

Continuous Dependence Results for Subdifferential Inclusions

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In this paper we examine the dependence on a parameter of the solution set of a class of nonlinear evolution inclusions driven by subdifferential operators. We prove that under mild hypotheses on the data, the solution set depends continuously on the parameter for both the Vietoris and Hausdorff topologies. Then we use these results to study the variational stability of the class of semilinear parabolic optimal control problems and we also indicate how our work incorporates the stability analysis of differential variational inequalities.

Key words: Subdifferential, fixed points, G -convergence, optimal control, Vietoris and Hausdorff continuities, differential variational inequalities

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1. Introduction

Let $T = [0, b]$ and H a separable Hilbert space. We consider the following parametrized family of evolution inclusions of subdifferential type:

$$-\dot{x}(t) \in \partial\varphi(x(t), \lambda) + F(t, x(t), \lambda) \text{ a.e., } x(0) = x_0(\lambda). \quad (1)$$

Denote the set of strong solutions (see Section 2) of (1) by $S(\lambda) \subseteq C(T, H)$. The purpose of this note is to study continuity properties of the multifunction $\lambda \rightarrow S(\lambda)$. Analogous continuous dependence results were obtained earlier by Vasilev [21] and Lim [9] for differential inclusions in \mathbb{R}^n and by Tolstonogov [19] and Papageorgiou [12], who considered differential inclusions in Banach spaces, but without subdifferential operators present. In fact, their hypotheses are such that preclude the application of their work to multivalued partial differential equations and to distributed parameter optimal control problems. More recently, Kravvaritis and Papageorgiou [8] considered evolution inclusions of subdifferential type and under more restrictive hypotheses on the data established that the solution multifunction $S(\cdot)$ has a closed graph (see Theorem 4.1 in Kravvaritis and Papageorgiou [8]).

In this paper, under general hypotheses on the data (weaker than those in Theorem 4.1 of Kravvaritis and Papageorgiou [8]), we prove that $S(\cdot)$ is continuous for both the Vietoris and Hausdorff metric topologies (see Theorems 3.2 and 3.3). Then we

use these results to establish a sensitivity result for a class of semilinear parabolic distributed parameter optimal control problems.

2. Preliminaries

In what follows, $T = [0, r]$, equipped with the Lebesgue measure dt , and H is a separable Hilbert space. Throughout this paper we will use the following notations:

$$P_{f(c)}(H) = \{A \subseteq H: \text{nonempty, closed (convex)}\}$$

$$P_{(w)k(c)}(H) = \{A \subseteq H: \text{nonempty, (weakly-) compact (convex)}\}.$$

A multifunction $F: T \rightarrow P_f(H)$ is said to be *measurable* if, for all $x \in H$, $t \rightarrow d(x, F(t)) = \inf\{\|x - v\| : v \in F(t)\}$ is a measurable \mathbb{R}_+ -valued function. By S_F^1 we will denote the set of selectors of $F(\cdot)$ that belong to the Lebesgue-Bochner space $L^1(H)$; i.e., $S_F^1 = \{f \in L^1(H): f(t) \in F(t) \text{ a.e.}\}$. This set may be empty. For a measurable $F(\cdot)$, it is nonempty if and only if $t \rightarrow \inf\{\|v\| : v \in F(t)\} \in L^1_+$.

Let $\varphi: H \rightarrow \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. We will say that $\varphi(\cdot)$ is *proper*, if it is not identically $+\infty$. Assume that $\varphi(\cdot)$ is proper, convex and lower semicontinuous. It is customary to denote this family of $\bar{\mathbb{R}}$ -valued functions by $\Gamma_0(H)$. By *dom* φ , we denote the *effective domain* of $\varphi(\cdot)$; i.e., $\text{dom } \varphi = \{x \in H: \varphi(x) < \infty\}$. The *subdifferential* of $\varphi(\cdot)$ at x is the set $\partial\varphi(x) = \{x^* \in H: (x^*, y - x) \leq \varphi(y) - \varphi(x) \text{ for all } y \in \text{dom } \varphi\}$, where (\cdot, \cdot) denotes the inner product of H . If $\varphi(\cdot)$ is Gateaux differentiable at x , then $\partial\varphi(x) = \{\varphi'(x)\}$. We say that $\varphi(\cdot)$ is of *compact type*, if for every $\lambda \in \mathbb{R}$ the level set $\{x \in H: \|x\|^2 + \varphi(x) \leq \lambda\}$ is compact. Also for $\mu > 0$, we define $J_\mu = (I + \mu\partial\varphi)^{-1}$ (the resolvent of $\partial\varphi(\cdot)$). It is well known (see for example the book of Brezis [3]) that, for all $\mu > 0$, $D(J_\mu) = H$ and furthermore $J_\mu(\cdot)$ is nonexpansive.

Let X a Banach space and $\{A_n, A\}_{n \geq 1} \subseteq 2^X \setminus \{\emptyset\}$. Let s - denote the strong topology on X and w - the weak topology on X . We define:

$$s\text{-}\underline{\lim} A_n = \{x \in X: \lim d(x, A_n) = 0\}$$

$$= \{x \in X: x = s\text{-}\lim x_n, x_n \in A_n, n \geq 1\},$$

$$s\text{-}\overline{\lim} A_n = \{x \in X: \underline{\lim} d(x, A_n) = 0\}$$

$$= \{x \in X: x = s\text{-}\lim x_{n_k}, x_{n_k} \in A_{n_k}, n_1 < n_2 < \dots < n_k < \dots\},$$

$$w\text{-}\overline{\lim} A_n = \{x \in X: x = w\text{-}\lim x_{n_k}, x_{n_k} \in A_{n_k}, n_1 < n_2 < \dots < n_k < \dots\}.$$

It is clear from the above definitions that we always have $s\text{-}\underline{\lim} A_n \subseteq s\text{-}\overline{\lim} A_n \subseteq w\text{-}\overline{\lim} A_n$. If $s\text{-}\underline{\lim} A_n = s\text{-}\overline{\lim} A_n = A$, then we say that the A_n 's converge to A in the *Kuratowski sense* and denote it by $A_n \xrightarrow{K} A$ as $n \rightarrow \infty$. If $s\text{-}\underline{\lim} A_n = w\text{-}\overline{\lim} A_n = A$, then

we say that the A_n 's converge to A in the *Kuratowski-Mosco sense*, denoted by $A_n \xrightarrow{K-M} A$.

