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On a Singular Perturbation Problem for Rotating Magnetic Fluids

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As a singular counterpart to the Newton and Plateau equilibrium figures of rotating stars or drops, respectively, we investigate magnetic drops rotating in weakly nonhomogeneous fields. Existence and uniqueness of a family of equilibrium figures near to the sphere is established.

Key words: *Free boundary value problems, rotating fluids, magnetizable drops.*
AMS subjekt classification: 35R35, 76W05

1 Indroduction

A spherical and magnetizable drop rotating in a homogeneous magnetic field becomes an (instable) equilibrium figure if its angular velocity meets a certain value ω_0. In the present paper we shall discuss existence and uniqueness of another nontrivial family of equilibrium shapes near to the sphere subject to magnetic fields varying in a neighbourhood of a homogeneous field and rotation rates near to ω_0. While less important from the viewpoint of physics because also leading to instable solutions, the problem considered here constitutes an interesting singular counterpart to two classical problems in nonlinear analysis and in applied mathematics that deal with rotating fluids. We have in mind in this connection the Newton equilibrium figures of rotating stars held together by gravitational forces (see e.g. [12]) and the Plateau rotating drop under surface tension [14]. In both cases the second variation of energy computed at a sphere leads to an elliptic operator of order zero or two, respectively. In contrast, the second energy variation of a rotating ball magnetized by a homogeneous field turns out to be degenerated elliptic only and of first order. Our approch in this case rests on a detailed study of its special structure and on an implicit function theorem argument well suited to that linearization.

In our treatment we neglect any coupling between rotation and magnetization (c.f. also [15] for equilibrium shapes of rotating drops under various forces). The physical interesting case of a magnetic field caused by surface distributed currents treated in [7] will be examined in a forthcoming paper. Concerning equilibrium shapes of nonrotating magnetic fluids with surface tension see [5]. Viscous drops in exterior fields are discussed

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in [4]. Further we mention [9] (and the references given therein) where free boundary problems arising in electromagnetic shaping of a liquid metal are studied.

The following discussion is completely local. Due to the degeneracy of our problem calculus of variations as used by Auchmuty [2], [3], Friedman [8] and Albano, Gonzales [1] to reconstruct the Newton and Plateau figures and their generalizations seems to be less promising to get solutions here.

2 The mathematical setting

Consider a drop Ω of an incompressible, magnetizable fluid under the influence of some exterior magnetic field

$$\vec{H}_a = \nabla h, \quad \Delta h = 0 \quad \text{in} \quad \mathbb{R}^3.$$

Let the pressure outside Ω be constant. We suppose the free boundary $\Sigma = \partial\Omega$ to be a graph over the unit sphere S^2:

$$\Sigma = \Sigma(u) = \{y \in \mathbb{R}^3 : y = x(1+u(x)),\ x \in S^2\}, \quad |u(x)| < 1.$$

If we assume the linear law

$$\vec{M} = \chi \vec{H}, \quad \chi = \frac{\mu - 1}{4\pi}$$

between the magnetization $\vec{M}$ inside the drop and the induced field $\vec{H}$, with a permeability constant $\mu \neq 1$, then

$$E = E(u,h) = -\frac{1}{2}\int_{\Omega(u)} \vec{H}_a \vec{M}\, dx \tag{2.1}$$

gives the magnetic energy of any virtual drop configuration $\Omega = \Omega(u)$ (see [6], [11]). Here the exterior domain $C\Omega$ is assumed to be magnetic neutral, i.e. $\mu = 1$ in $C\Omega$.

Let the drop rotate around the x_3–axis with constant angular velocity ω, then

$$G = G(u,\omega) = -\frac{\omega^2}{2}\int_{\Omega(u)} (x_1^2 + x_2^2)\, dx \tag{2.2}$$

is the corresponding rotational energy. According to the principle of virtual work an equilibrium shape of the drop is characterized by the variational equations

$$\mathcal{E}' - \lambda\Omega' = 0, \tag{2.3}$$

where

$$\mathcal{E} = \mathcal{E}(u,\omega,h) = E(u,h) + G(u,\omega). \tag{2.4}$$

In (2.3) the Lagrangian multiplier λ counts for incompressibility which implies constant volume $\Omega(u)$. Here and in the following Ω denotes both the domain and its volume; a prime is always denoting variation with respect to u. Hence we have to solve (2.3) for (u,ω,λ) subject to a volume constraint, which we normalize to be

$$\Omega(u) = 4\pi/3. \tag{2.5}$$

Solutions of (2.3) must satisfy the Euler–Lagrange equations

$$(\sigma_{ij}^{-} - \sigma_{ij}^{+})\, n_i n_j - \frac{\omega^2}{2}(x_1^2 + x_2^2) = \lambda \quad \text{along} \quad \Sigma(u),$$

where $n = (n_1, n_2, n_3)$ is the outer normal to Σ and $\sigma_{ij}^{-}, \sigma_{ij}^{+}$ denote the limit values of the magnetic stress tensor $\sigma_{ij} = \mu(H_i H_j - \frac{1}{2} H_k H_k \delta_{ij})/4\pi$ from the inside or the outside, respectively. Particularly, the total body force must vanish. In order to guarantee this in advance we assume the exterior field both rotational symmetric with respect to the x_3–axis and symmetric with respect to the (x_1, x_2)–plane, i.e.

$$h = h(r, x_3) = -h(r, -x_3) \tag{2.6}$$

when referred to cylindrical coordinates (θ, r, x_3). At the same time we are looking for axisymmetric solutions with $x_3 = 0$ as a symmetry plane:

$$u = u(x_3) = u(-x_3). \tag{2.7}$$

In the following $H^s(S^2)$ denote the Sobolev spaces of real measurable functions on S^2 with derivatives up to the order s in $L^2(S^2)$ normed by $\|u\|_s^2 = \|u\|_{L^2(S^2)}^2 + \|\Lambda^s u\|_{L^2(S^2)}^2$, where $\Lambda = (-\Delta_2)^{1/2}$ denotes the square root of the Laplacian Δ_2 on S^2. Finally, let Z be the Banach space of bounded and harmonic functions in $Q = \{x \in \mathbb{R}^3 : |x| < 2\}$ and let $\|h\|_Z = \sup_{x\in Q} |h(x)|$ be the norm of a function $h \in Z$.

We are now in position to state our main result.

Theorem: *Let $s > 3$ and*

$$h_0(x) = x_3, \quad \omega_0^2 = \frac{9}{4\pi}\left(\frac{1-\mu}{2+\mu}\right)^2, \quad \lambda_0 = \frac{9\mu}{8\pi}\frac{1-\mu}{(2+\mu)^2}.$$

Then for suitable constants $C, \varepsilon > 0$ and any harmonic function h in $\mathbb{R}^3$ satisfying (2.6) and $\|h - h_0\|_Z \le \varepsilon$, there exists a unique solution $(u,\omega,\lambda) \in H^s(S^2) \times \mathbb{R}^2$ of (2.3),(2.5) which satisfies (2.7) and $\|u\|_s$, $|\omega - \omega_0|$, $|\lambda - \lambda_0| \le C\|h - h_0\|_Z$.

Remark: As ε may depends on s one should note that the Theorem does not imply immediately $u \in C^\infty$. This would require further study.

3 Analyticity of $\mathcal{E}'$

Let W denote the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|\cdot\|_W^2$, $\|\varphi\|_W^2 = \int_{\mathbb{R}^3} |\nabla\varphi|^2\,dx$. It is well known that W imbeds into $L^6(\mathbb{R}^3)$ continuously. As usual $H^s(B^\pm)$ denotes the Sobolev spaces of real measurable functions on $B^- = \{x \in \mathbb{R}^3 : |x| < 1\}$ or $B^+ = \mathbb{R}^3 \setminus \overline{B^-}$ with derivatives up to order s in L^2. Further, $\partial_i = \partial/\partial x_i$ is used for partial differentiations and a capital $D = a_i\partial_i$, $a_i \in C_0^\infty(\mathbb{R}^3)$ always refers to a differential operator, which acts tangentially along S^2. Particularly, in the following D_3 denotes an arbitrary operator of this type with $D_3\big|_{S^2} = \partial_3 - x_3x_i\partial_i$.

To compute the energy functional (2.1) we need the magnetic field $\vec{H}$ which is induced by $\vec{H}_a$. Using the ansatz $\vec{H} = \nabla(h+\psi)$, $\psi \in W$, according to the laws of magnetostatics we have to determine ψ as a solution of the transmission problem

$$\Delta\psi = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \Sigma(u),$$

$$\psi^- - \psi^+ = 0, \quad \mu\frac{\partial\psi^-}{\partial n} - \frac{\partial\psi^+}{\partial n} = (1-\mu)\frac{\partial h}{\partial n} \quad \text{on} \quad \Sigma(u).$$

Here upper signs denote inner or outer limits as already introduced above. Obviously ψ can equivalently be characterized as a solution of

$$\min\{F(u,h;\varphi) : \varphi \in W\}, \tag{3.1}$$

where

$$F(u,h;\varphi) = \frac{1}{2}\int_{\mathbb{R}^3} \mu|\nabla\varphi|^2\,dx + (\mu-1)\int_{\Omega(u)} \nabla h\nabla\varphi\,dx. \tag{3.2}$$

Positive definiteness of (3.2) implies existence and uniqueness of its extremal $\psi \in W$. Remembering (2.1) we get

$$E(u,h) = -\frac{1}{4\pi}\big(F_1(u,h) + F_2(u,h)\big),$$

where

$$F_1(u,h) = \frac{\mu-1}{2}\int_{\Omega(u)} |\nabla h|^2\,dx, \quad F_2(u,h) = F\big(u,h,\psi(u,h)\big). \tag{3.3}$$

In the following let $U^s \subseteq H^s(S^2)$ be a neighbourhood of zero, which we assume sufficiently small, if necessary.

