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An Existence Theorem for Control Problems with Unbounded Control Domain

H. KRAUT

Control problems of Dieudonné-Rashevsky type for multiple integrals, generalized in the sense of E.J. McShane and L.C. Young, are considered. For the case of unbounded control domains an existence theorem for optimal generalized processes is proved.

Key words: *Generalized controls, unbounded control domains, multiple integrals*
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The question of the existence of an optimal solution of a given programming problem can often be answered only under strong assumptions, as for example convexity. To answer this question for optimal control problems the sets of feasible solutions were enlarged and the problems generalized (cf., e.g., [9, 14, 15, 17]), so that the existence of an optimal generalized process could be proved under weak assumptions; where the importance of the generalized processes exceeds this effect and lies also in their applications [19]. Most of the existing literature refers to bounded control sets; a few papers consider the existence of optimal generalized processes in case of an unbounded control domain but most in case of one-dimensional t-variable [3, 9, 11, 13, 16].

The existence of an optimal generalized process has also been proved for problems of Dieudonné-Rashevsky type involving multiple integrals, in which there is now a multidimensional t-variable. However a bounded control domain is always assumed.

Our paper shows the existence of optimal generalized processes for control problems of Dieudonné-Rashevsky type for multiple integrals and with unbounded control domains, where our assumptions are similar to those of [9].

Let us consider the following generalized control problem:

(P) Minimize $J(x,\mu) = \int_\Omega \int_U f(t,x(t),v)\, d\mu_t(v)\, dt \qquad (\Omega \subset \mathbb{R}^m,\ m \geq 1)$

subject to all generalized processes $(x,\mu) \in W_p^{1,n}(\Omega) \times \mathcal{M}_U$ $(p > m)$ satisfying the

state constraints	$x(t) \in \overline{G} \subset \mathbb{R}^n$ for all $t \in \overline{\Omega}$
control constraints	$\operatorname{supp} \mu_t \subseteq U \subseteq \mathbb{R}^r$ for a.a. $t \in \Omega$
process equation	$x_{t_\alpha}(t) = \int_U g_\alpha(t,x(t),v)\, d\mu_t(v)$ a.e. in Ω
boundary conditions	$x(s) = z(s)$ on $\partial\Omega$

H. Kraut: Breisgaustr. 33, D(*Ost*) 7010 - Leipzig

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with continuous functions f and g_α ($\alpha = 1,\dots,m$) and with $\mu \in \mathcal{M}_U$ if $\mu = \{\mu_t\}_{t\in\Omega}$ is a family of probability measures on U having the property that for each continuous function φ on $\Omega \times U$ the function $h_\varphi(t) = \int_U \varphi(t,v)\,d\mu_t(v)$ is Lebesgue measurable.

Without loss of generality, we will restrict the boundary conditions to the case $x(s) = 0$ on $\partial\Omega$; the general case $x(s) = z(s)$ on $\partial\Omega$ can be reduced to this case easily. Furthermore let Ω be a Lipschitz domain in the sense of [7,10]. Than we can put the process equation and the boundary conditions together and write them as variational equation

$$\int_\Omega\int_U \left(\chi^\alpha_{t_\alpha}(t)x(t) + \chi^\alpha(t)g_\alpha(t,x(t),v)\right)d\mu_t(v)\,dt = 0 \quad \text{for all } \chi \in C^{nm}_\infty(\Omega)$$

which is sometimes more convenient to use.

We shall say that a function f defined on $A \times U$ is of *slower growth* than a function g, also defined on $A \times U$ (or the function g is of *faster growth* than f) uniformly on A if $g \geq 0$ and for each $\varepsilon \geq 0$ there is a bounded subset $U_\varepsilon \subset U$ so that $|f| \leq \varepsilon g$ on $A \times (U \setminus U_\varepsilon)$ (cf. [9]).

Now we can formulate our main existence theorem.

