of u_1 and u_2 yields $J_1(u_1) = J_1(v_1)$ and $J_2(u_2) = J_2(v_2)$, and by the previous theorem there follows $u_1 = v_1$ and $u_2 = v_2$ ■

Now let $\{\lambda_n\}_{n \geq 1}$ be a positive sequence with $\lambda_n \rightarrow \lambda$. We denote by u_n and u the absolute minima to the functionals J_{λ_n} and J_λ , and by R_n and R the corresponding representation functions from (12) ($n \in \mathbb{N}$).

Theorem 7.3: *Under the foregoing assumptions*

$$u_n \rightarrow u \text{ weakly in } H^{1,2}(\Omega) \text{ and a.e. ,} \quad (21)$$

and

$$R_n(\phi) \rightarrow R(\phi) \text{ for all } \phi \in (0, \phi_A) . \quad (22)$$

Proof: We can show (21) and (22) for any $\phi \in (0, \phi_A)$ as in [4 : Lemma 10.4]. To prove (22) for $\phi = \phi_A$ we set $A_1 = (r_1 \cos \phi_A, r_1 \sin \phi_A)$ with $\overline{AA_1} = N \cap \{\phi = \phi_A\}$, where $A = A_1$ is possible, and assume that, for a subsequence $\{R'_n\}_{n \geq 1}$, $R'_n(\phi_A) \rightarrow R(\phi_A) + \beta$ as $n \rightarrow \infty$. Now if $0 > \beta$ and $r_1 > \beta + R(\phi_A)$ we get a contradiction to Theorem 6.1. If this is not true but we have $\beta \neq 0$, one proves as in [4: Lemma 10.4] that along a segment of length β lying on $\{\phi = \phi_A\}$ there follows $\frac{1}{\nu} \frac{\partial u}{\partial \nu} = \lambda$, ν - inner normal to $\{u > 0\}$, and, by analytic continuation, on the whole straight line $\{\phi = \phi_A\}$. Hence $\beta = 0$ ■

In the previous theorem we have shown that the point of separation A_λ of the free streamline on $\{\phi = \phi_A\}$ depends continuously on the parameter λ . To find a solution of the cavity problem in the pipe it suffices to prove that the sets $M^+ = \{\lambda \mid R_\lambda(\phi_A) > r_A\}$ and $M^- = \{\lambda \mid R_\lambda(\phi_A) \leq r_A\}$, where $r = R_\lambda(\phi)$ is the representation from (12), are not empty. Then we can choose a parameter λ^* such that $A = A_{\lambda^*}$.

We choose a point $A_0 = (x_0, y_0)$ on $\{\phi = \phi_A\}$ and set $\lambda_0 = \frac{(y_0 + r_0)^2}{cr_0 h_2^2}$ with $r_0 = \min\{\frac{y_0}{2}; h_2 - y_0\}$ and the number c from Lemma 5.2. Since we have $u_\lambda(x, y) \leq \frac{r^2}{h^2}$ in Ω (u_λ denotes the minimum to J_λ), it follows $\frac{1}{2\pi r r_0} \int_{\partial B_r(A_0)} u_\lambda \leq c \lambda_0 y_0$. But this implies $u_\lambda(A_0) = 0$ by Lemma 5.2. Since we can choose A_0 near A' it follows $M^+ \neq \emptyset$. Now let u_0 be the minimum to J_0 . Since a free boundary cannot occur we get $u_0 > 0$ in Ω . By Theorem 7.2 it follows that $M^- \neq \emptyset$. Thus we have proved the existence of a solution of the axially symmetric cavity problem in a pipe.

Completion of the Proof of Theorem 2.3: The property (7) implying smooth fit of the free boundary on the obstacle can be shown as in [3: Theorem 9.1]. We still have to investigate the behaviour of the free boundary in the infinity. Let (u, λ, Γ) be the above constructed solution. Because of $J_\lambda(u) < +\infty$ we have

$$\int_{\Omega \setminus \overline{D}} y \left| \frac{\nabla u}{y} - \lambda eI(\{u > 0\}) \right|^2 < +\infty \text{ if } \lambda \geq \frac{2}{h^2}$$

and

$$\int_{\Omega \setminus \bar{D}} y \left| \frac{\nabla u}{y} - \frac{2}{h^2} eI(\{u > 0\}) \right|^2 < +\infty \text{ if } \lambda < \frac{2}{h^2}.$$

First we consider the case $\lambda \geq \frac{2}{h^2}$. Setting $w_n(x, y) = u(x + x_n, y)$ for any sequence $x_n \rightarrow +\infty$ we get for an arbitrary number $k > 0$

$$\int_0^{h+k} \int_{-k}^k y \left| \frac{\nabla w_n}{y} - \lambda eI(\{w_n > 0\}) \right|^2 dx dy =: I_n \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