Lemma 3.1: $F_1 \in C^\omega(U^s \times Z, \mathbb{R})$, $F_1' \in C^\omega\big(U^s \times Z, H^s(S^2)\big)$, *provided that* $s > 1$.

Remark: Here and in the following $F' \in C^\omega$ means existence of $\tilde{F}' \in C^\omega$ such that $F'(u,h)\{v\} = \int_{S^2} \tilde{F}'(u,h)(x)\,v(x)\,dS^2(x)$.

Proof of Lemma 3.1: Expand $h \in Z$ and its derivatives $\partial_i h$ into power series

$$h(x) = \sum_\alpha a_\alpha x^\alpha, \quad \partial_i h(x) = \sum_\alpha a_{\alpha,i} x^\alpha, \tag{3.4}$$

then

$$\sum_\alpha |a_\alpha| r^{|\alpha|}, \quad \sum_\alpha |a_{\alpha,i}| r^{|\alpha|} \le C(r) \|h\|_Z \tag{3.5}$$

for any $r \in (0,2)$ with C depending on r only. F_1 may be written as

$$F_1(u,h) = \frac{\mu - 1}{2} \int_{S^2} \int_0^{1+u(x)} |\nabla h(rx)|^2 r^2 \, dr dS^2(x).$$

Now, insert (3.4) into this expression. If we recall that the spaces $H^s(S^2)$ form Banach algebras provided that $s > 1$ we obtain analyticity of F_1 as asserted. Similarly, the second assertion follows from

$$F_1'(u,h)\{v\} = \frac{\mu - 1}{2} \int_{S^2} \left|\nabla h(x(1 + u(x)))\right|^2 (1 + u(x))^2 v(x) \, dS^2(x). \square$$

In order to prove analyticity of F_2, we proceed by transforming (3.1) into a variational problem on a fixed domain. For this purpose we extend $u \in H^s(S^2)$ to functions $T^\pm u \in H^{s+1/2}(B^\pm)$ by solving the boundary value problems

$$(\Delta - 1)T^\pm u = 0 \quad \text{in} \quad B^\pm, \quad T^\pm u = u \quad \text{on} \quad S^2.$$

Elliptic regularity theory implies

$$T^\pm \in \mathcal{L}\big(H^s(S^2), H^{s+1/2}(B^\pm)\big) \quad \text{if} \quad s \ge 1/2,$$

see [13]. This extends immediately to any commutator $[D, T^\pm] = DT^\pm - T^\pm D$ which belongs to a tangential derivative $D = a_i \partial_i$:

$$[D, T^\pm] \in \mathcal{L}\big(H^s(S^2), H^{s+1/2}(B^\pm)\big)$$

because of

$$[D, T^\pm]u = 0 \quad \text{on} \quad S^2,$$

$$(\Delta - 1)[D, T^\pm]u = \big((\Delta a_i)\partial_i + 2(\partial_j a_i)\partial_i \partial_j\big) T^\pm u \quad \text{in} \quad B^\pm.$$

Finally, choose $\varrho \in C_0^\infty(\mathbb{R}^3)$, such that $\varrho = 1$ on S^2 and define

$$\tilde{u}(x) = (Tu)(x) = \begin{cases} (T^- u)(x) & \text{if} \quad x \in B^- \\ \varrho(x)(T^+ u)(x) & \text{if} \quad x \in B^+, \end{cases}$$

then

$$T, [D, T] \in \mathcal{L}\big(H^s(S^2), H^{s+1/2}(B^- \cup B^+)\big) \quad \text{if} \quad s \ge 1/2. \tag{3.6}$$

For $u \in U^s$, $s > 2$ the extension $\tilde{u}$ leads to one-to-one mappings

$$x \mapsto \theta(x) := x\,(1 + \tilde{u}(x)) \tag{3.7}$$

from B^- or B^+ onto $\Omega(u)$ or $\mathbb{R}^3 \setminus \overline{\Omega(u)}$, respectively. In the following we use the notation

$$g_{ij} = \partial_i \theta_l \, \partial_j \theta_l, \quad g = \det(g_{ij}), \quad (g^{ij}) = (g_{ij})^{-1}.$$

Let $\tilde{\mu} = \mu$ in B^- and $\tilde{\mu} = 1$ in B^+.

Lemma 3.2: *Let $s > 2, u \in U^s$ and $f \in L^{6/5}(\mathbb{R}^3), f_i \in L^2(\mathbb{R}^3)$, then there is a unique weak solution $\varphi = \varphi(u, f, f_i) \in W$ of*

$$\partial_i(\tilde{\mu}\sqrt{g}g^{ij}\partial_j\varphi) = f + \partial_i f_i \quad \text{in} \quad \mathbb{R}^3.$$

If, in addition, $f \in H^{s-3/2}(B^- \cup B^+), f_i \in H^{s-1/2}(B^- \cup B^+)$, then the first derivatives

$$\partial_i\varphi \in H^{s-1/2}(B^- \cup B^+), \quad i = 1, 2, 3 \tag{3.8}$$

depend on u, f, f_i analytically.

Proof: In virtue of

$$\left|\int_{\mathbb{R}^3} f\psi\,dx\right| \le \|f\|_{L^{6/5}}\|\psi\|_{L^6} \le c_1\,\|f\|_{L^{6/5}}\|\psi\|_W,$$

$$\left|\int_{\mathbb{R}^3} f_i\partial_i\psi\,dx\right| \le \|\psi\|_W \left(\sum_{i=1}^{3}\|f_i\|_{L^2}^2\right)^{1/2},$$

and

$$\int_{\mathbb{R}^3} \tilde{\mu}\sqrt{g}g^{ij}\partial_i\psi\partial_j\psi\,dx \ge c_2\,\|\psi\|_W^2$$

with some positive constants c_1, c_2 one obtains existence as well as uniqueness of φ. The relation (3.8) is easily seen in its simplest case $u = 0$, i.e. $g_{ij} = \delta_{ij}$. Now, the spaces $H^{s-1/2}(B^- \cup B^+)$ form Banach algebras if $s > 2$, hence

$$\sqrt{g}g^{ij} - \delta_{ij} \in C^\omega(U^s, H^{s-1/2}(B^- \cup B^+)). \tag{3.9}$$

Having in mind the supports of $\sqrt{g}g^{ij} - \delta_{ij}$ to be equibounded one gets (3.8) just as the analytical dependence of $\partial_i\varphi$ by a perturbation argument via implicit function theorem. □

In a next step put $H(u,h)(x) = h\big(x(1 + \tilde{u}(x))\big)$, $x \in B^-$ with $h \in Z$ and $u \in U^s$. Then (3.4), (3.5) imply

$$H \in C^\omega\big(U^s \times Z, H^{s+1/2}(B^-)\big) \quad \text{if} \quad s > 1. \tag{3.10}$$

According to (3.7) the variational integral (3.2) transforms

$$F(u,h;\varphi) = \frac{1}{2}\int_{\mathbb{R}^3} \tilde{\mu}\sqrt{g}g^{ij}\partial_i\varphi\partial_j\varphi\,dx + (\mu - 1)\int_{B^-} \sqrt{g}g^{ij}\partial_i H\partial_j\varphi\,dx. \tag{3.11}$$

If we maintain the earlier notation $\psi = \psi(u,h)$ for the solution of (3.1) also with respect to the new variables (3.11), we get

Lemma 3.3: *Let $s > 2$ and D tangential to S^2, then*

$$\partial_i\psi,\ [D,\partial_i\psi] \in C^\omega(U^s \times Z, H^{s-1/2}(B^- \cup B^+)), \quad i = 1,2,3. \tag{3.12}$$

Remark: Here and in the following $[D,\partial_i\psi]$ is used as an abbreviation for the mapping $(u,h) \mapsto D\partial_i\psi(u,h) - \partial_i(\psi'(u,h)\{Du\})$. According to the first inclusion in (3.12) the expression $[D,\partial_i\psi](u,h)$ originally will be defined for $(u,h) \in (U^s \cap H^{s+1}(S^2)) \times Z$ only. Then the second one states the possibility of a $C^\omega(U^s \times Z, H^{s-1/2}(B^- \cup B^+))$-continuation of that map.

Proof of Lemma 3.3: In the distributional sense ψ satisfies

$$\partial_i(\tilde{\mu}\sqrt{g}g^{ij}\partial_j\psi) = \partial_i((1-\tilde{\mu})\sqrt{g}g^{ij}\partial_j H) \tag{3.13}$$

in $\mathbb{R}^3$. Hence the regularity of $\partial_i\psi$ follows from (3.9), (3.10) and Lemma 3.2.