Theorem: *Let (x^k,μ^k) be a minimizing sequence of Problem* (P), *whose trajectories all lie in a bounded closed subset $A \subseteq \overline{\Omega} \times \overline{G}$, and assume that there is a continuous function $\Gamma = \Gamma(t,\xi,v)$ on $A \times U$ having the following properties*:

(i) *The integrals*

$$\int_U \Gamma(t,x^k(t),v)\,d\mu^k_t(v) \qquad \text{and} \qquad \int_\Omega\left(\int_U \Gamma(t,x^k(t),v)\,d\mu^k_t(v)\right)^p dt \quad (k \in \mathbb{N})$$

exist (*i.e. they are finite*) *for a.a. $t \in \Omega$ resp. exist and are bounded by a constant M_Γ independent of k.*

(ii) *The function Γ is of faster growth than 1 uniformly on A.*

(iii) The functions $f \geq 0$ and g_α *are of slower growth than Γ uniformly on A.*

Then there exists an admissible generalized process (x^o,μ^o) so that $J(x^o,\mu^o) = \inf J(x,\mu)$.

Before we start the proof of the theorem we want to prove the following lemma which is needed.

Lemma: *Let $\{\mu^k\}$ be a sequence of (real) Radon measures on Ω which are absolutely continuous with respect to the Lebesgue measure μ_L, and are such that $|\mu^k|(\Omega) < M$ for all k, for a constant $M > 0$ which is independent of k. Then there exist a subsequence $\{k'\}$ and a (real) Radon measure ν on Ω, absolutely continuous with respect to μ_L, such that $\lim_{k'\to\infty}\mu^{k'}(E) = \nu(E)$ for each μ_L-integrable subset E of Ω.*

Proof: Let $\{\mu^k\}$ be such a sequence of Radon measures. Since every (real) Radon measure on Ω is the difference of two positive measures, we can without loss of generality assume that all μ^k are positive. By our assumption there exists a subsequence $\{\mu^{k'}\}$ which is weakly convergent (cf. [5: p. 137]). Let our sequence $\{\mu^k\}$ be already of this property. Then by [4: p. 105] the sequence $\{\mu^k\}$ is bounded in the vector space $M_R(\Omega)$, the vector space of all real measures on Ω. Therefore, by virtue of [4: Theorem

13.15.4] there exists an increasing subseqeunce $\{\mu^{k'}\}$ such that $\nu = \sup_{k'} \mu^{k'}$ is in $M_R(\Omega)$. Then by [4: Theorem 13.15.9] for each function f that is $\mu^{k'}$ integrable for all k' and for which $\sup_{k'} \int_\Omega |f|\, d\mu^{k'} < \infty$ we have $\int_\Omega f\, d\nu = \lim_{k'\to\infty} \int_\Omega f\, d\mu^{k'}$.

Now let E be an arbitrary μ_L-measurable subset of Ω and χ_E its characteristic function. Then the above proves the lemma, since $\nu(E) = \int_\Omega \chi_E\, d\nu$ and $\mu^{k'}(E) = \int_\Omega \chi_E\, d\mu^{k'} < M$ for all k' ∎

Proof of the Theorem: Let (x^k, μ^k) be a minimizing sequence of Problem (P), so that the trajectories $\{x^k(t) |\ t \in \Omega\}$ $(k \in \mathbb{N})$ lie in a bounded closed subset $A \subseteq \overline{\Omega} \times \overline{G}$. Then by assumptions (i) and (ii) it follows that the sequence $\{x^k\}$ is bounded in the Sobolev space $W_p^{1,n}(\Omega)$ and thus weakly compact, so that, because of Sobolev's imbedding theorem, there exists a subsequence $\{x^{k'}\}$ which converges uniformly on $\overline{\Omega}$ to a continuous function x^0, with $(t, x^0(t)) \in A$ for all t. For the sake of simplicity we assume that our starting sequence is already of such type. As we shall see later on, the function x^0 is the state function of a minimizing generalized process in (P).

Now we want to construct the corresponding generalized control μ^0. First, with no loss of generality we assume that $\Gamma \geq 1$. Let ψ be a continuous function on $A \times U$ such that the integrals

$$\int_U \psi(t, x^k(t), v)\, d\mu_t^k(v) \quad \text{exist for a.a. } t \in \Omega \quad (k \in \mathbb{N}).$$

Now by

$$\psi^{[k]}(e) = \int_e \int_U \psi(t, x^k(t), v)\, d\mu_t^k(v)\, dt \quad (k \in \mathbb{N})$$

we define absolutely continuous (with respect to μ_L), countable additive set functions (measures) on Ω (cf. [12]), where e is an arbitrary measurable subset of Ω. By assumption (i) there exists a real constant M_Γ such that

$$\int_\Omega \int_U \Gamma(t, x^k(t), v)\, d\mu_t^k(v)\, dt < M_\Gamma \quad (k \in \mathbb{N}). \tag{1}$$