Let Λ be a complete metric space. A multifunction $G:\Lambda \rightarrow P_f(X)$ is said to be *upper semicontinuous* (resp. *lower semicontinuous*) if for all $U \subseteq X$ nonempty, open, the set $G^+(U) = \{\lambda \in \Lambda: G(\lambda) \subseteq U\}$ (resp. the set $G^-(U) = \{\lambda \in \Lambda: G(\lambda) \cap U \neq \emptyset\}$) is open in Λ . A multifunction $G(\cdot)$ which is both upper semicontinuous and lower semicontinuous, is said to be *continuous* or *Vietoris continuous*, to emphasize that it is continuous into the hyperspace $P_f(X)$ equipped with the Vietoris topology (see Klein and Thompson [7]). If $\overline{G(\Lambda)} = \overline{\bigcup_{\lambda \in \Lambda} G(\lambda)}$ is compact in X , then $G(\cdot)$ is Vietoris continuous if and only if for $\lambda_n \rightarrow \lambda$ in Λ , we have $G(\lambda_n) \xrightarrow{K} G(\lambda)$. This follows from Remarks 1.6 and 1.8 of DeBlasi and Myjak [4].

On $P_f(X)$ we can define a generalized metric, known in the literature as *Hausdorff metric*, by

$$h(A, B) = \max \left[\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right].$$

Recall that $(P_f(X), h)$ is a complete metric space. A multifunction $G:\Lambda \rightarrow P_f(X)$ is said to be *Hausdorff continuous* (*h-continuous*), if it is continuous from Λ into the metric space $(P_f(X), h)$. On $P_k(X)$ the Vietoris and Hausdorff metric topologies coincide (see Klein and Thompson [7, Corollary 4.2.3, p. 41]). So a multifunction $G:\Lambda \rightarrow P_k(X)$ is Vietoris continuous if and only if it is *h-continuous* (see DeBlasi and Myjak [4, Remark 1.9]). From Theorem 3.3 of Papageorgiou [13], we know that if $F:T \times X \rightarrow P_f(X)$ is a multifunction such that $F(\cdot, x)$ is measurable and $F(t, \cdot)$ is *h-continuous*, $F(\cdot, \cdot)$ is jointly measurable. Finally a multifunction $G:\Lambda \rightarrow P_f(X)$ is said to be *d-continuous* if, for all $x \in X$, $\lambda \rightarrow d(x, G(\lambda))$ is continuous. Clearly if $G(\cdot)$ is *h-continuous*, then it is *d-continuous*, too.

The following theorem was first proved by the author (see [12, Theorem 3.1]) and recently improved by Rybinski (see [18, Theorem 1 and the remark on page 33]). Here we state the improved version obtained by Rybinski [18].

Theorem 2.1: *If X is a Banach space, $K \in P_{*k}(X)$, $F_n, F:K \rightarrow P_{fc}(K)$ are h -Lipschitz multifunctions with the same Lipschitz constant $k \in (0, 1)$ such that if $x_n \xrightarrow{L} x$, then $F_n(x_n) \xrightarrow{K-M} F(x)$, then if $L_n = \{x \in X: x \in F_n(x)\}$ and $L = \{x \in X: x \in F(x)\}$, we have $L_n \xrightarrow{K} L$ as $n \rightarrow \infty$.*

Remark: The fixed point sets L_n, L are nonempty by Nadler's fixed point theorem [11].

Let Λ be a complete metric space (the parameter space), $T = [0, b]$ and H a separable Hilbert space. The following hypothesis concerning $\varphi(x, \lambda)$ will be in effect

throughout this work:

- $H(\varphi)$ $\varphi: H \times \Lambda \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a function such that
- (i) for every $\lambda \in \Lambda$, $\varphi(\cdot, \lambda)$ is proper, convex, lower semicontinuous (i.e. $\varphi(\cdot, \lambda) \in \Gamma_0(H)$) and of compact type,
 - (ii) if $\lambda_n \rightarrow \lambda$ in Λ , then for every $\mu > 0$, we have $(I + \mu\partial\varphi(\cdot, \lambda_n))^{-1}x \rightarrow (I + \mu\partial\varphi(\cdot, \lambda))^{-1}x$ for every $x \in H$.

Also we will make the following hypothesis concerning the initial condition $x_0(\lambda)$ of (1):

H_0 $\lambda \rightarrow x_0(\lambda)$ is continuous from Λ into H and for all $\lambda \in \Lambda$, $x_0(\lambda) \in \text{dom}\varphi(\cdot, \lambda)$.

Given $g \in L^2(H)$, consider the following evolution inclusion:

$$-\dot{x}(t) \in \partial\varphi(x(t), \lambda) + g(t) \text{ a.e., } x(0) = x_0(\lambda). \tag{2}$$

From Brezis [3, Theorem 3.6, p. 72], we know that (2) has a unique strong solution $p(g, \lambda)(\cdot) = x(\cdot) \in C(T, H)$, and in addition since $x_0(\lambda) \in \text{dom}\varphi(\cdot, \lambda)$, we have $\|\dot{x}\|_{L^2(H)} \leq \|g\|_{L^2(H)} + \varphi(x_0, \lambda)^{1/2}$, and $\varphi(x(\cdot), \lambda)$ is absolutely continuous on T . So we can define the solution map $p: L^2(H) \times \Lambda \rightarrow C(T, H)$ by $(g, \lambda) \rightarrow p(g, \lambda)(\cdot)$. The following continuity result concerning $p(\cdot, \cdot)$ can be found in Attouch [1, Theorem 3.74, p. 388].

Theorem 2.2: *If hypotheses $H(\varphi)$ and H_0 hold, then the solution map $p: L^2(H) \times \Lambda \rightarrow C(T, H)$ is continuous.*

By a *strong solution* of evolution inclusion (1) we mean a function $x \in C(T, H)$ such that $x(\cdot)$ is absolutely continuous on any compact subinterval of $(0, b)$, $x(t) \in \text{dom}\varphi(\cdot, \lambda)$ a.e. and $-\dot{x}(t) \in \partial\varphi(x(t), \lambda) + f(t)$ a.e., $f(\cdot) \in L^2(H)$, $f(t) \in F(t, x(t), \lambda)$ a.e., $x(0) = x_0(\lambda)$. We will denote by $S(\lambda) \subseteq C(T, H)$ the set of all strong solutions of the multivalued Cauchy problem (1).

An important selection theorem that we will use in the sequel is that of Aumann and can be found in Wagner [22, Theorem 5.10]. It says that if $G: T \rightarrow 2^H \setminus \{\emptyset\}$ is a multifunction such that $\text{Gr}G = \{(t, v) \in T \times H: v \in G(t)\} \in B(T) \times B(H)$ (i.e. $G(\cdot)$ is graph measurable), then we can find $g: T \rightarrow H$, a Lebesgue measurable function, such that $g(t) \in G(t)$ for all $t \in T$.

A particular case of Theorem 3.1 in Papageorgiou [14] tells us that if $\{f_n, f\}_{n \geq 1} \subseteq L^1(H)$, $f_n \xrightarrow{w} f$ in $L^1(H)$ and, for all $n \geq 1$ and almost all $t \in T$, $\|f_n(t)\| \leq \theta_t$ where $\theta_t > 0$, then $f(t) \in \overline{\text{conv } w\text{-}\overline{\lim} \{f_n(t)\}_{n \geq 1}}$ a.e.