Next, let $D = a_i\partial_i$ be tangential along S^2 and $D^* = -\partial_i(a_i\cdot)$ its formal adjoint. Concerning the second inclusion in (3.12) it is sufficient to prove

$$\partial_i[D,\psi] \in C^\omega(U^s \times Z, H^{s-1/2}(B^- \cup B^+)).$$

Equation (3.13) implies for all $\varphi \in C_0^\infty(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3} \tilde{\mu}\sqrt{g}g^{ij}\partial_j\psi\partial_i D^*\varphi\,dx = \int_{\mathbb{R}^3}(1-\tilde{\mu})\sqrt{g}g^{ij}\partial_j H\partial_i D^*\varphi\,dx.$$

Now, assume $u \in U^s \cap H^{s+1}(S^2), h \in Z$. Then $\psi(u,h) \in W$, $\partial_i\psi(u,h) \in H^{s+1/2}(B^- \cup B^+)$, hence $D\psi(u,h) \in W$. Integration by parts leads to

$$\begin{aligned}\partial_i(\tilde{\mu}\sqrt{g}g^{ij}\partial_j D\psi) &= -\partial_i(D(\tilde{\mu}\sqrt{g}g^{ij})\partial_j\psi) - \partial_i(\tilde{\mu}\sqrt{g}g^{ij}([D,\partial_j]\psi))\\ &- \partial_i D((\tilde{\mu}-1)\sqrt{g}g^{ij}\partial_j H) - \partial_i\partial_l a_l\sqrt{g}g^{ij}(\tilde{\mu}\partial_j\psi + (\tilde{\mu}-1)\partial_j H)\\ &+ \partial_l(\partial_i a_l\sqrt{g}g^{ij}(\tilde{\mu}\partial_j\psi + (\tilde{\mu}-1)\partial_j H)).\end{aligned} \tag{3.14}$$

On the other hand, according to (3.13), $\psi' = \psi'(u,h)\{Du\} \in W$ satisfies

$$\begin{aligned}&\partial_i(\tilde{\mu}\sqrt{g}g^{ij}\partial_j\psi') =\\ &-\partial_i(\tilde{\mu}\,(\sqrt{g}g^{ij})'\{Du\}\,\partial_j\psi) - \partial_i((\tilde{\mu}-1)\,(\sqrt{g}g^{ij}\partial_j H)'\{Du\}).\end{aligned} \tag{3.15}$$

Hence by subtraction of (3.15) from (3.14) we obtain that $[D,\psi] \in W$ satisfies an equation of the form

$$\partial_i(\tilde{\mu}\sqrt{g}g^{ij}\partial_j[D,\psi]) = f + \partial_i f_i,$$

where

$$\begin{aligned}f &= -\partial_i\partial_l a_l\sqrt{g}g^{ij}(\tilde{\mu}\partial_j\psi + (\tilde{\mu}-1)\partial_j H),\\ f_i &= -\tilde{\mu}[D,\sqrt{g}g^{ij}]\partial_j\psi - (\tilde{\mu}-1)[D,\sqrt{g}g^{ij}\partial_j H] - \tilde{\mu}\sqrt{g}g^{ij}[D,\partial_j]\psi\\ &\quad + \partial_l a_i\sqrt{g}g^{lj}(\tilde{\mu}\partial_j\psi + (\tilde{\mu}-1)\partial_j H).\end{aligned}$$

By (3.6) we conclude as an important supplement to (3.9), (3.10)

$$[D,\sqrt{g}g^{ij}] \in C^\omega(U^s, H^{s-1/2}(B^- \cup B^+)),$$

$$[D,\partial_j H] \in C^\omega(U^s \times Z, H^{s-1/2}(B^-)),$$

and Lemma 3.2 implies also the second inclusion in (3.12). □

Lemma 3.4: *If $s > 2$, then*

(i) $F_2 \in C^\omega(U^s \times Z, \mathbb{R})$, *and* (ii) F_2', $[D, F_2'] \in C^\omega(U^s \times Z, H^{s-1}(S^2))$.

Proof: (i) follows from (3.3), (3.11) and Lemma 3.3 immediately. Concerning (ii) we get from (3.11) by differentiation

$$F_2' = \frac{1}{2}\int_{\mathbb{R}^3} \tilde{\mu}(\sqrt{g}g^{ij})'\partial_i\psi\partial_j\psi\,dx + (\mu-1)\int_{B^-}(\sqrt{g}g^{ij}\partial_i H)'\partial_j\psi\,dx. \tag{3.16}$$

Let A_{ij} denote the algebraic complement of $\partial_j\theta_i$ in the Jacobian of (3.7). Then

$$(\sqrt{g})' = A_{kl}\partial_l\theta_k', \quad (g^{ij})' = -\frac{1}{\sqrt{g}}(A_{kj}g^{il} + A_{ki}g^{jl})\partial_l\theta_k',$$

$$(A_{ij})' = \frac{1}{\sqrt{g}}(A_{kl}A_{ij} - A_{kj}A_{il})\partial_l\theta_k'$$

and we obtain after some calculations

$$(\sqrt{g}g^{ij})'\partial_i\psi\partial_j\psi = \partial_l\big((A_{kl}g^{ij} - 2A_{ki}g^{jl})\partial_i\psi\partial_j\psi\,\theta_k'\big),$$
$$(\sqrt{g}g^{ij})'\partial_i H\partial_j\psi = \partial_l\big((A_{kl}g^{ij} - A_{kj}g^{il} - A_{ki}g^{jl})\partial_i H\partial_j\psi\,\theta_k'\big).$$

Inserting this and $\theta_k'\{v\} = x_k v$ into (3.16) we get after integration by parts

$$F_2'(u,h)\{v\} = \int_{S^2} \tilde{F}_2'(u,h)(x)\,v(x)\,dS^2(x)$$

with

$$\begin{aligned}\tilde{F}_2'(u,h) = {} & \frac{\mu}{2}x_k x_l(A_{kl}g^{ij} - 2A_{ki}g^{jl})^-\partial_i\psi^-\partial_j\psi^- \\ & - \frac{1}{2}x_k x_l(A_{kl}g^{ij} - 2A_{ki}g^{jl})^+\partial_i\psi^+\partial_j\psi^+ \\ & + (\mu-1)x_k x_l(A_{kl}g^{ij} - A_{kj}g^{il} - A_{ki}g^{jl})^-\partial_i H\partial_j\psi^- \\ & + (\mu-1)x_k x_i(\sqrt{g}g^{ij})^-(\partial_k h)(\theta(x))\partial_j\psi^-.\end{aligned} \tag{3.17}$$

Then (ii) follows by the trace mapping theorem in view of Lemma 3.3 and

$$[D,(\sqrt{g}g^{ij})^-],\ [D,(A_{kl}g^{ij} - A_{kj}g^{il} - A_{ki}g^{jl})^\pm] \in C^\omega(U^s, H^{s-1}(S^2)).\ \square$$

The remaining parts G, Ω in (2.2)–(2.4) obey similar properties as formulated for F_1 in Lemma 3.1. Without giving details, we summarize a final result in

Proposition 3.5: *If $s > 2$, then*
(i) $\mathcal{E} \in C^\omega(U^s \times Z \times \mathbb{R}, \mathbb{R})$ *and* (ii) $\mathcal{E}'$, $[D, \mathcal{E}'] \in C^\omega(U^s \times Z \times \mathbb{R}, H^{s-1}(S^2))$.

Remark: Our special choice of the extension operator T simplifies the considerations, but of course the results in Lemma 3.3 (without the commutator relation), Lemma 3.4 and Proposition 3.5 are independent of this choice.

4 Analysis of $\mathcal{E}'$, $\mathcal{E}''(0, \omega, h_0)$

In this section we compute the first and second variations of $\mathcal{E}$ at $u = 0$ and $h = h_0$, i.e. for a ball magnetized by a homogeneous exterior field. The following formal expansion procedure is observed to be valid actually in virtue of the preceeding results. To simplify both notation and computation we shall work here with an extension $T \in \mathcal{L}(H^s(S^2), H^{s+1/2}(\mathbb{R}^3))$ different from (3.6). In addition we assume that this extension $\tilde{u} = Tu$ satisfies $\partial_0 \tilde{u} := x_i \partial_i \tilde{u} = 0$ on S^2.

We start with the expansions

$$\Omega'(u) = 1 + 2u + O(u^2), \tag{4.1}$$

$$G'(u, \omega) = \frac{\omega^2}{2}\left((x_3^2 - 1) + 4(x_3^2 - 1)u + O(u^2)\right). \tag{4.2}$$

and note that by (3.3)

$$F_1(u, h_0) = \frac{\mu - 1}{2}\Omega(u). \tag{4.3}$$

Expansion of F_2 requires a more extensive analysis. Let $\psi(u, h_0) = \sum_{n=0}^{\infty} \psi_n(u^n)$ be the power series expansion of ψ in the sense of Lemma 3.3. Then inserting of

$$\sqrt{g}g^{ij} = (1 + \tilde{u} + \partial_0\tilde{u})\delta_{ij} - (x_i\partial_j\tilde{u} + x_j\partial_i\tilde{u}) + O(u^2),$$

$$\sqrt{g}g^{ij}\partial_j H = (1 + 2\tilde{u} + \partial_0\tilde{u})\delta_{i3} - x_i\partial_3\tilde{u} + O(u^2)$$

into (3.13) leads to

$$\partial_i((1 + \tilde{u} + \partial_0\tilde{u})\partial_i\psi_0) - \partial_i((x_i\partial_j\tilde{u} + x_j\partial_i\tilde{u})\partial_j\psi_0) + \Delta\psi_1 = O(u^2)$$

in $B^\pm$ subject to the transmission conditions $\psi^- - \psi^+ = 0$ and

$$\begin{aligned}((1 + u)x_i - \partial_i u)(\mu\partial_i\psi_0^- - \partial_i\psi_0^+) + \mu\partial_0\psi_1^- - \partial_0\psi_1^+ \\ = (1 - \mu)(x_3(1 + 2u) - \partial_3 u) + O(u^2)\end{aligned}$$

on S^2. This implies

$$\begin{gathered}\Delta\psi_0 = 0 \quad \text{in} \quad B^\pm, \\ \psi_0^- - \psi_0^+ = 0, \quad \mu\partial_0\psi_0^- - \partial_0\psi_0^+ = (1 - \mu)x_3 \quad \text{on} \quad S^2,\end{gathered} \tag{4.4}$$

