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Two-Sided Nevanlinna-Pick Interpolation for a Class of Matrix-Valued Functions

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Families of Matrix-valued analytic functions $W(\rho, P)$ depending on two parameters ρ and P are introduced. These include as special cases the Schur and Carathéodory functions, as well as classes of functions studied by the authors in [1] and by D. Alpay and H. Dym in [6]. A two-sided Nevanlinna-Pick interpolation problem is defined and solved in $W(\rho, P)$, using the fundamental matrix inequality method.

Key words: *Nevanlinna-Pick interpolation, reproducing kernel spaces, fundamental matrix inequalities*

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1 Introduction

In the present work we pursue our investigations of interpolation problems [1,8] using the fundamental matrix inequality approach. We introduce families of functions $W(\rho, P)$, depending on two parameters ρ and P, which encompass most classical cases (such as $\mathbb{C}^{n\times n}$-valued Schur and Carathéodory functions) and a number of new cases. We define in the classes $W(\rho, P)$ a two-sided Nevanlinna-Pick type interpolation problem for which the description of the solutions is given in terms of a linear fractional transformation.

To introduce the families $W(\rho, P)$, some notations and definitions are first called for. The symbol I_n denotes the identity matrix in the space $\mathbb{C}^{n\times n}$ of $n \times n$ matrices with complex entries, and, throughout the paper J denotes the matrix $\begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}$. Given two functions a and b analytic in an open connected subset Ω of $\mathbb{C}$, we set

$$\Omega_+ = \{\lambda \in \Omega,\ |a(\lambda)| > |b(\lambda)|\},$$

$$\Omega_- = \{\lambda \in \Omega,\ |a(\lambda)| < |b(\lambda)|\},$$

and

$$\Omega_0 = \{\lambda \in \Omega,\ |a(\lambda)| = |b(\lambda)|\}.$$

A function $\rho_\omega(\lambda)$ jointly analytic in λ and ω^* in Ω belongs, by definition, to the class D_Ω if it can be written as

$$\rho_\omega(\lambda) = a(\lambda)a(\omega)^* - b(\lambda)b(\omega)^*, \tag{1.1}$$

where the two sets Ω_+, Ω_- are nonempty. It follows (see [6]) that there is a point μ such that $|a(\mu)| = |b(\mu)| \neq 0$, and in particular, Ω_0 is nonempty.

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Such functions and associated reproducing kernel spaces were studied in [5] and [6] and seem to be a convenient framework to incorporate within a single theory both the "line" and "circle" cases (which correspond respectively to $\rho_\omega(\lambda) = -2\pi i(\lambda - \omega^*)$ and $\rho_\omega(\lambda) = 1 - \lambda\omega^*$). The representation (1.1) is essentially unique: indeed, if $\rho_\omega(\lambda) = c(\lambda)c(\omega^*) - d(\lambda)d(\omega)^*$ is another representation of ρ_ω with c and d two functions analytic in Ω, there exists a $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$-unitary matrix M such that $(c(\lambda), d(\lambda)) = (a(\lambda), b(\lambda))M$.

Let us recall that a $\mathbf{C}^{n\times n}$-valued function $K(z,\omega)$ defined for z, ω in some set E is said to be a positive kernel in E if $K(z,\omega) = K(\omega, z)^*$ (where A^* denotes the adjoint of the matrix A) and if, furthermore, for every choice of integer N and of $\omega_1, \ldots, \omega_N$ in E, the Hermitian block matrix with ij block $K(\omega_i, \omega_j)$ is nonnegative.

The following lemmas are easy corollaries of the corresponding results for matrices.

Lemma 1.1. *Let $K(z,w)$ be a $\mathbf{C}^{n\times n}$-valued nonnegative kernel on E and let $A(z)$ be a $\mathbf{C}^{r\times n}$-valued function on E. Then the function $A(z)K(z,w)A(w)^*$ is a positive kernel on E.*

Lemma 1.2. *Let D be a strictly positive $r \times r$ matrix and let $A(z)$ and $K(z,w)$ be functions respectively $\mathbf{C}^{r\times n}$- and $\mathbf{C}^{n\times n}$-valued, and defined for z, w in some set E. Then, the function*

$$\hat{K}(z,w) = \begin{pmatrix} D & A(w)^* \\ A(z) & K(z,w) \end{pmatrix}$$

is positive if and only if $K(z,w) - A(z)D^{-1}A(w)^*$ is a positive kernel on E.