It follows that there is a function w_0 such that $w_n \rightarrow w_0$, weakly in $H^{1,2}(\{0 \leq y \leq h\})$,

$$\int_0^{h+k} \int_{-k}^k y \left| \frac{\nabla w_0}{y} - \lambda eI(\{w_0 > 0\}) \right|^2 = \liminf_{n \rightarrow +\infty} I_n = 0$$

and

$$\nabla w_0 = \lambda e y I(\{w_0 > 0\}) \text{ a.e. in } \{0 \leq y \leq h\}. \tag{23}$$

Further in view of Theorem 5.4 we can choose a subsequence $\{w'_n\}_{n \geq 1}$ converging to w_0 uniformly in compact subsets of $\{0 \leq y \leq h\}$. From the inequality (19) we then get $w_0(X) \geq w(X)$. Together with (23) this yields $w_0(X) = \max\{0; 1 + \frac{\lambda}{2}(y^2 - h^2)\}$, and finally $\lim_{\sigma \rightarrow +\infty} u(x, y) = \max\{0; 1 + \frac{\lambda}{2}(y^2 - h^2)\}$.

In an analogous way we prove in the case $\lambda < \frac{2}{h^2}$ that $\lim_{\sigma \rightarrow +\infty} u(x, y) = \frac{2}{h^2}$ and $\lim_{\sigma \rightarrow +\infty} \frac{1}{y} \nabla u(x, y) = \frac{2}{h^2}$. The last convergence property shows that an infinite cavity cannot occur in this case.

Next suppose that the cavity is finite in the border case $\lambda = \frac{2}{h^2}$. Let $\{\lambda_n\}_{n \geq 1}$ be a monotonously decreasing sequence which tends to the limit λ and $\{\Gamma_n\}_{n \geq 1}$ the corresponding sequence of free boundaries of the minima to J_{λ_n} . Because of Theorem 7.3 the Γ_n -s converge for sufficiently large x (say $x > x_0$) to the axis $\{y = 0\}$. This implies

$$\lim_{y \rightarrow 0} \frac{u_y(x, y)}{y} = \lambda \text{ for all } x > x_0. \tag{24}$$

On the other hand it follows by the maximum principle for u that the left-hand side of (24) is less than the number $\frac{2}{h^2}$, which contradicts our assumption. Thus in the case $\lambda = \frac{2}{h^2}$ the cavity is also infinite. Now the convergence property of the free boundary in (B2) follows by Lemma 5.2. Thus we have proved the existence of a solution of the cavity problem in a pipe which has the properties from Theorem 2.3 ■

Now to get a solution of the cavity Problem 2.2 our procedure is as follows: Let $\{h_n\}_{n \geq 1}$ be a sequence with $h_n \rightarrow +\infty$ and $\{(u_n, \lambda_n, \Gamma_n)\}_{n \geq 1}$ the corresponding sequence of solutions of the cavity problem in a pipe with radius $h = h_n$. We set $v_n(X) = \frac{h^2}{2} u_n(X)$ in Ω and $\mu_n = \frac{h^2 \lambda_n}{2}$. We have to prove the boundedness of the μ_n -s.

Lemma 7.5: *Let $(u_k, \lambda_k, \Gamma_k)$, $k = 1, 2$, be two solutions of the cavity problems in the pipe with radius $h = h_k$, and assume that $h_2 > h_1$. Then there follows*

$$\lambda_2 h_2^2 \leq \lambda_1 h_1^2 . \tag{25}$$

Proof: Let $r = R_k(\phi)$, $k = 1, 2$, be the representation functions from (12) to Γ_k . We assume that (25) is not true. Then for sufficiently small $\phi > 0$ the inequality $R_2(\phi) > R_1(\phi)$ is valid. Therefore we can infer a function u_2^α , $0 \leq \alpha < 1$, (with the corresponding free boundary Γ_2^α and its representation $r = R_2^\alpha(\phi)$) such that

$$u_2^\alpha(X) = \frac{u_2(\alpha X)}{\alpha^2} \text{ for } \alpha X \in \{u_2 > 0\} ,$$

$$R_2^\alpha(\phi) \geq R_2(\phi) \text{ for } \phi \in (0, \phi_A) \text{ and } P \in \Gamma_2^\alpha \cap \Gamma_1 .$$

Setting $w = h_1^2 u_1 - h_2^2 u_2^\alpha$, we get $w \geq 0$ on $\partial(\{u_1 > 0\} \cap \{u_2^\alpha > 0\})$ and by the maximum principle in the whole domain $\{u_1 > 0\} \cap \{u_2^\alpha > 0\}$ and $\frac{\partial w}{\partial \nu}(P) > 0$, (ν - inner normal). But this implies $\lambda_2 h_2^2 < \lambda_1 h_1^2$ contradicting our assumption ■