as well as

$$\begin{aligned}\Delta\psi_1 &= 2\nabla\tilde{u}\nabla\psi_0 + \Delta\tilde{u}\partial_0\psi_0 + 2\partial_j\tilde{u}\partial_0\partial_j\psi_0 \quad \text{in} \quad B^\pm,\\ \psi_1^- - \psi_1^+ &= 0, \quad \mu\partial_0\psi_1^- - \partial_0\psi_1^+ = (1-\mu)(2x_3u - \partial_3 u)\\ &\quad + (\partial_i u - x_i u)(\mu\partial_i\psi_0^- - \partial_i\psi_0^+) \quad \text{on} \quad S^2,\end{aligned} \tag{4.5}$$

by comparison of coefficients. One realizes at once

$$\psi_0 = \begin{cases} Ax_3 & \text{in} \quad B^- \\ Ax_3/|x|^3 & \text{in} \quad B^+, \quad A := (1-\mu)/(2+\mu)\end{cases} \tag{4.6}$$

to be the unique solution of (4.4). Substituting of (4.6) into (4.5) yields

$$\begin{aligned}&\Delta\psi_1 = A\Delta(x_3\tilde{u}) \quad \text{in} \quad B^-, \quad \Delta\psi_1 = -2A\Delta(x_3\tilde{u}/|x|^3) \quad \text{in} \quad B^+,\\ &\psi_1^- - \psi_1^+ = 0, \quad \mu\partial_0\psi_1^- - \partial_0\psi_1^+ = (1-\mu)x_3u - 3A\partial_3 u \quad \text{on} \quad S^2.\end{aligned}$$

This will be reduced by

$$\psi_1 = A(x_3\tilde{u} + \varphi) \quad \text{in} \quad B^-, \quad \psi_1 = A(-2x_3\tilde{u}/|x|^3 + \varphi) \quad \text{in} \quad B^+ \tag{4.7}$$

to the problem

$$\begin{aligned}&\Delta\varphi = 0 \quad \text{in} \quad B^\pm,\\ &\varphi^- - \varphi^+ = -3x_3u, \quad \mu\partial_0\varphi^- - \partial_0\varphi^+ = 6x_3u - 3D_3u \quad \text{on} \quad S^2.\end{aligned} \tag{4.8}$$

Now by $\partial_i H = \partial_i\theta_3$, $(\partial_k h)(\theta(x)) = \delta_{3k}$ for $h = h_0$ we see from (3.17)

$$\begin{aligned}\tilde{F}_2'(u,h_0) &= \frac{1}{2}x_kx_l(A_{kl}g^{ij} - 2A_{ki}g^{jl})(\mu\partial_i\psi^-\partial_j\psi^- - \partial_i\psi^+\partial_j\psi^+)\\ &\quad + (\mu-1)x_kx_lg^{-1/2}(A_{kl}A_{3i} - A_{ki}A_{3l})\partial_i\psi^-.\end{aligned}$$

Taking into account

$$\begin{aligned}x_kx_l(A_{kl}g^{ij} - 2A_{ki}g^{jl}) &= \delta_{ij} - 2x_ix_j + x_i\partial_j u - x_j\partial_i u + O(u^2),\\ x_kx_lg^{-1/2}(A_{kl}A_{3i} - A_{ki}A_{3l}) &= (1+u)(\delta_{3i} - x_3x_i) + O(u^2) \quad \text{on} \quad S^2,\end{aligned}$$

we find

$$\begin{aligned}\tilde{F}_2'(u,h_0) &= \frac{1}{2}(\delta_{ij} - 2x_ix_j)\big(\mu\partial_i\psi_0^-(\partial_j\psi_0^- + 2\partial_j\psi_1^-) - \partial_i\psi_0^+(\partial_j\psi_0^+ + 2\partial_j\psi_1^+)\big)\\ &\quad + (\mu-1)\big((1+u)D_3\psi_0^- + D_3\psi_1^-\big) + O(u^2).\end{aligned}$$

Thus we obtain

$$\begin{aligned}\tilde{F}_2'(u,h_0) &= \frac{A^2}{2}(9x_3^2 - \mu - 5) - 3A^2(6x_3^2 + \frac{\mu-4}{3})u\\ &\quad - 3A^2(\partial_3\varphi^+ - 3x_3D_3u) + O(u^2),\end{aligned}$$

when (4.6)-(4.8) is inserted. Finally, remembering (4.1)-(4.3) we get

$$\mathcal{E}'(0,h_0,\omega) = \Big(\frac{\omega^2}{2} - \frac{9A^2}{8\pi}\Big)x_3^2 - \Big(\frac{\omega^2}{2} + \frac{9(\mu-1)}{8\pi(\mu+2)^2}\Big),$$

$$\mathcal{E}''(0,h_0,\omega)u = \frac{3A^2}{4\pi}\Big(\Big(6x_3^2 + \frac{\mu-4}{3}\Big)u + \partial_3\varphi^+ - 3x_3D_3u\Big) - \frac{\mu-1}{4\pi}u - 2\omega^2(1-x_3^2)u.$$

Particularly

$$\mathcal{E}'(0,h_0,\omega_0) - \lambda_0\Omega'(u) = 0. \tag{4.9}$$

In order to get some insight into the mapping properties of $\mathcal{E}''(0,h_0,\omega)$ we will determine, in a first step, its leading part. It is not difficult to establish (e.g. by expansion into spherical harmonics), that

$$\varphi^+ \sim \frac{1}{\mu+1}\big(\Lambda^{-1}(\mu\partial_0\varphi^- - \partial_0\varphi^+) - \mu(\varphi^- - \varphi^+)\big)$$
$$\sim \frac{3}{\mu+1}(\mu x_3 u - \Lambda^{-1}D_3u)$$

solves (4.8) modulo ($\sim$) an operator of order -1. Hence

$$\partial_3\varphi^+ = D_3\varphi^+ + x_3\partial_0\varphi^+$$
$$\sim -\frac{3}{\mu+1}(D_3^2\Lambda^{-1}u + \mu x_3^2\Lambda u) + 3x_3D_3u,$$

which implies

$$\mathcal{E}''(0,h_0,\omega) \sim -\frac{9A^2}{4\pi(\mu+1)}(D_3^2\Lambda^{-1} + \mu x_3^2\Lambda).$$

Its restriction to axisymmetric u is of particular interest. Because of $D_3^2 \sim (x_3^2-1)\Lambda^2$ in this case, one gets

$$\mathcal{E}''(0,h_0,\omega) \sim \frac{9A^2}{4\pi}\Big(\frac{1}{1+\mu} - x_3^2\Big)\Lambda. \tag{4.10}$$

Note that (4.10) degenerates on the two parallels $x_3^2 = 1/(1+\mu)$.

To solve the linearized equations we need some additional information about the lower-order terms in (4.10). This will be easily achieved by expansion into spherical harmonics. Now, the solution φ of (4.8) reads as

$$\varphi = \sum_{n=0}^{\infty}\varphi_n^-|x|^nY_n \quad \text{in} \quad B^-, \quad \varphi = \sum_{n=0}^{\infty}\varphi_n^+|x|^{-(n+1)}Y_n \quad \text{in} \quad B^+,$$

where

$$\varphi_n^- = -3(n+1)\sqrt{\frac{2n+1}{2n+3}}\frac{u_{n+1}}{(\mu+1)n+1},$$
$$\varphi_n^+ = \varphi_n^- + 3\left(\frac{n}{\sqrt{2n-1}\sqrt{2n+1}}u_{n-1} + \frac{n+1}{\sqrt{2n+1}\sqrt{2n+3}}u_{n+1}\right).$$

Here, as usual, $Y_n = Y_n(x) = \sqrt{\frac{2n+1}{4\pi}} P_n(x_3)$, $x = (x_1, x_2, x_3) \in S^2$ denote the zonal spherical harmonics, P_n the Legendre polynomials and u_n the Fourier coefficients in the expansion $u = \sum_{n=1}^{\infty} u_n Y_n$ of u. Because of

$$\partial_3 \varphi^+ = \sum_{n=0}^{\infty} n \sqrt{\frac{2n-1}{2n+1}} \varphi_{n-1}^+ Y_n,$$

$$x_3 D_3 u = \sum_{n=0}^{\infty} \Big\{ -\frac{(n-2)(n-1)n}{(2n-1)\sqrt{2n-3}\sqrt{2n+1}} u_{n-2} + \frac{n(n+1)}{(2n-1)(2n+3)} u_n + \frac{(n+1)(n+2)(n+3)}{(2n+3)\sqrt{2n+1}\sqrt{2n+5}} u_{n+2} \Big\} Y_n,$$

one finally obtains

$$\mathcal{E}''(u, h_0, \omega_0)\{u\} = \sum_{n=0}^{\infty} (\alpha_n u_{n-2} + \beta_n u_n + \alpha_{n+2} u_{n+2}) Y_n. \tag{4.11}$$

Here the coefficients α_n, β_n satisfy the asymptotics

$$\alpha_n = -\frac{9A^2}{16\pi} n + \alpha + O(\frac{1}{n}), \quad \beta_n = -\frac{9A^2}{8\pi} \frac{\mu-1}{\mu+1} n + \beta + O(\frac{1}{n}), \tag{4.12}$$

which is crucial in the following. Concerning α_n, the exact calculation shows

$$\alpha_n = -\frac{9A^2}{4\pi} \frac{n(n-1)(n-3)}{(2n-1)\sqrt{2n+1}\sqrt{2n-3}},$$

whence

$$\alpha_n \neq 0 \quad \text{for} \quad n \neq 0, 1, 3. \tag{4.13}$$

5 An abstract existence theorem

In this section we consider an abstract equation of the form

$$Ax = R(x) \tag{5.1}$$

in which A is a linear operator with a special structure motivated by our results about the second variation of the energy in Section 4; R is a nonlinear, in some sense small perturbation.

