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Smooth Interpolating Curves and Surfaces Generated by Iterated Function Systems

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We construct C^1- and C^2-interpolating fractal functions using a certain class of iterated function systems. An estimate for the box dimension of the graph of nonsmooth fractal functions generated by this new class is presented. We then generalize this construction to bivariate functions thus obtaining C^1-interpolating fractal surfaces. Finally, C^n-interpolating fractal surfaces are constructed via integration over C^0 fractal surfaces.

Key words: *Iterated function systems, fractal functions and surfaces, attractors, box dimensions*
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1 Introduction

Continuous fractal interpolation functions and surfaces are useful tools for interpolating and approximating highly complex sets or images. Unlike the classical approximations that treat each image component as a single entity arising from a random assemblage of objects, fractal approximations consider the image component as an interrelated single system. At this point we refer to [1] or [2] for a more complete overview of the techniques involved.

The graphs of these fractal functions and fractal surfaces are attractors of iterated function systems whose maps are affine functions, and provide examples of nowhere differentiable uni- or bivariate real-valued functions. The usefulness of such functions in interpolation and approximation theory is hampered by the lack of a degree of differentiability as is so often required from interpolants or approximants.

We introduce classes of iterated function systems whose attractors are C^1- and C^2-interpolating functions and C^n-interpolating surfaces, $n \geq 1$. (Here we abuse common 'language' by referring to these attractors as smooth fractal functions. But we think of a fractal as a set that is generated by a recursive procedure — random or deterministic — yielding a high degree of geometric selfness at all scales of approximation.) These new classes of smooth fractal curves and surfaces have all the power and advantages of their continuous analogs but provide now a new means of smoothly interpolating and approximating highly complex images.

The outline of this paper is as follows: In Section 2 we briefly review some basic results from the theory of iterated function systems, fractal functions and surfaces. Then we consider a broader class of iterated function systems yielding smooth fractal functions provided certain conditions apply. At the end of this section we present upper and lower bounds for the box dimension of those fractal curves in this new class that fail to be smooth. In Section 3 we introduce smooth fractal surfaces. This is done in two ways: Firstly, by extending the results obtained in Section 2 to bivariate functions. This yields a new class of fractal surfaces that are smooth. Secondly, by integrating over C^0 fractal surfaces. This method gives C^n-interpolating fractal surfaces, for any $n \in \mathbb{N}$.

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2 Smooth Fractal Functions

Before we commence with the construction of smooth fractal curves, let us recall some definitions and results from the theory of iterated function systems and fractal interpolation functions.

Let $X := [0,1] \times \mathbb{R}$, let N be an integer greater than one, and let $w_i : X \to X$, $i = 1, \ldots, N$, be a collection of contractions on (X, d), where d denotes a metric on X. We set $\mathbf{w} := \{w_i : i = 1, \ldots, N\}$. The pair $(X, \mathbf{w})$ is called an *iterated function system* on (X, d). If there exists a non-negative constant $s \in (-1, 1)$ such that $d(w_i(x), w_i(x')) \leq s\, d(x, x')$, for all $x, x' \in X$ and $i = 1, \ldots, N$, then $(X, \mathbf{w})$ is called a *hyperbolic iterated function system* with *contractivity constant* s. It is well-known that every hyperbolic iterated function system possesses a unique compact attractor. For, if $\mathcal{H}(X)$ denotes the set of all non-empty compact subsets of X and h the Hausdorff metric on $\mathcal{H}(X)$, then the map $W : \mathcal{H}(X) \to \mathcal{H}(X)$, $A \mapsto W(A) := \bigcup_{i=1}^N w_i(A)$, is a contraction with contractivity s on the complete metric space $(\mathcal{H}(X), h)$. Hence W has a unique fixed point A^* and

$$A^* = \bigcup_{i=1}^{N} w_i(A^*). \tag{1}$$

Equation (1) expresses the fact that the fractal A^* is a finite union of self-images (at every scale). In [1] and [4] special attractors were generated; namely, attractors that are graphs of continuous univariate real-valued functions. This construction was generalized in [5] and [8]: the attractors are fractal surfaces in $\mathbb{R}^3$, i.e., the graphs of bivariate real-valued functions defined on certain two-dimensional simplicial complexes. For the sake of completeness and to set notation and terminology, let us review these constructions.

Let $\mathcal{J} := \{(x_j, y_j) \in X : 0 = x_0 < x_1 < \cdots < x_N = 1; j = 0, 1, \ldots, N\}$ be a given set of data or interpolation points. Define $w_i(x, y) \equiv (u_i(x), v_i(x, y))$, where

$$u_i(x) := b_i x + x_{i-1}, \qquad v_i(x, y) := a_i x + s_i y + c_i$$

and

$$u_i(0) = x_{i-1}, \ \ u_i(1) = x_i, \ \ v_i(0, y_0) = y_{i-1}, \ \ v_i(1, y_N) = y_i,$$

for all $i = 1, \ldots, N$. The coefficients a_i, b_i, and c_i are then given by

$$a)\ \ a_i = (y_i - y_{i-1}) - s_i(y_N - y_0) \qquad b)\ \ b_i = x_i - x_{i-1} \qquad c)\ \ c_i = y_{i-1} - s_i y_0.$$

The s_i have to satisfy $0 \leq |s_i| < 1$, but are otherwise arbitrary parameters. With the maps w_i as defined above, $(X, \mathbf{w})$ is clearly an iterated function system. If we introduce the norm $\|\cdot\|_\theta$ on $\mathbb{R}^2$, $\|(x, y)\|_\theta := |x| + \theta\,|y|$, for some $0 < \theta < \min_i\{(1 - b_i)/(1 + |c_i|)\}$, then each w_i becomes a contraction in the norm $\|\cdot\|_\theta$. The unique attractor G is the graph of a continuous function $f : I \to \mathbb{R}$ satisfying $f(x_j) = y_j$, for all $j = 0, 1, \ldots, N$ (here and in what follows, I always denotes the unit interval $[0, 1]$). This function is called a *fractal interpolation function*. By equation (1), G is self-affine, i.e., it is a finite union of affine images of itself. To show that G is the graph of a continuous function interpolating $\mathcal{J}$, one defines an operator $T : \hat{C}(I) \to \mathbb{R}^I$ by

$$(T\varphi)(x) := v_i(u_i^{-1}(x), \varphi(u_i^{-1}(x))), \ \ x \in u_i(I), \tag{2}$$

where $\hat{C}(I) := \{\varphi \in C^0(I) : \varphi(x_j) = y_j, j = 0, 1, \ldots, N\}$. One proceeds by showing that T maps into $\hat{C}(I)$, is well-defined and contractive in the sup-norm with contractivity $s = \max_i\{|s_i|\}$. For a more detailed introduction to fractal interpolation functions we refer the reader to [1, 2, 4].

