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On the Existence of Optimal Solutions for Time-Optimal Semilinear Parabolic Boundary Control Problems

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A class of time-optimal semilinear parabolic control problems with state constraints and a generalized terminal condition is considered. For the derivation of solvability results, which is the main aim of the paper, we present two methods. The first method works with the complete continuity of the state mapping, whereas in the second one theorems about the separation of convex sets and measurable selection are applied for overcoming the complete continuity of the state mapping. The connection with a family of associated fixed-time problems is very helpful in both cases. This connection between time-optimal control problems and related problems with fixed time are also interesting in their own right, especially they can be used to get optimality conditions for the original time-optimal control problem.

Key words: *Time-optimal control, semilinear parabolic equations, existence theory*
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1 Introduction

Time-optimal boundary control problems for parabolic equations have been investigated in the literature by some authors from several points of view: Optimality conditions, relations to associated problems with fixed time and also existence of optimal solutions are the most interesting theoretical problems. We mention, for instance, Fattorini [6], Friedman [8], Lempio [15], Mackenroth [16] and Tröltzsch [23,24]. In contrast to most of the earlier papers, we study problems with semilinear parabolic differential equations. Moreover, in many of these papers problems with a terminal condition of the type $w(T) \in W_T \subset X$ are investigated. Here the state w is interpreted as an abstract function of time t with values in a Banach space X (the state space) and W_T denotes a convex, bounded and closed subset of X. Now we will generalize this to a condition which also explicitly depends on the control u. This is done in Section 2, together with the introduction of all basic notions and notations for the formulation of the problem. Section 3 contains some analytical results, concerning mainly the state equation, but they are also used in the further investigations.

Moreover, there are some essential relations to associated fixed-time problems known for linear time-optimal control problems. One of the most important is the following:

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Let T_0 be the infimum of all times T, for which a control $u(\cdot)$ and an associated state $w(u)$ exists, satisfying the terminal condition at T and all other restrictions of the problem. Then the time-optimal control problem is solvable if and only if the control problem with fixed time T_0 is solvable and $w(T_0) \in \partial W_{T_0}$ is fulfilled for an optimal state w.

It is shown in Section 4 that analogous relations hold also in the semilinear case under very general assumptions, such as continuity of the states w with respect to t and a similar property concerning the dependence of the terminal condition on u. Nevertheless, as we will see in Section 5, for the proof of existence results we need more restrictive conditions, for instance, *uniform* continuity with respect to t and a special structure of the nonlinear boundary condition. These conditions are necessary in order to guarantee an analogue to $w(T_0) \in \partial W_{T_0}$ as well as a completely continuous state mapping, the main tool for proving the solvability of the control problem with fixed time T_0. In Section 6 we point out that with additional effort the last condition can be weakened, if the terminal condition does not explicitly depend on the control u. The consequences are especially fruitful in the case $u = u(x,t)$.

Finally, let us finish with some auxiliary remarks on the literature concerning the topic of the paper. The difficult case $\text{int}W_T = \emptyset$ for time-optimal control problems with linear equation, in particular, if W_T consists of only one element, was studied by Fattorini [6] and Schmidt [19]. The problem of reaching a steady-state target in shortest time was discussed by Schmidt in [20]. For similar investigations involving distributed controls, we refer to Balakrishnan [1], Fattorini [5], Friedman [9] and Hoppe [14].

Roubiček [18] considered time-optimal control problems with a nonlinear boundary condition and general state restrictions, too, but he used a set of uniformly Lipschitz continuous controls and monotonicity assumptions for handling the equation. Semilinear equations with measurable controls were investigated in a paper by Schmidt [21] using the approach of weak solutions for the definition of generalized states .

2 The time-optimal control problem

We investigate control problems with the following semilinear parabolic state equation, defined in a bounded domain $D \subset R^N$ with boundary $S = \partial D$:

$$\begin{array}{rcll} \partial w/\partial t + (L(t)w(t,\cdot))(x) &=& f(t,x) & \text{in } (0,T_E] \times D, \\ w(0,x) &=& w_0(x) & \text{on } \bar{D}, \\ \partial w/\partial \eta + \alpha w(t,x) &=& b(u,w(t,x),t,x) & \text{on } (0,T_E] \times S. \end{array} \tag{1}$$

Here $T_E > 0, \alpha \geq 0$ are fixed, f, w_0 and b are continuous functions of all arguments in their natural domain of definition. Futhermore, $\{L(t)\}_{t\in[0,T_E]}$ denotes a family of uniformly elliptic linear operators and $\partial/\partial\eta$ is the usual outward conormal derivative. The measurable and essentially bounded controls u are assumed to satisfy the inclusion

$$u(t,x) \in [a,b], \; a,b \in R, \text{a.e. on } [0,T_E] \times S.$$

Because of the specifics of time-optimal control problems we need control restrictions (and the related set of admissible controls) not only on $[0,T_E]$ but also on some subintervals. Therefore, we take for convenience a bounded, convex and closed subset

$U_{ad} \subset L_\infty(S)$ and denote the set of admissible controls connected with an interval $[0,T]$ by

$$L_\infty(0,T;U_{ad}) = \{u \in L_\infty((0,T)\times S) \mid \ u(t,\cdot) \in U_{ad} \text{ a.e. on } [0,T]\}.$$

Moreover, it is useful to work with the notion of abstract functions and to interprete the set of admissible controls as a subset of some associated L_p-space, more precisely: We take a fixed $p > n+1$ and regard $L_\infty(0,T;U_{ad})$ as a subset of $L_p(0,T;L_p(S))$.

Remark 2.1: Obviously $L_\infty(0,T;U_{ad}) = L_\infty(0,T_E;U_{ad})|_{[0,T]}$, for all $T \in (0,T_E]$.

In order to define a generalized solution of (1), the integral equation method is used in the paper. To this aim we suppose the boundary S and the family $\{L(t)\}_{t\in[0,T_E]}$ of operators to be sufficiently smooth, so that a Green's function $G(x,y,t,s)$ of equation (1) (subject to homogeneous boundary conditions) exists.

We call $w = w(u)$ a *generalized solution* of (1) (or a generalized state) *associated with a given control* $u \in L_\infty((0,T_E)\times S)$, if w is continuous on $[0,T_E]\times \bar{D}$ and the integral equation

$$w(t,x) = r(t,x) + \int_0^t \int_S G(x,y,t,s) b(u(s,y), w(s,y), s, y)\, dS_y ds \tag{2}$$

holds on $[0,T_E]\times \bar{D}$, where

$$r(t,x) = \int_D G(x,y,t,0) w_0(y)\, dy + \int_0^t \int_D G(x,y,t,s) f(s,y)\, dy ds.$$

This method to define generalized solutions was extensively studied by v. Wolfersdorf [25] and frequently used by several authors, especially in control theory.