Now we can choose a subsequence $\{(u'_n, \lambda'_n, \Gamma'_n)\}_{n \geq 1}$ such that $u'_n \rightarrow u$ weakly in $H^{1,2}(\Omega_\infty)$ and a.e. in Ω_∞ , $\Gamma'_n \rightarrow \Gamma$ in measure and $\lambda'_n \rightarrow \lambda$. The triple (u, λ, Γ) is a solution of Problem 2.2 which has the property (7). The free boundary is starlike with respect to the origin and has a representation $\Gamma : R = R(\phi)$ for all $\phi \in (0, \phi_A)$, with continuous values in $[0, +\infty]$. We set $\phi_0 = \max\{\phi \mid 0 \leq \phi \leq \phi_A, R(\phi) = +\infty\}$. Because of $R(\phi_A) = r_A$ it follows $\phi_0 < \phi_A$ and $\lim_{\phi \downarrow \phi_0} R(\phi) = +\infty$. We have to show that $\phi_0 = 0$. To this we consider the functions $u_R(X) = \frac{u(RX)}{R^2}$, $R > 0$. Because of $u(X) \leq \frac{r^2}{2}$ we also have $u_R(X) \leq \frac{r^2}{2}$. Therefore there is a sequence $R_n \rightarrow +\infty$ such that $u_{R_n} \rightarrow u_0$ uniformly in compact subsets of $\{y \geq 0\}$. Clearly it is $u_0 = 0$ by Lemma 5.2. On the other hand we have $Lu_0 = 0$, $0 \leq u_0(X) \leq \frac{r^2}{2}$ in Σ and $u_0 = 0$ on $\partial \Sigma$, where Σ denotes the circular segment $\{\phi_0 < \phi < \pi\}$. As in [5: pp. 154], this implies $u_0 = 0$. We can conclude analogously as in case of the problem in the pipe by use of the sequence $\{u'_n\}_{n \geq 1}$ that $\lambda \leq 1$ and that $\lambda = 1$ in the case of an infinite cavity. Thus we have proved the existence of the Cavity Problem 2.2 satisfying the conditions of Theorem 2.4 ■

Remark 7.6: Let $A_1 = (x_1, y_1)$ be a point on the straight line \overline{OA} . We consider solutions of our cavity problems with the point of separation A replaced by A_1 . Then if A_1 lies near O the cavities are finite.

Proof: First let (u, λ, Γ) be a solution of the problem in the pipe. Then if u_0 is the (unique) solution of the following boundary problem

$$Lu_0 = 0, \quad 0 \leq u_0 \leq 1 \text{ in } \Omega, \quad u_0 = 0 \text{ on } S, \quad u_0 = 1 \text{ on } H,$$

it follows $\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} |\nabla u_0(\epsilon A)| = 0$. On the other hand we have by the maximum principle

$$\lambda y_1 = \frac{\partial u}{\partial \nu}(A_1) < \frac{\partial u_0}{\partial \nu}(A_1), \quad \nu - \text{inner normal} .$$

If A_1 is near O we conclude that $\lambda < \frac{2}{R^2}$, which implies that the cavity is finite. Now let (u, λ, Γ) be a solution of the Cavity Problem 2.2. Then instead of u_0 we use the function u_1 satisfying

$Lu_1 = 0$, $0 \leq u_1 \leq \frac{1}{2}$ in Ω_∞ and $u_1 = 0$ on S , and derive in an analogous manner that $\lambda < 1$ which implies that the cavity is finite ■

REFERENCES

- [1] ALT, H.W. and L.A. CAFFARELLI: *Existence and regularity for a minimum problem with free boundary*. J. Reine Angew. Math. **325** (1981), 105 - 144.
- [2] ALT, H.W., CAFFARELLI, L.A. and A. FRIEDMAN: *Asymmetric jet flows*. Commun. Pure Appl. Math. J. **35** (1982), 25 - 68.
- [3] ALT, H.W., CAFFARELLI, L.A. and A. FRIEDMAN: *Axially symmetric jet flows*. Arch. Rat. Mech. Anal. **82** (1983), 97 - 149.
- [4] BROCK, F.: *Axially symmetric flow with finite cavities II*. Z. Anal. Anw. **12** (1993) (to appear).
- [5] CAFFARELLI, L.A. and A. FRIEDMAN: *Axially symmetric infinite cavities*. Indiana Univ. Math. **31** (1982), 135 - 160.
- [6] FRIEDMAN, A.: *Variational principles and free-boundary problems*.
- [7] CARABEDIAN, P.R., LEWY, H. and M. SCHIFFER: *Axially symmetric cavitation flow*. Ann. Math. **56** (1952), 560 - 602.
- [8] HILBIG, H.: *Existenzsätze für einige Totwasserprobleme der Hydrodynamik*. Dissertation B. Leipzig: Universität 1983.
- [9] SERRIN, J.: *Existence theorems for some hydrodynamical free boundary problems*. J. Rat. Mech. Anal. **1** (1952), 1 - 48.
- [10] VOGEL, TH. and A. FRIEDMAN: *Cavitation flow in a channel with oscillatory wall*. Nonlin. Anal., Theory, Methods Appl. **7** (1983), 1175 - 1192.

Received 01.12.1988; in revised form 25.11.1991