Fractal surfaces are defined in an analogous fashion: Given is a set of interpolation points $\mathcal{J} = \{(x_j, y_j, z_j) : (x_j, y_j) \in \sigma^2,\ z_j \in \mathbb{R}$, and $z_j = 0$ if $(x_j, y_j) \in \partial\sigma^2, j = 0, 1, \ldots, m\}$ in $\mathbb{R}^3$. Here σ^2 denotes the standard 2-simplex in $\mathbb{R}^3$. (This definition follows the developments in [8] rather than [5]. There a more general initial set-up is considered.) Suppose that $\sigma^2 = \bigcup_{i=1}^N \sigma_i^2$ with $\mathring{\sigma}_i^2 \cap \mathring{\sigma}_{i'}^2 = \emptyset$, for $i \neq i'$, and such that each $\sigma_i^2 = u_i(\sigma^2)$, for some unique affine map u_i, and such that the vertices of σ_i^2 are in $\{(x_j, y_j) : j = 0, 1, \ldots, m\}$. We define maps $v_i : \sigma^2 \times \mathbb{R} \to \mathbb{R}$ by $v_i(x, y, z) := a_i x + b_i y + s_i z + c_i$, for given $|s_i| < 1$, and the a_i, b_i, and c_i are uniquely determined by $v_i(x_j, y_j, z_j) = z_{\ell(i,j)}$. Here we defined the labelling map $\ell : \{1, \ldots, N\} \times \{1, 2, 3\} \to \{1, \ldots, m\}$ such that $\{(x_{\ell(i,j)}, y_{\ell(i,j)}) : j = 1, 2, 3\}$ are the vertices of σ_i^2.

Now define an operator T on $\hat{C}(\sigma^2)$, the set of all $\varphi \in C^0(\sigma^2, \mathbb{R})$ with $\varphi(x_j, y_j) = z_j$, $j = 0, 1, \ldots, m$, by

$$T : \hat{C}(\sigma^2) \to \mathbb{R}^{\sigma^2}, \qquad (T\varphi)(x, y) := v_i(u_i^{-1}(x, y), \varphi(u_i^{-1}(x, y))), \tag{3}$$

for $(x, y) \in u_i(\sigma_i^2)$, $i = 1, \ldots, N$. This operator T maps $\hat{C}(\sigma^2)$ into itself, is well-defined and contractive in the sup-norm with contractivity $s = \max_i\{|s_i|\}$. Its unique fixed point is a fractal function $f : \sigma^2 \to \mathbb{R}$ satisfying $f(x_j, y_j) = z_j$, $j = 0, 1, \ldots, m$. The graph of f is called a *fractal surface*. At this point we refer the interested reader to [5] and [8] for an elaborate description of these fractal surfaces. Let us note that the graph of f, as constructed above, is the attractor of the iterated function system $(\sigma^2 \times \mathbb{R}, \mathrm{w})$, where $w_i = (u_i, v_i)$.

2.1 C^1- and C^2-interpolating fractal functions. We now proceed with the construction of smooth fractal curves. We assume without loss of generality that the attractor of the iterated function system is contained in $X = I \times [-1, 1]$. Let $\mathcal{J} := \{(x_j, y_j) : 0 = x_0 < x_1 < \cdots < x_N = 1,\ y_j \in \mathbb{R},\ j = 0, 1, \ldots, N; 1 < N \in \mathbb{N}\}$ be a given set of interpolation points. We define affine maps $u_i : I \to I$ by

$$u_i(x) := b_i x + x_{i-1}, \quad x \in u_i(I), \tag{4}$$

where $b_i = x_i - x_{i-1}$, $i = 1, \ldots, N$. Now let $K_i(\xi, \eta)$ be a symmetric bilinear form on $\mathbb{R}^2$. For $\xi = (x, y) \in \mathbb{R}^2$ let $v_i(\xi) := K_i(\xi, \xi) + d_i$, $d_i \in \mathbb{R}$ and $i = 1, \ldots, N$. Then $v_i : X \to \mathbb{R}$, and it can be written as

$$v_i(x, y) = a_i x^2 + 2 s_i x y + t_i y^2 + d_i. \tag{5}$$

The coefficients a_i and d_i are uniquely determined by requiring that

$$v_i(0, y_0) = y_{i-1}, \qquad v_i(1, y_N) = y_i. \tag{6}$$

We thus obtain

$$a_i = y_i - y_{i-1} - 2 s_i y_N - t_i(y_0^2 + y_N^2) \qquad \text{and} \qquad d_i = y_{i-1} - t_i y_0^2. \tag{7}$$