In the sequel we shall consider the states also as abstract functions (or trajectories) with respect to the time t. According to the definition above we get $w \in C([0,T_E];C(\bar{D}))$ ($w(t) \in C(\bar{D})$ describes the state of the system at time t).

For the definition of the terminal condition we introduce a functional f_0 which is continuous on $C(\bar{D})$ as well as a family of functionals $\{g_T\}_{T\in(0,T_E]}$ with the following properties:

(A1) (i) g_T is continuous on $L_p(0,T;L_p(S))$,

(ii) for each $u \in L\infty(0,T_E;U_{ad})$, the function

$$h_u(t) = \begin{cases} 0 & , \ t = 0 \\ g_t(R_t u) & , \ t > 0 \end{cases}$$

is continuous on $[0,T_E]$, where $R_t u$ is the restriction of u to $[0,t]$ (for an easy example compare problem (E) below).

Now we can formulate the following time–optimal control problem:

$$
(\mathrm{P})\begin{cases}\text{Minimize } T \text{ subject to the terminal condition} \\ \qquad f_0(w(u;T)) + g_T(u) \le r & (3) \\ \text{and subject to constraints on the control and the state} \\ \qquad u \in L_\infty(0,T;U_{ad}) & (4) \\ \qquad f_i(w(u;t)) \le c_i(t), \quad i = 1(1)m, \quad \forall t \in [0,T], & (5) \end{cases}
$$

where the f_i are continuous functionals on $C(\bar{D})$, $c_i \in C[0,T_E]$, $r \in R$ is fixed and $w = w(u,\cdot)$ is the solution of equation (2) on $[0,T]$.

In order to illustrate this general class of problems, we give an easy example. A heating problem for a domain $D \subset R^3$ with sufficiently smooth surface S can be described as follows (w(u;t,x) is the temperature in the body):

$$
(\mathrm{E})\begin{cases}\text{Minimize T subject to the terminal condition} \\ \qquad \|w(u;T) - z(\cdot)\|_{L_2}^2 + \lambda \int_0^T \int_S u(t,x)^2 dS_x dt \le r \\ \text{and to constraints on the control and the state} \\ \qquad |u(t,x)| \le 1 \quad \text{a.e. on } (0,T) \times S, \\ \qquad |w(t,x_1) - w(t,x_2)| \le c \quad \text{on } [0,T]\,, \end{cases}
$$

where w is the solution of the initial boundary-value problem

$$
\begin{aligned} w_t &= \Delta w && \text{in } (0,T] \times D, \\ w(0) &= w_0\,, \\ \partial w / \partial \eta &= \alpha(w)[u - w] && \text{on } (0,T] \times S \end{aligned}
$$

($\alpha(\cdot) \in C^1(R)$, $z \in L_2(D)$ and $w_0 \in C(\bar{D})$ given, $x_1, x_2 \in D$, $r, \lambda \ge 0$ are fixed, u is the outward temperature, viewed as the control).

Obviously, the presence of state constraints does not cause essential difficulties for the investigation of existence of time-optimal controls. Nevertheless, this kind of constraints is included in the statement of the problem, because we shall be also interested in optimality conditions.

For problem (P) we define a set $\mathbf{T}_r$ of 'admissible times'

$$\mathbf{T}_r = \{T > 0 : \text{there exists } u,\ w(u) \text{ such that conditions (3)-(5) are satisfied } \}.$$

Moreover, we suppose in what follows that the following assumptions are fulfilled, which are natural for a meaningful problem:

(A2) (i) $\mathbf{T}_r \cap [0, T_E) \neq \emptyset$

(ii) $f_0(w_0) > r$

(iii) $f_i(w_0) \leq c_i(0)$, $i = 1(1)m$.

Finally we define

$$T_0 = \inf \{T | \, T \in \mathbf{T}_r\}.$$

T_0 is said to be the *optimal time*. If $T_0 \in \mathbf{T}_r$, then the control problem is said to be *solvable* and any associated control u_0 and state $w(u_0)$ satisfying conditions (3)-(5) is called an *optimal pair*.

Remark 2.2: Condition (ii) of (A2) does not guarantee $T_0 > 0$.

Remark 2.3: The problem (P) can be formulated in a similar manner also for controls $u = u(t)$ depending only on the time. In this case, the use of spaces of abstract functions is not necessary with respect to the control (now the embedding $L_\infty(0, T; U_{ad}) \subset L_p(0, T)$, p as above, holds). This simplifies the presentation, but it is more difficult than for $u = u(t, x)$ to show the existence of optimal controls by convexity and measurable selection (see Section 6).

3 Existence and uniqueness of generalized states

Before discussing the solvability of problem (P), we shall investigate the integral equation (2). The following theorem answers the question of existence and uniqueness of a solution (corresponding to a fixed $u \in L_\infty(0, T_E; U_{ad})$).

Theorem 3.1: *Assume that the conditions*

(i) *there exists a continuous partial derivative with respect to the second variable of the function $b(y, v, x, t)$,*

(ii) *w_0 can be extended to a continuously differentiable function on a neighbourhood of the boundary S*

are satisfied, together with the properties of $L(t)$, f, w_0 and b stated above. Then, for an arbitrarily given $u \in L_\infty(0, T_E; U_{ad})$, a unique solution $w(u)$ of the integral equation (2) exists locally with respect to the time t. If additionally an a priori estimate of the kind

$$\|w(u; t)\|_{C(\bar{D})} \leq K_0 \tag{6}$$

is fulfilled for all t on which $w(u; \cdot)$ exists, then a unique solution $w(u)$ of equation (2) exists globally, i.e. on $[0, \infty)$.

Proof: The main idea is based on a fixed-point technique, which is a powerful tool to handle semilinear evolution equations in the theory of semigroups (Goldstein [12], Haraux [13]). In order to reduce the technical effort, we will only briefly sketch the essential steps of the application to our problem. For a detailed treatment see Eppler [3, Chapter 1.3], a survey is also contained in Eppler [4].

At first we have continuity of $r(\cdot, \cdot)$ on $[0, T_E] \times \bar{D}$ by condition (ii) (see Friedman [7, Chapter 5.3]) and properties of integral operators on D with Green's function as the kernel (see Tröltzsch [23, Chapter 5.6]). Moreover, it is useful to split equation (2) into an integral equation on $[0, T_E] \times S$ and an integral representation on $[0, T_E] \times D$.

