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Norms of Best Smoothness in Orlicz Spaces

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Equivalent norms of best smoothness are constructed for large classes of Orlicz sequence and function spaces.

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1. Introduction

It is well known [1, 12] that the usual norm in the spaces l_p and $L_p\,(p > 1, p$ not even) is $E\,(p)$-times uniformly differentiable, and the Taylor remainder term is of order $p - E\,(p)$, where

$$E\,(p) = \begin{cases} p-1 & \text{if p is an integer} \\ [p] & \text{otherwise} \end{cases}.$$

Moreover, this order can not be improved by equivalent renorming [1]. For the Orlicz spaces l_M and L_M, the exact order, up to equivalent Orlicz renorming, of the reminder term after differentiation of the norm was found in [9]. This order gives information about the type of l_M, L_M (see, e.g., [6: Section 1.e.16]).

Recently, the best order of Frechet and uniformly Frechet differentiability of the norm (up to equivalent renorming) in Orlicz sequence and function spaces was found in [10]. As usual, in every case an appropriate Orlicz function is constructed so the corresponding Orlicz norm, equivalent to the initial one, is of highest order of differentiability. Our aim is a further investigation of the smoothness of this "good" norm, which is related to a precise estimation of the remainder term after the last derivative. It turns out that in many cases it is also norm of best smoothness. We note that, in a separable Banach space, the existence of an equivalent norm (or more generally bump function) from some smoothness class implies the existence of a partition of unity from the same class (see, e.g., [13: Section 3.1.6]).

Some of the results contained in this paper were announced in a talk given by the author at the 17-th Winter School on Abstract Analysis, Srni, CzechoSlovakia, 1990.

2. Preliminaries

We begin with some notations and definitions. In the sequel X, Y denote Banach spaces, $S(X)$ the unit sphere of X, $B\,(x;r)$ the ball centered at x of radius r, and $\mathbf{N}$ the set of all naturals, $\mathbf{R}$ of all reals, $\mathbf{R}^+$ of all positive reals. Everywhere differentiability is understood in Frechet sense.

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We denote by $B^j(X,Y)$ the space of all continuous symmetric j-linear forms

$$T : \underbrace{X \times X \times \ldots \times X}_{j-times} \to Y$$

equipped with the norm

$$\|T\|_1 = \sup\{\|T(x_1,\ldots,x_j)\|;\ x_i \in X, \|x_i\| \le 1\ \ (i = 1,\ldots,j)\}.$$

In the next we use the notation $x^{(j)} = \underbrace{(x,\ldots,x)}_{j-times}$ for $x \in X$. An equivalent norm (see, e.g., [13: Section 1.3.8]) is given by

$$\|T\| = \sup\left\{\left\|T\left(x^{(j)}\right)\right\|; x \in X, \|x\| \le 1\right\}$$

and $\|T\| \le \|T\|_1 \le \kappa\|T\|, \kappa = (2j)^j/j!$. If $Y = \mathbf{R}$, the space of all continuous symmetric j-linear functionals on X is denoted $B^j(X)$.

Definition 1: A map $f : X \longmapsto Y$ is said to be *k-times differentiable at $x \in X$* if there exist $T_j \in B^j(X,Y)$ $(j = 1,\ldots,k)$ such that

$$f(x+th) = f(x) + \sum_{j=1}^{k} \frac{t^j}{j!} T_j\left(h^{(j)}\right) + o_x\left(|t|^k\right)$$

uniformly for h in the unit sphere $S(X)$ of X, i.e. given $\varepsilon > 0$ there is a $\delta > 0$ independent of $h \in S(X)$ such that $\left|f(x+th) - \sum_{j=0}^{k} \frac{t^j}{j!} T_j\left(h^{(j)}\right)\right| < c(x)\varepsilon|t|^k$ provided $|t| < \delta$. T_j is called *j-th derivative of f at x* and is denoted $D^j f(x)$ or $f^{(j)}(x)$.

Let $\omega : \mathbf{R}^+ \longmapsto \mathbf{R}^+$ be a non-decreasing function with $\lim_{t \mapsto 0} \omega(t) = 0$, and k the greatest integer such that $\omega(t) = o\left(t^k\right)$. Set $\omega_1(t) = \omega(t)/t^k$.

Definition 2: A map $f : X \longmapsto Y$ is called *H^ω-smooth* in $V \subset X$ (see, e.g., [2]) if f is k-times continuously differentiable and for every $x \in V$ there exist δ, $A(x) > 0$ such that

$$\left\|f^{(k)}(y) - f^{(k)}(z)\right\| \le A(x)\ \omega_1(\|y - z\|), \tag{1}$$

provided $y, z \in B(x;\delta) \cap V$.