In order for $w_i(x, y) = (u_i(x), v_i(x, y))$, $i = 1, \ldots, N$, to be contractive on X we have to require that $v_i(\,\cdot\,, y)$ is Lipschitz for all $y \in \mathbb{R}$, and $v_i(x, \,\cdot\,)$ contractive for all $x \in I$. Then

$$\begin{aligned} |v_i(x, y) - v_i(x', y)| &= |a_i(x^2 - x'^2) + 2 s_i(x - x')| \\ &\le |2a_i + 2s_i|\, |x - x'| < \ell\, |x - x'|, \end{aligned}$$

for all $x, x' \in I$, all $y \in [-1, 1]$, and $\ell > \max_i\{|2a_i + 2s_i|\}$. Also,

$$\begin{aligned} |v_i(x, y) - v_i(x, y')| &= |2 s_i x(y - y') + t_i(y^2 - y'^2)| \\ &= |2s_i + t_i(y + y')|\, |y - y'| \le r\, |y - y'|, \end{aligned}$$

for all $x \in I$, $y, y' \in [-1, 1]$, whenever

$$\max\{|s_i| + |t_i| : i = 1, \ldots N\} \le r < 1/2. \tag{8}$$

Hence the s_i and t_i, as long as they satisfy inequality (8), are free parameters. Now let $0 < \theta < (1 - \max_i\{|b_i|\})/\ell$. It is straight-forward to show that each w_i is contractive in the complete normed linear space $(X, \|\cdot\|_\theta)$, where $\|(x, y)\|_\theta := |x| + \theta\, |y|$, $(x, y) \in X$. Hence the iterated function system (X, w) has a unique attractor G.

We will show that under certain conditions G is the graph of a C^1-function interpolating $\mathcal{J}$. To this end, let $\hat{C}^1(I)$ denote the complete metric space (in the C^1-topology) consisting of all $\varphi \in C^1(I,\mathbb{R})$ such that $\varphi(x_j) = y_j$, $j = 0, 1, \ldots, N$, and $\varphi'(0) = \alpha$ and $\varphi'(1) = \beta$, for some given $\alpha, \beta \in \mathbb{R}$. Define an operator $T : \hat{C}^1(I) \to \mathbb{R}^I$ by

$$(T\varphi)(x) := v_i(u_i^{-1}(x), \varphi(u_i^{-1}(x))), \quad x \in u_i(I), \tag{9}$$

$i = 1, \ldots, N$. Then

$$(T\varphi)' = \frac{2}{b_i}\left(a_i u_i^{-1}(\cdot) + s_i\varphi(u_i^{-1}(\cdot)) + s_i u_i^{-1}(\cdot)\varphi'(u_i^{-1}(\cdot)) + t_i\varphi(u_i^{-1}(\cdot))\varphi'(u_i^{-1}(\cdot))\right).$$

If we require that

$$(T\varphi)'(0) = \varphi'(0), \quad (T\varphi)'(1) = \varphi'(1), \quad \lim_{x \to x_i^-}(T\varphi)'(x) = \lim_{x \to x_i^+}(T\varphi)'(x), \tag{10}$$

for $i = 1, \ldots, N-1$, then this together with equations (4) and (6) implies that $T\varphi \in C^1(I,\mathbb{R})$. Furthermore, since u_i and $v_i(x, \cdot)$ are contractive, T is contractive on $\hat{C}^1(I)$ in the C^1-topology with contractivity factor $2\max_i\{|s_i| + |t_i|\}$. Thus, by the Contractive Mapping Theorem, T has a unique fixed point $f : I \to [-1, 1]$ in $\hat{C}^1(I)$. Moreover, $f(x_j) = y_j$, for all $j = 0, 1, \ldots, N$. We refer to f as a *C^1-interpolating fractal function.*

In the special case where $y_0 = y_N = 0$, the following conditions on the s_i and on α have to hold for f to be of class C^1:

$$s_i = -\frac{a_i}{\varphi'(1)} \quad (i = 1, \ldots, N-1), \qquad s_N = \frac{b_N}{2} - \frac{a_N}{\varphi'(1)}, \qquad \varphi'(0) = \alpha = 0.$$

The coefficients t_i, as long as they satisfy (8), are free parameters. Figures 1 — 3 show examples of such C^1-interpolating fractal functions. The set of interpolation points for Figures 1 and 2 is $\mathcal{J} = \{(0,0), (1/2,1/2), (1,0)\}$.

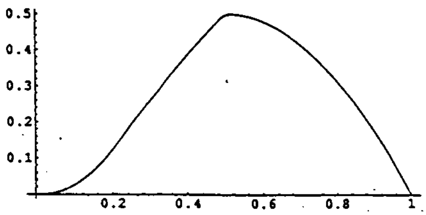

Figure 1: $s_1 = \frac{1}{4}$, $s_2 = 0$, $t_1 = -t_2 = \frac{5}{16}$, $\varphi'(1) = -2$.

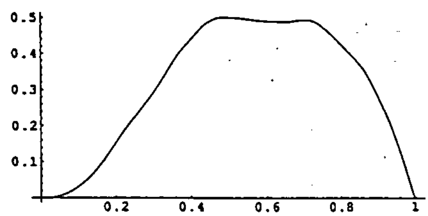

Figure 2: $s_1 = \frac{3}{10}$, $s_2 = \frac{1}{5}$, $t_1 = \frac{1}{10}$, $t_2 = \frac{1}{5}$, $\varphi'(1) = -10$.

It is also possible to obtain C^2-interpolating fractal functions: one has to require the additional conditions

$$\lim_{x \to x_i^-}(T\varphi)''(x) = \lim_{x \to x_i^+}(T\varphi)''(x) \quad (i = 1, \ldots, N-1), \qquad (T\varphi)''(1) = \varphi''(1). \tag{11}$$

If we denote by $\hat{C}^2(I)$ the complete metric space consisting of all $\varphi \in \hat{C}^1(I)$ satisfying $\varphi''(1) = \gamma$, for some fixed $\gamma \in \mathbb{R}$, and define T as in equation (9), we see that T is a contraction on $\hat{C}^2(I)$ in the C^2-topology and that its unique fixed point is a C^2-function $f : I \to [-1, 1]$ interpolating $\mathcal{J}$.