In the next step we take a fixed admissible control u and investigate the nonlinear operator $\mathbf{A}_u(\tau)$, defined on $C([0,\tau]; C(S))$ by

$$(\mathbf{A}_u(\tau)p)(t,x) = r(t,x) + \int_0^t \int_S G(x,y,t,s) b(u(s,y), p(s,y), s, y)\, dS_y ds.$$

The linear boundary integral operators $\mathbf{S}(t,s)$, defined by

$$(\mathbf{S}v)(x) = \int_S G(x,y,t,s) v(y)\, dS_y,$$

are continuous from $L_\infty(S)$ to $C(S)$ for all $0 \le s < t \le \tau$ and it holds $\|\mathbf{S}(t,s)\| \le c(t-s)^{-1/2}$ (Tröltzsch [23,chapter 5.6]). Together with condition (i) this implies the continuity of $\mathbf{A}_u(\tau)$ and the validity of $\|\mathbf{A}_u(\tau)p - \mathbf{A}_u(\tau)q\|_C \le c\tau^{1/2}\|p-q\|_C$ with some constant c. Similarly it can be shown that $\mathbf{A}_u(\tau)$ maps the set

$$M(\tau) = \{p(\cdot) \in C([0,\tau]; C(S)) \mid \|p(t) - w_0\|_{C(S)} \le d \text{ for all } t \in [0,\tau]\}$$

into itself for some $d > 0$, if τ is small enough. Hence, the local existence of a unique solution $w(u)$ of equation (2) is ensured by the Banach fixed–point theorem.

Additionally we have to show a continuation principle, which is obvious in semigroup theory: If for a given $u \in L_\infty(0, T_E; U_{ad})$ the associated state $w(u)$ exists on an interval $[0, t_0]$, $0 < t_0 < T_E$, this existence interval can be extended to an interval $[0, t_1]$, with $t_1 > t_0$.

To this aim we choose the state $w(u; t_0)$ at t_0 as the new initial condition, 'start' with an equation similar to (1) and apply the local existence result. By means of formula

$$\int_D G(x,y,t,t_0) G(y,z,t_0,s)\, dy = G(x,z,t,s),\ s < t_0 < t,$$

it is possible to prove that $\bar{r}$, with

$$\bar{r}(t,x) = \int_D G(x,y,t_0+t,t_0) w(u;t_0,y) dy + \int_0^t \int_D G(x,y,t_0+t,t_0+s) f(t_0+s,y) dy ds,$$

is continuous on $[0, t_1 - t_0] \times \bar{D}$ ($w(u; t_0)$ does not satisfy the regularity condition (ii) on w_0 in general) and that the 'composed' state is in fact a solution of (2) on the interval $[0, t_1]$.

Finally, the length of the continuation intervals is bounded below by a positive constant, if estimate (6) is valid. Consequently, the solution of (2) exists on a time interval of arbitrary length ∎

Applying Theorem 3.1 to the example (E) stated in Section 2, we get the following corollary.

Corollary 3.2: *For $\alpha \in C^1$ and an initial state w_0 satisfying condition (ii) of Theorem 3.1, the generalized solution of the state equation exists locally. Furthermore, in one of the following cases an a priori estimate (6) holds for all $u \in L_\infty(0, T_E; U_{ad})$, implying globally existing trajectories:*

(i) $|\alpha(w)| \leq K$, *for all* $w \in R$,

or

(ii) $\alpha(w) > 0$, *for all* $w \in R$; *the function f (see (1)) satisfies a Hölder condition* $|f(t,x_1) - f(t,x_2)| \leq \bar{c}|x_1 - x_2|^\beta$, *with* $\beta \in (0,1)$, *for all* $x_1, x_2 \in \bar{D}$ *uniformly for all* $t \in [0,T_E]$; *the maximum principle is satisfied for the parabolic differential operator* $\partial/\partial t - L(t)$.

In order to derive an a priori bound (6) for $w(u)$ with condition (i), we get a linear integral inequality for the absolute value of $w(u;\cdot,\cdot)$. The solution of the associated integral equation is an upper bound for $|w(u;\cdot,\cdot)|$. In the case of condition (ii) we use the maximum principle for the derivation of an a priori estimate (6) *uniformly* for all $u \in C([0,T_E];C(S))$. Conclusion (i) of Lemma 3.3 ensures the existence of generalized states for all admissible controls. For more details see Eppler [3, Chapter 1.4-1.5], or Eppler [4].

Although the question of global existence and uniqueness of the states is not essential for solvability investigations of problem (P), this result is interesting in its own right. Moreover, we shall see that a priori bounds play also an important role for other results. At first we obtain the following conclusions.

Lemma 3.3: *Assume that the a priori estimate (6) is valid uniformly for all* $u \in L_\infty(0,T_E;U_{ad})$ *with the same* K_0. *Then*

(i) *the "state mapping"* $u \mapsto w(u)$ *is continuous from* $L_\infty(0,T_E;U_{ad})$ *(endowed with the* L_p*-norm, compare Section 2) into the state space* $C([0,T_E];C(\bar{D}))$,

(ii) *the trajectory set* $Q(0,T_E) = \{w(u) | u \in L_\infty(0,T_E;U_{ad})\}$ *is a set of equicontinuous abstract functions.*

Proof: For $p > N+1$ a linear operator defined by

$$w(t,x) \mapsto \int_0^t \int_S G(x,y,t,s)w(s,y)\,dS_y ds$$

is completely continuous from $L_p(0,T_E;L_p(S))$ into $C([0,T_E];C(S))$ or $C([0,T_E];C(\bar{D}))$ (cf. Tröltzsch [23, Chapter 5.6]). We take a sequence $\{u_n\} \subset L_\infty(0,T_E;U_{ad})$ with $u_n \to_{L_p} u_0 \in L_\infty(0,T_E;U_{ad})$ and show that every subsequence $\{u_{n'}\} \subset \{u_n\}$ contains a subsequence $\{u_{n''}\} \subset \{u_{n'}\}$ with $w(u_{n''}) \to_C w(u_0)$, which would imply statement (i) of the Lemma. The continuity of function b, the uniform a priori estimate (6) and $\{u_{n'}\} \subset L_\infty(0,T_E;U_{ad})$ ensure the uniform boundedness a.e. of the functions $b_{n'}$, $b_{n'}(t,x) = b(u_{n'}(t,x), w(u_{n'};t,x), t, x)$. Hence, we have a subsequence $\{u_{n''}\}$ satisfying

$$b_{n''} = b(u_{n''}, w(u_{n''})) \rightharpoonup_{L_p} \bar{b} \in L_p(0,T_E;L_p(S)).$$

