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Boundary Theorems of the Gehring-Lohwater and Plessner Type for Polyanalytic Functions

M. B. BALK and W. TUTSCHKE

Using methods of the theory of generalized analytic functions, the present paper investigates the boundary behaviour of non-holomorphic functions (such as polyanalytic ones) provided the areolar derivative $\partial f/\partial \bar{z}$ belong to an L_p-space, $p > 2$.

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1. Introduction. Important classical statements concerning the boundary behaviour of analytic functions (such as the theorems of Lindelöf, Gehring-Lohwater, Fatou, Luzin-Privalov a.o.) in their generally accepted formulations cease to be true if we pass from the class of analytic functions to the wider class of polyanalytic functions. The search of conditions permitting to extend boundary values results to broader classes of complex functions (specifically to polyanalytic functions) was the subject of studies of some authors (see [1 - 4]). In the present note we try to make these results more general using some ideas due to I. N. Vekua [5] and to A. Nagel and W. Rudin [1, 2].

2. Terminology and notations. Hereafter G denotes a Jordan region having on its boundary ∂G some rectifiable simple Jordan arc Γ. We shall assume that any two different points of Γ can be connected by some rectifiable simple (open) Jordan arc belonging to G. Suppose that S is some set in $\mathbb{C}$ and ζ is some point on the boundary ∂S, $\zeta \notin S$. Let f be some function defined in S. Then $C(f,\zeta,S)$ will denote the *cluster set* of f in the point ζ over the set S (cf. [6]), i.e. the set of all numbers A satisfying the condition: there exists in S a sequence of points (z_n) such that, for $n \to \infty$, $z_n \to \zeta$ and $f(z_n) \to A$.

Let d be a fixed complex number; let E be some set of points in $\mathbb{C}$, and the numbers e are its elements; hereafter we denote as $E + a$ the set of all points z of the form $z = e + d$, $e \in E$.

We make also use of notations from I. N. Vekua's book [5]; specifically the relation of the form $g \in L_p(\overline{G})$ means that $\iint_G |g(z)|^p dx\,dy < +\infty$, and the relation $g \in L_p(G)$ means that for every region S compactly belonging to G (i.e. $S \cup \partial S \subset G$) there exists a constant $M(S,g)$ such that $\iint_S |g(z)|^p dx\,dy < M(S,g)$.

3. The basic lemma. We shall make use of some connection existing (under certain conditions) between the cluster set of a given non-analytic function and that of some auxiliary analytic function. The results of I.N. Vekua [5] concerning the properties of the Pompeiu integral (see below) will play a crucial part in our discussion.

Lemma: *Suppose that*

(i) *G is a Jordan region (in $\mathbb{C}$), and ζ is a point on its boundary ∂G*

(ii) *S is a subset of G such that $\partial G \cap \partial S = \{\zeta\}$*

(iii) *f is a function defined and continuous in G*

(iv) *$\partial f/\partial \bar{z}$ (the areolar derivative in the sense of Sobolev-Vekua) exists a.e. in G*

(v) *$\partial f/\partial \bar{z} \in L_p(\bar{G})$ for some p, $2 < p \leq +\infty$.*

Then there exists a function h holomorphic in G and a constant d such that

$$C(f,\zeta,S) = C(h,\zeta,S) + d. \tag{1}$$

Proof: It will be given here only for the case $2 < p < +\infty$; the proof in the case $p = +\infty$ is quite similar (but simpler).

Consider the Pompeiu integral

$$\varphi(z) = (T_G g)[z] = -\frac{1}{\pi}\iint_G \frac{g(t)}{t - z}\, d\xi\, d\eta \tag{2}$$

where $g = \partial f/\partial \bar{z}$, $\xi + i\eta = t$. It follows from (2), by virtue of the Hölder inequality, that (see I.N. Vekua [5: Chap. 1, § 6, formula (6.2)]) the function φ satisfies for any $z_1, z_2 \in \mathbb{C}$ the condition

$$|\varphi(z_1) - \varphi(z_2)| \leq \text{const}\, |z_1 - z_2|^{\alpha}, \quad \alpha = (p-2)/p.$$