The norm in the left-hand side of (1) is understood as the norm of the k-linear continuous symmetric form $f^{(k)}(y) - f^{(k)}(z)$ from $B^k(X,Y)$. The space of all H^ω-smooth functions on V is denoted $H^\omega(V,Y)$. If the norm in a Banach space is H^ω -smooth in $X \setminus \{0\}$, then X is called *H^ω-smooth.* A H^ω-smooth map (space) with $\omega(t) = t^p$ is called *H^p-smooth.*

Let us recall some definitions and facts about Orlicz spaces which will be necessary in what follows. An even convex continuous function M, defined and non-decreasing on $[0,\infty)$, is called

Orlicz function if $M(0) = 0$, $M(\infty) = \infty$. Let (S, Σ, μ) be a measure space. The space of all equivalent classes of μ-measurable functions x on S such that

$$\int_S M(x(s)/\lambda)\, d\mu(s) = \widetilde{M}(x/\lambda) < \infty$$

for some positive λ with the norm

$$\|x\| = \inf\left\{\lambda > 0; \widetilde{M}(x/\lambda) \leq 1\right\}$$

is a Banach space, which is called the *Orlicz space generated by M* and denoted by $L_M(S, \Sigma, \mu)$. The subspace of $L_M(S, \Sigma, \mu)$ which consists of all x such that $\widetilde{M}(\lambda x) < \infty$ for every $\lambda > 0$ is denoted $H_M(S, \Sigma, \mu)$.

The most interesting Orlicz spaces considered usually in the literature are the sequence spaces l_M, h_M and the pairs of function spaces $L_M(0,1)$, $H_M(0,1)$ and $L_M(0,\infty)$, $H_M(0,\infty)$ corresponding to the cases: S is a countable union of atoms of equal mass, $S = [0,1]$ or $S = [0,\infty)$, and μ the usual Lebesgue measure. We note that if the Orlicz function M satisfies the Δ_2-*condition* at 0 (at ∞, at 0 and ∞), i.e. there exists $k > 0$ such that

$$M(2t) \leq kM(t),\ t \in [0,1] \quad (t \in [1,\infty), t \in [0,\infty)),$$

the spaces l_M and h_M ($L_M(0,1)$ and $H_M(0,1)$, $L_M(0,\infty)$ and $H_M(0,\infty)$) coincide. Obviously l_M, $L_M(0,1)$ and $L_M(0,\infty)$ essentially depend on the behaviour of the function M near 0, ∞, and 0 and ∞, respectively. It is well known (see, e.g., [5]) that if two Orlicz functions M and N are equivalent ($M \sim N$) at 0 (at ∞, at 0 and ∞), i.e.

$$c^{-1}M\left(c^{-1}t\right) \leq N(t) \leq cM(ct),\ t \in [0,1] \ (t \in [1,\infty), t \in [0,\infty))$$

for some positive constant c, then $h_N\,(H_N(0,1), H_N(0,\infty))$ is isomorphic to $h_M\,(H_M(0,1), H_M(0,\infty))$. Using this result equivalent norms in h_M, $H_M(0,1)$ or $H_M(0,\infty)$ are usually constructed through Orlicz functions, equivalent to M at 0, at ∞ or at 0 and ∞, respectively.

Now we recall that the Boyd indices for h_M, $H_M(0,1)$ and $H_M(0,\infty)$ can be expressed by the formulas (see, e.g., [6: Section 2.b.5])

$$\alpha_M^0 = \sup\left\{p;\ \sup\left\{\frac{M(uv)}{u^p M(v)};\ u,v \in (0,1]\right\} < \infty\right\},$$

$$\alpha_M^\infty = \sup\left\{p;\ \sup\left\{\frac{u^p M(v)}{M(uv)};\ u,v \in [1,\infty)\right\} < \infty\right\},$$

$$\alpha_M = \min\left(\alpha_M^0, \alpha_M^\infty\right),$$

respectively.

A detailed study of the problem of the isomorphic embeddings of the h_M spaces into Orlicz spaces is contained in [7] and [8]. Here we only mention that always $\alpha_M \geq 1$ and that $h_{\alpha_M^0}$ is isomorphic to a subspace of h_M. Finally we consider a class of smooth Orlicz functions that was introduced in [10].

Definition 3: *$AC^k, k \in \mathbb{N}$ is the class of all functions M such that:*

i) $\alpha_M > k$;
ii) the k-th derivative $M^{(k)}$ *of* M *is absolutely continuous in every finite interval;*
iii) $t^{k+1}\left|M^{(k+1)}(t)\right| \leq cM(ct)$ *a.e. in* $[0,\infty)$ *for some* $c > 0$. (2)

It is not hard to check that every $M \in AC^k$ satisfies for fixed $a \in (k, \alpha_M)$ the inequalities

$$M(\lambda t) \leq c_1 \lambda^a M(t), \lambda \in [0,1], t \in [0,\infty) \tag{3}$$

and

$$t^i \left|M^{(i)}(t)\right| \leq c_1 M(c_1 t), t \in [0,\infty) \ (i = 1,\dots,k), \tag{4}$$

where c_1 is a constant depending on a and M. Without loss of generality we shall assume in the sequel that $c_1 = c \geq 1$, i.e. that for a fixed $a \in (k, \alpha_M)$, M satisfies (2), (3) and (4) with the same constant $c \geq 1$.

3. Properties of the class AC^k

In this section we make a further investigation of the class AC^k in order to improve some estimates from [10]. We shall often use the following simple inequalities implied by the convexity of M:

$$uM(v) \leq M(uv) + M(v), \tag{5}$$

$$M(\max(|u|,|v|)) \leq (M(2u) + M(2v))/2, \tag{6}$$

for any real u, v.