This implies $w(u_{n''}) \to_C \bar{w}$,

$$\bar{w}(t,x) = r(t,x) + \int_0^t \int_S G(x,y,t,s)\bar{b}(s,y)\,dS_y ds.$$

Moreover, we have $b_{n''} \to_{L_p} b(u_0,\bar{w})$, because of $u_{n''} \to_{L_p} u_0$, $w(u_n) \to_C \bar{w}$ and the continuity of b. Consequently, it holds $\bar{b} = b(u_0,\bar{w})$ and therefore $\bar{w} = w(u_0)$, i.e., $w(u_{n''}) \to_C w(u_0)$.

For (ii) we need precompactness of the set $Q(0,T_E)$ with respect to $C([0,T_E];C(\bar{D}))$. This can be easily derived, because the set

$$B = \{b \in L_p(0,T_E;L_p(S)) |\ b = b(u,w(u)),\ u \in L_\infty(0,T_E;U_{ad})\}$$

is bounded (hence, weakly precompact) in $L_p(0,T_E;L_p(S))$ and the Volterra integral operator with Green's function as the kernel is compact for $p > N+1$ from $L_p(0,T_E;L_p(S))$ into $C([0,T_E];C(\bar{D}))$ ∎

Remark 3.1: Obviously the same holds with T_E replaced by $T \in (0,T_E)$. Similarly, the precompactness of the sets of final states $Q(T) = \{w(u;T)|\ u \in L_\infty(0,T;U_{ad})\}$ in $C(\bar{D})$ can be proved, but all sets $Q(0,T)$ and $Q(T)$ are not closed in general. Analogous results can be proved in a similar way in the case $u = u(t)$ (control depending only on time).

Finally, we state another result on the sets $Q(T)$, being of technical interest for the proofs in Sections 5 and 6.

Lemma 3.4: *If a uniform a priori estimate is satisfied for the states, then the set-valued mapping $T \mapsto Q(T)$, $(Q(0) = \{w_0\})$, is continuous on $[0,T_E]$ with respect to the usual Hausdorff metric in $2^{C(\bar{D})}$. Moreover, the set $Q = \bigcup_{T\in[0,T_E]} Q(T)$ is precompact.*

Proof: The Hausdorff continuity of the set-valued mapping $T \mapsto Q(T)$ follows immediately from the precompactness of $Q(0,T_E)$ (cf. Lemma 3.3). Consequently, the continuity of $T \mapsto Q(T)$, the precompactness of the sets $Q(T)$ and the compactness of $[0,T_E]$ ensure the precompactness of Q (cf. Eppler [3, Chapter 2.4]) ∎

4 Associated fixed–time problems and optimality conditions

In this chapter we investigate the relations between a family of problems with fixed final time T and the time-optimal control problem (P). These connections are very useful for the derivation of optimality conditions as well as for proving the existence of time-optimal solutions.

For all $T \in (0,T_E]$ the problems are defined as follows :

$$(\mathrm{P(T)}) \left\{ \begin{array}{lll} f_0(w(u;T)) + g_T(u) & \rightarrow & \inf \\ u \in L_\infty(0,T;U_{ad}) & & \\ f_i(w(u;t)) \le c_i(t), & i = 1(1)m, & \forall t \in [0,T], \\ \text{w(u) is the solution of} & \text{equation (2).} & \end{array} \right.$$

According to the definition above, we introduce the optimal value function

$$F(T) = \left\{ \begin{array}{ll} f_0(w_0) & ,\ T = 0 \\ \inf\{(P(T))\} & ,\ 0 < T \le T_E. \end{array} \right.$$

Lemma 4.1: *For the optimal value function $F(T)$ the following assertions are true:*

(i) $F(T) \geq r$, *for all* $T \leq T_0$,

(ii) $F(T) \leq r$, *for all* $T \in \mathbf{T}_r$,

(iii) the solvability of all problems $(P(T))$ *implies* $F(T) > r$, *for all* $T < T_0$,

(iv) problem (P) *is solvable if and only if* $F(T_0) = r$ *holds and* $(P(T_0))$ *is solvable.*

Proof: Conclusions (i)-(iii) can be immediately derived from the definitions of T_0, $\mathbf{T}_r$ and the optimal value function F. In order to prove conclusion (iv), we need only (A1) and the continuity of the states, hence the continuity of the functions

$$\varphi_u(t) := f_0(w(u;t)) + h_u(t), \tag{7}$$

the 'trace function' of the terminal condition along a given $u \in L_\infty(0;T_E;U_{ad})$. Obviously, $F(T_0) = r$ and the solvability of $(P(T_0))$ implies the solvability of problem (P). For the opposite direction we take an optimal control $u_0 \in L_\infty(0,T_0;U_{ad})$ of problem (P) and show that u_0 is also optimal for problem $(P(T_0))$ and $\varphi_{u_0}(T_0) = F(T_0) = r$ is satisfied. To this aim we assume $\varphi_{u_0}(T_0) < r$. The continuity of φ_{u_0} and $\varphi_{u_0}(0) = f_0(w_0) > r$ (cf. condition (ii) of (A2)) ensure the existence of a $\hat{T} < T_0$ with $\varphi_{u_0}(\hat{T}) = r$. This contradicts the definition of T_0, because u_0 and $w(u_0)$ are also admissible on the interval $[0,\hat{T}]$. Analogously we get the optimality of u_0 with respect to problem $(P(T_0))$ ∎

Remark 4.1: The analogue to $F(T_0) = r$ for a 'classical terminal condition' is $w(T_0) \in \partial W_{T_0}$. Moreover, the proof of the equivalence relation (iv) is totally analogous to the linear case (cf. Tröltzsch [23, chapter 5.5]). However, as we will see in Sections 5 and 6, we need additional assumptions in order to guarantee the solvability of $(P(T_0))$ as well as the relation $F(T_0) = r$.

Remark 4.2: It is necessary to remark that (A1) and the continuity of the states do not guarantee $T_0 > 0$ (cf. Remark 2.2). Each of the following conditions is sufficient for $T_0 > 0$:

(i) The solvability of (P).

