Zeitschrift für Analysis
und ihre Anwendungen
Volume 12 (1993), 179–182
Heldermann Verlag Berlin

λ-Convergence and λ-Conullity

W. BEEKMANN and S.-C. CHANG

The notions of $\lambda-$convergence and $\lambda-$summability was first defined a couple of decades ago by G. Kangro and has been further studied by Kangro and his school at Tartu University. In analogy to the usual notions of coregularity and conullity of conservative (matrix–) methods, he has introduced, *inter alia*, $\lambda-$coregularity and $\lambda-$conullity for $\lambda-$conservative matrices. Here we compare $\lambda-$conullity with the general notion of conullity of an $FK-$space with respect to a subspace which was introduced in [2]. To this end we show that the space c^λ of all $\lambda-$convergent sequences is of type c_D, a summability domain. As a consequence, we prove that the space of all sequences that are $\lambda-$summable by a given matrix A is a summability domain c_E for some matrix E.

Key words: *$\lambda-$convergence, $\lambda-$summability, conullity of $FK-$spaces*
AMS subject classification: 40 D 09, 40 C 05

1. Notations

Given an infinite matrix $A = (a_{nk})_{n,k=1,2,\ldots}$ the summability domain c_A of A is defined by

$$c_A = \Big\{x = (x_k) \,|\, Ax = \Big(\sum_{k=1}^{\infty} a_{nk}x_k\Big) \text{ exists and } Ax \in c\Big\},$$

where c is the space of convergent sequences. As usual (see [5]) the sequences x in c_A will be called *summable* A, and we write $\lim_A x = \lim Ax = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}x_k$ for $x \in c_A$. It is well-known that c_A is an $FK-$space and that each continuous linear functional f on c_A has a representation

$$f(x) = \mu \lim_A x + t(Ax) + sx \qquad (x \in c_A), \tag{1}$$

where $\mu \in \mathbb{C}$, $t = (t_n) \in \ell = \{(t_n) | \sum |t_n| < \infty\}$, and $s = (s_k) \in c_A^\beta = \{(s_k) | sx = \sum s_k x_k$ converges for all $(x_k) \in c_A\}$ (the $\beta-$dual of c_A). If the coefficient μ in (1) is uniquely determined (by A and f) we denote it by $\mu_A(f)$, otherwise we define $\mu_A(f) = 0$. (In fact, if the coefficient of $\lim_A$ is not unique, then there exists a representation (1) of f with $\mu = 0$.) By $\mu_A^\perp$ we denote the set of all functionals in c_A such that $\mu_A(f) = 0$, and A is said to be $\mu-$unique, if $\mu_A^\perp \neq c_A'$.

Let $\lambda = (\lambda_n)$ be a (real) sequence with $0 < \lambda_1 \leq \cdots \leq \lambda_{n-1} \leq \lambda_n \to \infty$. Then

$$c^\lambda := \{x = (x_k) \in c \,|\, \lim_{n\to\infty} \lambda_n(x_n - \lim x) \text{ exists}\}$$

W. Beekmann: FernUniv.-Gesamthochschule, FB Math., Lützowstr. 125, D - 5800 Hagen
S.-C. Chang: Brock Univ., Dep. Math., St. Catharines, Ontario, Canada L2S 3A1. Compl. during a visit of FernUniv. Hagen, supp. by NSERC of Canada grant # A 9201

ISSN 0232-2064

is the space c^λ of $\lambda-$ *convergent sequences* (sequences which converge with "speed λ"), and, given the matrix A,

$$c_A^\lambda := \{x \in c_A \mid Ax \in c^\lambda\}$$

is the subspace (of c_A) of sequences which are *summable A with speed λ*. Both, c^λ and c_A^λ, are $FK-$spaces, see Kangro [3].

We shall assume that c_A^λ contains φ, the set of finite sequences, which is spanned by the unit sequences $e^k = (0,\ldots,0,1,0,\ldots)$ with "1" in the k-th position. Hence the limit $a_k := \lim_A e^k = \lim_{n\to\infty} a_{nk}$ exists for all $k \in \mathbb{N}$.

It is noted in [4] that each $f \in (c_A^\lambda)'$, the continuous dual space of c_A^λ, can be expressed as

$$f(x) = \mu_1 \lim_A x + \mu_2 \lim_n \gamma_n(x) + \sum_k t_k\gamma_k(x) + sx, \tag{2}$$

where $\mu_1, \mu_2 \in \mathbb{C}$, $t = (t_k) \in \ell$, $\gamma_n(x) = \lambda_n(\sum_k a_{nk}x_k - \lim_A x)$, and $s = (s_k) \in (c_A^\lambda)^\beta$, the $\beta-$dual of c_A^λ.