Also from Lemma A.5 of Brezis [3] we know that if $m \in L^1(T, \mathbb{R})$, $m \geq 0$ a.e. $a \in \mathbb{R}_+$ and $u \in C(T, \mathbb{R})$ satisfy $\frac{1}{2}u^2(t) \leq \frac{1}{2}a^2 + \int_0^t m(s)u(s)ds$ for all $t \in T$, then we have $|u(t)| \leq a + \int_0^t m(s)ds$ for all $t \in T$.

Finally from Lemma δ , p. 71 of Papageorgiou [16] we know that if $S:\Lambda \rightarrow P_k(C(T, H))$ is a multifunction such that for each $K \subseteq \Lambda$ compact, the restriction of S on K is upper semicontinuous, then $S(\cdot)$ is upper semicontinuous.

3. Continuous dependence results

In this section we study continuity properties of the solution multifunction $S(\cdot)$. For this, we will need the following hypothesis on the orientor field $F(t, x, \lambda)$:

- $H(F)$ $F:T \times H \times \Lambda \rightarrow P_{wkc}(H)$ is a multifunction such that
- (i) $t \rightarrow F(t, x, \lambda)$ is measurable,
 - (ii) $h(F(t, x, \lambda), F(t, y, \lambda)) \leq k_B(t) \|x - y\|$ a.e. for all $\lambda \in B \subseteq \Lambda$, B compact and with $k_B(\cdot) \in L^1_+$,
 - (iii) $\lambda \rightarrow F(t, x, \lambda)$ is d -continuous,
 - (iv) $|F(t, x, \lambda)| = \sup\{\|v\| : v \in F(t, x, \lambda)\} \leq \alpha_B(t) + \beta_B(t) \|x\|$ a.e. for all $\lambda \in B \subseteq \Lambda$, B compact and with $\alpha_B(\cdot), \beta_B(\cdot) \in L^2_+$.

Because of hypothesis $H(F)$ above we know that, for every $\lambda \in \Lambda$, $S(\lambda)$ is nonempty and compact in $C(T, H)$ (see Kravvaritis and Papageorgiou [8, Theorem 3.1] and Papageorgiou [15, Theorem 4.1]).

Theorem 3.1: *If hypotheses $H(\varphi)$, $H(F)$, H_0 hold and $\lambda_n \rightarrow \lambda$ in Λ , then $S(\lambda_n) \xrightarrow{K} S(\lambda)$ in $C(T, H)$ as $n \rightarrow \infty$.*

Proof: Let $B \subseteq \Lambda$ be a nonempty, compact subset. First we will derive an a priori bound for the elements in $\bigcup_{\lambda \in B} S(\lambda)$. To this end, let $\lambda \in B$, $x(\cdot) \in S(\lambda)$ and let $u_\lambda(\cdot) \in C(T, H)$ be the unique solution of the Cauchy problem

$$-\dot{u}_\lambda(t) \in \partial\varphi(u_\lambda(t), \lambda) \text{ a.e., } u(0) = x_0(\lambda).$$

Exploiting the monotonicity of the subdifferential operator, we have

$$(-\dot{x}(t) + u_\lambda(t), u_\lambda(t) - x(t)) \leq (f(t), u_\lambda(t) - x(t)) \text{ a.e.}$$

with $f \in L^2(H)$, $f(t) \in F(t, x(t), \lambda)$ a.e. and $-\dot{x}(t) \in \partial\varphi(x(t), \lambda) + f(t)$ a.e. Then we have

$$\frac{1}{2} \frac{d}{dt} \|x(t) - u_\lambda(t)\|^2 \leq \|f(t)\| \cdot \|x(t) - u_\lambda(t)\| \text{ a.e.}$$

$$\frac{1}{2} \|x(t) - u_\lambda(t)\|^2 \leq \int_0^t \|f(s)\| \cdot \|x(s) - u_\lambda(s)\| ds.$$

Apply Lemma A.5, p. 157 of Brezis [3] (see Section 2) to get

$$\begin{aligned} \|x(t) - u_\lambda(t)\| &\leq \int_0^t \|f(s)\| ds \leq \int_0^t (\alpha_B(s) + \beta_B(s) \|x(s)\|) ds \\ \|x(t)\| &\leq \|u_\lambda\|_\infty + \int_0^t (\alpha_B(s) + \beta_B(s) \|x(s)\|) ds. \end{aligned}$$

From Theorem 2.2, we know that we can find $\theta_B > 0$ such that $\|u_\lambda\|_\infty \leq \theta_B$ for all $\lambda \in B$. Hence, we get

$$\|x(t)\| \leq \theta_B + \int_0^t (\alpha_B(s) + \beta_B(s)) \|x(s)\| ds, \quad t \in T.$$

Invoking Gronwall's inequality, we deduce that there exists $M_B > 0$ such that, for all $x \in \bigcup_{\lambda \in B} S(\lambda)$, we have $\|x\|_{C(T,H)} \leq M_B$. Hence without any loss of generality, we may assume that $|F(t, x, \lambda)| = \sup\{\|v\| : v \in F(t, x, \lambda)\} \leq \psi_B(t) = \alpha_B(t) + \beta_B(t)M_B$ a.e., $\psi_B(\cdot) \in L^2_+$ for all $\lambda \in B$. Then let $K_B = \{h \in L^1(H) : \|h(t)\| \leq \psi_B(t) \text{ a.e.}\}$ (viewed as a subset of $L^1(H)$) and consider the multifunction $R: K_B \times B \rightarrow P_{fc}(K_B)$ defined by $R(f, \lambda) = S^1_{F(\cdot, p(f, \lambda)(\cdot), \lambda)}$.

On $L^1(H)$, consider the norm $\|g\|_B = \int_0^r \exp[-L \int_0^t k_B(s) ds] \|g(t)\| dt$, $L > 0$, which is clearly equivalent to the usual one. Our claim is that for $L > 1$, the family $\{R(\cdot, \lambda)\}_{\lambda \in B}$ is h -Lipschitz for this norm $\|\cdot\|_B$, with the same Lipschitz constant $\eta_B \in (0, 1)$. To this end let $f, g \in K_B$ and let $v \in R(g, \lambda)$. Let

$$\Gamma(t) = \{u \in F(t, p(f, \lambda)(t), \lambda) : \|v(t) - u\| = d(v(t), F(t, p(f, \lambda)(t), \lambda))\}.$$

Note that for every $t \in T$, $\Gamma(t) \neq \emptyset$ since by hypothesis $H(F)$, F is $P_{wkc}(H)$ -valued. Then observe that

$$Gr\Gamma = \{(t, u) \in GrF(\cdot, p(f, \lambda)(\cdot), \lambda) : \|v(t) - u\| - d(v(t), F(t, p(f, \lambda)(t), \lambda)) = 0\}.$$