(ii) The uniform a priori estimate (6) (which implies the equicontinuity of the trajectories, cf. Lemma 3.3) together with the nonnegativity of all functionals $g_T(\cdot)$.

(iii) A uniform a priori estimate (6) together with the equicontinuity of the functions $h_u(\cdot)$ at $t = 0$.

By means of conclusion (iv) of Lemma 4.1 we are now able to derive optimality conditions for problem (P) via the fixed-time problem $(P(T_0))$. In doing so, we obviously need additional differentiability and regularity assumptions on the objective, the state equation and the state restrictions. Moreover, having in mind applications, for instance, for the derivation of bang-bang or generalized bang-bang principles, it is much more suited to suppose more structural details, especially on f_0 and g_T. This can be done for first order necessary conditions along the lines of the theory developed by Tröltzsch [23]. Because there are no essential difficulties on this way, we refer only to Eppler [3, Chapter 3]. With respect to time t we clearly get no additional differential-type condition for our kind of time-optimal control problems (roughly speaking, according to Lemma 4.1 the related 'optimality condition' is $F(T_0) = r$).

In a similar manner it is possible to invoke the theory of second order necessary and sufficient optimality conditions, which was investigated recently by Goldberg [10], and Goldberg and Tröltzsch [11].

5 Existence by completely continuous state mapping

A well known difficulty for the existence of optimal controls is the noncompactness (with respect to the embedding space) of the set of admissible controls if it is defined as above. However, $L_\infty(0,T;U_{ad})$ is closed, bounded and convex for all $T \in (0,T_E]$ in the corresponding L_p–space, hence weakly compact. Therefore, one way to overcome this difficulty consists in proving the *complete continuity* of the state mapping. This main idea was realized by Sperber [22] for optimal control problems governed by semilinear parabolic equations.

Lemma 5.1: *Assume that the function b occuring in the boundary condition has the form*

$$b(y,v,t,x) = y\, b_1(v,t,x) + b_2(v,t,x), \tag{8}$$

(i.e., b is affine linear in y) with continuous functions b_1, b_2 having continuous partial derivatives b_{1v} and b_{2v}. If furthermore the uniform a priori estimate (6) holds, then the state mapping $u \mapsto w(u)$ is continuous from $L_\infty(0,T;U_{ad})$ (endowed with the weak topology of L_p) to the state space $C([0,T];C(\bar{D}))$ ($T \in (0,T_E]$ arbitrarily chosen).

Proof: We proceed similar to the proof of conclusion (i) of Lemma 3.3 (clearly with $u_n \rightharpoonup_{L_p} u_0$ instead of $u_n \to_{L_p} u_0$), but now we are able to show that $u_n \rightharpoonup_{L_p} u_0$ and $w(u_n) \to \bar{w}$ implies $b(u_n, w(u_n)) \rightharpoonup_{L_p} b(u_0, \bar{w})$ by using condition (8) and the concrete duality product in $L_p(0,T;L_p(S))$ ∎

Remark 5.1: The complete continuity of the state mapping implies the compactness of the sets $Q(0,T)$, $Q(T)$ and Q.

These results, together with conclusion (iv) of Lemma 4.1, are basic tools to show the solvability of problem (P). Moreover, as we had already discussed in Section 4, we need even more in order to guarantee also $F(T_0) = r$.

Theorem 5.2: *Additionally to the assumptions of Lemma 5.1 we assume that*

(i) *for all $T \in (0,T_E]$ the functionals $g_T(\cdot)$ are weakly lower semicontinuous on $L_\infty(0,T;U_{ad})$ (in L_p-sense, this is fulfilled, e.g., for convex and continuous functionals),*

(ii) *the functions $h_u(\cdot)$ $(u \in L_\infty(0,T_E;U_{ad}))$ are equicontinuous on $[0,T_E]$.*

Then (P) is solvable.

Proof: At first we note that the assumptions ensure $T_0 > 0$ (cf. Remark 4.2). Now we take a sequence $\{T_n\} \subset \mathbf{T}_r \cap [T_0,T_E]$ with $T_n \to T_0$ (which is always possible, cf. (A2)) and associated admissible pairs $(u_n, w(u_n))$, i.e., $u_n \in L_\infty(0,T_n;U_{ad})$, $w(u_n)$ satisfies (5) on $[0,T_n]$ and

$$\varphi_{u_n}(T_n) = f_0(w(u_n;T_n)) + g_{T_n}(u_n) \le r \tag{9}$$

holds for all n. In a first step we show

$$F(T_0) = r. \tag{10}$$

The restrictions of u_n and $w(u_n)$ to the interval $[0, T_0]$ (in the sequel also denoted by $u_n, w(u_n)$) are obviously admissible for problem $(P(T_0))$, hence $\liminf \varphi_{u_n}(T_0) \geq F(T_0) \geq r$ (the second inequality is implied by relation (i) of Lemma 4.1). One the other hand, the assumptions guarantee the equicontinuity of the states $w(u_n)$ and the uniform continuity of f_0 on the compact set Q (cf. Remark 5.1). Therefore the "terminal trace functions" (cf. (7)) $\varphi_u(\cdot)$ $(u \in L_\infty(0, T_E; U_{ad}))$ are equicontinuous on $[0, T_E]$ and it holds $\lim |\varphi_{u_n}(T_0) - \varphi_{u_n}(T_n)| = 0$. Together with (9), this yields immediately

$$F(T_0) \leq \limsup \varphi_{u_n}(T_0) = \limsup \varphi_{u_n}(T_n) \leq r .$$

Thus, (10) holds and $\{(u_n, w(u_n))\}$ is a minimizing sequence for $(P(T_0))$.

It remains to prove the solvability of $(P(T_0))$. Because of the weak compactness of $L_\infty(0, T_0; U_{ad})$ with respect to $L_p(0, T_0; L_p(S))$, we have at least one weak accumulation point $u_0 \in L_\infty(0, T_0; U_{ad})$ of the sequence $\{u_n\}$. Moreover, by Lemma 5.1 we deduce $w(u_{n'}) \longrightarrow_C w(u_o)$ for all subsequences $\{u_n\} \subset \{u_n\}$ with $u_{n'} \rightharpoonup_{L_p} u_0$. Together with the admissibility of all states $w(u_n)$ (with respect to (5)), this ensures that $w(u_0)$ also satisfies the state restrictions on $[0, T_0]$. Furthermore, the weak lower semicontinuity of g_{T_0} guarantees $\varphi_{u_o}(T_0) \leq \lim \varphi_{u_{n'}}(T_0) = F(T_0) = r$, that means, $(u_0, w(u_0))$ is an optimal pair for $(P(T_0))$ and also for (P) ∎

Theorem 5.3: *If there exists a $\bar{u} \in L_\infty(0, T_E; U_{ad})$ with $f_i(w(\bar{u}; t)) \leq c_i(t), i = 1(1)m$, for all $t \in [0, T_E]$, then the assumptions of Lemma 5.1 and assumption (i) of Theorem 5.2 are sufficient for the solvability of all problems $(P(T))$, $T \in (0, T_E]$.*

Roughly speaking, the proof consists of the second part of the preceding one (notice that the admissible sets of all problems $(P(T))$ are non-empty by means of the additional assumption).