Hence φ is continuous in $\mathbb{C}$. But (see [5]) $\partial\varphi/\partial\bar{z} = g$ a.e. in G. Hence $\partial(f - \varphi)/\partial\bar{z} = 0$ a.e. in G. Since the function $f - \varphi$ belongs to $C(G)$ we conclude, by virtue of the Weyl lemma (see I.N. Vekua [5: Chap. 1, § 5, Theorem 1.15]) that there must exist a function h holomorphic in G such that

$$f = h + \varphi. \tag{3}$$

Let $\varphi(\zeta) = d$; let A be an arbitrary number belonging to the set $C(h,\zeta,S)$. Then there must exist in S a sequence (z_n) such that $z_n \to \zeta$, $h(z_n) \to A$ while $n \to +\infty$. But then, as it is seen from (3), $f(z_n) \to A + d$. Hence $C(f,\zeta,S) \subset C(h,\zeta,S) + d$. Similarly it is easy to see that $C(h,\zeta,S) + d \subset C(f,\zeta,S)$. Hence (1) is true ∎

4. A theorem of the Gehring-Lohwater type. In [7] (see also [6: Chap. 1, Theorem 2.3.1]) F. W. Gehring and A.J. Lohwater proved a remarkable statement which can be formulated in such way:

Suppose that

(i) *the function f is analytic in the open circular sector S with the vertex ζ*

(ii) *there exists in S such path γ_1 leading to ζ that $\mathrm{Re} f(z) \to A$ while $z \to \zeta$, $z \in \gamma_1$*

(iii) *there exists in S such path γ_2 leading to ζ that $\mathrm{Im} f(z) \to B$ while $z \to \zeta$, $z \in \gamma_2$*

(iv) *f is bounded in S.*

Then, for every choice of a sector S' (closed in S) which is an inner sector for S and has the same vertex ζ, $f(z) \to A + iB$ while $z \to \zeta$, $z \in S'$.

In the case of polyanalytic functions f of order $n > 1$ this statement does not hold anymore and a corresponding example can easily be constructed (see [3: Section 3.3]). But what condition should be added to the conditions (ii) - (iv) of the Gehring-Lohwater statement in order to retain the conclusion of this statement for polyanalytic functions of arbitrary order? Hereafter we give a possible answer (even for a more general class of functions than polyanalytic ones).

Theorem 1: *Suppose that*

(i) *f is bounded and continuous in the open circular sector S with the vertex ζ*

(ii) *f satisfies the conditions* (ii) - (iv) *of the Gehring-Lohwater theorem above*

(iii) $\partial f/\partial \bar{z}$ *(in the sense of Sobolev-Vekua) exists in a.e. point of S and belongs to the class $L_p(\bar{S})$, for some $p > 2$.*

Then the conclusion of the Gehring-Lohwater theorem remains true for the function f (i.e. for every choice of a sector S' inner to S and having the same vertex ζ as S we have $f(z) \to A + iB$ while $z \to \zeta$, $z \in S'$).

Proof: By virtue of the Lemma the function f can be presented in the form (3): $f(z) = h(z) + \varphi(z)$ $(z \in S)$ where h is holomorphic in S and φ is continuous in $\mathbb{C}$. Denote $\varphi(\zeta)$ as $d = d_1 + id_2$. As it is seen from (3) the function h satisfies in S the conditions (i) - (iv) of the Gehring-Lohwater theorem (h is holomorphic and bounded in S; if (z_n) is some sequence and $z_n \to \zeta$ along γ_1, then $\operatorname{Re} h(z_n) \to A - d_1$; if $z_n \to \zeta$ along γ_2, then $\operatorname{Im} h(z_n) \to B - d_2$. Therefore for any choice of a closed (in S) inner to S sector S' we have for any sequence (z_n) belonging to S': if $z_n \to \zeta$ while $n \to +\infty$, then $h(z_n) \to (A - d_1) + i(B - d_2) = A + iB - d$. Therefore $f(z_n) = h(z_n) + \varphi(z_n) \to A + iB$. Thus Theorem 1 is proved ∎

5. A theorem of the Plessner type. The well-known Plessner boundary values theorem (see, e.g., [6: Theorem 8.2]) is usually formulated for functions analytic (or even meromorphic) in a disc. Hereafter we shall make use of a somewhat more general formulation of this theorem. First we shall remind some terminology (see [6: Chap. 8, § 2]). Suppose that G is a Jordan region, Γ is a rectifiable arc on its boundary. Clearly there exists a tangent in almost every point of Γ (i.e. in every point of Γ - except possibly a set of zero linear measure on Γ). Let ζ be one such point; denote by Δ any circular sector with the vertex ζ having the properties

a) $\Delta \subset G$

b) the boundary radii of Δ do not belong to the tangent to Γ in the point ζ.