Because of hypotheses $H(F)(i)$ and (ii) and Theorem 3.3 of Papageorgiou [13] (see Section 2), $GrF(\cdot, p(f, \lambda)(\cdot), \lambda) \in B(T) \times B(H)$, where $B(T)$ (resp. $B(H)$) is the Borel σ -field of T (resp. of H). Furthermore, $(t, u) \rightarrow \|v(t) - u\| - d(v(t), F(t, p(f, \lambda)(t), \lambda))$ is clearly measurable in $t \in T$ and continuous in $u \in H$ (i.e. a Caratheodory function), thus jointly measurable. Therefore $Gr\Gamma \in B(T) \times B(H)$. Apply Aumann's selection theorem (see Wagner [22, Theorem 5.10 or Section 2]), to get $u: T \rightarrow H$ measurable such that $u(t) \in \Gamma(t)$ a.e. Then we have

$$\begin{aligned} d_B(v, R(f, \lambda)) &\leq \|v - u\|_B \\ &= \int_0^r \|v(t) - u(t)\| \exp \left[-L \int_0^t k_B(s) ds \right] dt \\ &= \int_0^b d(v(t), F(t, p(f, \lambda)(t), \lambda)) \exp \left[-L \int_0^t k_B(s) ds \right] dt \end{aligned}$$

$$\begin{aligned} &\leq \int_0^r h(F(t, p(g, \lambda)(t), \lambda), F(t, p(f, \lambda)(t), \lambda)) \exp \left[-L \int_0^t k_B(s) ds \right] dt \\ &\leq \int_0^r k_B(t) \| p(g, \lambda)(t) - p(f, \lambda)(t) \| \exp \left[-L \int_0^t k_B(s) ds \right] dt. \end{aligned}$$

As in the beginning of the proof, by exploiting the monotonicity of the subdifferential operator and by using Lemma A.5, p. 157 of Brezis [3] (see Section 2), we get

$$\| p(g, \lambda)(t) - p(f, \lambda)(t) \| \leq \int_0^t \| g(s) - f(s) \| ds \text{ for all } (t, \lambda) \in T \times B.$$

So we have

$$\begin{aligned} d_B(v, R(f, \lambda)) &\leq \int_0^r k_B(t) \exp \left[-L \int_0^t k_B(s) ds \right] \int_0^t \| g(s) - f(s) \| ds dt \\ &= -\frac{1}{L} \int_0^r \left(\int_0^t \| g(s) - f(s) \| ds \right) d \left(\exp \left[-L \int_0^t k_B(s) ds \right] \right) \\ &\leq \frac{1}{L} \int_0^r \exp \left[-L \int_0^t k_B(s) ds \right] \| g(s) - f(s) \| ds \quad (\text{by integration by parts}) \\ &\leq \frac{1}{L} \| g - f \|_B. \end{aligned}$$

Similarly for $w \in R(f, \lambda)$, we can get $d_B(w, R(g, \lambda)) \leq \frac{1}{L} \| g - f \|_B$, i.e. $\{R(\cdot, \lambda)\}_{\lambda \in B}$ is h -Lipschitz with constant $\frac{1}{L}$, for the $\| \cdot \|_B$ -norm.

Next, let $[f_n, \lambda_n] \rightarrow [f, \lambda]$ in $(K_B, \| \cdot \|_B) \times B$ imply $[f_n, \lambda_n] \rightarrow [f, \lambda]$ in $L^1(H) \times B$. We will show that $R(f_n, \lambda_n) \xrightarrow{K-M} R(f, \lambda)$. To this end, let $u \in R(f, \lambda)$ and set $\gamma_n(t) = d(u(t), F(t, p(f_n, \lambda_n)(t), \lambda_n))$. Then

$$\begin{aligned} \gamma_n(t) &\leq d(u(t), F(t, p(f, \lambda)(t), \lambda_n)) + h(F(t, p(f, \lambda)(t), \lambda_n), F(t, p(f_n, \lambda_n)(t), \lambda_n)) \\ &\leq d(u(t), F(t, p(f, \lambda)(t), \lambda_n)) + k_B(t) \| p(f, \lambda)(t) - p(f_n, \lambda_n)(t) \| \text{ a.e.} \end{aligned}$$

Because of hypothesis $H(F)(iii)$, we have $d(u(t), F(t, p(f, \lambda)(t), \lambda_n)) \rightarrow 0$ as $n \rightarrow \infty$. Also because of Theorem 2.2, we have $\| p(f, \lambda)(t) - p(f_n, \lambda_n)(t) \| \rightarrow 0$ as $n \rightarrow \infty$, uniformly on T .

Therefore, we get $\gamma_n(t) \rightarrow 0$ a.e. as $n \rightarrow \infty$. As before via Aumann's selection theorem, we can find $u_n(\cdot) \in K_B$ such that $u_n(t) \in F(t, p(f_n, \lambda_n)(t), \lambda_n)$ a.e. and

$$\begin{aligned} \|u(t) - u_n(t)\| &\leq \gamma_n(t) + \frac{1}{n} \text{ a.e.}, \\ u_n(t) &\xrightarrow{s} u(t) \text{ a.e. in } H \text{ as } n \rightarrow \infty, \\ u_n &\xrightarrow{s} u \text{ in } (L^1(H), \|\cdot\|_B). \end{aligned}$$

Since $u_n \in R(f_n, \lambda_n)$, $n \geq 1$ we have established that

$$R(f, \lambda) \subseteq s\text{-}\overline{\lim} R(f_n, \lambda_n). \quad (3)$$

Next, let $v \in w\text{-}\overline{\lim} R(f_n, \lambda_n)$. Denoting subsequences with the same index as original sequences, we know that we can find $v_n \in R(f_n, \lambda_n)$ such that $v_n \xrightarrow{w} v$ in $L^1(H)$. Apply Theorem 3.1 of [14] (see also Section 2), to get

$$v(t) \in \overline{\text{conv}} w\text{-}\overline{\lim} \{f_n(t)\}_{n \geq 1} \subseteq \overline{\text{conv}} w\text{-}\overline{\lim} F(t, p(f_n, \lambda_n)(t), \lambda_n) \text{ a.e.}$$

Note that, for any $v \in H$, we have

$$\begin{aligned} d(v, F(t, p(f, \lambda)(t), \lambda_n)) \\ \leq d(v, F(t, p(f_n, \lambda_n)(t), \lambda_n)) + h(F(t, p(f, \lambda)(t), \lambda_n), F(t, p(f_n, \lambda_n)(t), \lambda_n)) \\ \leq d(v, F(t, p(f_n, \lambda_n)(t), \lambda_n)) + k_B(t) \|p(f, \lambda)(t) - p(f_n, \lambda_n)(t)\| \text{ a.e.} \end{aligned}$$