Remark 5.2: The structure of the boundary condition and the special choice of the functionals $g_T(\cdot)$ in our example (E) allows us to apply directly Theorems 5.2 and 5.3 in order to get the existence of optimal solutions for the problem (E) as well as for the associated problems with fixed final time.

6 Existence by separation of convex sets and measurable selection theorems

Whereas the question of existence is solved for (E), we want to present another approach for proving the existence of optimal controls, overcoming the complete continuity of the state mapping, that means, to overcome the affine-linear structure of the boundary condition with respect to the control. For control problems governed by ordinary differential equations this was done by means of theorems on the separation of convex sets and measurable selection (see, for instance, Macki and Strauss [17]). The main purpose of this section is to apply similar techniques to parabolic problems.

Unfortunately, the differences between the two cases $u = u(t,x)$ and $u = u(t)$ now become essential, in contrary to the preceding investigations. This concerns the formulation of the assumptions and their consequences as well as a different effort in the proofs.

Theorem 6.1: *Let K_0 be the constant of the uniform a priori estimate (6). We assume:*

(i) *There exists a uniform a-priori bound (6).*

(ii) *The terminal condition does not explicitly depend on the control u (i.e., $g_T(u) \equiv 0$, for all $T \in (0,T_E]$).*

(iii) (a) *In the case $u = u(t,x)$: the sets $M_{t,x}(v) = \{b(y,v,t,x)|\ y \in [a,b]\}$ are convex subsets of R, for all $(t,x) \in [0,T_E] \times S$ and all $v \in R$ with $|v| \leq K_0$.*

(b) *In the case $u = u(t)$: the sets $M_t(v) = \{b(y,v,t,\cdot) \in C(S)\ |\ y \in [a,b]\}$ are convex subsets of $C(S)$, for all $t \in [0,T_E]$ and all $v(\cdot) \in C(S)$ with $\|v\|_{C(S)} \leq K_0$.*

Then problem (P) is solvable.

Proof: In the first part we can follow the lines of the proof of Theorem 5.2. This causes even lower effort, as the part $g_T(u)$ is now vanishing. Nevertheless, the equicontinuity of the states and the precompactness of the set Q, hence the uniform continuity of f_0 on $\bar{Q}$, is necessary in this situation, too. As a result we also get a minimizing sequence $\{(u_n, w(u_n))\}$ for problem $(P(T_0))$ and (cf. (10))

$$\lim f_0(w(u_n;T_n)) = \lim f_0(w(u_n;T_0)) = F(T_0) = r.$$

Concerning the second part, we recall (cf. Section 3) that the a priori estimate (6) together with the essential boundedness of the controls ensure the boundedness a.e. of the functions $b_n(\cdot,\cdot)$, defined by

$$b_n(t,x) = b(u_n(t,x), w(u_n;t,x), t, x), \quad (t,x) \in [0,T_0] \times S.$$

Therefore, $\{b_n\}$ is uniformly bounded in $L_p((0,T_0) \times S)$ and without loss of generality we have $b_n \rightharpoonup_{L_p} \bar{b}$. Moreover, we deduce from the compactness of the linear boundary integral operator, with Green's function as the kernel, that

$$w(u_n) \longrightarrow_C \bar{w}, \tag{11}$$

where (compare (2))

$$\bar{w}(t,x) = r(t,x) + \int_0^t \int_S G(x,y,t,s)\bar{b}(s,y)dS_y ds, \ (t,x) \in [0,T_0] \times D.$$

If we were able to show the existence of a control $u_0 \in L_\infty(0,T_0;U_{ad})$ such that $\bar{b} = b(u_0,\bar{w})$ (in detail: $\bar{b}(t,x) = b(u_0(t),\bar{w}(t,x),t,x)$ or $\bar{b}(t,x) = b(u_0(t,x),\bar{w}(t,x),t,x)$ a.e. on $[0,T_0] \times S$), then, altogether we have $\bar{w} = w(u_0)$, $f_0(w(u_0;T_0)) = r$ from (10) and (11), and $w(u_0)$ satisfies (5) on $[0,T_0]$ (all $w(u_n)$ are admissible). This ensures the optimality of $(u_0, w(u_0))$ for problem $(P(T_0))$.

The final step of the proof must be performed separately for the two cases $u = u(t,x)$ and $u = u(t)$.

In the case $u = u(t,x)$ we introduce the set

$$M_1 = \{(t,x) \in [0,T_0] \times S|\ \ \bar{b}(t,x) \notin M_{t,x}(\bar{w}(t,x))\}.$$

We are able to show that

(i) M_1 is Lebesque–measurable

(ii) $\text{mes}(M_1) > 0$ is a contradiction to $b_n \rightharpoonup_{L_p} \bar{b}$,

with the help of our convexity assumption (iii)/(a).

For the proof of (i) we introduce the functions $\underline{m}(\cdot,\cdot)$ and $\overline{m}(\cdot,\cdot)$, defined by

$$\underline{m}(t,x) = \min\{z|\ z \in M_{t,x}(\bar{w}(t,x))\} = \min\{b(y,\bar{w}(t,x),t,x)|\ y \in [a,b]\},$$

$$\overline{m}(t,x) = \max\{z|\ z \in M_{t,x}(\bar{w}(t,x))\} = \max\{b(y,\bar{w}(t,x),t,x)|\ y \in [a,b]\},$$

for all $(t,x) \in [0,T_0] \times S$. The continuity of $\bar{w}$ on $[0,T_0] \times S$ and the uniform continuity of $b(\cdot,\cdot,\cdot,\cdot)$ on the compact set $[a,b] \times [-K_0,K_0] \times [0,T_0] \times S$ guarantee the continuity, hence Lebesgue measurability of $\underline{m}(\cdot,\cdot)$ and $\overline{m}(\cdot,\cdot)$. Moreover, we deduce from (iii)/(a)

$$M_{t,x}(\bar{w}(t,x)) \;=\; [\underline{m}(t,x),\, \overline{m}(t,x)]$$

and $M_1 = \underline{M}_1 \cup \overline{M}_1$ with

$$\underline{M}_1 \;=\; \{(t,x)|\ \bar{b}(t,x) < \underline{m}(t,x)\}, \quad \overline{M}_1 \;=\; \{(t,x)|\ \bar{b}(t,x) > \overline{m}(t,x)\}.$$

The Lebesgue measurability of M_1 now follows obviously from the measurability of the functions $\bar{b}$, $\underline{m}$ and $\overline{m}$.