Since (x^k, μ^k) is a minimizing sequence there exists a real constant M_0 such that

$$\int_\Omega \int_U |f(t, x^k(t), v)|\, d\mu_t^k(v)\, dt < M_0 \quad (k \in \mathbb{N}). \tag{2}$$

By assumption (iii) the functions g_α $(\alpha = 1, \dots, m)$ are of slower growth than Γ uniformly on A. Hence for a positive ε there is a bounded subset U_ε, such that $|g_\alpha| \leq \varepsilon\Gamma$ $(\alpha = 1, \dots, m)$ on $A \times (U \setminus U_\varepsilon)$, while g_α $(\alpha = 1, \dots, m)$ is bounded on $A \times U_\varepsilon$. Hence by (1) there exist real constants M_α such that

$$\int_\Omega \int_U |g_\alpha(t, x^k(t), v)|\, d\mu_t^k(v)\, dt < M_\alpha \quad (k \in \mathbb{N};\ \alpha = 1, \dots, m). \tag{3}$$

Let ψ be Γ, f, g_α $(\alpha = 1, \dots, m)$ or an arbitrarily chosen function of $\overset{\circ}{C}(A \times U)$, the set of continuous functions with a compact support in $A \times U$. Then because of our lemma we can choose a subsequence $\{k'\}$ such that $\lim_{k'\to\infty} \psi^{[k']}(e) = \tilde{\psi}(e)$ for all measurable $e \subseteq$

Ω, where $\tilde{\psi}$ is a Radon measure absolutely continuous with respect to μ_L. Then there exists a summable function ψ' such that $\tilde{\psi}(e) = \int_e \psi'(t)dt$ for all e [12]. Let our starting sequence $\{k\}$ be already of this type. Furthermore there exists a countable subset Φ of $\overset{\circ}{C}(A \times U)$ such that every function ψ in $\overset{\circ}{C}(A \times U)$ can be uniformly approximated by functions in Φ. Since we have the above assertion for each $\varphi \in \Phi$ and f, Γ, g_α, respectively, by the diagonal process we can select a subsequence $\{k'\}$ so that $\lim_{k' \to \infty} \psi^{[k']}(e) = \tilde{\psi}(e)$ for all measurable subsets $e \subseteq \Omega$, for f, Γ, g_α $(\alpha = 1, \ldots, m)$ and all $\varphi \in \Phi$. We assume that our starting sequence is already of this type.

Now let ψ be a function of $\overset{\circ}{C}(A \times U)$, e a measurable subset of Ω and $\varepsilon > 0$ arbitrarily chosen. Then there exist a function $\varphi \in \Phi$ and a constant $k_0(\varepsilon)$ such that

$$
\begin{aligned}
&|\psi^{[k]}(e) - \psi^{[m]}(e)| \\
&\quad \le |\psi^{[k]}(e) - \varphi^{[k]}(e)| + |\varphi^{[k]}(e) - \varphi^{[m]}(e)| + |\varphi^{[m]}(e) - \psi^{[m]}(e)| \\
&\quad \le \varepsilon + 2\varepsilon\mu_L(e) \qquad \text{for } m, k > k_0(\varepsilon).
\end{aligned}
$$

Then $\lim_{k' \to \infty} \psi^{[k']}(e) = \tilde{\psi}(e)$ exists for each measurable subset $e \subseteq \Omega$ and the above considerations are valid (cf. [6]). Now let ψ be an arbitrary function in $\overset{\circ}{C}(A \times U)$. According to the above there is for each $\tau \in \Omega$ a positive linear functional φ_o on $\overset{\circ}{C}(A \times U)$ defined by $\varphi_o(\psi, \tau) = \psi'(\tau)$. By Riesz' representation theorem there exists a unique corresponding Borel measure μ_τ^o on $A \times U$ depending on the parameter τ and such that $\varphi_o(\psi, \tau) = \int_{A \times U} \psi(t, \xi, v)\, d\mu_\tau^o(t, \xi, v)$. Then by construction of $\varphi_o(\cdot, \tau)$, putting $B(\tau, \varepsilon) = \{t \in \Omega \mid |\tau - t| \le \varepsilon\}$ we have