Let f be some function given in G. The point ζ is called a *Plessner point* (for the function f) if for any choice of a sector Δ of the mentioned type the cluster set $C(f,\zeta,\Delta)$ contains all points of the extended complex plane $\overline{\mathbb{C}}$. The point ζ is called a *Fatou point* for the function f if for every choice of a sector Δ the cluster set $C(f,\zeta,\Delta)$ contains only one point (clearly the same point for all choices of Δ); in other words, the function f has in the point ζ an angular limit (a limit along non-tangent paths leading to ζ). It is quite obvious that the classical Plessner theorem admits such equivalent reformulation:

Suppose that

(i) *G is some Jordan region in the complex plane $\mathbb{C}$ and Γ is some rectifiable arc on its boundary ∂G*

(ii) *f is an analytic function (or even meromorphic) in G.*

Then Γ can be presented in the form $\Gamma = P(f) \cup F(f) \cup N(f)$ where $P(f)$ is the set of all Plessner points (of the function f on Γ), $F(f)$ is the set of all Fatou points on Γ, and $N(f)$ is some "meager" set of points (a set of zero measure on Γ).

The basic Lemma permits to extend this statement to some polyanalytic functions - and even to functions of a more general type:

Theorem 2: *Suppose that*

(i) *G is some Jordan region having on its boundary ∂G some rectifiable arc Γ*

(ii) *the function f is continuous in G*

(iii) *in a.e. point of G there exists the areolar derivative $\partial f/\partial \bar{z}$, and $\partial f/\partial \bar{z} \in L_p(\overline{G})$ for some p, $2 < p \le +\infty$.*

Then Γ can be presented in the form

$$\Gamma = P(f) \cup F(f) \cup N(f), \tag{4}$$

where $P(f)$, $F(f)$, $N(f)$ have the same meaning as in the given above version of the classical Plessner theorem.

Proof: By virtue of the basic Lemma the function f can be presented in the form $f = h + \varphi$ where h is holomorphic in G and φ is continuous in G. In accordance with Plessner's theorem we have $\Gamma = P(h) \cup F(h) \cup N(h)$ where $P(h)$ is a set of Plessner points for the function h and $F(h)$ is the set of Fatou points for h. Let $\zeta \in P(h)$ and $\varphi(\zeta) = d$. Then for every choice of a sector Δ (in G) with the mentioned above properties we have

$$C(h,\zeta,\Delta) = \overline{\mathbb{C}} \quad \text{and} \quad C(f,\zeta,\Delta) = C(h,\zeta,\Delta) + d.$$

Hence $C(f,\zeta,\Delta) = \overline{\mathbb{C}} + d = \overline{\mathbb{C}}$. Thus every Plessner point for h on Γ is also a Plessner point for f. Quite similarly, every Fatou point for h on Γ is a Fatou point for f. Hence the presentation (4) holds for the function f: $\Gamma = P(f) \cup F(f) \cup N(f)$ with $P(f) = P(h)$, $F(f) = F(h)$ and $N(f) = N(h)$ ∎

Corollary 1: *Suppose that*

(i) *G is some Jordan region having on its boundary a rectifiable arc Γ*

(ii) *some function f is continuous and bounded in G*

(iii) *the areolar derivative $\partial f/\partial \bar{z}$ is defined in G and is bounded in G (or at least belongs to the class $L_p(\overline{G})$ for $p > 2$).*

Then f has an angular limit in a.e. point of the arc Γ.

Corollary 2: *Suppose that the*

(i) *(non-analytic) function f belongs to the Hardy space $H^1(D)$ in the unit disc D*

(ii) the *areolar derivative $\partial f/\partial \bar{z}$ exists in D and belongs to the space $L_p(\overline{D})$ with $p > 2$.*

Then the function f has a (finite) angular limit in a.e. point of the circumference ∂D.

Remark: Provided f is a C^1-function in a rectangular domain G, in Rudin's book [5] a decomposition of type (3) is proved by using the Green formula. Investigating the oscillation of the Pompeiu integral at boundary points, one gets for such G the following Nagel-Rudin theorem:

The Fatou theorem is true for bounded C^1-functions f whose derivative $\partial f/\partial \bar{z}$ belongs to the space $L_p(G)$ with $p > 1$.

Unlike that the application of Weyl's lemma (instead of Green's formula) in the present paper yields a Fatou theorem for a conciderably more general class of functions:

It is sufficient that f belongs to $C(G)$, while $\partial f/\partial \bar{z} \in L_p(G)$ where a stronger restriction on p, $p > 2$, seems to be unavoidable, whereas G must not necessarily be a rectangular domain.