We refer to f as a C^2-*interpolating fractal function*. Using again the special case $y_0 = y_N = 0$ as an example, we obtain

$$(T\varphi)''(0) = \frac{2a_i}{b_i^2}, \qquad t_i = \frac{a_{i+1}\left(\frac{b_i}{b_{i+1}}\right)^2 + a_i\left(1 + \frac{\varphi''(1)}{\varphi'(1)}\right)}{\varphi'(1)[\varphi'(1) + \varphi''(1)]} \qquad (i = 1, \ldots, N-1),$$

$$t_N = \frac{b_N^2\varphi''(1) - 2a_N - 2s_N(2\varphi'(1) - \varphi''(1))}{2\varphi'(1)[\varphi'(1) + \varphi''(1)]}.$$

Figure 4 shows the graph of a C^2-interpolating fractal function.

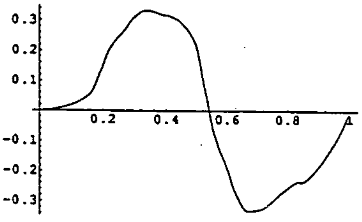

Figure 3: $\mathcal{J} = \{(0,0), (\frac{1}{3}, \frac{1}{3}), (\frac{2}{3}, -\frac{1}{3}), (1,0)\}$,
$s_1 = -\frac{2}{9}$, $s_2 = \frac{4}{9}$, $s_3 = \frac{1}{9}$;
$t_1 = -t_2 = t_3 = \frac{1}{10}$, $\varphi'(1) = 18$

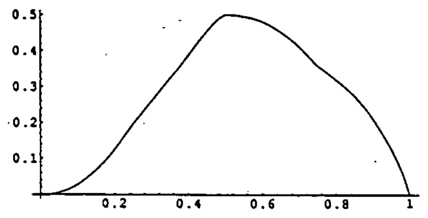

Figure 4: $\mathcal{J} = \{(0,0), (\frac{1}{2}, \frac{1}{2}), (1,0)\}$,
$s_1 = s_2 = \frac{1}{4}$, $t_1 = -t_2 = \frac{1}{32}$,
$\varphi'(1) = -4 = -(T\varphi)''(0) = \varphi''(1)$.

Remark. Since f is the unique fixed point of T, we have

$$f = a_i(u_i^{-1}(\cdot))^2 + s_i u_i^{-1}(\cdot) f(u_i^{-1}(\cdot)) + t_i[f(u_i^{-1}(\cdot))]^2 + d_i,$$

and if we set $s_i = s$ and $t_i = 0$; for all $i = 1, \ldots, N$, then it can be shown that the set $\mathcal{V}_0 := \{f : \mathbb{R} \to \mathbb{R}$: for all $j \in \mathbb{Z}$ there exists a fractal interpolation function g on $[j, j+1]$ with $s = s_i$ and $b_i = 1/N$ such that $f\big|_{(j,j+1)} = g\big|_{(j,j+1)}\}$ is a linear space and that the mapping $(y_0, y_1, \ldots, y_N) \mapsto f$ is linear. If one defines linear spaces $\mathcal{V}_k$, $k \in \mathbb{Z}$, by $f \in \mathcal{V}_k$ if and only if $f(N^{-k}\cdot) \in \mathcal{V}_0$, then a nested sequence of linear spaces is obtained. This sequence of spaces can then be used to define a multiresolution analysis on $L^2(\mathbb{R})$ (see [7]). The author will discuss this approach in a forth-coming paper.

The set of fractal functions generated by (4) and (5) and which satisfy (6) is denoted by $\mathcal{K}^0$. We denote by $\mathcal{K}^1$ those fractal functions in $\mathcal{K}^0$ that also satisfy (10), and by $\mathcal{K}^2$ those in $\mathcal{K}^1$ that obey (11). Clearly, $\mathcal{K}^0 \supset \mathcal{K}^1 \supset \mathcal{K}^2$. Next we present a formula for the box dimension of graph(f) when $f \in \mathcal{K}^0 \setminus \mathcal{K}^1$.

2.2 Estimates for the box dimension of graph f, $f \in \mathcal{K}^0 \setminus \mathcal{K}^1$. Let us briefly recall the definition of box dimension of a bounded set $E \subset \mathbb{R}^m$. The *upper* and *lower box dimensions* of E are defined by

$$\overline{\dim}_B E := \limsup_{\epsilon \to 0+} \frac{\log \mathcal{N}_\epsilon E}{-\log \epsilon} \qquad \text{and} \qquad \underline{\dim}_B E := \liminf_{\epsilon \to 0+} \frac{\log \mathcal{N}_\epsilon E}{-\log \epsilon}, \tag{12}$$

respectively, where $\mathcal{N}_\epsilon E$ denotes the minimum number of ϵ-cubes necessary to cover E. If $\underline{\dim}_B E = \overline{\dim}_B E$, then their common value is called the *box dimension of* E and denoted by $\dim_B E$. We present an upper bound for $\overline{\dim}_B E$ and a lower bound for $\underline{\dim}_B E$ in the case where E = graph f, and $f \in \mathcal{K}^0 \setminus \mathcal{K}^1$ is generated by N maps with $b_i = 1/N$, $i = 1, \ldots, N$.

In [6] it is shown that in the above-mentioned case it is sufficient to consider covers $\mathcal{C}$ of graph f which are of the form $\mathcal{C} = \{[(k-1)/N^n, k/N^n] \times [a, a + 1/N^n] : k, n \in \mathbb{N}, a \in \mathbb{R}\}$. It should also be clear that one can replace the continuous variable ϵ in (12) by any sequence $\{\epsilon_n\}$ with $\epsilon_n \downarrow 0$ and $\log \epsilon_{n+1}/\log \epsilon_n \to 1$.

The following lemma is needed in the proof of the next theorem.