$$
|\varphi_o(\psi, \tau)| \le \left| \lim_{\varepsilon \to o} \frac{\tilde{\psi}(B(\tau, \varepsilon))}{\mu_L(B(\tau, \varepsilon))} \right| \le \limsup_{\varepsilon \to o} \left| \frac{\tilde{\psi}(B(\tau, \varepsilon))}{\mu_L(B(\tau, \varepsilon))} \right| \le \limsup_{\varepsilon \to o} \left| \lim_{k \to \infty} \frac{\psi^{[k]}(B(\tau, \varepsilon))}{\mu_L(B(\tau, \varepsilon))} \right|
$$

$$
\le \limsup_{\varepsilon \to o} \frac{\left| \lim_{k \to \infty} \int_{B(\tau, \varepsilon)} \int_U \psi(t, x^k(t), v)\, d\mu_t^k(v)\, dt \right|}{\mu_L(B(\tau, \varepsilon))}
$$

$$
\le \limsup_{\varepsilon \to o} \frac{\limsup_{k \to \infty} \int_{B(\tau, \varepsilon)} \max_{v \in U} |\psi(t, x^k(t), v)|\, dt}{\mu_L(B(\tau, \varepsilon))}.
$$

Further, [4: Theorem 13.8.3/p. 123] implies the continuity of the maximal value function (cf. [1: p. 116]) and the uniform convergence $x^k \to x^o$. Thus we can continue the above estimation as follows:

$$
|\varphi_o(\psi, \tau)| \le \limsup_{\varepsilon \to o} \frac{\int_{B(\tau, \varepsilon)} \limsup_{k \to \infty} \max_{v \in U} |\psi(t, x^k(t), v)|\, dt}{\mu_L(B(\tau, \varepsilon))} \le \limsup_{\varepsilon \to o} \frac{\int_{B(\tau, \varepsilon)} \max_{v \in U} |\psi(t, x^o(t), v)|\, dt}{\mu_L(B(\tau, \varepsilon))}
$$

$$
\le \limsup_{\varepsilon \to o} \max_{v \in U,\, t \in B(\tau, \varepsilon)} |\psi(t, x^o(t), v)| \le \max_{v \in U} |\psi(\tau, x^o(\tau), v)|.
$$

Thus μ_τ^o is a Borel measure on $A \times U$ concentrated on $\{(\tau, x^o(\tau))\} \times U$ (for all $\tau \in \Omega$). Then

μ_τ^o can be considered as a Radon measure on U and acting on an arbitrary function $g \in \mathring{C}(U)$ according to $\int_U g(v)\,d\mu_\tau^o(v) = \varphi_0(h(t,\xi)g(v),\tau)$, where $h \in \mathring{C}(A)$ and $h(\tau,x^o(\tau)) = 1$. Since $\psi_1^{[k]}(e) = \mu_L(e)$ for all k, with $\psi_1 \equiv 1$, we have $\tilde{\psi}_1(e) = \mu_L(e)$ and hence $\psi_1'(\tau) = 1 = \int_U 1\,d\mu_\tau^o(v)$ $(\tau \in \Omega)$, that is, μ_τ^o is a probability measure.

Now let ψ be continuous on $A \times U$ and of slower growth than Γ uniformly on A, and ρ_l a continuous function on $\mathbb{R}^r$, with

$$0 \le \rho_l(v) \le 1 \qquad \text{and} \qquad \rho_l(v) = \begin{cases} 1 & \text{if } v \in B(0,l) \\ 0 & \text{if } v \notin B(0,l+1). \end{cases} \tag{4}$$

Hence $\psi_l = \psi\rho_l$ is in $\mathring{C}(A \times U)$ and we have $\tilde{\psi}_l(e) = \int_e \psi_l'(t)\,dt = \int_e \int_U \psi_l(t,x^o(t),v)\,d\mu_t^o(v)\,dt$ for all measurable $e \subseteq \Omega$ and all $l \in \mathbb{N}$.