Let $k \in \mathbf{N}$. Put

$$F_k^M(u,v) = M(uv)\left(u^{k+1}M(v)\right)^{-1}.$$

For any $k \in \mathbf{N}$ and interval $I \subset \mathbf{R}^+$ we associate to M the function $r_{k,I}^M$ defined as

$$r_{k,I}^M(t) = t \sup\left\{F_k^M(u,v);\ (u,v) \in [t,1] \times I\right\}.$$

If $I = \mathbf{R}^+$, we simply choose $r_{k,I}^M = r_k^M$. We set $R_{k,I}^M(t) = t^k r_{k,I}^M(t)$ and $R_k^M(t) = t^k r_k^M(t)$. Obviously

$$r_k^M(t) \geq t,\ r_k^M(t) \geq M(t)/t^k,\ t \in (0,1], \tag{7}$$

$$M(uv) \leq u^{k+1}M(v)\, r_k^M(t)/t,\ t \in (0,1],\ (u,v) \in [t,1] \times \mathbf{R}^+. \tag{8}$$

The following properties of R_k^M will be useful.

Lemma 1: *Let* $M \in AC^k$. *Then* r_k^M, *and of course also* R_k^M, *are non-decreasing in* $[0,t_0]$ *for some* $t_0 \in (0,1)$.

Proof: According to (3) for a suitable $t_0 \in (0,1)$

$$M(uv) \leq u^k M(v),\ (u,v) \in [0,t_0] \times \mathbf{R}.$$

First we show that $r_k^M(t) \leq r_k^M(a)$ for any $t \in [a^2, a]$ and $a \in (0, t_0]$. Indeed, the above inequality implies

$$\sup\left\{F_k^M(u,v);\ (u,v) \in [t,a] \times \mathbf{R}^+\right\} \leq \left(\frac{a}{t}\right) sup\left\{F_k^M(u,v);\ (u,v) \in \left[a, \frac{a^2}{t}\right] \times \mathbf{R}^+\right\},$$

with $a^2/t \le 1$. Using this inequality and the representation

$$r_k^M(t) = t \max\left(\sup\left\{F_k^M(u,v);\ (u,v) \in [t,a] \times \mathbf{R}^+\right\}, \sup\left\{F_k^M(u,v);\ (u,v) \in [a,1] \times \mathbf{R}^+\right\}\right)$$

we immediately obtain $r_k^M(t) \le r_k^M(a)$. Let now $0 < t_1 < t_2 \le t_0$. Then $t_2^{2^j} \le t_1 \le t_2^{2^{j-1}}$ for some $j \in \mathbf{N}$ and the sequence of inequalities

$$r_k^M(t_1) \le r_k^M\left(t_2^{2^{j-1}}\right) \le r_k^M\left(t_2^{2^{j-2}}\right) \le \dots \le r_k^M(t_2)$$

completes the proof. ■

We note that $r_k^M(\lambda t) \le \lambda r_k^M(t)$, $\lambda \ge 1$.

Lemma 2: *If $M \sim N$ at 0 and ∞, then $R_k^M \sim R_k^N$ at 0.*

Proof: Without loss of generality we may assume that $M(1) = N(1) = 1$. Let $c^{-1}M(c^{-1}t) \le N(t) \le cM(ct)$ for some $c \ge 1$. Then

a) for $c^{-2} \le u \le 1, v \in \mathbf{R}^+$,

$$\frac{N(uv)}{u^{k+1}N(v)} \le c^{2(k+1)}\frac{M(u_1v_1)}{u_1^{k+1}M(v_1)},\quad u_1 = v_1 = 1;$$

b) for $t \le u \le c^{-2},\ v \in \mathbf{R}^+$,

$$\frac{N(uv)}{u^{k+1}N(v)} \le \frac{c^2M(cuv)}{u^{k+1}M(c^{-1}v)} = c^{2(k+2)}\frac{M(u_1v_1)}{u_1^{k+1}M(v_1)},\ t \le c^2u = u_1 \le 1, v_1 = c^{-1}v,$$

which implies

$$\sup\left\{F_k^N(u,v);(u,v) \in [t,1] \times \mathbf{R}^+\right\} \le c^{2(k+2)} \sup\left\{F_k^M(u,v);(u,v) \in [t,1] \times \mathbf{R}\right\}$$

and symmetrically

$$\sup\left\{F_k^M(u,v);(u,v) \in [t,1] \times \mathbf{R}^+\right\} \le c^{2(k+2)} \sup\left\{F_k^N(u,v);(u,v) \in [t,1] \times \mathbf{R}\right\}.$$

Thus $c^{-2(k+2)}R_k^M(t) \le R_k^N(t) \le c^{2(k+2)}R_k^M(t), t \in (0,1]$. ■

Lemma 3: *For any real a and $b,t \in (0,1]$, the inequality*

$$b^{k+1}M(a) \le (M(a) + M(ab/t))\,R_k^M(t) \tag{9}$$

holds.