Definition 1.3. *Let Ω be a connected subset of $\mathbf{C}$ and let ρ_ω be in D_Ω. Let P be a $\mathbf{C}^{2n\times 2n}$-valued function analytic in Ω_+ and with nonidentically vanishing determinant. The class $W(\rho, P)$ consists of the $\mathbf{C}^{n\times n}$-valued functions S meromorphic in Ω_+ and such that the function*

$$K_S(\lambda,\omega) = (S(\lambda), I_n)\frac{P(\lambda)JP(\omega)^*}{\rho_\omega(\lambda)}(S(\omega), I_n)^* \tag{1.2}$$

is positive in Ω_+.

The classes $W(\rho, P)$ encompass a wide range of cases, some of which are detailed in Section 2. In Section 3 we study the main properties of the elements in $W(\rho, P)$. This section provides the necessary background to Sections 4, 5 and 6, where the following interpolation problem is studied (in the statement, $\mathbf{C}^{r\times n}$ denotes the space of $r \times n$ matrices with complex entries).

Definition 1.4. *Let $W(\rho, P)$ be as in Definition 1.3. The interpolation problem $IP(\rho, P)$ consists of the following: given $N \in \mathbf{N}$, given integers $r_i, s_i \in \mathbf{N}$, given matrices*

$h_i, c_i \in \mathbf{C}^{r_i \times n}$, $f_i, d_i \in \mathbf{C}^{s_i \times n}$ and $\gamma_i \in \mathbf{C}^{r_i \times s_i}$, $i \in \{1, \ldots, N\}$ and given $\omega_1, \ldots, \omega_N$ in Ω_+ such that

$$b(\omega_i)a'(\omega_i) - a(\omega_i)b'(\omega_i) \neq 0 \ (i = 1, \ldots, N),$$

(1) find necessary and sufficient conditions which ensure the existence of a function $S \in W(\rho, P)$ analytic in ω_i and such that

$$h_i S(\omega_i) = c_i, \quad f_i S(\omega_i)^* = d_i, \quad h_i S'(\omega_i) f_i^* = \gamma_i \tag{1.3}$$

for $i = 1, \ldots, N$ and

(2) describe the set of all solutions.

Our approach to solve this interpolation problem relies on Potapov's method of the fundamental matrix inequality suitably adapted to the present framework. This method was developed by V. Potapov and his coworkers to solve matrix-valued versions of the Nevanlinna-Pick interpolation problem for Schur and Nevanlinna functions (see [11,13–15]; the definitions of Schur and Nevanlinna functions are reviewed in the next section). As will be made clearer in the sequel, the problem $IP(\rho, P)$ is a matrix version of the classical Nevanlinna–Pick problems for Schur functions.

In Section 4 we give necessary and sufficient conditions for the problem $IP(\rho, P)$ to be solvable. As is often the case in interpolation theory, a necessary condition for the problem $IP(\rho, P)$ to be solvable is the nonnegativity of a certain block matrix K, the so called *informative matrix* of the problem (defined in (4.2)-(4.4)); its strict positivity (under some additional requirements) is a sufficient condition for the problem to be solvable. Under the assumption that $K > 0$, a description of the solutions to $IP(\rho, P)$ is given in Section 5 using a linear fractional transformation. Such a description is still possible when $K \geq 0$; this is treated in Section 6.

The interpolation problems $IP(\rho, P)$ could presumably be solved using other approaches to interpolation: we have in particular in mind reproducing kernels methods [3, 4, 6, 9, 10], methods based on operator theory [2] or methods based on the theory of rational functions [7]. This suggests a number of problems which will be treated elsewhere.

2 Examples

In this section we list a number of examples of classes $W(\rho, P)$ for various choices of ρ and P. We first focus on the case of constant matrices P.

Example 2.1. *The case $P(\lambda) = I_n$. When $\rho_\omega(\lambda) = 1 - \lambda\omega^*$, the family $W(\rho, P)$ is equal to the class of $\mathbf{C}^{n \times n}$-valued Schur functions, i.e. $\mathbf{C}^{n \times n}$-valued functions analytic and contractive in the unit disk $\mathbf{D}$.*

For $\rho_\omega(\lambda) = -2\pi i(\lambda - \omega^*)$, we have the analogue class for functions analytic in the open half plane $\mathbf{C}_+$. More generally, for ρ in D_Ω, the function S is in $W(\rho, P)$ if and only if the operator of multiplication by S is a contraction from H_ρ^m into itself, when H_ρ^m denote the reproducing kernel Hilbert space of $\mathbf{C}^{m\times 1}$–valued functions analytic in Ω_+ with reproducing kernel $I_n/\rho_\omega(\lambda)$ (see [6] for details). One–sided interpolation problems with derivative in the classes $W(\rho, I_n)$ were solved in [6].