Then by passing to the limit as $n \rightarrow \infty$ and using Theorem 2.2 together with hypothesis $H(F)(iii)$, we get

$$d(v, F(t, p(f, \lambda)(t), \lambda)) \leq \underline{\lim} d(v, F(t, p(f_n, \lambda_n)(t), \lambda_n)) \text{ a.e.}$$

Invoking Theorem 2.2 (iv) of Tsukada [20], we get

$$\begin{aligned} w\text{-}\overline{\lim} F(t, p(f_n, \lambda_n)(t), \lambda_n) &\subseteq F(t, p(f, \lambda)(t), \lambda) \text{ a.e.} \\ v(t) &\in F(t, p(f, \lambda)(t), \lambda) \text{ a.e.} \\ v &\in R(f, \lambda). \end{aligned}$$

Thus we have established that

$$w\text{-}\overline{\lim} R(f_n, \lambda_n) \subseteq R(f, \lambda). \quad (4)$$

From (3) and (4) above, we have that if $[f_n, \lambda_n] \rightarrow [f, \lambda]$ in $(L^1(H), \|\cdot\|_B) \times B$, then $R(f_n, \lambda_n) \xrightarrow{K\text{-}M} R(f, \lambda)$.

Let $\Phi(\lambda_n) = \{f \in K_B : f \in R(f, \lambda_n)\}$ and $\Phi(\lambda) = \{f \in K_B : f \in R(f, \lambda)\}$. From Theorem 2.1, we have

$$\Phi(\lambda_n) \xrightarrow{K} \Phi(\lambda) \text{ in } L^1(H) \text{ as } n \rightarrow \infty.$$

But since $\psi_B(\cdot) \in L^2_+$ (see the definition of K_B), we can easily see that

$$\Phi(\lambda_n) \xrightarrow{K} \Phi(\lambda) \text{ in } L^2(H) \text{ as } n \rightarrow \infty.$$

Since the solution map $p(\cdot, \cdot): L^2(H) \times \Lambda \rightarrow C(T, H)$ is continuous, we get

$$p(\Phi(\lambda_n), \lambda_n) \xrightarrow{K} p(\Phi(\lambda), \lambda) \text{ in } C(T, H) \text{ as } n \rightarrow \infty.$$

But note that $S(\lambda_n) = p(\Phi(\lambda_n), \lambda_n)$ and $S(\lambda) = p(\Phi(\lambda), \lambda)$. So we have $S(\lambda_n) \xrightarrow{K} S(\lambda)$ in $C(T, H)$ as $n \rightarrow \infty$. □

If we strengthen hypothesis $H(\varphi)$ using Theorem 3.1 above, we can have the Vietoris continuity of the multifunction $S: \Lambda \rightarrow P_k(C(T, H))$. The strengthened version of $H(\varphi)$ that we will need is the following:

- $H(\varphi)'$ $\varphi: H \times \Lambda \rightarrow \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a function such that
- (i) for every $\lambda \in \Lambda$, $\varphi(\cdot, \lambda)$ is proper, convex, lower semicontinuous (i.e. $\varphi(\cdot, \lambda) \in \Gamma_0(H)$),
 - (ii) if $\lambda_n \rightarrow \lambda$ in Λ , then for every $\mu > 0$ we have $(I + \mu \partial \varphi(\cdot, \lambda_n))^{-1}x \rightarrow (I + \mu \partial \varphi(\cdot, \lambda))^{-1}x$ for every $x \in H$,
 - (iii) if $B \subseteq \Lambda$ is compact, then $\overline{\bigcup_{\lambda \in B} \{x \in H: \|x\|^2 + \varphi(x, \lambda) \leq \theta\}}$ is compact for every $\theta > 0$ and $\{\varphi(x_0(\lambda), \lambda): \lambda \in B\}$ is bounded.

Theorem 3.2: *If hypotheses $H(\varphi)'$, $H(F)$ and H_0 hold, then $S: \Lambda \rightarrow P_k(C(T, H))$ is Vietoris continuous.*

Proof: First, note that for any $\lambda \in \Lambda$ and any compact set C containing $x_0(\lambda)$, we have $\inf \{\varphi(x, \lambda): x \in C\} = \varphi(\hat{x}, \lambda)$ for some $\hat{x} \in C$ (Weierstrass theorem). Since $\partial(\varphi(x, \lambda) - \varphi(\hat{x}, \lambda)) = \partial \varphi(x, \lambda)$, we may assume without any loss of generality that, for every $\lambda \in \Lambda$, $\varphi(\cdot, \lambda) \geq 0$.

Let $B \subseteq \Lambda$ compact and let $V_B = \{h \in L^2(H): \|h(t)\| \leq \psi_B(t) \text{ a.e.}\}$, where $\psi_B(\cdot) \in L^2_+$ is as in the proof of Theorem 3.1. Let $W = p(V_B, B)$, where $p(\cdot, \cdot)$ is the solution map. Our claim is that W is relatively compact in $C(T, H)$. So let $x \in W$ and $0 \leq t \leq t' \leq r$. We have

$$\begin{aligned} \|x(t') - x(t)\| &= \left\| \int_t^{t'} \dot{x}(s) ds \right\| \leq \int_t^{t'} \|\dot{x}(s)\| ds \\ &\leq \left[\int_0^r \chi_{[t, t']}(s)^2 ds \right]^{1/2} \left[\int_0^r \|\dot{x}(s)\|^2 ds \right]^{1/2}. \end{aligned}$$

But from Theorem 3.6, p. 72 of Brezis [3] (see Section 2), we have

$$\left[\int_0^r \|\dot{x}(s)\|^2 ds \right]^{1/2} \leq \|\psi_B\|_2 + \sup_{\lambda \in B} \varphi(x_0, \lambda) = M < \infty$$

(see hypothesis $H(\varphi)'(iii)$). So we get $\|x(t') - x(t)\| \leq M(t' - t)^{1/2}$, i.e. W is equicontinuous.