First we fix some notations. Let X, Y be real, separable and infinite-dimensional Hilbert spaces with the scalar products $(\cdot,\cdot)_X, (\cdot,\cdot)_Y$ and norms $\|\cdot\|_X = (\cdot,\cdot)_X^{1/2}, \|\cdot\|_Y = (\cdot,\cdot)_Y^{1/2}$, respectively. We choose complete orthonormal systems $\{e_n\}_{n\geq 1}, \{f_n\}_{n\geq 2}$ of X and Y, respectively. For any $x \in X$ we denote its Fourier coefficients with respect to $\{e_n\}_{n\geq 1}$ by x_n; analogously for any $y \in Y$.

Now, let $A \in \mathcal{L}(X,Y)$ be a bounded linear operator with

$$Ax = \sum_{n=2}^{\infty} (\alpha_n x_{n-1} + \beta_n x_n + \gamma_n x_{n+1}) f_n \tag{5.2}$$

and sequences $\{\alpha_n\}_{n\geq 2}, \{\beta_n\}_{n\geq 2}, \{\gamma_n\}_{n\geq 2}$ of real numbers with the asymptotic behaviour

$$\alpha_n = 1 + \frac{\alpha}{n} + O(\frac{1}{n^2}),\ \beta_n = a + \frac{\beta}{n} + O(\frac{1}{n^2}),\ \gamma_n = 1 + \frac{\gamma}{n} + O(\frac{1}{n^2}), \tag{5.3}$$

where $a, \alpha, \beta, \gamma \in \mathbb{R}$,

$$|a| < 2, \quad \gamma - \alpha < 1 \tag{5.4}$$

and

$$\alpha_n \neq 0 \quad \text{for all} \quad n \geq 2. \tag{5.5}$$

Further, let $B_r \subseteq X$ be the closed ball of radius r centered at $0 \in X$. We denote the linear subspaces of X, Y consisting of elements with only a finite number of nonvanishing Fourier coefficients by $\bar{X}$ and $\bar{Y}$, respectively. Finally, let $L : \bar{X} \to \bar{Y}$ be the linear operator with

$$Lx = \sum_{n=2}^{\infty} n(x_{n-1} - x_{n+1}) f_n. \tag{5.6}$$

Then we can formulate

Proposition 5.1: *To every operator $A \in \mathcal{L}(X,Y)$ with (5.2)–(5.5) one can find a positive number c, depending only on A, with the following property: If $r > 0$ and $R : B_r \subseteq X \to Y$ is a sequentially weakly continuous map with*

$$\|R(x)\|_Y,\ (R(x), Lx)_Y / r \leq cr \quad \textit{for all} \quad x \in B_r \cap \bar{X}, \tag{5.7}$$

then there exists a solution $x \in B_r$ of the equation (5.1).

Remark: (5.7) are the smallness conditions for the nonlinear term R mentioned above.

It is easy to see, that for a proof of Proposition 5.1 we can without loss of generality assume: $Y = \{x \in X : x_1 = 0\}$ and $f_n = e_n$ for all $n \geq 2$. Then in the scalar products and norms we can drop the indices X, Y. First we give some properties of operators $A \in \mathcal{L}(X,Y)$ with (5.2)–(5.5).

Lemma 5.2: *One has $A\bar{X} = \bar{Y}$.*

Proof: The inclusion $A\bar{X} \subseteq \bar{Y}$ follows immediately from the structure (5.2) of A, and the reverse relation is a consequence of (5.5). □

For $x \in \bar{X}$ and $s \in \mathbb{R}$ we define norms $\|\cdot\|_s$ by $\|x\|_s^2 = \sum_{n=1}^{\infty} n^{2s} x_n^2$, $\|x\|_0 \equiv \|x\|$. Let X_s, Y_s be the completion of $\bar{X}$ and $\bar{Y}$, respectively, with respect to the norms $\|\cdot\|_s$. Then we have the compact imbeddings $X_{s_1} \hookrightarrow X_{s_2}$ for $s_2 < s_1$ and $|(x,y)| \leq \|x\|_s \|y\|_{-s}$ for $x, y \in \bar{X}$. Obviously L extends to an operator $L \in \mathcal{L}(X_s, Y_{s-1})$ for all $s \in \mathbb{R}$. Because of (5.4) we can choose $\tilde{\alpha}, \tilde{\beta} \in \mathbb{R}$ with

$$-1 < 2\tilde{\alpha} < \alpha - \gamma, \tag{5.8}$$

$$|a|(1+2\tilde{\alpha})/2 < \tilde{\beta} < 1 + 2\tilde{\alpha} < 1. \tag{5.9}$$

Define, besides L, a second linear operator $\tilde{L} \in \mathcal{L}(X_s, Y_{s-1})$ by

$$\tilde{L}x = \sum_{n=2}^{\infty}((n+\tilde{\alpha})x_{n-1} + \tilde{\beta}x_n - (n-\tilde{\alpha})x_{n+1})e_n. \tag{5.10}$$

Note that $2|\tilde{\alpha}| + |\tilde{\beta}| < 2$ and therefore

$$\|Lx - \tilde{L}x\| \leq 2\|x\|. \tag{5.11}$$

Lemma 5.3: *There exist positive numbers c_1, c_2, such that*

$$(Ax, \tilde{L}x) \geq c_1\|x\|^2 - c_2\|x\|\|x\|_{-1} \quad \textit{for all} \quad x \in X_1. \tag{5.12}$$

Proof: It remains to prove (5.12) for all $x \in \bar{X}$. For two quadratic functionals B, B' on $\bar{X}$ we write $B \sim B'$, if there is a number c with $|B(x) - B'(x)| \leq c\|x\|\|x\|_{-1}$ for all $x \in \bar{X}$. Using this notation, we have

$$\sum_{n=2}^{\infty} n(x_{n-1} + ax_n + x_{n+1})(x_{n-1} - x_{n+1}) \sim \sum_{n=1}^{\infty}(2x_n^2 + ax_nx_{n+1}).$$

After inserting the expressions (5.2), (5.3) and (5.10) for A and $\tilde{L}$, respectively, we obtain

$$(Ax, \tilde{L}x) \sim \sum_{n=1}^{\infty} x_n^2(2 + 2\tilde{\alpha} + a\tilde{\beta} + \alpha - \gamma) \\ + \sum_{n=1}^{\infty} x_nx_{n+1}(a + 2a\tilde{\alpha} + 2\tilde{\beta}) + \sum_{n=1}^{\infty} x_nx_{n+2}(2\tilde{\alpha} - \alpha + \gamma).$$

Therefore an application of the Cauchy-Schwarz inequality gives

$$(Ax, \tilde{L}x) \geq \{(2 + 2\tilde{\alpha} + a\tilde{\beta} + \alpha - \gamma) - (a + 2a\tilde{\alpha} + 2\tilde{\beta}) \\ + (2\tilde{\alpha} - \alpha + \gamma)\}\|x\|^2 - c\|x\|\|x\|_{-1}$$

because of (5.8), (5.9), i.e. (5.12) with $c_1 = (2-a)(1 + 2\tilde{\alpha} - \tilde{\beta}) > 0$. □

Lemma 5.4: *One has* $\ker A = \{0\}$. *Moreover, there exists a right inverse operator $A^{-1} \in \mathcal{L}(Y_1, X)$.*

Remark: In solving the linear equation $Ax = y$ one looses some regularity. This indicates, that the standard contracting mapping priciple is not suitable to handle the nonlinear equation (5.1).

Proof of Lemma 5.4: *Step 1.* By (5.12) we conclude that

$$\|x\| \leq c(\|Ax\|_1 + \|x\|_{-1}) \quad \text{for all} \quad x \in \bar{X}. \tag{5.13}$$

The linear operator $A|_{\bar{X}} : \bar{X} \subseteq X \to Y_1$ is closable and we denote by $\bar{A} : D(\bar{A}) \subseteq X \to Y_1$ its smallest closed extension. Obviously (5.13) holds for all $x \in D(\bar{A})$. Because of the compact embedding $X \hookrightarrow X_{-1}$ we obtain by a standard argument $\dim \ker \bar{A} < \infty$ and $\operatorname{im}\bar{A}$ is a closed set in Y_1, see e.g. [10, §7, Theorem 7.1]. From Lemma 5.1 we get $\operatorname{im}\bar{A} = Y_1$. Therefore $\bar{A}$ is a Fredholm operator with nonpositive index and there exists an operator $A^{-1} \in \mathcal{L}(Y_1, X)$ with $AA^{-1}y = y$ for all $y \in Y_1$.

Step 2. By (5.3) we can find an integer $n_0 \geq 3$ with $\alpha_n > 0$ for all $n \geq n_0$. We define another operator $A_0 \in \mathcal{L}(X, Y)$ with the structure (5.2) by choosing sequences $\{\alpha_n^0\}_{n\geq 2}, \{\beta_n^0\}_{n\geq 2}, \{\gamma_n^0\}_{n\geq 2}$ with

$$\begin{aligned} \alpha_n^0 = \gamma_n^0 = 1,\ \beta_n^0 = a \quad &\text{for} \quad n \geq n_0, \\ \alpha_n^0 = \alpha_n,\ \beta_n^0 = \gamma_n^0 = 0 \quad &\text{for} \quad n < n_0. \end{aligned}$$

For $\lambda \in [0,1]$ the operator $\lambda A_0 + (1-\lambda)A$ possesses again the structure (5.2) and the conditions (5.3)–(5.5) are satisfied. By Step 1 and constancy of the index of a Fredholm operator against bounded perturbations we conclude $\dim \ker A_0 = \dim \ker A$.