Proof: If $b/t \le 1$, then (7) implies

$$\left(b^{k+1}/R_k^M(t) - 1\right)M(a) \le \left((b/t)^{k+1} - 1\right)M(a) \le 0 \le M(ab/t).$$

Suppose now $b/t > 1$. Obviously $t \leq t/b < 1$, and (9) immediately follows from

$$r_k^M(t) \geq t\frac{M(a)}{(t/b)^{k+1} M(ab/t)} = \frac{b^{k+1} M(a)}{t^k M(ab/t)}.$$

Thus Lemma 3 is proved. ∎

Lemma 4: *Let $M \in AC^k$. Then for any real u, v with $u^2 + v^2 \neq 0$ we have*

$$\left|M^{(k)}(u+v) - M^{(k)}(u)\right| \leq c_1 |v| M(2c\xi)/\xi^{k+1},$$

where $c_1 = 2^{k+1}c$, $\xi = \max(|u|, |v|)$.

Proof: Suppose first that $|v| < |u|/2$. In this case $\xi = |u|$, and using (2) we have

$$\left|M^{(k)}(u+v) - M^{(k)}(u)\right| = \left|\int_u^{u+v} M^{(k+1)}(t)\,dt\right|$$

$$\leq c\int_{\min(|u|,|u+v|)}^{\max(|u|,|u+v|)} \frac{M(ct)}{|t|^{k+1}}dt \leq c\frac{|v| M(c(|u|+|v|))}{(|u|-|v|)^{k+1}} \leq 2^{k+1}c|v|\frac{M(2c\xi)}{\xi^{k+1}}.$$

If $|v| \geq |u|/2$, then $|v|/\xi \geq 1/2$ and using (4) and (3) we obtain

$$\left|M^{(k)}(u+v) - M^{(k)}(u)\right| \leq c\left(\frac{M(c(u+v))}{|u+v|^k} + \frac{M(cu)}{|u|^k}\right)$$

$$\leq \frac{c|v|}{2^{k-1}\xi^{k+1}}\left(\left(\frac{2\xi}{|u+v|}\right)^k M(c(u+v)) + \left(\frac{2\xi}{|u|}\right)^k M(u)\right) \leq 2^{2-k}c|v|\frac{M(2c\xi)}{\xi^{k+1}}.$$

Thus Lemma 4 is proved. ∎

We associate to every $M \in AC^k$ and $x \in X = L_M(S, \Sigma, \mu)$ the symmetric i-linear forms $(i = 1, 2, \ldots, k)$ defined by

$$\widetilde{M}_i(x;\ y_1, y_2, \ldots, y_i) = \int_S M^{(i)}(x(s)) \prod_{k=1}^{i} y_k(s)\,d\mu(s)$$

and the symmetric $(i-j)$-linear forms $(0 \leq j < i)$

$$\widetilde{M}_{i,j}(x;\ y_1, y_2, \ldots, y_{i-j}) = \int_S M^{(i)}(x(s))\,x^j(s) \prod_{k=1}^{i-j} y_k(s)\,d\mu(s),$$

where $y_1, y_2, \ldots, y_i \in X$. Obviously $\widetilde{M}_{i,0} = \widetilde{M}_i$ and $M_0 = \widetilde{M}$.

Lemma 5: *Let $M \in AC^k$. Then $\widetilde{M}_{i,j}(x) \in B^{i-j}(X)$ for every $x \in X$ $(i = 1, 2, \ldots, k;\ 0 \leq j < i)$ and*

$$\left\|\widetilde{M}_{i,j}(x)\right\| \leq c^{k+1}\left(\widetilde{M}(cx) + c\right).$$

Proof: It is sufficient to show that for fixed $x \in X$

$$\sup\left\{\left|\widetilde{M}_{i,j}\left(x;\ h^{(i-j)}\right)\right|;\ \|h\| \le 1/c\right\} < \infty \ (i = 1,2,\dots,k;\ 0 \le j < i).$$

Denote $S_1 = \{s \in S;\ 0 \le |x(s)| \le |h(s)|\}$ and $S_2 = S \setminus S_1$. Using (3) and (4) we have

$$\begin{aligned}\left|\widetilde{M}_{i,j}\left(x;\ h^{(i-j)}\right)\right| &\le \int_S \left|M^{(i)}(x(s))\right| \ |x(s)|^j\,|h(s)|^{i-j}\,d\mu(s)\\ &\le c\left(\int_{S_1} M(cx(s))(|h(s)|/|x(s)|)^{i-j}\,d\mu(s) + \int_{S_2} M(cx(s))\,d\mu(s)\right)\\ &\le c^2\int_{S_1} M(ch(s))\,d\mu(s) + c\int_{S_2} M(cx(s))\,d\mu(s) \le c\left(\widetilde{M}(cx) + c\right). \ \blacksquare\end{aligned}$$

The next lemma essentially shows that $M^{(k)} \in H^{\omega}(\mathbf{R}^+)$, $\omega(t) = r_k^M(t)$.

Lemma 6: *Let $M \in AC^k$. For any real u, v, w, t such that $u^2 + v^2 \ne 0$, $|t| \le 1/4c$, the inequalities*

$$\left|\left(M^{(k)}(u+tv) - M^{(k)}(u)\right)u^i w^{k-i}\right| \le c_2\left(M(4cu) + M(v) + M(w)\right) r_k^M(|t|) \qquad (10)$$

for $0 \le i \le k-1$ and

$$\left|\left(M^{(k)}(u+tv) - M^{(k)}(u)\right)u^k\right| \le c_2\left(M(4cu) + M(v)\right) r_k^M(|t|) \qquad (10')$$

hold, where $c_2 = 2c_1(2c)^{k+1}$.