Notice, finally, that the arguments of Nagel and Rudin (cf. [2]) can be applied, too, if f is continuous and the weak derivative $\partial f/\partial \bar{z}$ belongs to L_p with $p > 1$. For that sake it is enough to note that the Green formula remains true also for such functions, as it follows from formula (7.1) in [5: Chapter I, §7].

The used in this note general approach of presentation of a function f with an "L_p-bounded" areolar derivative (in some region G) as a sum of two functions - one holomorphic in G and the second continuous in the closure $G \cup \partial G$ - is applicable in a lot of other cases beyond the considered above. We shall bring here as an example a statement of the Meier type (c.f. [6: Theorem 8.8]). First we ought to remind the reader (see [6: Section 8.5]) that a point ζ belonging to the unit circumference $\Gamma = \{z: |z| = 1\}$ is called a *Meier point* for the function f given in the unit disc D if the cluster set $C(f,\zeta,D)$ does not coincide with the extended complex plane and for every choice of a chord S_ζ having ζ as one of its ends the cluster set along this chord $C(f,\zeta,S_\zeta)$, coincides with $C(f,\zeta,D)$.

The following statement of the Meier type is true.

Theorem 3: *Suppose that*

(i) *f is a continuous function in the unit disc D*

(ii) *the areolar derivative $\partial f/\partial \bar{z}$ exists a.e. in D and belongs to the class $L_p(\bar{D})$, $p > 2$. Then $\Gamma = P(f) \cup M(f) \cup N(f)$ where $M(f)$ is the set of all Meier points on Γ, $P(f)$ is the set of all Plessner points on Γ, and $N(f)$ is some "topologically meager" set (a set of the first Baire category on Γ).*

The **proof** of this statement is quite similar to the reasoning in the proof of Theorem 2 ■

REFERENCES

[1] NAGEL, A. and W. RUDIN: *Local boundary behaviour of bounded holomorphic functions*. Can. J. Math. **30** (1978), 583 - 592.

[2] RUDIN, W.: *Function Theory in the Unit Ball of $\mathbb{C}^n$*. Berlin - Heidelberg - New

York: Springer - Verlag 1980.

[3] BALK, M. B. and V. P. VASSILENKOV: *On boundary properties of polyanalytic functions in Jordan regions.* Izv. Vyssh. Uch. Zav., Math. (in Russian; in print).

[4] BALK, M. B.: *Polyanalytic Functions.* Berlin: Akademie-Verlag 1991.

[5] VEKUA, I. N.: *Generalized Analytic Functions.* Moscow: Fizmatgiz 1959.(in Russian).

[6] COLLINGWOOD, E. F. and A. J. LOHWATER: *The Theory of Cluster Sets.* Cambridge: Univ. Press 1966.

[7] GEHRING, F. W. and A. J. LOHWATER: *On the Lindelöf theorem.* Math. Nachr. **19** (1958), 165 - 170.

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Prof. Dr. Mark Benevich Balk
Pedagogical Institute Smolensk
Przhevalski str. 4
214000 - Smolensk, Russia

Prof. Dr. Wolfgang Tutschke
Fischerinsel 10
D (*Ost*) - 1020 Berlin

Book review

H. M. FARKAS and I. KRA: **Riemann Surfaces** (Graduate Texts in Mathematics: Vol. 71). Berlin - Heidelberg - New York: Springer-Verlag 1992; XVI + 364 pp., 27 fig.

The present book provides a rich representation of the theory of Riemann surfaces beginning with more elementary facts (function theoretic and topological foundations, differentials and divisors) and ending at the frontier of present-day research. A long chapter is devoted to uniformization theory (for arbitrary Riemann surfaces) containing the solution of the Dirichlet problem via subharmonic functions and Perron's method, discontinuous groups, Riemann-Roch theorem and the correspondence between Riemann surfaces and algebraic function fields in one variable. The main part of the book is concerned with compact Riemann surfaces containing among others embedding in projective 3-space, Torelli's theorem on the determination of the conformal class of a Riemann surface by its period matrix, automorphism groups, theta functions and some examples, especially on hyperelliptic and non-hyperelliptic surfaces, quadratic differentials and Prym differentials.

The book contains many exercises and additional remarks. It is accessible to anyone acquainted with elementary function theory and algebra and can be highly recommended.

Leipzig G. EISENREICH