Now we want to show that for each $\varepsilon > 0$ there exists an $l_0(\varepsilon)$ such that

$$|\psi^{[k]}(e) - \psi_l^{[k]}(e)| \le \varepsilon \text{ for all measurable } e \subseteq \Omega \text{ and all } k \tag{5}$$

if $l \ge l_0(\varepsilon)$. Let $\varepsilon > 0$ be arbitrarily given. Then we set $\bar{\varepsilon} = \varepsilon/M_\Gamma$ and choose for $U_{\bar{\varepsilon}}$ (cf. (iii)) $l_0(\varepsilon)$ so large that $U_{\bar{\varepsilon}} \subset B(0,l_0(\varepsilon))$ and following $U \setminus B(0,l) \subset U \setminus U_{\bar{\varepsilon}}$ for all $l \ge l_0(\varepsilon)$. Since ψ is of slower growth than Γ uniformly on A we get

$$\begin{aligned}
&|\psi^{[k]}(e) - \psi_l^{[k]}(e)| \\
&\le \left| \int_e \int_U \psi(t,x^k(t),v)\,d\mu_t^k(v)\,dt - \int_e \int_U \psi_l(t,x^k(t),v)\,d\mu_t^k(v)\,dt \right| \\
&\le \left| \int_e \int_{U \setminus B_{l+1}} \psi(t,x^k(t),v)\,d\mu_t^k(v)\,dt + \int_e \int_{B_{l+1}\setminus B_l} (\psi(1-\rho_l))(t,x^k(t),v)\,d\mu_t^k(v)\,dt \right| \\
&\le \int_e \int_{U \setminus B_{l+1}} |\psi(t,x^k(t),v)|\,d\mu_t^k(v)\,dt + \int_e \int_{B_{l+1}\setminus B_l} |\psi(t,x^k(t),v)|\,d\mu_t^k(v)\,dt \\
&\le \int_e \int_{U \setminus B_l} |\psi(t,x^k(t),v)|\,d\mu_t^k(v)\,dt \le \bar{\varepsilon} M_\Gamma \le \varepsilon \quad \text{if } l \ge l_0(\varepsilon),
\end{aligned}$$

where B_l denotes $B(0,l)$.

Furthermore we want to prove $\int_\Omega \int_U \Gamma(t,x^o(t),v)\,d\mu_t^o(v)\,dt \le M_\Gamma$. Let ρ_l be defined as before in (4) and $\Gamma_l = \Gamma\rho_l$. Then we have $0 \le \Gamma_l \le \Gamma_{l+1}$ for all $l \in \mathbb{N}$ and $\lim_{l\to\infty} \Gamma_l(t,\xi,v) = \Gamma(t,\xi,v)$ on $A \times U$. Hence

$$\int_\Omega \int_U \Gamma_l(t,x^k(t),v)\,d\mu_t^k(v)\,dt \le \int_\Omega \int_U \Gamma(t,x^k(t),v)\,d\mu_t^k(v)\,dt \le M_\Gamma \quad \text{for all } k,l$$

and following

$$\lim_{k\to\infty} \int_\Omega \int_U \Gamma_l(t,x^k(t),v)\,d\mu_t^k(v)\,dt \le \int_\Omega \int_U \Gamma_l(t,x^o(t),v)\,d\mu_t^o(v)\,dt \le M_\Gamma \quad \text{for all } l$$

since $\Gamma_l \in \mathring{C}(A \times U)$. By the theorem of Beppo Levi (cf. [8: p. 305]) it follows that

$$M_\Gamma \ge \lim_{l\to\infty} \int_\Omega \int_U \Gamma_l(t,x^o(t),v)\,d\mu_t^o(v)\,dt$$

$$\ge \int_\Omega \int_U \lim_{l\to\infty} \Gamma_l(t,x^o(t),v)\,d\mu_t^o(v)\,dt \ge \int_\Omega \int_U \Gamma(t,x^o(t),v)\,d\mu_t^o(v)\,dt.$$

Now we can show that $\lim_{k\to\infty} \psi^{[k]}(e) = \int_e \int_U \psi(t,x^o(t),v)\,d\mu_t^o(v)\,dt$. Let $\varepsilon > 0$ be arbitrarily given and $L \ge l_o(\varepsilon)$ chosen according to (5). Since $\psi_L \in \overset{\circ}{C}(A\times U)$ we can find a $k_o(\varepsilon, l_o(\varepsilon)) = k_o(\varepsilon)$ such that, for $k \ge k_o(\varepsilon)$,