In order to show (ii), we assume without loss of generality $\text{mes}(\overline{M}_1) > 0$. Assuming this, we get the existence of a constant $\delta > 0$ and of a subset $\overline{M}_1^\delta \subset \overline{M}_1$ with $\text{mes}(\overline{M}_1^\delta) > 0$, such that

$$\bar{b}(t,x) \geq \delta + \overline{m}(t,x) \geq \delta + \sup_{n\geq 1} b(u_n(t,x),\bar{w}(t,x),t,x), \qquad \forall (t,x) \in \overline{M}_1^\delta$$

(the last inequality follows by the definition of $\overline{m}$). The uniform continuity of b and $w(u_n) \rightarrow_C \bar{w}$ ensure

$$|b(u_n(t,x),w(u_n;t,x),t,x) \;-\; b(u_n(t,x),\bar{w}(t,x),t,x)| \leq \delta/2,$$

for all $(t,x) \in [0,T_0] \times S$, $n \geq N_0(\delta)$, and therefore

$$\bar{b}(t,x) \geq \delta/2 + \sup_{n\geq N_0(\delta)} b(u_n(t,x),w(u_n;t,x),t,x), \qquad \forall (t,x) \in \overline{M}_1^\delta.$$

Integration over $\overline{M}_1^\delta$ yields

$$\int_{\overline{M}_1^\delta} \bar{b}(t,x)\, dS_x dt \geq \text{mes}(\overline{M}_1^\delta)\delta/2 \;+ \int_{\overline{M}_1^\delta} b_n(t,x)\, dS_x dt, \quad \forall n \geq N_0.$$

This is a contradiction to $b_n \rightharpoonup_{L_p} \bar{b}$, because it holds $\chi_{\overline{M}_1^\delta} \in L_q((0,T_0)\times S)\,(1/q+1/p=1)$ for the characteristic function $\chi(\cdot)$ of the set $\overline{M}_1^\delta$. Consequently, we get

$$\bar{b}(t,x) \in \{b(y,\bar{w}(t,x),t,x)|\ y \in [a,b]\} \text{ a.e. on } [0,T_0] \times S,$$

that means, there exists at least one selection $\bar{u}(t,x) \in [a,b]$ with

$$\bar{b}(t,x) \;=\; b(\bar{u}(t,x),\bar{w}(t,x),t,x) \text{ a.e. on } [0,T_0] \times S.$$

The application of the Fillipov lemma (cf. Macki and Strauss [17]) guarantees a *measurable* selection $u_0(\cdot,\cdot)$, by setting

$$u_0(t,x) = \begin{cases} a, & (t,x) \in M_1 \\ \min\{y \in [a,b] | \bar{b}(t,x) = b(y,\bar{w}(t,x),t,x), & (t,x) \notin M_1. \end{cases}$$

This completes the proof of Theorem 6.1 for the case $u = u(t,x)$.

In the case $u = u(t)$ we regard $\bar{b}: [0,T-0] \mapsto L_p(S)$ as an abstract L_p-function with respect to t and work with the set (compare (iii)/(b))

$$M_2 = \{t \in [0;T_0] | \bar{b}(t) \notin M_t(\bar{w}(t))\}.$$

Now we have to derive $\text{mes}(M_2) = 0$. At first we remark that the convex sets $M_t(\bar{w}(t))$ are compact subsets of $C(S)$ for all $t \in [0,T_0]$ (as images of the continuous mappings $y \mapsto b(y,\bar{w}(t),t,\cdot)$ over the compact set $[a,b] \subset R$) , and the set–valued mapping $t \mapsto M_t(\bar{w}(t))$ is Hausdorff continuous (with respect to $C(S)$) on $[0,T_0]$. After an embedding of the function $\bar{b}$ and the sets $M_t(\bar{w}(t))$ into $L_2(0,T_0;L_2(S))$ and $L_2(S)$, respectively, these properties are also valid with respect to the new metric. Obviuosly it holds

$$M_2 = \{t \in [0,T_0] | \bar{d}(t) := d_H[\bar{b}(t), M_t(\bar{w}(t))] = \min_{v \in M_t} \|\bar{b}(t) - v\|_{L_2} > 0\},$$

where $d_H[\cdot,\cdot]$ denotes the usual Hausdorff distance. The function $\bar{d}(\cdot)$ is measurable, because convexity and compactness of the sets $M_t(\cdot)$ as well as the Hausdorff continuity of the set–valued mapping $t \mapsto M_t(\cdot)$ guarantee the continuity of the function $\hat{d}(x,t) = d_H[x, M_t(\bar{w}(t))]$ with respect to $x(\cdot) \in L_2(S)$ and $t \in [0,T_0]$. Hence M_2 is measurable and we assume $\text{mes}(M_2) > 0$.

The convexity and compactness of the sets $M_t(\bar{w}(t))$ imply for all $t \in [0,T_0]$ the existence of a closed hyperplane, which strongly separates $\bar{b}(t)$ from $M_t(\bar{w}(t))$, i.e., for all $t \in M_2$ there exists a $p(t) \in L_2(S)$ and an $\alpha(t) > 0$, such that

$$(p(t),\bar{b}(t))_{L_2} \geq \alpha(t) + (p(t),v)_{L_2}, \ \forall v \in M_t(\bar{w}(t)).$$