Proof: Lemma 4 implies

$$\left|\left(M^{(k)}(u+tv) - M^{(k)}(u)\right)u^i w^{k-i}\right| \le c_1\,|tv|\,|w|^{k-i}\,M(2c\xi)/\xi^{k+1-i} \qquad (11)$$

where $\xi = \max(|u|, |tv|)$. It is clear that to estimate the right-hand side of (11) it suffices to consider only positive u, v, w, t, and $M(1) = 1$. We separate the following cases:

a) $w \le 2c\xi$. Using (5) and (6) we obtain

$$\begin{aligned}&tvw^{k-i}M(2c\xi)/\xi^{k+1-i} \le (2c)^{k+1-i}\,tvM(2c\xi)/(2c\xi)\\ &\le (2c)^{k+1-i}\,t\left(M(2c\xi) + M(v)\right) \le 3(2c)^{k+1-i}\,t\left(M(4cu) + M(v)\right).\end{aligned}$$

b) $2c\xi \le w \le 2c\xi/t$. Now

$$\frac{tvw^{k-i}M(2c\xi)}{\xi^{k+1-i}} \le (2c)^{k+1-i}\,t\left(\left(\frac{v}{2c\xi}\right)^{k+1-i} + \left(\frac{w}{2c\xi}\right)^{k+1-i}\right)M(2c\xi). \qquad (12)$$

If $i \ne 0$ using once more (5) and (6) we obtain

$$\begin{aligned}\frac{tvw^{k-i}M(2c\xi)}{\xi^{k+1-i}} &\le (2c)^{k+1-i}\,t\left(2M(2c\xi) + M(v) + M(w)\right)\\ &\le 2(2c)^{k+1-i}\,t\left(M(4cu) + M(v) + M(w)\right).\end{aligned}$$

If $i = 0$ we continue the estimation in (12) using Lemma 3:

$$\begin{aligned}\frac{tvw^{k-i}M(2c\xi)}{\xi^{k+1-i}} &\le (2c)^{k+1}t^{-k}\left(\left(\frac{tv}{2c\xi}\right)^{k+1}+\left(\frac{tw}{2c\xi}\right)^{k+1}\right)M(2c\xi)\\ &\le (2c)^{k+1}(2M(2c\xi)+M(v)+M(w))\,r_k^M(t)\\ &\le 2(2c)^{k+1}(M(4cu)+M(v)+M(w))\,r_k^M(t).\end{aligned}$$

Finally we consider

c) $w \ge 2c\xi/t$. This case is quite easy. Indeed

$$\begin{aligned}\frac{tvw^{k-i}M(2c\xi)}{\xi^{k+1-i}} &\le \left(\frac{w}{\xi}\right)^{k-i}M(2c\xi)=(2c)^{k-i}t^{i-k}\left(\frac{wt}{2c\xi}\right)^{k-i}M(2c\xi)\\ &\le (2c)^{k-i}t^{i-k}M(wt)\le(2c)^{k-i}t^{i}r_k^M(t)\,M(w),\end{aligned}$$

where we used (8) in the last inequality.

Combining the estimates obtained in the cases a), b) and c) it is easy to get (10) with $c_2 = 2c_1(2c)^{k+1}$. The proof of (10′) is the same. Thus Lemma 6 is proved. ∎

Corollary 1: *Let $M \in AC^k$. Then for every $x, h \in X$ and any $t \in (0, 1/4c)$ the estimate*

$$\left|\widetilde{M}(x+th)-\sum_{j=0}^{k}\frac{t^j}{j!}\widetilde{M}_j\left(x;\ h^{(j)}\right)\right|\le c_3\left(\widetilde{M}(4cx)+\widetilde{M}(h)\right)R_k^M(t), \tag{13}$$

holds, where $c_3 = 2c_2/k!$.

Proof: Obviously, for every $s \in S$:

$$\left|M(x(s)+th(s))-\sum_{j=0}^{k}\frac{(th(s))^j}{j!}M^{(j)}(x(s))\right|\le\frac{|th(s)|^k}{k!}\left|M^{(k)}(x(s)+\theta_s th(s))-M^{(k)}(x(s))\right|$$

for some $\theta_s \in (0,1)$. Lemma 6 applied for $i = 0$, $u = x(s)$, $v = \theta_s h(s)$, $w = h(s)$ gives

$$\left|M(x(s)+th(s))-\sum_{j=0}^{k}\frac{(th(s))^j}{j!}M^{(j)}(x(s))\right|\le\frac{2c_2}{k!}(M(4cx(s))+M(h(s)))\,R_k^M(t).$$

Now to obtain (13) we only have to integrate over S the last inequality. ∎

Remark 1: Corollary 1 is a quantitative improvement of Lemma 4 in [10], where only $o\left(|t|^k\right)$ instead of $|t|^k r_k^M(|t|)$ in the right-hand side of (13) was given. The estimate (13) implies, of course, that $\widetilde{M}_i : X \to B^i(X)$ is $(k-i)$-times differentiable in X for $i = 0, \ldots, k-1$ ($B^0(X) = \mathbf{R}^+$) and $D^j\widetilde{M} = \widetilde{M}_j$ $(j = 1, 2, \ldots, k)$, $D^i\widetilde{M}_j = \widetilde{M}_{i+j}$ $(i + j \le k)$.