Furthermore, using once more Theorem 3.6 of Brezis [3] (see Section 2), we have

$$\|\dot{x}(t)\|^2 + \frac{d}{dt}\varphi(x(t), \lambda) = (h(t), \dot{x}(t)),$$

$$\frac{d}{dt}\varphi(x(t), \lambda) \leq (h(t), \dot{x}(t)) \text{ a.e.,}$$

$$\begin{aligned} \varphi(x(t), \lambda) &\leq \varphi(x_0, \lambda) + \int_0^t \|h(s)\| \|\dot{x}(s)\| ds \\ &\leq \varphi(x_0, \lambda) + \|h\|_2 \|\dot{x}\|_2 \leq \varphi(x_0, \lambda) + \|\psi_B\|_2 M \leq M_1 \end{aligned}$$

for all $\lambda \in B$ (see hypothesis $H(\varphi)'(iii)$). Thus

$$\overline{W(t)} = \overline{\{x(t): x(\cdot) \in W\}} \subseteq \overline{\bigcup_{\lambda \in B} \{v \in H: \|v\|^2 + \varphi(v, \lambda) \leq M_1\}} \in P_k(H)$$

(see hypothesis $H(\varphi)'(3)$). Therefore by the Arzela-Ascoli theorem, we deduce that \overline{W} is compact in $C(T, H)$ and $S(\lambda) \subseteq \overline{W}$ for all $\lambda \in B$. Combining this fact with Theorem 3.1 above, we get that $S|_B$ is Vietoris continuous. Since $B \subseteq \Lambda$ was an arbitrary compact subset, from Lemma δ , p. 71 of [16] (see Section 2) and Remark 1.7 of DeBlasi and Myjak [4] we conclude that $S(\cdot)$ is Vietoris continuous \square

Finally, recalling that the Vietoris and Hausdorff metric topologies coincide on $P_k(C(T, H))$ (see Section 2), we also have

Theorem 3.3: *If hypotheses $H(\varphi)', H(F)$ and H_0 hold, then $S: \Lambda \rightarrow P_k(C(T, H))$ is h -continuous.*

4. Sensitivity analysis in optimal control

In this section, we use the previous theorems to study the variational stability of a class of nonlinear distributed parameter optimal control problems.

So let $T = [0, r]$ and $Z = [0, b]$. Let Λ be a complete metric space (the parameter space). We consider the following parametrized parabolic optimal control problem:

$$\begin{aligned} &\int_0^b \eta(z, x(r, z), \lambda) dz \rightarrow \inf = m(\lambda) \\ \text{subject to } &\frac{\partial x(t, z)}{\partial t} - \frac{\partial}{\partial z}(z(z, \lambda) \frac{\partial x}{\partial z}) = f(t, z, x(t, z), \lambda) u(t, z) \text{ a.e.} \end{aligned} \tag{5}$$

$$x(0, z) = x_0(z, \lambda), x(t, 0) = x(t, b) = 0 \text{ and } |u(t, z)| \leq v(t, z, \lambda) \text{ a.e.}$$

$u(\cdot, \cdot)$ -measurable.

We will need the following hypotheses on the data of (5):

$H(a)$ $0 < m_1 \leq a(t, z) \leq m_2$ a.e.

$H(f)$ $f: T \times Z \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R}$ is a function such that

(i) $(t, z) \rightarrow f(t, z, x, \lambda)$ is measurable,

(ii) $|f(t, z, x, \lambda) - f(t, z, x', \lambda)| \leq k_B(t, z) |x - x'|$ a.e. with $k_B \in L^1(T \times Z)$, $\lambda \in B \subseteq \Lambda$, $B = \text{compact}$,

(iii) $\lambda \rightarrow f(t, z, x, \lambda)$ is continuous,

(iv) $|f(t, z, x, \lambda)| \leq a_B(t, z) + c_B(t, z) |x|$ a.e. with $a_B \in L^2(T \times Z)$, $c_B \in L^\infty(T \times Z)$, $\lambda \in B \subseteq \Lambda$, $B = \text{compact}$.

$H(r)$ $(t, z) \rightarrow v(t, z, \lambda)$ is measurable, $\lambda \rightarrow v(t, z, \lambda)$ is continuous and $|v(t, z, \lambda)| \leq \theta_B(t, z)$ a.e. with $\theta_B(\cdot, \cdot) \in L^\infty(T \times Z)$, $\lambda \in B \subseteq \Lambda$, $B = \text{compact}$.

$H(\eta)$ $\eta: Z \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R}$ is an integrand such that

(i) $z \rightarrow \eta(z, x, \lambda)$ is measurable,

(ii) $(x, \lambda) \rightarrow \eta(z, x, \lambda)$ is continuous,

(iii) $|\eta(z, x, \lambda)| \leq \psi_{1B}(z) + \psi_{2B}(z) |x|^2$ a.e. with $\psi_{1B}(\cdot) \in L^2_+$, $\psi_{2B} \in L^2_+$, $\lambda \in B \subseteq \Lambda$, $B = \text{compact}$.

H_0 $x_0(\cdot, \lambda) \in H^1_0(Z)$ and $\lambda \rightarrow x_0(\cdot, \lambda)$ is continuous from Λ into $L^2(Z)$.

H_c If $\lambda_n \rightarrow \lambda$ in Λ , then $\frac{1}{a(\cdot, \lambda_n)} \xrightarrow{w} \frac{1}{a(\cdot, \lambda)}$ in $L^2(Z)$.

Let $Q(\lambda) \subseteq C(T, L^2(Z))$ be the set of optimal trajectories of (5).

Theorem 4.1: *If hypotheses $H(a), H(f), H(r), H(\eta), H_0$ and H_c hold, then for every $\lambda \in \Lambda$, $Q(\lambda) \neq \emptyset$, $Q: \Lambda \rightarrow P_k(C(T, L^2(Z)))$ is upper semicontinuous and $m: \Lambda \rightarrow \mathbb{R}$ is continuous.*

Proof: Let $H = L^2(Z)$ and $A_H(x, \lambda) = -\frac{\partial}{\partial x} (a(z, \lambda) \frac{\partial x}{\partial z})$ with $D(A_H(\cdot, \lambda)) = \{x \in H^1_0(Z) : \frac{\partial}{\partial x} (a(z, \lambda) \frac{\partial x}{\partial z}) \in L^2(Z)\}$. Then from Attouch [1, p. 379], we know that $A_H(\cdot, \lambda)$ is maximal monotone and linear on $L^2(Z)$ and furthermore, $A_H(\cdot, \lambda) = \partial\varphi(\cdot, \lambda)$, where

$$\varphi(x, \lambda) = \begin{cases} 1/2 \int_Z a(z, \lambda) (\frac{\partial x}{\partial z})^2 dz & \text{if } x \in H^1_0(Z) \\ +\infty & \text{otherwise.} \end{cases}$$

Because of hypothesis H_c and using Theorem 29 of Zhikov, Kozlov and Oleinik [23], we have that if $\lambda_n \rightarrow \lambda$ in Λ , then $A_H(\cdot, \lambda_n) \xrightarrow{G} A_H(\cdot, \lambda)$ as $n \rightarrow \infty$, and this by Theorem 3.62, p. 365 of Attouch [1], tells us that

$$(I + \mu \partial \varphi(\cdot, \lambda_n))^{-1} \rightarrow (I + \mu \partial \varphi(\cdot, \lambda))^{-1} \text{ as } n \rightarrow \infty,$$

for all $\mu \in \mathbb{R}$. Let $\hat{f}: T \times H \times \Lambda \rightarrow H$ defined by $\hat{f}(t, x, \lambda)(\cdot) = f(t, \cdot, x(\cdot), \lambda)$ and $\hat{U}(t, \lambda) = \{u \in L^2(Z): |u(z)| \leq v(t, z, \lambda) \text{ a.e.}\}$. Set

$$F(t, x, \lambda) = \hat{f}(t, x, \lambda) \hat{U}(t, \lambda) \in P_{wkc}(L^2(Z)).$$