Lemma 1. *Assume that the free parameters s_i and t_i satisfy inequality (8). Let $B_1 := 2\sum_{i=1}^N |s_i|$. If $B_1 > 1$ and $\mathcal{J}$ is not collinear, then $\lim_{n\to\infty} N^{-n}\mathcal{N}(n) = \infty$.*

Proof. The proof is essentially the same as the one for the corresponding lemma in [6] with only minor changes ($|a_i| \to 2|s_i|$, for instance) and will not be repeated here ■

Theorem 1. *Let $f \in \mathcal{K}^0 \setminus \mathcal{K}^1$ be the unique fixed point of the operator T defined by (9) with $b_i = 1/N$, for all $i = 1,\ldots,N$, and let G be its graph. Suppose that the free parameters s_i and t_i satisfy inequality (8). Let $B_1 := 2\sum_{i=1}^N |s_i|$ and $B_2 := 2\sum_{i=1}^N(|s_i| + |t_i|)$. If $\mathcal{J}$ is not collinear and if $B_1 > 1$, then*

$$1 + \log_N B_1 \le \underline{\dim}_B G \le \overline{\dim}_B G \le 1 + \log_N B_2.$$

If either $\mathcal{J}$ is collinear or $B_2 \le 1$, then $\dim_B G = 1$.

Proof. Let $C_n \in C$ be a minimal cover of G consisting of $\mathcal{N}(n)$ $1/N^n \times 1/N^n$- squares with disjoint interiors. Consider the intervals $I_{n,k} := [(k-1)/N^n, k/N^n]$, $k = 1,\ldots,N^n$. Then $G\big|_{I_{n,k}}$ is contained in a rectangle R of width $1/N^n$ and height $h = \max\{f\big|_{I_{n,k}}\} - \min\{f\big|_{I_{n,k}}\}$. Let $\mathcal{N}(n,k)$ denote the least number of squares from C_n needed to cover R. Note that $\mathcal{N}(n) = \sum_{k=1}^{N^n}\mathcal{N}(n,k)$. Let $i \in \{1,\ldots,N\}$. The image of R under the map w_i is then contained in a rectangle of width $1/N^{n+1}$ and height $h_i = \max\{f\big|_{I_{n,k,i}}\} - \min\{f\big|_{I_{n,k,i}}\}$, where $I_{n,k,i} := u_i(I_{n,k})$. Setting $y_k = \max\{f\big|_{I_{n,k}}\}$ and $y_{k-1} = \min\{f\big|_{I_{n,k}}\}$, one easily shows that

$$h_i = \left|\frac{a_i k^2}{N^{2n}} + \frac{2s_i k y_k}{N^n} + t_i y_k^2 - \frac{a_i(k-1)^2}{N^{2n}} - \frac{2s_i(k-1)y_{k-1}}{N^n} - t_i y_{k-1}^2\right|.$$

Hence, if $\mathcal{N}(n+1,k,i)$ denotes the minimum number of $1/N^{n+1} \times 1/N^{n+1}$-squares from C_{n+1} needed to cover $w_i(R)$, we have

$$\begin{aligned}\mathcal{N}(n,k,i) &\le N^{n+1}\left[2|t_i|h + \tfrac{2|s_i|}{N^n}kh + \tfrac{|a_i|}{N^{2n}}(2k-1) + \tfrac{2|s_i|}{N^n}|y_{k-1}|\right]\\ &\le [N(2|t_i| + 2|s_i|)]\mathcal{N}(n) + \tfrac{|a_i|}{N^{n-1}}(2k-1) + 2N|s_i|.\end{aligned}$$

Since the cardinality of a minimal cover $C_{n+1} \in C$ is given by $\mathcal{N}(n+1) = \sum_{i=1}^N\sum_{k=1}^{N^n}\mathcal{N}(n,k,i)$, we have

$$\mathcal{N}(n+1) \le (NB_2)\mathcal{N}(n) + \sum_{i=1}^N(|a_i| + 2|s_i|)N^{n+1}.$$

Thus, by induction on n,

$$\mathcal{N}(n) \le (NB_2)^n\mathcal{N}(1) + c_1 N^{n+1}(1 + B_2 + \cdots + B_2^{n+1}),$$

where $c_1 := \sum_{i=1}^N(|a_i| + 2|s_i|)$. Therefore, if $B_2 \le 1$, $\mathcal{N}(n) \le c_2 n N^n$, where $c_2 := \mathcal{N}(1) + c_1$, and thus $\overline{\dim}_B G \le 1$, i.e., $\dim G = 1$. If $B_2 > 1$, we have $\mathcal{N}(n) \le c_3(B_2 N)^n$, where $c_3 := \mathcal{N}(1) + c_1/(1 - B_2)$. Thus $\overline{\dim}_B G \le 1 + \log_N B_2$.

Now let us obtain the given lower bound for $\underline{\dim}_B G$. We note that — after possibly interchanging the min and max in the definition of y_{k-1} and y_k — $G\big|_{I_{n,k}}$ must contain a rectangle of height at least

$$|v_i((k-1)/N^n, y_k) - v_i(k/N^n, y_{k-1})|.$$

This height, however, is at least equal to

$$\begin{aligned}
&|t_i(y_k^2 - y_{k-1}^2) + 2s_i(y_k - y_{k-1})| - \left|\tfrac{a_i(k-1)^2}{N^{2n}} - \tfrac{a_i k^2}{N^{2n}}\right| - \left|\tfrac{2s_i}{N^n}\right| \\
&= |y_k - y_{k-1}|\,|t_i(y_k + y_{k-1}) + 2s_i| - \left|\tfrac{(2k-1)a_i}{N^{2n}}\right| - \left|\tfrac{2s_i}{N^n}\right| \\
&\geq 2|s_i|\,|y_k - y_{k-1}| - \left|\tfrac{(2k-1)a_i}{N^{2n}}\right| - \left|\tfrac{2s_i}{N^n}\right|.
\end{aligned}$$

Hence, after summing over k and i, and by induction on n, we obtain

$$\mathcal{N}(n+1) \geq \left(\sum_{i=1}^{N} 2|s_i|\right)^n \mathcal{N}(1) - c_4 N^{n+1},$$

for some $c_4 > 0$. By Lemma 1 we can choose n large enough to ensure that the right-hand side of the above inequality is positive. Therefore, $\mathcal{N}(n) \geq c_5(NB_1)^n$, for some $c_5 > 0$. Thus, $\underline{\dim}_B G \geq 1 + \log_N B_1$ ■

3 Smooth Fractal Surfaces

In this section we present two methods for constructing smooth fractal surfaces. The first one is an extension of the method given in the previous section, the second one defines smooth fractal surfaces as indefinite integrals of C^0- fractal surfaces.