Step 3. It remains to prove $\ker A_0 = \{0\}$. Let x with $A_0 x = 0$ be given. According to our definition of A_0 it follows

$$\begin{aligned} x_{n-1} + a x_n + x_{n+1} = 0 \quad &\text{for} \quad n \geq n_0, \\ \alpha_n x_{n-1} = 0 \quad &\text{for} \quad n < n_0. \end{aligned} \tag{5.14}$$

From $\alpha_n \neq 0$ we get immediately $x_1 = x_2 = \ldots = x_{n_0-2} = 0$. For solutions of the difference equation (5.14) the series $\sum_{n\geq n_0-1} x_n^2$ diverges or is equal to zero. This completes the proof. □

Lemma 5.5: *There exist constants* $c_1, c_2 > 0$, *such that for all* $x \in X$ *with* $Ax \in Y_1$ *one has* $\|x\|^2 \leq c_1(Ax, Lx) + c_2\|Ax\|\|x\|$.

Proof: In view of Lemma 5.3 and (5.11) it remains to prove, that there exists to every $\varepsilon > 0$ a constant $c = c(\varepsilon) > 0$ with $\|x\|_{-1} \leq \varepsilon\|x\| + c(\varepsilon)\|Ax\|$ for all $x \in X$. But this is a consequence of the compact embedding $X \hookrightarrow X_{-1}$ and $\ker A = \{0\}$. □

Now, we define operators $P_\varepsilon \in \mathcal{L}(X_s, X_{s+2})$ by $P_\varepsilon x = \sum_{n=1}^\infty x_n e_n/(1+\varepsilon n^2)$ for $\varepsilon \in (0,1]$. Then one has

$$\|P_\varepsilon\| \le \|x\| \quad \text{and} \quad \|x - P_\varepsilon x\|_{-2} \le \varepsilon\|x\|. \tag{5.15}$$

Moreover, a simple calculation gives

$$(P_\varepsilon L - LP_\varepsilon)x = -\sum_{n=2}^{\infty}\Big(\frac{\varepsilon n(2n-1)x_{n-1}}{1+\varepsilon(n-1)^2} + \frac{\varepsilon n(2n+1)x_{n+1}}{1+\varepsilon(n+1)^2}\Big)\frac{e_n}{1+\varepsilon n^2},$$

and therefore

$$\|(P_\varepsilon L - LP_\varepsilon)x\| \le 4\|x\|. \tag{5.16}$$

Proof of Proposition 5.1: Let $R : B_r \subseteq X \to Y$, $r > 0$ be a map, which satisfies the assumptions of Proposition 5.1 with the constant $c = 1/(24\max\{c_1, c_2\})$, where c_1, c_2 are the constants in Lemma 5.5. Because of the weak continuity of R the condition (5.7) holds even for all $x \in X_1$. We consider the fixed point problem $x = T_\varepsilon(x)$, $x \in B_r$, $\varepsilon \in (0,1]$, with $T_\varepsilon(x) = A^{-1}P_\varepsilon R(P_\varepsilon x)$. Obviously $T_\varepsilon : B_r \subseteq X \to X$ is a compact operator. Assume, that $x = \lambda T_\varepsilon(x)$, $x \in B_r$, $0 < \lambda < 1$, then we have $Ax = \lambda P_\varepsilon R(P_\varepsilon x)$, and therefore $Ax \in Y_1$. According to Lemma 5.5 we get

$$\begin{aligned}\|x\|^2 &\le c_1(Ax, Lx) + c_2\|Ax\|\|x\| \\ &\le c_1\lambda(P_\varepsilon R(P_\varepsilon), Lx) + c_2\lambda\|P_\varepsilon R(P_\varepsilon x)\|_r.\end{aligned} \tag{5.17}$$

By (5.15) and (5.16) we obtain

$$\|P_\varepsilon R(P_\varepsilon x)\| \le \|R(P_\varepsilon x)\| \le cr, \tag{5.18}$$

$$\begin{aligned}(P_\varepsilon R(P_\varepsilon x), Lx) &= (R(P_\varepsilon x), LP_\varepsilon x) + (R(P_\varepsilon x), (P_\varepsilon L - LP_\varepsilon)x) \\ &\le cr^2 + 4\|R(P_\varepsilon x)\|\|x\| \le 5cr^2.\end{aligned} \tag{5.19}$$

Finally, by (5.17)–(5.19) we conclude

$$\|x\|^2 \le 6\max\{c_1, c_2\}cr^2 \le r^2/4, \quad \text{i.e.} \quad x \in B_{r/2}.$$

According to the Leray–Schauder principle for every $\varepsilon \in (0,1]$ there exists a solution $x_\varepsilon \in B_{r/2}$ of the equation $x = T_\varepsilon(x)$. We choose sequences $\varepsilon_n \to 0, x_{\varepsilon_n} \rightharpoonup x \in B_{r/2}$ and obtain by (5.15) and the weak continuity of R

$$P_{\varepsilon_n}x_{\varepsilon_n} \rightharpoonup x,\ P_{\varepsilon_n}R(P_{\varepsilon_n}x_{\varepsilon_n}) \rightharpoonup R(x),\ Ax_{\varepsilon_n} \rightharpoonup Ax$$

with $\rightharpoonup$ denoting weak convergence in X. Consequently we have $Ax = R(x)$. □

Next we wish to formulate Proposition 5.1 as an implicit function theorem in order to solve a nonlinear equation

$$F(x, y) = 0, \tag{5.20}$$

where y as an element of a normed space Z is given. Let $U \subseteq X, V \subseteq Z$ be neighbourhoods of 0. We investigate (5.20) under the following assumptions:

(i) Let $F, f \in C^1(U \times V, Y)$ be mappings with $F(0,0) = f(0,0) = 0$ and $J \in \mathcal{L}(X,Y)$ such that $(F(x,y), Lx)_Y = (f(x,y), Jx)_Y$ for all $x \in U \cap \bar{X}, y \in V$.

(ii) The Frechet derivative $A := F_x(0,0) \in \mathcal{L}(X,Y)$ is an operator with the structure (5.2)–(5.5).

(iii) The mapping $F(\cdot, y) : U \subseteq X \to Y$ is sequentially weakly continuous for every $y \in V$.

Proposition 5.6: *Under the assumptions* (i)–(iii) *there exists for all y in a sufficiently small neighbourhood of* 0 *a solution $x = x(y)$ of equation (5.20) with $\|x\|_X \leq c\|y\|_Z$, where the constant c is independent of y.*

Remark: Uniqueness of the solution can be proved under some additional assumptions, but we refrain from doing so at the abstract level.

Proof of Proposition 5.6: By $F \in C^1(U \times V, Y)$ and the mean value theorem we can find a small number $r_1 > 0$ and a constant c_1, such that for $\|x\|_X, \|y\|_Z \leq r_1$ one has $\|F(0,y)\|_Y \leq c_1\|y\|_Z$, and moreover with the constant c from Proposition 5.1

$$\|F(x,y) - F(0,y) - F_x(0,0)x\|_Y \leq c\|x\|_X/2.$$

A similar estimate holds for f. Let $y \in V$ with $\|y\|_Z \leq r_1$ and $r := 2c_1\|y\|_Z/c < r_1$ be given. We write the equation (5.20) in the form (5.1) with $R(x) = F(x,y) - F_x(0,0)x$. Then we have

$$\|R(x)\|_Y \leq c_1\|y\|_Z + c\|x\|_Y/2 \leq c(2c_1\|y\|_Z/c) = cr \quad \text{for} \quad x \in B_r.$$

From (i) we obtain

$$(F_x(x,y)x, Lx)_Y = (f_x(x,y)x, Jx)_Y \quad \text{for} \quad x \in U \cap \bar{X},$$

and

$$(F_x(0,0)x, Lx)_Y = (f_x(0,0)x, Jx)_Y \quad \text{for} \quad x \in \bar{X}.$$

Without loss of generality we may assume $\|J\|_{\mathcal{L}(X,Y)} \leq 1$ and get

$$\begin{aligned} |(R(x), Lx)_Y| &= |(f(x,y) - f_x(0,0)x, Jx)_Y| \\ &\leq \|f(x,y) - f_x(0,0)x\|_Y \|x\|_X \leq cr^2 \end{aligned}$$

for $x \in B_r \cap \bar{X}$, and Proposition 5.1 gives the assertion. □

6 An important relation

One main step in applying Proposition 5.6 to our problem of Section 2 is the verification of the assumption (i). To this purpose we state the following proposition. We use the notations and results of Section 3. The proof is based on the inclusions

$$E',\ [D, E'] \in C^\omega(U^s \times Z, H^{s-1}) \quad \text{for} \quad s > 2, \tag{6.1}$$

(see Lemmata 3.1, 3.4), the symmetry

$$\big(E''(u,h)\{v\}, w\big)_0 = \big(E''(u,h)\{w\}, v\big)_0, \tag{6.2}$$

and a suitable partial integration. For brevity we write $H^s = H^s(S^2)$.

Proposition 6.1: *Let $s \geq 3$ be an integer. For every differential operator D there exists a mapping $e \in C^\omega(U^{s+1/2} \times Z, H^{s-1/2})$, such that $\big(E'(u,h), Du\big)_s = \big(e(u,h), u\big)_s$ for all $u \in U^{s+1/2} \cap C^\infty$, $h \in Z$.*

Remark: Note the important (and at first glance somewhat surprising) fact that E', e possess the same mapping properties.