Corollary 2: *We have $\widetilde{M}_{k,j} \in H^\omega\left(X, B^{k-j}(X)\right)$ $(j = 0, 1, \ldots, k)$, where $\omega(t) = r_k^M(t)$.*

Proof: We shall prove that for any y, z from the ball $B(x;\ 1/8c)$ the inequalities

$$\left\|\widetilde{M}_{k,j}(y) - \widetilde{M}_{k,j}(z)\right\| \le c_4\varphi(x)\, r_k^M(\|y-z\|),\ j = 0,1,2,\ldots,k-1, \tag{14}$$

$$\left|\widetilde{M}_{k,k}(y) - \widetilde{M}_{k,k}(z)\right| \le c_4\varphi(x)\, r_k^M(\|y-z\|), \tag{14'}$$

where $c_4 = \kappa 8^{k-2}c^{2k}c_2$, $\varphi(x) = (8c\|x\|+1)^{k-2}\left(\widetilde{M}(8cx) + 8c + 1\right)$, hold. Indeed, let $h \in S(X)$, $s \in S$. Obviously

$$\begin{aligned}&\left|\widetilde{M}_{k,j}\left(y;\ h^{(k-j)}\right) - \widetilde{M}_{k,j}\left(z;\ h^{(k-j)}\right)\right|\\ &\qquad \le \left|\int_S M^{(k)}(y(s))\left(y^j(s) - z^j(s)\right) h^{k-j}(s)\, d\mu(s)\right|\\ &\qquad + \int_S \left|M^{(k)}(y(s)) - M^{(k)}(z(s))\right| |z(s)|^j\, |h(s)|^{k-j}\, d\mu(s).\end{aligned}$$

The second member of the last sum is easily estimated for $j < k$ using (10) for $u = z(s)$, $v = ((y(s) - z(s)))/\|y-z\|$, $t = \|y-z\|$, $w = h(s)$ and (6):

$$\begin{aligned}&\int_S \left|M^{(k)}(y(s)) - M^{(k)}(z(s))\right| |z(s)|^j\, |h(s)|^{k-j}\, d\mu(s)\\ &\le c_2\left(\widetilde{M}(4cz) + \widetilde{M}((y-z)/\|y-z\|) + \widetilde{M}(h)\right) r_k^M(\|y-z\|)\\ &\le \frac{c_2}{2}\left(\widetilde{M}(8cx) + 5\right) r_k^M(\|y-z\|).\end{aligned}$$

To obtain (14) for $j < k$ with $c_4 = \kappa 8^{k-2}c^{2k}c_2$ and $\varphi(k) = (8c\|x\|+1)^{k-1}\left(\widetilde{M}(8cx) + 8c + 1\right)$, it is enough to estimate the first member of (15) in the following way:

$$\begin{aligned}&\left|\int_S M^{(k)}(y(s))\left(y^j(s) - z^j(s)\right) h^{k-j}(s)\, d\mu(s)\right|\\ &\le \left\|\widetilde{M}_k(y)\right\|_1 \|y-z\| \sum_{i=0}^{j-1} \|y\|^i\, \|z\|^{j-i-1}\, \|h\|^{k-j}\\ &\le \kappa c^{k+1}\left(\widetilde{M}(cy) + c\right)(\|x\| + 1/8c)^{j-1}\, \|y-z\|\\ &\le \kappa 8^{k-1}c^{2k}\left(\frac{\widetilde{M}(2cx) + \widetilde{M}(2c(y-x))}{2} + c\right)(8c\|x\|+1)^{k-2}\, \|y-z\|\\ &\le \kappa 8^{k-2}c^{2k}\left(4\widetilde{M}(2cx) + 8c + 1\right)(8c\|x\|+1)^{k-2}\, \|y-z\|.\end{aligned}$$

We used Lemma 5, the relation between the norms $\|\cdot\|$ and $\|\cdot\|_1$ in $B^k(X)$ and the convexity of M. The proof of (14′) uses (10′) and is practically the same. ∎

Remark 2: Obviously $\widetilde{M}_j \in H^1(X, B^j(X))$ and $\widetilde{M}_{i,j} \in H^1(X, B^{i-j}(X))$ for $0 \le j \le i < k$.

4. Main result

We are ready to prove the following

Theorem 1: *Let $M \in AC^k$ and (S, Σ, μ) be a measure space. Then $X = H_M(S, \Sigma, \mu)$ is H^ω-smooth, where $\omega(t) = R_k^M(t)$.*

Proof: Set $n(x) = \|x\|$. Using Remark 1 and the implicit function theorem applied to the equation $\widetilde{M}(x/n(x)) - 1 = 0$ we obtain as in [10, Theorem 6] $n'(x) = \widetilde{M}_{1,0}(x/n(x)) / \widetilde{M}_{1,1}(x/n(x))$, which, by an easy induction argument, implies that n is k-times differentiable in $X \setminus \{0\}$. What we have to prove in addition is that $n^{(k)} \in H^{\omega_1}\left(X \setminus 0, B^k(X)\right)$, $\omega_1 = r_k^M$. To this end we need some more information about the k-th derivative of the norm.