3.1 Construction via iterated function systems. Let $Q = [0,1] \times [0,1]$, let $\mathbf{e}_1 = (1,0)$, $\mathbf{e}_2 = (0,1)$, and let N be a fixed integer greater than one. Let $\Gamma = \{(m/N)\mathbf{e}_1 + (n/N)\mathbf{e}_2 : m,n \in \mathbb{Z}\}$ be a lattice in $\mathbb{R}^2$. Suppose that for each lattice point $(x_j, y_j) \in \Gamma \cap Q$ we are given a real number z_{ij}, $i,j \in \{0,1,\ldots,N\}$. The set $\mathcal{J} := \{(x_j, y_j, z_{ij}) : i,j = 0,1,\ldots,N\}$ can be thought of as a given set of data or interpolation points on Q. We will define a smooth fractal surface interpolating $\mathcal{J}$.

Let $u_{ij} : Q \to Q$ be given by

$$u_{ij}(x,y) = \begin{pmatrix} \frac{1}{N} & 0 \\ 0 & \frac{1}{N} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{i-1}{N} \\ \frac{j-1}{N} \end{pmatrix},$$

and let $v_{ij} : Q \times \mathbb{R} \to \mathbb{R}$ be defined as

$$v_{ij}(x,y,z) = A_{ij}x^2 + B_{ij}y^2 + C_{ij}z^2 + D_{ij}xy + E_{ij}yz + F_{ij}zx + G_{ij},$$

such that

$$\begin{aligned}
v_{ij}(0,0,z_{0,0}) &= z_{i-1,j-1}, & v_{ij}(0,1,z_{0,N}) &= z_{i-1,j}, \\
v_{ij}(1,0,z_{N,0}) &= z_{i,j-1}, & v_{ij}(1,1,z_{N,N}) &= z_{i,j},
\end{aligned} \tag{13}$$

$i,j = 1,\ldots,N$, and that the following join-up conditions are satisfied:
For $j = 1,\ldots,N$ and $y \in [(j-1)/N, j/N]$

$$\begin{aligned}
v_{ij}(0,y,\varphi(0,y)) &= v_{i-1,j}(1,y,\varphi(1,y)), & i &= 2,\ldots,N, \\
v_{ij}(1,y,\varphi(1,y)) &= v_{i+1,j}(0,y,\varphi(0,y)), & i &= 1,\ldots,N-1,
\end{aligned} \tag{14}$$

and for $i = 1,\ldots,N$ and $x \in [(j-1)/N, j/N]$

$$\begin{aligned}
v_{ij}(x,0,\varphi(x,0)) &= v_{i,j-1}(x,1,\varphi(x,1)), & j &= 2,\ldots,N, \\
v_{ij}(x,1,\varphi(x,1)) &= v_{i,j+1}(x,0,\varphi(x,0)), & j &= 1,\ldots,N-1.
\end{aligned} \tag{15}$$

Here φ denotes any C^0-function interpolating $\mathcal{J}$.

Conditions (13), (14), and (15) uniquely determine some of the coefficients $A_{ij}, \ldots, G_{ij}$. For instance, if $z_{0,0} = z_{0,N} = z_{N,0} = z_{N,N} = 0$, we obtain

$$\begin{aligned} A_{ij} &= z_{i,j-1} - z_{i-1,j-1}, & B_{ij} &= z_{i-1,j} - z_{i-1,j-1} \\ D_{ij} &= (z_{ij} - z_{i,j-1}) - (z_{i-1,j} - z_{i-1,j-1}), & G_{ij} &= z_{i-1,j-1}, \end{aligned} \tag{16}$$

for $i, j = 1, \ldots, N$. If $\varphi \equiv 0$ on ∂Q, then $A_{ij} = B_{ij} = D_{ij} \equiv 0$, and the join-up conditions are automatically satisfied. If $\varphi(0, y) \equiv \varphi(1, y)$ and $\varphi(x, 0) \equiv \varphi(x, 1)$, then in addition to equation (16) we also have to have

$$\begin{aligned} A_{ij} &= A_{i,j-1}, & B_{ij} &= B_{i-1,j} \\ C_{ij} &= C_{i-1,j} = C_{i,j-1}, & E_{ij} &= F_{ij} = 0 \end{aligned}$$

in order for the join-up conditions to be satisfied.

Now let $\hat{C}^0(Q) := \{\varphi \in C^0(Q, \mathbb{R}) : \varphi(x_j, y_j) = z_{ij},\ i, j = 0, 1, \ldots, N\}$. We define a mapping $T : \hat{C}^0(Q) \to \mathbb{R}^Q$ by

$$(T\varphi)(x, y) := v_{ij}(u_{ij}^{-1}(x, y), \varphi(u_{ij}^{-1}(x, y))), \quad (x, y) \in u_{ij}(Q). \tag{17}$$

Suppose, without loss of generality, that $\|\varphi\| \le 1$ on Q and that $s := 2\max_{i,j}|C_{ij}| + \max_{i,j}|E_{i,j}| + \max_{i,j}|F_{ij}| < 1$.

Theorem 2. *T maps $\hat{C}^0(Q)$ into itself, is well-defined and contractive in the* sup-norm *with contractivity s.*

Proof. The results follow inmmediately from the definition of T, conditions (13), (14), and (15) and the assumption on s ■

The unique fixed point of T is the graph of a C^0-function $f : Q \to \mathbb{R}$ that interpolates $\mathcal{J}$. The graph of f is called a *fractal surface*. The following figures display the fifth level approximation of two of these surfaces with $z_{00} = z_{02} = z_{20} = z_{22} = 0$, $z_{01} = z_{10} = z_{12} = z_{21} = 1/2$, $z_{11} = 1$.