We use the negativ–norms

$$\|u\|_{-s} = \sup_{v \in H^s, v \neq 0} |(u,v)_0| / \|v\|_s, \quad s \geq 0 \tag{6.3}$$

and denote by H^{-s} the completion of $L^2(S^2)$ with respect to the norm $\|\cdot\|_{-s}$. Then we have the compact and dense imbeddings $H^s \hookrightarrow H^{s'}$ for all real $s > s'$.

Lemma 6.2: *Let $A \in \mathcal{L}(H^s, H^{s-1}), s \geq 1$ be a linear operator, such that*

$$(Au, v)_0 = \pm (Av, u)_0 \quad \textit{for} \quad u, v \in H^s. \tag{6.4}$$

Then there exists an extension of A to an operator in $\mathcal{L}(H^{s'}, H^{s'-1})$ for all $s' \in [-s+1, s]$. Moreover, if $A \in \mathcal{L}(H^s, H^{s-1})$ depends analytically on a parameter of a normed space, then the extensions likewise.

Proof: The first assertion is trivial if $s' = s$ and is a consequence of the definition (6.3) and the assumption (6.4) if $s' = -s+1$. Therefore interpolation gives the assertion for all $s' \in [-s+1, s]$; further, we have $\|A\|_{\mathcal{L}(H^{s'}, H^{s'-1})} \leq C\|A\|_{\mathcal{L}(H^s, H^{s-1})}$ for $s' \in [-s+1, s]$, where the constant is independent of A. Now, let $A = A(\theta) \in \mathcal{L}(H^s, H^{s-1})$ depend analytically on a parameter $\theta \in N$, where N is a normed space. Then we have an expansion of the form $A = \sum_{n=0}^\infty A_n(\theta^n)$ in $\mathcal{L}(H^s, H^{s-1})$ with bounded n–linear maps A_n from N^n into $\mathcal{L}(H^s, H^{s-1})$. By comparison of coefficients we find from (6.4) the equalities $(A_n(\theta^n)u, v)_0 = \pm(A_n(\theta^n)v, u)_0$ and as above we obtain for $s' \in [-s+1, s]$

$$\|A_n(\theta^n)\|_{\mathcal{L}(H^{s'}, H^{s'-1})} \leq C\|A_n(\theta^n)\|_{\mathcal{L}(H^s, H^{s-1})}.$$

This easily gives the second assertion. □

Lemma 6.3: *We have for* $s > 2$, $s' \in [-s+1, s]$

$$E''(\cdot,\cdot)\{\cdot\} \in C^\omega(U^s \times Z \times H^{s'}, H^{s'-1}), \tag{6.5}$$

and for $s > 3$, $s' \in [-s+2, s-1]$

$$DE''(\cdot,\cdot)\{\cdot\} - E''(\cdot,\cdot)\{D\cdot\} \in C^\omega(U^s \times Z \times H^{s'}, H^{s'-1}). \tag{6.6}$$

Proof: The first assertion (6.5) follows immediately from (6.1), Lemma 6.2 and the symmetry (6.2). Further, every differential operator D possesses a representation in the form $Du = \tilde{D}u + au$, $a \in C_0^\infty(\mathbb{R}^n)$, where $\tilde{D}$ is anti-symmetric, i.e. $(\tilde{D}u, v)_0 = -(\tilde{D}u, v)_0$ for all $u, v \in H^1$. Because of (6.5), in proving (6.6) we can therefore assume that D is an anti-symmetric differential operator. By differentiation of $[D, E']$ with respect to u we obtain

$$DE''(u,h)\{v\} - E''(u,h)\{Dv\} = \big([D,E'](u,h)\big)'\{v\} + E'''(u,h)\{Du, v\}.$$

Hence (6.1) gives the assertion (6.6) for $s' = s-1$; note that $s-1 > 2$. Further (6.2) and the anti-symmetry of D gives

$$\big(DE''(u,h)\{v\} - E''(u,h)\{Dv\}, w\big)_0 = \big(DE''(u,h)\{w\} - E''(u,h)\{Dw\}, v\big)_0,$$

and application of Lemma 6.2 finishes the proof. □

In the following Lemma $D^{s'} = D_1 D_2 \dots D_{s'}$ denotes an arbitrary differential operator of order s', where $D_1, \dots, D_{s'}$ are first order differential operators tangential to S^2.

Lemma 6.4: *If* $s > 3$ *and* s' *an integer with* $1 \le s' \le 2s-1$, *then*

$$[D^{s'}, E'] \in C^\omega(U^s \times Z, H^{s-s'}). \tag{6.7}$$

Proof: We apply induction on s'. The statement is contained in (6.1) for $s' = 1$. So assume it true for an integer s' with $1 \le s' \le 2s-2$. If $u \in U^s$, then $D^{s'}u \in H^{s-s'}$ and $s - s' \in [-s+2, s-1]$. Hence by Lemma 6.3, (6.6) we conclude that the mapping

$$(u,h) \mapsto DE''(u,h)\{D^{s'}u\} - E''(u,h)\{D^{s'+1}u\}$$

belongs to $C^\omega(U^s \times Z, H^{s-s'-1})$. This is also true for $D[D^{s'}, E']$ by induction hypotheses. Therefore

$$[D^{s'+1}, E'](u,h) = D\big([D^{s'}, E'](u,h)\big) + DE''(u,h)\{D^{s'}u\} - E''(u,h)\{D^{s'+1}u\}$$

gives (6.7) also for $s'+1$. □

Proof of Proposition 6.1: Let $s \geq 3$ be an integer. We write the scalar product of H^s in the form

$$(u,v)_s = (u,v)_0 + (\tilde{D}_{i_1}\tilde{D}_{i_2}\dots\tilde{D}_{i_s}u, \tilde{D}_{i_1}\tilde{D}_{i_2}\dots\tilde{D}_{i_s}v)_0, \tag{6.8}$$

where $\tilde{D}_1, \tilde{D}_2, \tilde{D}_3$ are the anti–symmetric differential operators

$$\tilde{D}_1 = x_1\partial_2 - x_2\partial_1, \quad \tilde{D}_2 = x_2\partial_3 - x_3\partial_2, \quad \tilde{D}_3 = x_3\partial_1 - x_1\partial_3.$$

We use the abreviation $f_1(u,h) \sim f_2(u,h)$, if $f_1, f_2 : (U^{s+1/2} \cap C^\infty) \times Z \to \mathbb{R}$ are functionals such that the difference $f_1 - f_2$ possesses an extension in $C^\omega(U^{s+1/2} \times Z, \mathbb{R})$. Then we have according to Lemma 6.4

$$\left(D^s E'(u,h), DD^s u\right)_0 \sim \left(E''(u,h)\{D^s u\}, DD^s u\right)_0. \tag{6.9}$$

Applying partial integration, Lemma 6.3, (6.6) and the symmetry (6.2) we get

$$\begin{aligned}\left(E''(u,h)\{D^s u\}, DD^s u\right)_0 &\sim -\left(DE''(u,h)\{D^s u\}, D^s u\right)_0 \\ &\sim -\left(E''(u,h)\{DD^s u\}, D^s u\right)_0 \\ &\sim -\left(E''(u,h)\{D^s u\}, DD^s u\right)_0.\end{aligned} \tag{6.10}$$

From (6.9), (6.10) we obtain $\left(D^s E'(u,h), DD^s u\right)_0 \sim 0$, and therefore by (6.8) we have $\left(E''(u,h), Du\right)_s \sim 0$. Thus there exists $\varrho \in C^\omega(U^{s+1/2} \times Z, \mathbb{R})$, such that

$$\left(E'(u,h), Du\right)_s = \varrho(u,h) \quad \text{for all} \quad u \in U^{s+1/2} \cap C^\infty, h \in Z.$$

According to the Riesz representation theorem we can find a mapping $e \in C^\omega(U^{s+1/2} \times Z, H^{s-1/2})$ with

$$\varrho'(u,h)\{v\} = \left(e(u,h), v\right)_s \quad \text{for all} \quad u \in U^{s+1/2}, v \in H^{s+1/2}, h \in Z.$$

Because of $\varrho(0,h) = 0$ we have

$$\varrho(u,h) = \int_0^1 \varrho'(tu,h)\{u\}\,dt = \Big(\int_0^1 e(tu,h)\,dt, u\Big)_s,$$

and this proves the assertion of Proposition 6.1. □

7 Proof of the Theorem

In order to solve the equations (2.3), (2.5), we first look at the volume constraint. From the expansion (4.1) it is easy to see that for every

$$v \in U_0^s := \left\{u \in U^s : \int_{S^2} u\,dS^2(x) = 0\right\}, \quad s > 1$$

there exists a unique $r = r(v) \in \mathbb{R}$ in a sufficiently small neighbourhood of $0 \in \mathbb{R}$ with $\Omega(v + r(v)) = 4\pi/3$. Moreover one has

$$r \in C^\omega(U_0^s, \mathbb{R}), \quad r(v) = \int_{S^2} v^2\, dS^2(x) + O(v^3). \tag{7.1}$$

Conversely, if $u \in U^s$ with $\Omega(u) = 4\pi/3$, then $u = v + r(v)$ with a unique $v \in U_0^s$. Therefore, instead of (2.3), (2.5), we can consider equivalently the equation

$$\mathcal{F}(w,h) = 0, \quad w = (v, \omega, \lambda) \in U_0^s \times \mathbb{R}^2$$

with

$$\mathcal{F}(w,h) = \mathcal{E}'(v + r(v), h, \omega) - \lambda\Omega'(v + r(v)). \tag{7.2}$$