Most classical families of analytic functions for which Nevanlinna–Pick type interpolation problems are considered occur for constant P and $\rho_\omega(\lambda)$ equal to either $1 - \lambda\omega^*$ or $-2\pi i(\lambda - \omega^*)$.

Example 2.2. *Let* $P(\lambda) = \frac{1}{\sqrt{2}}\begin{pmatrix} I_n & I_n \\ iI_n & -iI_n \end{pmatrix}$ *and* $\rho_\omega(\lambda) = -i(\lambda - \omega^*)$. *Then* $K_S(\lambda, \omega) = \frac{S(\lambda) - S(\omega)^*}{\lambda - \omega^*}$. *A function* S *is in* $W(\rho, P)$ *if and only if it has a non-negative imaginary part in* $\mathbf{C}_+$, *i.e. if and only if it is a Nevanlinna function.*

Example 2.3. *Let* $P(\lambda) = \frac{1}{\sqrt{2}}\begin{pmatrix} I_n & I_n \\ -I_n & I_n \end{pmatrix}$ *and* $\rho_\omega(\lambda) = 1 - \lambda\omega^*$. *We then have* $K_S(\lambda, \omega) = \frac{S(\lambda) - S(\omega)^*}{1 - \lambda\omega^*}$. *The class* $W(\rho, P)$ *coincides with the set of* $\mathbf{C}^{n\times n}$*–valued functions analytic and with non-negative real part in* $\mathbf{D}$, *i.e. with the* $\mathbf{C}^{n\times n}$*–valued functions of the Carathéodory class in* $\mathbf{D}$.

The next examples consider the case of nonconstant P. We first recall that the function

$$\rho_\omega(\lambda) = -2\pi i(\lambda - \omega^*)(1 - \lambda\omega^*) \tag{2.1}$$

belongs to D_Ω with $\Omega = \mathbf{C}$ and

$$a(\lambda) = \sqrt{\pi}(\lambda + i(\lambda^2 + 1)), \quad b(\lambda) = \sqrt{\pi}(\lambda - i(\lambda^2 + 1)).$$

Moreover, the corresponding set Ω_+ consists of two connected components: the open upper half–disk $\mathbf{D}_+ = \mathbf{D} \cap \mathbf{C}_+$ and its reflection under the map $\lambda \to \frac{1}{\lambda}$ (see [5]).

Example 2.4. *Let* ρ_ω *be as in (2.1) and let* S *be a* $\mathbf{C}^{n\times n}$*–valued function analytic in the corresponding set* Ω_+. *Then the function*

$$K(\lambda, \omega) = \frac{I_n - S(\lambda)S(\omega)^*}{1 - \lambda\omega^*} + \frac{S(\lambda) - S(\omega)^*}{\lambda - \omega^*} \tag{2.2}$$

is of the form (1.2) with this choice of ρ_ω *and* $P(\lambda) = \frac{1}{\sqrt{2}}\begin{pmatrix} (z+i)I_n & (i-z)I_n \\ (1+iz)I_n & (iz-1)I_n \end{pmatrix}$.

Functions S for which the kernel (2.2) is positive in the open upper half–disk $\mathbf{D}_+$ were studied in our previous paper [1].

Example 2.5. *With the notation of the previous example, the function*

$$K(\lambda, \omega) = \frac{I_n - S(\lambda)S(\omega)^*}{i(\lambda - \omega^*)} + \frac{S(\lambda) - S(\omega)^*}{1 - \lambda\omega^*}$$

is of the form (1.2) with $P(\lambda)=\frac{1}{\sqrt{2}}\begin{pmatrix}\lambda I_n & I_n\\ I_n & \lambda I_n\end{pmatrix}$.