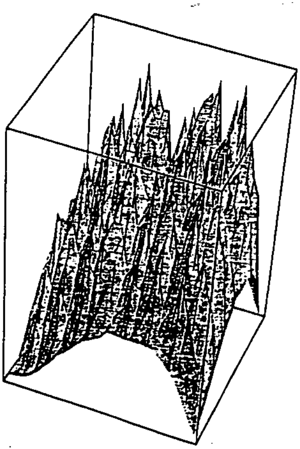

Figure 5: $C_{ij} = 9/20$.

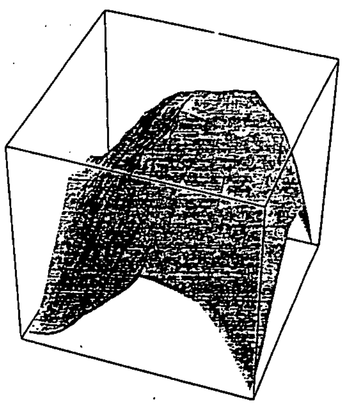

Figure 6: $C_{ij} = 1/4$.

If we impose the following C^1 join-up conditions, we can guarantee that f is a C^1-function: Suppose $\varphi \in \hat{C}^1(Q) := \{\psi \in C^1(Q, \mathbb{R}) : \psi(x_j, y_j) = z_{ij},\ i, j = 0, 1, \ldots, N\}$. Let

$$\nabla(v_{ij}(u_{ij}^{-1}(\cdot,\cdot), \varphi(u_{ij}^{-1}(\cdot,\cdot)))) = \nabla(v_{i,j-1}(u_{i,j-1}^{-1}(\cdot,\cdot), \varphi(u_{i,j-1}^{-1}(\cdot,\cdot)))), \tag{18}$$

for all $(x, y) \in [i/N, (i+1)/N] \times \{j/N\}$, and similarly for the three other edges. In the case where $z_{0,0} = z_{0,N} = z_{N,0} = z_{N,N} = 0$, this implies that $E_{ij} = F_{ij} = 0$ and that $\nabla\varphi|_{\partial Q} \equiv 0$. If we consider the class $\bar{C}^1(Q) := \{\varphi \in \hat{C}^1(Q) : \nabla\varphi|_{\partial Q} \equiv 0\}$, then we have the next result.

Theorem 3. *Let the mapping $T : \hat{C}^0(Q) \to \mathbb{R}^Q$ be defined as in (17). Suppose that condition (18) is satisfied. Then T maps $\bar{C}^1(Q)$ into itself, is well-defined and contractive in the C^1-topology with contractivity s.*

Proof. This follows directly from Theorem 2 and the above considerations ■

3.2 Smooth fractal surfaces via integration. In this subsection we consider fractal surfaces defined on Q that are generated by choosing u_{ij} as in Subsection 3.1, but require v_{ij} to be of the following form: $v_{ij}(\cdot,\cdot,z) - v_{ij}(0,0,z_0)$ is a symmetric quadratic form for all $z, z_0 \in \mathbb{R}$, and $v_{ij}(x,y,\cdot) - v_{ij}(x,y,z_0)$ is a linear form for all $(x,y) \in Q$ and $z_0 \in \mathbb{R}$. Furthermore, we require that conditions (13), (14), and (15) also hold for this choice of v_{ij}. In the special case $z_{0,0} = z_{N,0} = z_{0,N} = z_{N,N} = 0$, we obtain the same expressions for A_{ij}, B_{ij}, D_i, and G_{ij} as above, if

$$v_{ij}(x,y,z) = A_{ij}x^2 + B_{ij}y^2 + C_{ij}z + D_{ij}xy + G_{ij}.$$

Note that (14) and (15) follow whether $\varphi \equiv 0$ on ∂Q or $\varphi\Big|_{[0,1]\times\{0\}} \equiv \varphi\Big|_{[0,1]\times\{1\}}$ and $\varphi\Big|_{\{0\}\times[0,1]} \equiv \varphi\Big|_{\{1\}\times[0,1]}$, $\varphi \in \hat{C}^0(Q)$.

Defining an operator T as in (17) and assuming $s = \max_{i,j}|C_{ij}| < 1$, we obtain the following theorem whose proof is straight-forward.

Theorem 4. *The unique fixed point of T is a C^0-function $f : Q \to \mathbb{R}$ such that $f(x_j, y_j) = z_{ij}$, for all $i, j = 0, 1, \ldots, N$.*

The reason for choosing this particular form of the v_{ij} will become clear shortly. Let

$$\tilde{f}(x,y) := \tilde{z}_{0,0} + \int_0^x \int_0^y f(s,t)\,dt\,ds,$$

for some $\tilde{z}_{0,0} \in \mathbb{R}$. Denote the integral operator $\int_0^x \int_0^y (\cdot)dtds$ by $I_{(0,0)}^{(x,y)}(\cdot)$, and let $u_{ij}(x,y) = (\kappa_i(x), \lambda_j(y))$, where $\kappa_i(x) := (1/N)x + (i-1)/N$ and $\lambda_j(y) := (1/N)y + (j-1)/N$, $i, j = 0, 1, \ldots, N$. Then

$$\begin{aligned}
\tilde{f}(u_{ij}(x,y)) &= \tilde{z}_{0,0} + I_{(0,0)}^{(x_{i-1},y_{j-1})}(f) + I_{(x_{i-1},0)}^{(\kappa_i(x),y_{j-1})}(f) \\
&\quad + I_{(0,y_{i-1})}^{(x_{i-1},\lambda_j(y))}(f) I_{(x_{i-1},y_{j-1})}^{(\kappa_i(x),\lambda_j(y))}(f) \\
&= \tilde{z}_{0,0} + I_{(0,0)}^{(x_{i-1},y_{j-1})}(f) + I_{(x_{i-1},0)}^{(\kappa_i(x),y_{j-1})}(f) + I_{(0,y_{j-1})}^{(x_{i-1},\lambda_j(y))}(f) \\
&\quad + \tfrac{1}{N^2} I_{(0,0)}^{(x_{i-1},y_{j-1})}(f \circ u_{ij}).
\end{aligned}$$