First for sake of brevity we introduce the notation

$$\overline{M}_{i,j}(x) = \widetilde{M}_{i,j}(x/n(x)).$$

Using the equality

$$D\left(\frac{x}{n(x)}\right)(y) = \frac{y}{n(x)} - \frac{x}{n^2(x)} Dn(x;\ y) = \frac{y\overline{M}_{1,1}(x) - \overline{M}_{1,0}(x;\ y)}{n(x)\overline{M}_{1,1}(x)},$$

we obtain by induction

$$n^{(k)}(x) = \frac{\sum_{i=0}^{k} C_k^i (-1)^i \overline{M}_{k,i}(x) \overline{M}_{1,1}^{k-i}(x) \overline{M}_{1,0}^{i}(x) + P\left(\overline{M}_{i,j}(x)\right)}{n^{k-1}(x) \overline{M}_{1,1}^{k+1}(x)}, \tag{15}$$

where $P\left(\overline{M}_{i,j}(x)\right)$ is a polynomial with respect to $\overline{M}_{i,j}$ $(i < k)$ and $P\left(\overline{M}_{i,j}(x)\right) \in B^k(X)$ for fixed x.

Let $\omega_1 = r_k^M$. It is easy to check that $f \in H^{\omega_1}\left(X, B^k(X)\right)$, $g \in H^1(x, \mathbf{R}^+)$ imply $f/g \in H^{\omega_1}\left(X \setminus A;\ B^k(X)\right)$, where $A = \{x \in X : g(x) = 0\}$. Indeed, fix $x \notin A$. Then for sufficiently small $\delta > 0$,

$$\left\|\frac{f(y)}{g(y)} - \frac{f(z)}{g(z)}\right\| \le 6 \frac{\|f(y) - f(z)\| \|g(x)\| + \|g(y) - g(z)\| \|f(x)\|}{\|g(x)\|^2}, \tag{16}$$

for any $y, z \in B(x;\ \delta)$. Let now $x \neq 0$, $r = \min(\|x\|/2, 1/8c)$. As

$$\left\|\frac{y}{\|y\|} - \frac{z}{\|z\|}\right\| \le 4 \frac{\|y - z\|}{\|x\|}$$

for $y, z \in B(x;\ r)$, from (14) and (14′) and Lemma 1 it follows for any $y, z \in B(x;\ r)$ that

$$\begin{aligned} \left\|\overline{M}_{k,j}(y) - \overline{M}_{k,j}(z)\right\| &\le c_4 \varphi(x) r_k^M \left(\left\|\frac{y}{\|y\|} - \frac{z}{\|z\|}\right\|\right) \\ &\le c_4 \varphi(x) \max\left(\frac{4}{\|x\|}, 1\right) r_k^M (\|y - z\|)\ (j = 0, 1, \ldots, k). \end{aligned} \tag{17}$$

Obviously n and $\overline{M}_{i,j}$ are in $H^1(X, B^{i-j}(X))$ for every $i = 1, 2 \ldots, k-1$, $0 \le j \le i$ and therefore $n\overline{M}_{1,1} \in H^1(X, \mathbf{R}^+)$. Now from the representation (15) of $n^{(k)}$, (16) and (17) it follows that $n^{(k)} \in H^{\omega_1}\left(X \setminus \{0\}, B^k(X)\right)$. Thus Theorem 1 is proved. ∎

5. Smooth renormings in arbitrary Orlicz spaces

We shall treat in details only the case of sequence spaces. The cases $H_M(0,1)$, $H_M(0,\infty)$, and the general case $H_M(S,\Sigma,\mu)$ can be analogously treated. It is not hard to see that, without some kind of smoothness of the Orlicz function M, the condition $\alpha_M^0 > 1$ cannot ensure the differentiability of the usual norm in h_M. Nevertheless, equivalent smooth Orlicz renorming, i.e. one generated by a suitable Orlicz function, equivalent to M at 0, is possible. Namely, the following theorem holds true.

Theorem 2: *Let $\alpha_M^0 \in (1,\infty)$. Then in h_M there exists an equivalent H^ω-smooth norm, where $\omega(t) = R_{k,[0,1]}^M(t)$, $k = E(\alpha_M^0)$.*

Proof: As usual we suppose $M(1) = 1$, and since the behaviour of M at ∞ is unessential, we "correct" it near ∞ to a function N in such a way that $\alpha_N = \alpha_M^0$. For example:

$$N(t) = \begin{cases} M(t), & t \in [0,1] \\ t^{k+1}, & t \in [1,\infty) \end{cases}.$$

Put $N_2(t) = \int_0^t N_1(u)\exp(u/(u-t))\,du/u$, where $N_1(t) = \int_0^t N(u)\,du/u$. It is not hard to check that

$$(1/4)\,N(t/4) \le e^{-1}N(t/4) \le N_2(t) \le N(t),\ t \in [0,\infty). \tag{19}$$

This implies that h_{N_2} is isomorphic to h_M. Moreover, it is easily verified that $N_2 \in AC^k$ for any k, and we may apply Theorem 1: h_{N_2} is H^ω-smooth, $\omega(t) = R_k^{N_2}(t)$, $k = E(\alpha_{N_2}) = E(\alpha_M^0)$.