Because of (4.9) we have

$$\mathcal{F}(w_0, h_0) = 0, \quad w_0 = (0, \omega_0, \lambda_0).$$

Next we pay attention to the symmetries (2.6), (2.7). To this purpose we assume in the sequel, that the Banach space Z defined in Section 2 contains only harmonic functions with (2.7). Further, let $\bar{H}^s, \bar{H}_0^s$ be the Sobolev spaces of functions on S^2 with (2.6); the functions of $\bar{H}_0^s$ should have the mean value 0. We use the Hilbert spaces $X^s := \bar{H}_0^s \times \mathbb{R}^2$, $Y^s := \bar{H}^s$ and denote the norms simply by $\|\cdot\|_s$. With $V^s := (U^s \cap \bar{H}_0^s) \times \mathbb{R}^2$ we formulate

Lemma 7.1: *We have the following assertions.*
(i) *Let $s > 2$. Then we have $\mathcal{F} \in C^\omega(V^s \times Z, Y^{s-1})$ and the mapping $\mathcal{F}(\cdot, h) : V^s \subseteq X^s \to Y^{s-1}$ is sequentially weakly continuous for every fixed $h \in Z$.*
(ii) *Let $s \geq 3$ be an integer. Then there exists a mapping $f \in C^\omega(V^{s+1/2} \times Z, Y^{s-1/2})$ such that $\big(\mathcal{F}(w,h), x_3 D_3 v\big)_s = \big(f(w,h), v\big)_s$ for all $w = (v,\omega,\lambda) \in V^{s+1/2} \cap X^{s+3/2}$ and $h \in Z$.*

Proof: First of all we note that for functions v, h with the symmetry properties (2.6) and (2.7), respectively, $\mathcal{F}\big((v,\omega,\lambda), h\big)$ also possesses a symmetry of the form (2.6). Therefore $\mathcal{F} \in C^\omega(V^s \times Z, Y^{s-1}), s > 2$ is a consequence of the results of Section 3 and (7.1), (7.2). Particularly the mapping $\mathcal{F}$ is bounded and hence from the compact imbeddings $X^s \hookrightarrow X^{s'}$ for $s > s'$ the weakly continuity of the mapping $\mathcal{F}(\cdot, h) : V^s \to Y^{s-1}$ follows. The second assertion is a consequence of Proposition 6.1. □

In the Hilbert spaces X^s, Y^s we can choose complete orthonormal systems $\{e_{n,s}\}_{n\geq 1}$, $\{f_{n,s}\}_{n\geq 2}$ according to

$$\begin{aligned} e_{1,s} &= (0,0,1), \quad e_{2,s} = (0,1,0), \\ e_{n,s} &= (1 + \lambda_{2(n-2)}^{s/2})^{-1}(Y_{2(n-2)}, 0, 0) \quad \text{for} \quad n \geq 3, \end{aligned} \tag{7.3}$$

and

$$f_{n,s} = (1 + \lambda_{2(n-2)}^{s/2})^{-1} Y_{2(n-2)} \quad \text{for} \quad n \geq 2. \tag{7.4}$$

Thereby $\lambda_n = n(n+1)$ denotes the eigenvalues of the Laplacian Δ_2 on S^2.

Lemma 7.2: *The operator $A := \mathcal{F}_w(w_0, h_0)$ belongs to $\mathcal{L}(X^s, Y^{s-1})$ for $s \geq 1$ and possesses the structure (5.2)–(5.5) with respect to the orthonormal systems (7.3), (7.4).*

Proof: By (7.1), (7.2) and (4.2), (4.3) we conclude

$$\begin{aligned}\mathcal{F}_w(w_0, h_0)\{v, \omega, \lambda\} &= \mathcal{E}''(0, h_0, \omega_0)\{v\} - \lambda_0 \Omega''(0)\{v\} + \omega\omega_0 G'(0) - \lambda\Omega'(0) \\ &= \mathcal{E}''(0, h_0, \omega_0)\{v\} - 2\lambda_0 v + \omega\omega_0(x_3^2 - 1) - \lambda. \end{aligned} \tag{7.5}$$

Denoting the Fourier coefficients of $w = (v, \omega, \lambda) \in X^s$ with respect to the orthonormal system $\{e_{n,s}\}_{n\geq 1}$ by w_n we obtain from (4.11) a representation

$$\mathcal{F}_w(w_0, h_0)\{v, \omega, \lambda\} = \sum_{n=2}^{\infty}(\tilde{\alpha}_n w_{n-1} + \tilde{\beta}_n w_n + \tilde{\gamma}_n w_{n+1}) f_{n,s-1},$$

where the sequences $\{\tilde{\alpha}_n\}_{n\geq 2}, \{\tilde{\beta}_n\}_{n\geq 2}, \{\tilde{\gamma}_n\}_{n\geq 2}$ have by (4.12) the asymptotic behaviour

$$\tilde{\alpha}_n = -\frac{9A^2}{16\pi}\Big(1 + \frac{\tilde{\alpha}}{n} + O\big(\frac{1}{n^2}\big)\Big), \quad \tilde{\gamma}_n = -\frac{9A^2}{16\pi}\Big(1 + \frac{\tilde{\gamma}}{n} + O\big(\frac{1}{n^2}\big)\Big),$$

$$\tilde{\beta}_n = -\frac{9A^2}{16\pi}\Big(2\frac{\mu - 1}{\mu + 1} + \frac{\tilde{\beta}}{n} + O\big(\frac{1}{n^2}\big)\Big).$$

Moreover a simple calculation shows

$$\tilde{\gamma}_n = \tilde{\alpha}_{n+1}\frac{\big(1 + \lambda_{2(n-2)}^{(s-1)/2}\big)\big(1 + \lambda_{2(n-2)}^{s/2}\big)}{\big(1 + \lambda_{2(n-1)}^{(s-1)/2}\big)\big(1 + \lambda_{2(n-1)}^{s/2}\big)}$$

and hence $\tilde{\gamma} < \tilde{\alpha}$. Further we have $\tilde{\alpha}_n \neq 0$ for all $n \geq 2$. For $n = 2, 3$ this can be seen immediately by (7.5); for $n > 3$ it is a consequence of (4.13). Therefore all conditions (5.3)–(5.5) are satisfied. □

After this preparation we finish the proof of our Theorem. We start with the existence statement and apply Proposition 5.6. Fix an integer $s \geq 3$ and choose $X = X^{s+1/2}, Y = Y^{s-1/2}$. With respect to the orthonormal systems (7.3), (7.4) we obtain for the operator L corresponding to (5.6) from an expansion into spherical harmonics

$$Lw \equiv L(v, \omega, \lambda) = -\frac{1}{4}\Lambda(x_3 D_3 v) + L_1 w$$

and therefore

$$\big(\mathcal{F}(w, h), Lw\big)_Y = -\frac{1}{4}\big(\mathcal{F}(w, h), x_3 D_3 v\big)_s + \big(\mathcal{F}(w, h), L_2 w\big)_Y$$

with linear bounded operators $L_1, L_2 \in \mathcal{L}(X, Y)$. Hence the the assumptions (i)–(iii) of Proposition 5.6 are satisfied by the Lemmata 7.1, 7.2.

Next we proof the uniqueness statement. Let be $s > 3$. From the Lemmata 7.2, 5.3 we obtain for $A = \mathcal{F}_w(w_0, h_0)$ with a suitable constant c

$$\|\dot{w}\|_{3/2}^2 \le c\left(\left|\left(A\dot{w}, x_3 D_3 \dot{v}\right)_1\right| + \|A\dot{w}\|_{1/2}^2\right) \tag{7.6}$$

for all $\dot{w} = (\dot{v}, \dot{\omega}, \dot{\lambda}) \in X^{5/2}$. With the same method as in the proof of Proposition 6.1 we obtain from Lemma 6.3 the existence of a mapping $f \in C^\omega(V^s \times Z \times X^{3/2}, Y^{1/2})$ with

$$\left(\mathcal{F}_w(w,h)\{\dot{w}\}, x_3 D_3 \dot{v}\right)_1 = \left(f(w,h)\{\dot{w}\}, \dot{v}\right)_1 \tag{7.7}$$

for all $\dot{w} = (\dot{v}, \dot{\omega}, \dot{\lambda}) \in X^{5/2}$ and $(w,h) \in V^s \times Z$. Now let $w_1, w_2 \in V^s$ and $h \in Z$ with $\mathcal{F}(w_1, h) = \mathcal{F}(w_2, h)$. Then we have for $\dot{w} = w_1 - w_2$

$$A\dot{w} = -\int_0^1 \left(\mathcal{F}_w(w_1 + t\dot{w}, h) - \mathcal{F}_w(w_0, h_0)\right)\{\dot{w}\}\, dt,$$

and by (7.7) we get

$$\|A\dot{w}\|_{1/2}^2,\ \left|\left(A\dot{w}, x_3 D_3 \dot{v}\right)_1\right| \le c_1 \theta \|\dot{w}\|_{3/2}^2$$

with the abbreviation $\theta = \max\{\|w_1 - w_0\|_s, \|w_2 - w_0\|_s, \|h - h_0\|_Z\}$. By (7.6) we conclude $\|\dot{w}\|_{3/2}^2 \le c_2 \theta \|\dot{w}\|_{3/2}^2$. Hence if θ is sufficiently small, then we have $\dot{w} = 0$, i.e. $w_1 = w_2$. Thus all statements of the Theorem are proved. □

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