The matrix A is said to be $\lambda-$ *conservative* if $c^\lambda \subset c_A^\lambda$. This implies that the matrix

$$\mathcal{A} = (\alpha_{nk}) \text{ with } \alpha_{nk} = \frac{\lambda_n(a_{nk} - a_k)}{\lambda_k},$$

is conservative in the usual sense, in particular the column limits $\alpha_k = \lim_{n\to\infty} \alpha_{nk}$ and the limit of the row sums $\alpha = \lim_n \sum_k \alpha_{nk}$ exist. A $\lambda-$conservative matrix A is defined to be $\lambda-$ *conull* if

$$\chi(\mathcal{A}) := \alpha - \sum_{k=1}^{\infty} \alpha_k = 0,$$

and $\lambda-$ *coregular* if $\chi(\mathcal{A}) \neq 0$. Leiger [4] has shown that a $\lambda-$conservative matrix A is $\lambda-$conull if and only if $\lambda^{-1} := (\lambda_1^{-1}, \lambda_2^{-1}, \ldots) \in W(c_A^\lambda)$, i.e., if the sections $\lambda^{-1[n]} := (\lambda_1^{-1},\ldots,\lambda_n^{-1},0,0,\ldots)$ converge to λ^{-1} weakly in the $FK-$space c_A^λ.

On the other hand, given $FK-$spaces X and Y such that $\varphi \subset X \subset Y$, Beekmann and Chang [2] defined Y to be *conull with respect to X* if $f|_X \in \mu_X^\perp$ for all $f \in Y'$, where

$$\mu_X^\perp = \{g \in X' \mid \mu_D(g|_{c_D}) = 0 \text{ for all matrices } D \text{ with } c_D \subset X\}.$$

2. $\lambda-$conullity

The following assertion shows the close connection between the above notions.

Theorem 1: *Let A be a $\lambda-$ conservative matrix. Then A is $\lambda-$ conull if and only if c_A^λ is conull with respect to c^λ.*

For the proof we use the fact that the space c^λ is a summability domain, a result which is interesting in itself.

Theorem 2: *Let $\lambda = (\lambda_n)$, $0 < \lambda_1 \leq \cdots \leq \lambda_{n-1} \leq \lambda_n \to \infty$ be given. Then there exists a normal matrix $D = (d_{nk})$ (i.e., $d_{nk} = 0$ for $k > n$ and $d_{nn} \neq 0$) such that*

$$c^\lambda = c_D \quad \textit{and} \quad \lim_{n\to\infty} \lambda_n(x_n - \lim x) = \lim_D x.$$

Proof: By Zeller [6] there exists a regular normal matrix $\widetilde{D} = (\tilde{d}_{nk})$ such that $c_{\widetilde{D}} = c \oplus \lambda$ and $\lim_{\widetilde{D}} \lambda = 0$. Define $D = (d_{nk})$ by $d_{nk} = \tilde{d}_{nk}\lambda_k$. Then $x \in c_D$ if and only if $(\lambda_k x_k) \in c_{\widetilde{D}}$. The latter means

$$\lambda_k x_k = \alpha_k + \xi\lambda_k, \quad \text{i.e.,} \quad x_k = \frac{\alpha_k}{\lambda_k} + \xi \quad \text{for } k \in \mathbb{N},$$

where $\alpha = (\alpha_k) \in c$ and $\xi \in \mathbb{C}$. Thus $x \in c_D$ if and only if $x \in c^\lambda$, and in this case

$$\lim_D x = \lim_{\widetilde{D}} (\lambda_k x_k) = \lim_{\widetilde{D}} (\alpha_k + \xi\lambda_k) = \lim_{\widetilde{D}} (\alpha_k) = \lim \alpha.$$

Also, $\lim x = \xi$ and $\lim\limits_{n\to\infty} \lambda_n(x_n - \xi) = \lim\limits_{n\to\infty} \alpha_n = \lim_D x$ ∎

Corollary 3: *For $X = c^\lambda$ we have*

$$\mu_X^\perp = \left\{ f \in X' \;\middle|\; f(\lambda^{-1}) = \sum_{k=1}^{\infty} f(e^k)\lambda_k^{-1} \right\}.$$