Moreover, constructing a special $p(\cdot)$ explicitly, it is possible to get $p(\cdot) \in L_\infty(M_2;L_2(S))$ and $\alpha(\cdot) \in L_2(M_2)$. To this aim we introduce the abstract function $v: L_2(S) \times [0,T_0] \to L_2(S)$, defined by

$$v(x,t) = \bar{v}, \ \text{ with } \|\bar{v} - x\|_{L_2} = \hat{d}(x,t)$$

($\bar{v}$ is the orthogonal projection of $x \in L_2(S)$ onto the set $M_t(\bar{w}(t))$). The function $v(\cdot,\cdot)$ is well-defined and continuous with respect to both arguments. Therefore, the function $v_0(\cdot)$, defined by $v_0(t) = v(\bar{b}(t),t)$, is measurable on $[0,T_0]$ and the choice

$$p(t) := (\bar{b}(t) - v_0(t))/\bar{d}(t), \quad \alpha(t) := \bar{d}(t) = \|v_0(t) - \bar{b}(t)\|_{L_2}, \quad \forall t \in M_2$$

satisfies all necessary properties on $p(\cdot)$ and $\alpha(\cdot)$:

(i) $p(\cdot)$ is measurable on M_2 with respect to t,

(ii) $\|p(t)\|_{L_2} = 1$, for all $t \in M_2$, hence $p \in L_\infty(M_2;L_2(S))$ (with $p(t) := 0$, for all $t \in [0,T_0] \setminus M_2$ we get $p \in L_\infty(0,T_0;L_2(S))$),

(iii) $(p(t), v - v_0(t))_{L_2} \leq 0$ for all $v \in M_t(\bar{w}(t))$ and $t \in M_2$ yields $(\bar{b}(t), p(t))_{L_2} \geq \bar{d}(t) + (v, p(t))_{L_2}$, for all $v \in M_t(\bar{w}(t))$, $t \in M_2$.

Futhermore, $\text{mes}(M_2) > 0$ implies the existence of a set $M_2^\delta \subset M_2$ with $\text{mes}(M_2^\delta) > 0$ and $\bar{d}(t) \geq \delta > 0$, for all $t \in M_2^\delta$. Consequently, we have

$$(\bar{b}(t), p(t))_{L_2} \geq \delta + \sup_{v \in M_t} (v, p(t))_{L_2} \geq \delta + \sup_{n \geq 1} (b(u_n(t), \bar{w}(t), t)\,,\, p(t))_{L_2},$$

for all $t \in M_2^\delta$. Analogously to the case $u = u(t,x)$ we get from $w(u_n) \rightarrow_C \bar{w}$ and the continuity of b

$$(\bar{b}(t), p(t))_{L_2} \geq \delta/2 + \sup_{n \geq N_0} (b(u_n(t), w(u_n; t), t)\,,\; p(t))_{L_2}, \quad \forall t \in M_2^\delta.$$

Integration over M_2^δ now yields

$$\int_{M_2^\delta} (\bar{b}(t), p(t))_{L_2}\, dt \geq \text{mes}(M_2^\delta)\delta/2 + \int_{M_2^\delta} (b_n(t), p(t))_{L_2}\, dt, \quad \forall n \geq N_0,$$

in detail

$$\int_0^{T_0} \chi_{M_2^\delta}(t) \int_S \bar{b}(t,x) p(t,x) dS_x\, dt \geq \text{mes}(M_2^\delta)\delta/2 + \int_0^{T_0} \chi_{M_2^\delta}(t) \int_S b_n(t,x) p(t,x) dS_x\, dt$$

for all $n \geq N_0$. This is a contradiction to $b_n \rightharpoonup_{L_p} \bar{b}$, and $\text{mes}(M_2) = 0$ is proved, because it holds $p\chi_{M_2^\delta}(\cdot) \in L_\infty(0, T_0; L_2(S)) \subset L_q(0, T_0; L_q(S))$, with $p\chi_{M_2^\delta}(t,x) := p(t,x)\chi_{M_2^\delta}(t)$.

The application of the Fillipov lemma is also possible, although the situation is now more difficult, because an abstract measurable function occurs. However, we are also able to show the measurability of $u_0(\cdot)$,

$$u_0(t) = \begin{cases} a\,, & t \in M_2 \\ \min\, \{y \in [a,b] \mid \bar{b}(t) = b(y, \bar{w}(t), t)\}\,, & t \notin M_2, \end{cases}$$

because $\bar{b}(\cdot)$ has values in a *separable* Banach space. This is sufficient for the equivalence between the measurability definition of $\bar{b}$ via step functions (in accordance with $\bar{b} \in L_p(0, T_0; L_p(S))$ and the 'almost continuity' property of the Luzin theorem (cf. Dinculeanu [2]), more detailed: For all $n > 1$ there exist a closed subset $F_n \subset [0, T_0]$ with $\text{mes}([0, T_0] \setminus F_n) < 1/n$, such that the abstract function $\bar{b}$ is continuous on F_n. But the continuity of $\bar{b}$ on F_n is sufficient for the closedness of $M_r \cap F_n$ with $M_r := \{t \in [0, T_0] \mid u_0(t) \leq r\}$ for all $n > 1$ and $r \in R$, hence, sufficient for the measurability of M_r. This ensures the measurability of $u_0(\cdot)$ and completes the proof of Theorem 6.1 also for the case $u = u(t)$ ∎

Similarly, the following assertion holds.

Theorem 6.2: *If there exists a control $\bar{u} \in L_\infty(0, T_E; U_{ad})$ with associated admissible state $w(\bar{u})$ ($w(\bar{u})$ satisfies (5) on $[0, T_E]$), then the conditions of Theorem 6.1 are sufficient for the solvability of all problems $(P(T))$, $T \in (0, T_E]$.*

Proof: The restrictions of $(\bar{u}, w(\bar{u}))$ to the interval $[0, T]$ are obviously admissible for all problems $(P(T))$. Hence, we take without loss of generality a minimizing sequence and proceed similarly to the second part of the proof of Theorem 6.1, which is in fact the solvability proof for problem $(P(T_0))$ ∎

Remark 6.1: In the proof we have actually shown the closedness (hence compactness) of the sets $Q(0,T)$, implying also the compactness of all sets $Q(T)$ and hence of Q.

Remark 6.2: In contrary to Section 5 a weak accumulation point of the sequence $\{u_n\}$ is generally not an optimal control. On the other hand, the optimal control u_0 need not to be a weak accumulation point of the sequence $\{u_n\}$, whereas the state $w(u_0)$ is an accumulation point of $\{w(u_n)\}$.

Remark 6.3: Whereas (iii)/(b) is actually an additional assumption for a control depending only on the time (which is satisfied for instance if b has an affine-linear structure), assumption (iii)/(a) for the case $u = u(t,x)$ holds for *every* nonlinear continuous right-hand side of the boundary condition. Consequently, if the terminal condition does not depend explicitly on the control $u = u(t,x)$, the uniform a priori estimate is the unique basic assumption for the solvability of problem (P).

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