$$\left|\psi^{[k]}(e) - \int_e \int_U \psi(t,x^o(t),v)\,d\mu_t^o(v)\,dt\right|$$

$$\le \left|\psi^{[k]}(e) - \psi_L^{[k]}(e)\right| + \left|\psi_L^{[k]}(e) - \int_e \int_U \psi_L(t,x^o(t),v)\,d\mu_t^o(v)\,dt\right|$$

$$+ \left|\int_e \int_U \psi_L(t,x^o(t),v)\,d\mu_t^o(v)\,dt - \int_e \int_U \psi(t,x^o(t),v)\,d\mu_t^o(v)\,dt\right|$$

$$\le \varepsilon + \varepsilon + \int_e \int_{U\setminus B_L} |\psi(t,x^o(t),v)|\,d\mu_t^o(v)\,dt \le 2\varepsilon + \varepsilon M_\Gamma.$$

Hence $\lim_{k\to\infty} \psi^{[k]}(e) = \int_e \int_U \psi(t,x^o(t),v)\,d\mu_t^o(v)\,dt$ for all measurable subsets $e \subset \Omega$ and all functions $\psi \in C(A\times U)$ with slower growth than Γ uniformly on A. This shows also that we can extend all our previous considerations to such ψ, so that $\mu^o = \{\mu_t^o\}_{t\in\Omega}$ indeed is a generalized control.

Let now χ be an arbitrary function in $C_\infty^{nm}(\Omega)$ and $\psi(t,\xi,v) := \chi_{t_\alpha}^\alpha(t)\xi + \chi^\alpha(t) g_\alpha(t,\xi,v)$, which implies that $\psi^{[k]}(\Omega) = 0$ for all k, since the generalized processes (x^k,μ^k) are feasible. It follows that $\lim_{k\to\infty} \psi^{[k]}(\Omega) = \int_\Omega \int_U \psi(t,x^o(t),v)\,d\mu_t^o(v)\,dt = 0$. Hence (x^o,μ^o) is feasible and $J(x^o,\mu^o) \ge \inf J(x,\mu)$ in (P). To prove that (x^o,μ^o) is optimal let us assume $J(x^o,\mu^o) > \inf J(x,\mu)$ in (P). Then there exists an $\varepsilon > 0$ such that

$$J(x^o,\mu^o) - \varepsilon > \inf J(x,\mu) \text{ in (P)}. \tag{6}$$

As in the previous consideration the function $f_l = \rho_l f$ is in $\overset{\circ}{C}(A\times U)$ and we have $f_l \le f_{l+1} \le f$ for all l and $\lim_{l\to\infty} f_l(t,\xi,v) = f(t,\xi,v)$ on $A\times U$. Hence there is an $\bar{l}$ such that

$$\int_\Omega \int_U f_{\bar{l}}(t,x^o(t),v)\,d\mu_t^o(v)\,dt > \int_\Omega \int_U f(t,x^o(t),v)\,d\mu_t^o(v)\,dt - \varepsilon/2. \tag{7}$$

Furthermore we have

$$\begin{aligned} \int_\Omega \int_U f_{\bar{l}}(t,x^o(t),v)\,d\mu_t^o(v)\,dt &= \lim_{k\to\infty} \int_\Omega \int_U f_{\bar{l}}(t,x^k(t),v)\,d\mu_t^k(v)\,dt \\ &\le \lim_{k\to\infty} \int_\Omega \int_U f(t,x^k(t),v)\,d\mu_t^k(v)\,dt = \inf J(x,\mu). \end{aligned} \tag{8}$$

But then by (6) - (8) we get

$$\int_\Omega \int_U f_{\bar{l}}(t,x^o(t),v)\,d\mu_t^o(v)\,dt \le \inf J(x,\mu) < J(x^o,\mu^o) - \varepsilon < \int_\Omega \int_U f_{\bar{l}}(t,x^o(t),v)\,d\mu_t^o(v)\,dt - \frac{\varepsilon}{2}$$