Lemma 2 and (19) imply

$$4^{-2(k+2)}R_k^N(t) \le R_k^{N_2}(t) \le 4^{2(k+2)}R_k^N(t),\ t \in [0,1].$$

To finish we observe that $R_k^N = R_{k,[0,1]}^M$. Indeed, $R_k^N \ge R_{k,[0,1]}^M$ is obvious, and we only have to prove $R_k^N \le R_{k,[0,1]}^M$. Analogously to the proof of Lemma 2 we show that for any $(u,v) \in [t,1]\times\mathbf{R}^+$ there are u_1 and v_1, $(u_1,v_1) \in [t,1]\times[0,1]$ with $F_k^N(u,v) = F_k^M(u_1,v_1)$. If $v \le 1$ we take simply $u_1 = u$, $v_1 = v$. Let $v > 1$ and $uv \le 1$. Then

$$F_k^N(u,v) = (uv)^{-(k+1)}M(uv) = F_k^M(u_1,v_1),\, u_1 = uv, v_1 = 1.$$

Finally, if $uv > 1$, then $F_k^N(u,v) = 1 = F_k^M(u_1,v_1)$, $u_1 = v_1 = 1$. Theorem 2 is proved ∎

Remark 3: This result is of no interest if $M \sim t^{2p}$ at 0, $p \in \mathbf{N}$, because it is well known that in l_{2p} the usual norm is infinitely many times differentiable. On the other hand if $M \not\sim t^{2p}$ at 0, $p \in \mathbf{N}$, the best order of smoothness in h_M by equivalent renorming is not better than $t^{\alpha_M^0}$. Indeed, for $\alpha_M^0 \ne 2p$, $p \in \mathbf{N}$ this follows simply from the fact that $l_{\alpha_M^0}$ is isomorphic to a subspace of h_M, if we combine this with the result from [1] formulated in the Introduction. If $\alpha_M^0 = 2p$, $p \in \mathbf{N}$, but $M \not\sim t^{2p}$ at 0, it was shown in [11] that in h_M there is no equivalent α_M^0-times differentiable norm, i.e. any equivalent norm in h_M is again at most $H^{\alpha_M^0}$-smooth.

Corollary 3: *Let $M \not\sim t^{2p}$ at 0, $p \in \mathbf{N}$. Then*

a) *the order of smoothness* $R^M_{k,[0,1]}$, $k = E(\alpha^0_M)$ *cannot be improved with respect to power type orders;*

b) *if*

$$M(uv) \le cu^{\alpha^0_M} M(v), \; u, v \in [0,1], \tag{20}$$

then in h_M *there is an* $H^{\alpha^0_M}$*-smooth norm, i.e.norm of best order of smoothness.*

Proof: To obtain a) it is sufficient to observe that from the definition of α^0_M it follows for any $u, v \in [0,1]$ and fixed $\varepsilon > 0$ that $M(u,v) \le c_\varepsilon u^{\alpha^0_M - \varepsilon} M(v)$, for some $c_\varepsilon > 0$, which implies for $k = E(\alpha^0_M)$ that $R^M_{k,[0,1]}(t) \le c_\varepsilon t^{\alpha^0_M - \varepsilon}$, $t \in [0,1]$.

b) In this case $R^M_{k,[0,1]}(t) \le ct^{\alpha^0_M}$, $k = E(\alpha^0_M)$ ∎

Remark 4: The condition (20) is fulfilled for example if $M\left(t^{1/\alpha^0_M}\right)$ is quasi-convex. Results analogous to those from Theorem 2 and Corollary 3 b) can be obtained for the function spaces $H_M(0,1)$ and $H_M(0,\infty)$ and for general Orlicz spaces $H_M(S,\Sigma,\mu)$, as well, using the same techniques and results on embeddings of l_p spaces in Orlicz function spaces [8]. The corresponding orders of smoothness for $H_M(0,1)$ and $H_M(0,\infty)$ are respectively

$$\Re^M_k(t) = t^{k+1} \sup\left\{1/F^M_k(u,v);\; u \in [1, 1/t],\; v \in [1,\infty)\right\},\; k = E(\alpha^\infty_M)$$

and $R^M_k(t)$, $k = E(\alpha_M)$.

Remark 5: Very probably the orders of smoothness from Theorem 2 and Remark 4 are the best ones in general as they agree with those from [10] for the cases α^0_M, α^∞_M, $\alpha_M \in (1,2)$ that are the best possible up to arbitrary (not only Orlicz) equivalent renorming (see [3, 4]).

Finally we give some examples.

Examples: Let $M(t) = t^p(1+|lnt|)^q$, $p > 1$. Obviously M satisfies the Δ_2-condition at 0 and at ∞ and $\alpha^0_M = \alpha^\infty_M = p$. Therefore $h_M = l_M$, $H_M(0,1) = L_M(0,1)$ and

a) if $q < 0$:

$R^M_{E(p),[0,1]}(t) \le t^p$, l_M is H^p-smooth and the usual norm is norm of best smoothness;

$\Re^M_{E(p)}(t) \le 2/M(1/t)$ for small t and $L_M(0,1)$ is H^M-smooth.

b) if $q > 0$:

$R^M_{E(p),[0,1]}(t) \le 2M(t)$ for small t and l_M is H^M-smooth;

$\Re^M_{E(p)} \le ct^p$ and $L_M(0,1)$ is H^p-smooth and the usual norm is norm of best smoothness.

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