We will now check that $F(\cdot, \cdot)$ satisfies hypothesis $H(F)$. To this end, let $w \in H = L^2(Z)$ be given. Then we have

$$\begin{aligned} d(w, F(t, x, \lambda)) &= \inf \{ \|w - \hat{f}(t, x, \lambda)u\|_{L^2(Z)} : u \in \hat{U}(t, \lambda) \} \\ &= \inf \left[\int_Z |w(z) - f(t, z, x(z), \lambda)u(z)|^2 dz : u \in \hat{U}(t, \lambda) \right]^{1/2} \\ &= \left[\inf \int_Z |w(z) - f(t, z, x(z), \lambda)u(z)|^2 dz : u \in \hat{U}(t, \lambda) \right]^{1/2} \\ &= \left(\int_Z \inf [|w(z) - f(t, z, x(z), \lambda)u|^2 : u \in U(t, z, \lambda)] dz \right)^{1/2} \end{aligned}$$

(see Theorem 2.2 of Hiai and Umegaki [6])

$$= \left(\int_Z d(w(z), G(t, z, \lambda))^2 dz \right)^{1/2}$$

(with $G(t, z, \lambda) = f(t, z, x(z), \lambda)U(t, z, \lambda)$ and $U(t, z, \lambda) = [-v(t, z, \lambda), v(t, z, \lambda)]$). But note that because of hypotheses $H(f), H(r)$, it is clear that $(t, z) \rightarrow G(t, z, \lambda)$ is measurable and so all the transformations

$$t \rightarrow \left(\int_Z d(w(z), G(t, z, \lambda))^2 dz \right)^{1/2}, d(w, F(t, x, \lambda)), F(t, x, \lambda).$$

are measurable. Also note that because of hypothesis $H(f)(ii)$, if $x, y \in L^2(Z)$, we have

$$h(F(t, x, \lambda), F(t, y, \lambda)) \leq \| \hat{f}(t, x, \lambda) - \hat{f}(t, y, \lambda) \|_2 \| v \|_{\infty} r \leq \hat{k} \| x - y \|_2, \hat{k} > 0.$$

We will also show that, for every $w \in L^2(Z)$, $\lambda \rightarrow d(w, F(t, x, \lambda))$ is continuous. To this end, let $\lambda_n \rightarrow \lambda$ and let $u \in \hat{U}(t, \lambda)$. Because of hypothesis $H(r)$, clearly $\hat{U}(t, \cdot)$ is continuous and so we can find $u_n \in \hat{U}(t, \lambda_n)$, $u_n \xrightarrow{s} u$ in $L^2(Z)$. We have

$$\begin{aligned} d(w, F(t, x, \lambda_n)) &\leq \|w - \hat{f}(t, x, \lambda_n)u_n\|_2 \\ \overline{\lim} d(w, F(t, x, \lambda_n)) &\leq \|w - \hat{f}(t, x, \lambda)u\|_2 \end{aligned}$$

since $\lambda \rightarrow \hat{f}(t, x, \lambda)$ is continuous (hypothesis $H(f)$ (iii)). Since $u \in \hat{U}(t, \lambda)$ was arbitrary, we get

$$\overline{\lim} d(w, F(t, x, \lambda_n)) \leq d(w, F(t, x, \lambda)). \tag{6}$$

On the other hand, let $u_n \in \hat{U}(t, \lambda_n)$ be such that

$$d(w, F(t, x, \lambda_n)) = \|w - \hat{f}(t, x, \lambda_n)u_n\|_2.$$

Its existence follows from the fact that $\hat{U}(t, \lambda_n) \in P_{wkc}(L^2(Z))$. Because $\theta_B(\cdot, \cdot) \in L^\infty(T \times Z)$, $B = \{\lambda_n, \lambda\}_{n \geq 1}$ (see hypothesis $H(r)$), by passing to a subsequence if necessary, we may assume that $u_n \overset{w^*}{\rightharpoonup} u$ in $L^\infty(Z)$. Then, for every $p(\cdot) \in L^2(Z)$, we have

$$\begin{aligned} (\hat{f}(t, x, \lambda_n)u_n, p)_{L^2(Z)} &= \int_Z f(t, z, x(z), \lambda_n)u_n(z)p(z)dz \\ &\rightarrow (\hat{f}(t, x, \lambda)u, p)_{L^2(Z)} = \int_Z f(t, z, x(z), \lambda)u(z)p(z)dz \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence $\hat{f}(t, x, \lambda_n)u_n \overset{w^*}{\rightharpoonup} \hat{f}(t, x, \lambda)u$ in $L^2(Z)$ and clearly $u \in \hat{U}(t, \lambda)$. Recalling that the norm is weakly lower semicontinuous, we get

$$\begin{aligned} \|w - \hat{f}(t, x, \lambda)u\|_2 &\leq \underline{\lim} \|w - \hat{f}(t, x, \lambda_n)u_n\|_2 \\ d(w, F(t, x, \lambda)) &\leq \underline{\lim} d(w, F(t, x, \lambda_n)). \end{aligned} \tag{7}$$

From (6) and (7) above, we conclude that

$$\lambda \rightarrow d(w, F(t, x, \lambda)) \text{ is continuous and } \lambda \rightarrow F(t, x, \lambda) \text{ is } d\text{-continuous.}$$

Finally, note that

$$|F(t, x, \lambda)| \leq \|a_B(t, \cdot)\|_2 \|r\|_\infty + \|c_B\|_2 \|r\|_\infty \|x\|_2, \lambda \in B \subseteq \Lambda, B = \text{compact.}$$

So we have satisfied hypothesis $H(F)$.

Next let $\hat{\eta}: H \times \Lambda \rightarrow \mathbb{R}$ be defined by $\hat{\eta}(x, \lambda) = \int_Z \eta(z, x(z), \lambda)dz$. Using hypothesis $H(\eta)$, we can easily check that $\hat{\eta}(\cdot, \cdot)$ is in fact continuous. Now rewrite problem (5) in the following equivalent abstract form:

$$\hat{\eta}(x(b), \lambda) \rightarrow \inf = m(\lambda)$$

such that

$$-\dot{x}(t) \in \partial\varphi(x(t), \lambda) + F(t, x(t), \lambda) \text{ a.e., } x(0) = x_0(\lambda). \tag{8}$$

We know (see Theorem 3.1) that, for every $\lambda \in \Lambda$, problem (8) above has a nonempty set $S(\lambda)$ of admissible trajectories, which is compact in $C(T, L^2(Z))$. Since $\hat{\eta}(\cdot, \cdot)$ is continuous, we deduce that $Q(\lambda) \neq \emptyset$ for every $\lambda \in \Lambda$.