which is a contradiction. Hence (x^o,μ^o) is optimal ∎

Remark: Often the state domain is a compact subset of $\mathbb{R}^n$, for instance a Lipschitz domain, so that we can set $A = \overline{\Omega} \times \overline{G}$ in this case. But even if $\overline{G}$ is not bounded such a bounded closed subset $A \subseteq \overline{\Omega} \times \overline{G}$ exists. Then, if we first assume A to be only closed (but not necessary bounded), the other assumptions of the theorem imply that we can assume A to be bounded. Because of the process equation, assumptions (i) and (iii) there is a real constant M (analogous to (3)) such that $\int_U |x^k_{t_\alpha}(t)|^p dt < M$ ($k \in \mathbb{N}$) and by the equivalence of the different norms in the Sobolev spaces [18: p. 1033] it follows that the sequence $\{x^k\}$ is bounded in the Sobolev space $W_p^{1,n}(\Omega)$ and, as in the beginning of our proof, we get the uniform convergence of a subsequence to a continuous function x^0 on $\overline{\Omega}$, with $(t, x^0(t)) \in A$. Since Ω is bounded and x^0 is continuous x^0 is bounded. So, because of the uniform convergence, we can choose a subsequence and a bounded closed subset $B \subset A$ so that all trajectories of this subsequence and $x^0(t)$ lie in B.

If the functions g_α ($\alpha = 1,\dots,m$) are bounded on $\overline{\Omega} \times \overline{G} \times U$, each sequence of state functions $\{x^k\}$ of admissible generalized processes (x^k, μ^k) is bounded in the Sobolev space $W_\infty^{1,n}(\Omega)$. Thus, in this case, the choice $p = 1$ in assumption (i) is sufficient for our prove. Furthermore it is possible to replace the assumption $f \geq 0$ with the assumption that f is of slower growth than Γ uniformly on A.

If f is bounded below than (by adding a constant) we can satisfy the condition $f \geq 0$. If further the set of admissible generalized processes is non-empty, then the existence of a minimizing sequence of generalized processes is guaranteed and by our theorem even a minimal generalized process exists.

Example: Finally we want to give an example with $\Omega = [0,1]\times[0,1]$ and $U = \mathbb{R}^2$, which satisfies the assumptions of our theorem (but does not satisfy the assumptions made, for example, in [11]). Minimize

$$J(x,\mu) = \int_0^1\int_0^1\int_{\mathbb{R}^2}\left((1 - v_1^2 - v_2^2)^2 + x^2(t) - v_1^2 v_2^2\right)d\mu_t(v)\,dt$$

subject to all generalized processes $(x,\mu) \in W_3^1(\Omega)\times\mathcal{M}_{\mathbb{R}^2}$ satisfying the process equations $x_{t_\alpha}(t) = \int_{\mathbb{R}^2} v_\alpha d\mu_t(v)$ a.e. in Ω ($\alpha = 1,2$) and the boundary conditions $x(s) = 0$ on $\partial\Omega$. Since f is bounded below and the set of admissible generalized processes is non-empty the assumptions of the theorem are satisfied with $\Gamma(t,\xi,v) = |v|^{4/3}$ and $p = 3$. Here we do not need a special minimizing sequence to show that the assumptions are satisfied. The existence of a minimizing sequence is guaranteed. Let (x^k, μ^k) be an arbitrary minimizing sequence of the example. Then the sequence of

$$J(x^k,\mu^k) = \int_\Omega\int_U\left(1 + v_1^4 + v_2^4 + v_1^2 v_2^2 - 2v_1^2 - 2v_2^2 + (x^k)^2\right)d\mu_t^k(v)\,dt$$

is bounded. There follows the boundedness of the sequences of

$$\int_\Omega\int_U\left(v_1^2(v_1^2 - 2) + v_2^2(v_2^2 - 2) + v_1^2 v_2^2\right)d\mu_t^k(v)\,dt \quad \text{and} \quad \int_\Omega\int_U v_1^2 v_2^2\, d\mu_t^k(v)\,dt$$

(note the positivity of the summands for $v_1^2 \geq 2$ and $v_2^2 \geq 2$), respectively of

$$\int_\Omega\int_U v_1^2\, d\mu_t^k(v)\,dt \quad \text{and} \quad \int_\Omega\int_U v_2^2\, d\mu_t^k(v)\,dt.$$

too. Since the integrand of the objective functional can be written as $\Gamma - 2v_1 - 2v_2 - v_1^2 v_2^2 + x^2 + 1$, the sequence of

$$\int_0^1\int_0^1\int_{\mathbb{R}^2}\left(\Gamma^3(t,x^k(t),v)-2v_1^2-2v_2^2-v_1^2v_2^2\right)d\mu_t^k(v)\,dt$$

is bounded. Together with the above it shows that (i) of the theorem is satisfied ∎

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