Proof: Let D and $\widetilde{D}$ be defined as in the proof of Theorem 2, and let $f \in X' = c_D'$, then

$$f(x) = \mu \lim_D x + t(Dx) + sx \qquad (x \in X = c^\lambda = c_D)$$

with $\mu \in \mathbb{C}, t \in \ell, s \in c_D^\beta$, see (1). From the fact that $\sum f(e^k)\lambda_k^{-1}$ converges for each $f \in X'$, see Leiger [4], we get

$$\begin{aligned} f(\lambda^{-1}) - \sum_{k=1}^{\infty} f(e^k)\lambda_k^{-1} &= \mu \left(\lim_D \lambda^{-1} - \sum_{k=1}^{\infty} (\lim_D e^k)\lambda_k^{-1}\right) \\ &= \mu \left(\lim_{\widetilde{D}} e - \sum_{k=1}^{\infty} \lim_{\widetilde{D}} e^k\right) = \mu, \end{aligned}$$

since $\widetilde{D}$ is regular. Thus D is $\mu-$unique, and f is in $\mu_D^\perp$ if and only if $f(\lambda^{-1}) = \sum_k f(e^k)\lambda^{-1}$. From $X = c_D$ we have $\mu_X^\perp = \mu_D^\perp$, see [2], which proves the corollary ∎

Proof of Theorem 1: Now, this proof is very simple: By definition c_A^λ is conull with respect to c^λ if and only if $f|_{c^\lambda} \in \mu_{c^\lambda}^\perp$ for all $f \in (c_A^\lambda)'$, and from the corollary we infer that this is the case if and only if $f(\lambda^{-1}) = \sum_k f(e^k)\lambda_k^{-1}$ for all $f \in (c_A^\lambda)'$. The last condition means $\lambda^{-1} \in W(c_A^\lambda)$, i.e., A is $\lambda-$conull ∎

3. The space c_A^λ as a summability domain

Theorem 2 leads to the following representation of c_A^λ.

Theorem 4: *With the matrix D of Theorem 2 and $E = DA$ we have $c_A^\lambda = c_E$.*

Proof: If $x \in c_E$, then $Ex \in c$ and we see that $D^{-1}(Ex)$ exists and equals $(D^{-1}E)x = Ax$, since D is normal. Thus $Ex = D(Ax) \in c$, hence $Ax \in c_D = c^\lambda$, i.e., $x \in c_A^\lambda$.

Conversely, if $x \in c_A^\lambda$, then Ax exists and $Ax \in c^\lambda = c_D$ or $D(Ax) = (DA)x = Ex \in c$ or $x \in c_E$ ∎

By means of this theorem we can use standard results to characterize distinguished subsets of c_A^λ such as $S(c_A^\lambda)$, $W(c_A^\lambda)$, $F(c_A^\lambda)$, $B(c_A^\lambda)$ and $P(c_A^\lambda)$. For details we refer to [5] and [1].

Also we have a representation of the continuous linear functionals on c_E of the form, see (1),

$$f(x) = \tilde{\mu} \lim_E x + \tilde{t}(Ex) + \tilde{s}x$$

with $\tilde{\mu} \in \mathbb{C}$, $\tilde{t} \in \ell$ and $\tilde{s} \in c_E^\beta$. It can be realized that the connection with (2) is given by

$$\mu_2 = \tilde{\mu},\ \mu_1 = \tilde{t}(\widetilde{D}\lambda),\ t = (\tilde{t}\widetilde{D}) \quad \text{and} \quad s = \tilde{s}.$$

So, for the question of $\mu-$uniqueness, in (2) the coefficient μ_2 is essential.

The authors gratefully acknowledge fruitful discussions with T. Leiger of Tartu and K.–G. Große–Erdmann of Hagen.

References

[1] Beekmann, W. and S.-C. Chang: *Some distinguished subsets of $FK-$ spaces containing φ.* Results Math. **16** (1989), 190 - 198.

[2] Beekmann, W. and S.-C. Chang: *On distinguished subsets and conullity of $FK-$ spaces.* Israel Math. Conf. Proc. **4** (1991), 45 - 50.

[3] Kangro, G.: *The $\lambda-$perfectness of summability methods, and applications of it. I.* (Russian. Estonian and English summaries). Izvestija Akad. Nauk Eston. SSR, Fiz. Mat. **20** (1971), 111 - 120 .

[4] Leiger, T.: *The structure of fields of $\lambda-$ summability of matrices.* (Russian. German summary). Tartu Riikl. Ül. Toimetised **846** (1989), 50 - 60.

[5] Wilansky, A.: *Summability through Functional Analysis.* Amsterdam: North Holland 1984.

[6] Zeller, K.: *Merkwürdigkeiten bei Limitierungsverfahren. Einfolgenverfahren.* Archiv Math. **4** (1953), 1 - 5.

Received 15.08.1991