Next we will establish the continuity of the value function $m(\cdot)$. So let $\lambda_n \rightarrow \lambda$ in Λ . Let $x \in S(\lambda)$ be such that $m(\lambda) = \hat{\eta}(x, \lambda)$. From Theorem 3.1 we know that $S(\lambda_n) \xrightarrow{K} S(\lambda)$ in $C(T, L^2(Z))$ and so we can find $x_n \in S(\lambda_n)$, $n \geq 1$ such that $x_n \xrightarrow{s} x$ in $C(T, L^2(Z))$. Then we have

$$m(\lambda_n) \leq \hat{\eta}(x_n, \lambda_n), \overline{\lim} m(\lambda_n) \leq \lim \hat{\eta}(x_n, \lambda_n) = \hat{\eta}(x, \lambda) = m(\lambda). \quad (9)$$

Note that if $B \subseteq \Lambda$ is compact, then for any $\beta > 0$ we have that

$$\bigcup_{\lambda \in B} \{x \in H_0^1(Z) : \|x\|_2^2 + \varphi(x, \lambda) \leq \beta\}$$

is bounded in $L^2(Z)$. Since $H_0^1(Z)$ embeds compactly in $L^2(Z)$ (Sobolev embedding theorem), we have that

$$\overline{\bigcup_{\lambda \in B} \{x \in H_0^1(Z) : \|x\|_2^2 + \varphi(x, \lambda) \leq \beta\}}$$

is compact in $L^2(Z)$. Then from the proof of Theorem 3.2, we know that $\overline{\bigcup_{\lambda \in B} S(\lambda)} \in P_k(C(T, L^2(Z)))$. So if $\lambda_n \rightarrow \lambda$ in Λ , $B = \{\lambda_n, \lambda\}_{n \geq 1}$ and $x_n \in S(\lambda_n)$ is such that $m(\lambda_n) = \hat{\eta}(x_n, \lambda_n)$, by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{s} x$ in $C(T, L^2(Z))$. Then we have

$$\hat{\eta}(x, \lambda) = \lim \hat{\eta}(x_n, \lambda_n), m(\lambda) \leq \underline{\lim} m(\lambda_n). \quad (10)$$

From (9) and (10) above, we get the continuity of $m(\cdot)$. Using it, we can easily check that

$$s\text{-}\overline{\lim} Q(\lambda_n) \subseteq Q(\lambda), Q|_B \text{ is upper semicontinuous}$$

and this by Lemma δ of [16] (see Section 2) implies that $Q(\cdot)$ is upper semicontinuous. \square

Remark: Our result extends the work of Przylyuski [17], who considers linear systems and the parameter λ appears only on the control constraint set.

Our formulation of the problem also incorporates "differential variational inequalities" (see Aubin and Cellina [2, p. 264]). These are differential inclusions of the following form:

$$-\dot{x}(t) \in N_{K(\lambda)}(x(t)) + F(t, x(t), \lambda) \text{ a.e., } x(0) = x_0(\lambda). \quad (11)$$

Recall that the normal cone $N_{K(\lambda)}(x)$ to the closed, convex set $K(\lambda) \subseteq \mathbb{R}^k$ at the point x is defined to be the set $N_{K(\lambda)}(x) = \partial \delta_{K(\lambda)}(x)$, where $\delta_{K(\lambda)}(x) = 0$ if $x \in K(\lambda)$,

$\delta_{K(\lambda)}(x) = +\infty$ otherwise (indicator function of the set $K(\lambda)$). Also $N_{K(\lambda)}(x) = T_{K(\lambda)}(x)^* = \{v \in \mathbb{R}^k: (v, u) \leq 0 \text{ for all } u \in T_{K(\lambda)}(x)\}$, with $T_{K(\lambda)}(x)$ being the tangent cone to $K(\lambda)$ at the point x . In fact, problem (11) is equivalent to the following "projected differential inclusion" (see Aubin and Cellina [2]):

$$\dot{x}(t) \in \text{proj}(F(t, x(t), \lambda); T_{K(\lambda)}(x(t))) \text{ a.e., } x(0) = x_0(\lambda). \quad (12)$$

Here $\text{proj}(\cdot; T_{K(\lambda)}(x(t)))$ denotes the metric projection on the tangent cone $T_{K(\lambda)}(x(t))$ and $\text{proj}(F(t, x(t), \lambda); T_{K(\lambda)}(x(t))) = \cup[\text{proj}(z; T_{K(\lambda)}(x(t))): z \in F(t, x(t), \lambda)]$. In many applications like control theory, theoretical mechanics and mathematical economics, we encounter systems with state constraints. In describing the effect of the constraint on the dynamics of the system, it can be assumed in many cases that the velocity $\dot{x}(t)$ is projected at each time instant on the set of allowed directions toward the constraint set at the point $x(t)$. This is true for electrical networks with diode nonlinearities and for unilateral problems in mechanics. Also in Aubin and Cellina [2, Chapter 5, Section 6], the interested reader can find an example concerning monotone trajectories converging to Pareto minima in a problem of efficient allocation of resources (planning procedures) (see also Henry [5]). So inclusion (12) arises naturally in applications and (12) in turn is equivalent to (11), which fits in the general framework of this paper.

Note that if $K: \Lambda \rightarrow P_{fc}(\mathbb{R}^k)$ is continuous, then for $\lambda_n \rightarrow \lambda$ in Λ we have $\delta_{K(\lambda_n)}(\cdot) \xrightarrow{\tau} \delta_{K(\lambda)}(\cdot)$, where τ denotes the convergence in the epigraphical sense (see Mosco [10]). So by Theorem 3.66, p. 373 of Attouch [1], we have that for all $\mu > 0$,

$$(I + \mu \partial \delta_{K(\lambda_n)})^{-1} x = (I + \mu N_{K(\lambda_n)})^{-1} x \rightarrow (I + \mu \partial \delta_{K(\lambda)})^{-1} x = (I + \mu N_{K(\lambda)})^{-1} x$$

for all $x \in \mathbb{R}^k$. Thus if $S(\lambda_n), S(\lambda)$ are the solution sets for (11), by Theorem 3.1 we have $S(\lambda_n) \xrightarrow{K} S(\lambda)$ in $C(T, \mathbb{R}^k)$. Furthermore, if $K(\cdot)$ is $P_{kc}(\mathbb{R}^k)$ -valued, then for $B \subseteq \Lambda$ compact we have $K(B) \in P_k(\mathbb{R}^k)$ and so hypothesis $H(\varphi)'$ is satisfied. Thus, via Theorems 3.2 and 3.3, we can get that $S(\cdot)$ is Vietoris and Hausdorff continuous from Λ into $P_k(C(T, \mathbb{R}^k))$.

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