Since $f \circ u_{ij} = v_{ij}(\cdot, f)$, we have

$$\begin{aligned}
\tilde{f}(u_{ij}(x,y)) &= \left(\tilde{z}_{0,0} + I_{(0,0)}^{(x_{i-1},y_{j-1})}(f) + I_{(x_{i-1},0)}^{(\kappa_i(x),y_{j-1})}(f) + I_{(0,y_{j-1})}^{(x_{i-1},\lambda_j(y))}(f)\right) \\
&\quad + \tfrac{C_{ij}}{N^2} I_{(0,0)}^{(x,y)}(f) + \tfrac{1}{N^2} I_{(0,0)}^{(x,y)}(v_{ij}|_{z=0}) \\
&=: \tfrac{C_{ij}}{N^2}\tilde{f}(x,y) + R_{ij}(x,y).
\end{aligned}$$

Hence $\tilde{f}$ is the unique fixed point of the operator $\Psi : C^1(Q,\mathbb{R}) \to C^1(Q,\mathbb{R})$,

$$\Psi\varphi := \tilde{v}_{ij}(u_{ij}^{-1}(\cdot,\cdot), \varphi(u_{ij}^{-1}(\cdot,\cdot))), \tag{19}$$

where $\tilde{v}_{ij}(x,y,z) = R_{ij}(x,y,z) + \frac{C_{ij}}{N^2}z$, or equivalently, graph f is the unique attractor of the iterated function system $(Q \times \mathbb{R}, \tilde{\mathbf{w}})$ with $\tilde{\mathbf{w}} = \{\tilde{w}_{ij} : Q \times \mathbb{R} \to Q \times \mathbb{R} : \tilde{w}_{ij} = (u_{ij}, \tilde{v}_{ij}),\ i, j =$

$1, \ldots, N\}$. Since the operator $I_{(0,0)}^{(x,y)}(\cdot)$ is continuous, $\tilde{f}$ is continuous at its interpolation points $\{(x_j, y_j, \tilde{z}_{ij}) : i, j = 0, 1, \ldots, N\}$. To determine the $\tilde{z}_{ij}$, notice that $u_{ij}(0,0) = (x_{i-1}, y_{j-1}) = z_{i-1,j-1}$, and thus $\tilde{f}(u_{ij}(0,0)) = \tilde{z}_{0,0} + I_{(0,0)}^{(x_{i-1},y_{j-1})}(f) =: \tilde{z}_{i-1,j-1}$. Therefore,

$$\begin{aligned} \tilde{z}_{ij} &= \tilde{z}_{i-1,j-1} + \tfrac{C_{ij}}{N^2} I_{(0,0)}^{(1,1)}(f) + \tfrac{1}{N^2} I_{(0,0)}^{(1,1)}(v_{ij}\,|_{z=0}) + I_{(x_{i-1},0)}^{(x_i,y_{j-1})}(f) + I_{(0,y_{j-1})}^{(x_{i-1},y_j)}(f) \\ &= \tfrac{C_{ij}}{N^2}(\tilde{z}_{N,N} - \tilde{z}_{0,0}) + \tfrac{1}{N^2} I_{(0,0)}^{(1,1)}(v_{ij}\,|_{z=0}) + (\tilde{z}_{i-1,j} - \tilde{z}_{i-1,j-1}) + \tilde{z}_{i,j-1}. \end{aligned}$$

Hence the $\tilde{z}_{ij}$ can be expressed in terms of $\tilde{z}_{0,0}$, C_{ij}, and $I_{(0,0)}^{(1,1)}(v_{ij}\,|_{z=0})$, $(i,j) \neq (0,0)$. Let us summarize these results in a theorem.

Theorem 5. *Let graph f be a fractal surface generated by the iterated function system $(Q \times \mathbb{R}, \mathbf{w})$, where $w_{ij} = (u_{ij}, v_{ij})$ with*

$$u_{ij}(x,y) = \begin{pmatrix} \frac{1}{N} & 0 \\ 0 & \frac{1}{N} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{i-1}{N} \\ \frac{j-1}{N} \end{pmatrix},$$

and

$$v_{ij}(x,y,z) = A_{ij}x^2 + B_{ij}y^2 + C_{ij}z + D_{ij}xy + G_{ij},$$

such that $\max_{i,j} |C_{ij}| < 1$. *Let*

$$\tilde{f}(x,y) := \tilde{z}_{0,0} + \int_0^x \int_0^y f(s,t)\,dt\,ds, \qquad \text{for some } \tilde{z}_{0,0} \in \mathbb{R}.$$

Then graph $\tilde{f}$ is the attractor of the iterated function system $(Q \times \mathbb{R}, \tilde{\mathbf{w}})$ with $\tilde{w}_{ij} = (u_{ij}, \tilde{v}_{ij})$, where

$$\tilde{v}_{ij}(x,y,z) = \tilde{z}_{i-1,j-1} + \frac{C_{ij}}{N^2}z + \frac{1}{N^2}\int_0^x \int_0^y v_{ij}(s,t,0)\,dt\,ds, \quad i,j = 1, \ldots N.$$

Furthermore, the $\tilde{z}_{ij}$, $(i,j) \neq (0,0)$, are recursively and uniquely determined by $\tilde{z}_{0,0}$ which is a free parameter, C_{ij}, and $\int_0^x \int_0^y v_{ij}(s,t,0)\,dt\,ds$. Also, $\nabla \tilde{f}(x,y) = (g_y(x), h_x(y))$, where

$$g_y(x) = \int_0^y f(x,t)\,dt \;\text{ and }\; h_x(y) = \int_0^x f(s,y)\,ds.$$

Moreover, $\frac{\partial}{\partial x}\frac{\partial}{\partial y}\tilde{f} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}\tilde{f} = f$.

Proof. The last part of the theorem follows from Calculus ∎

It should now be clear how one can construct C^n-interpolating fractal surfaces, $n \in \mathbb{N}$: the above procedure can be iterated an arbitrary number of times.

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