Example 2.6. *Let* $\rho_\omega(\lambda)=-i(\lambda^2-\omega^{*2})$. *Then* ρ_ω *belongs to* D_Ω *with* $\Omega=\mathbf{C}$ *and the corresponding set* Ω_+ *is the quarter upper right plane* $\mathbf{C}_{++}=\{\lambda|Re\lambda>0,\ Im\lambda>0\}$ *and the function*

$$K(\lambda,\omega)=\frac{S(\lambda)-S(\omega)^*}{\lambda-\omega^*}+\frac{S(\lambda)+S(\omega)^*}{\lambda+\omega^*}$$

is of the form (1.2) with this choice of ρ_ω *and* $P(\lambda)=\frac{1}{\sqrt{2}}\begin{pmatrix}\lambda I_n & \lambda I_n\\ -iI_n & iI_n\end{pmatrix}$.

We leave to the reader to check that J-contractive functions (in either an open half -plane or the unit disk) are in some family $W(\rho,P)$ for adequate choices of P and ρ.

3 The classes $W(\rho,P)$

Let S be a $\mathbf{C}^{n\times n}$-valued function analytic in the open unit disk $\mathbf{D}$. Then, as is well known (see, e.g., [13]), for every choice of points $\lambda_1,\ldots,\lambda_N$ and $\omega_1,\ldots,\omega_P$ in $\mathbf{D}$ such that $\lambda_i\neq\omega_j$ the block matrix $\begin{pmatrix}A & B\\ B^* & D\end{pmatrix}$ with the block entries

$$A_{ij}=\frac{I_n-S(\lambda_i)S(\lambda_j)^*}{1-\lambda_i\lambda_j^*}\quad(i,j=1,\ldots,N)\tag{3.1}$$

$$B_{ij}=\frac{S(\lambda_i)-S(\omega_j)}{\lambda_i-\omega_j}\quad(i=1,\ldots,N;\ \ j=1,\ldots,P)\tag{3.2}$$

$$D_{ij}=\frac{I_n-S(\omega_i)^*S(\omega_j)}{1-\omega_j\omega_i^*}\quad(i,j=1,\ldots,P)\tag{3.3}$$

is nonnegative. The main objective of this section is to prove an analogous result in the classes $W(\rho,P)$. We first need some preliminary lemmas.

Lemma 3.1. *Let* S *be in a class* $W(\rho,P)$ *and let* $P=(p_{ij})$ *be the block decomposition of* P *into four* $\mathbf{C}^{n\times n}$*-valued functions. Then*

(i) the function $\lambda\to det(S(\lambda)p_{12}(\lambda)+p_{22}(\lambda))$ *is not identically vanishing in* Ω_+;

(ii) the kernel

$$\frac{I_n-R_S(\lambda)R_S(\omega)^*}{\rho_\omega(\lambda)}\tag{3.4}$$

with the function R_S *being defined by*

$$R_S(\lambda)=(S(\lambda)p_{12}(\lambda)+p_{22}(\lambda))^{-1}(S(\lambda)p_{11}(\lambda)+p_{21}(\lambda))\tag{3.5}$$

is positive in Ω_+. *In particular,* R_S *is analytic and takes contractive values in* Ω_+.

Proof: Let λ be in Ω_+ and let K_S be as in (1.2). Then

$$(S(\lambda)p_{12}(\lambda)+p_{22}(\lambda))(S(\lambda)p_{12}(\lambda)+p_{22}(\lambda))^*$$

$$=(S(\lambda)p_{11}(\lambda)+p_{21}(\lambda))(S(\lambda)p_{11}(\lambda)+p_{21}(\lambda))^*+\rho_\lambda(\lambda)K_S(\lambda,\lambda).$$

Therefore, if $h(S(\lambda)p_{12}(\lambda)+p_{22}(\lambda))=0$ for some h in $\mathbf{C}^{1\times n}$ it follows that $h(S(\lambda)p_{11}(\lambda)+p_{21}(\lambda))=0$, so that $h(S(\lambda),I)P(\lambda)=0$. Hence $detP(\lambda)=0$ at the given point λ. This concludes the proof of (i) since, by hypothesis, $detP\neq 0$ in Ω_+.

To prove (ii), let λ,ω be points in Ω_+ where the function $Sp_{12}+p_{22}$ is invertible. Then,

$$\frac{I_n-R_S(\lambda)R_S(\omega)^*}{\rho_\omega(\lambda)}=(S(\lambda)p_{12}(\lambda)+p_{22}(\lambda))^{-1}K_S(\lambda,\omega)(S(\omega)p_{12}(\omega)+p_{22}(\omega))^{-*} \quad (3.6)$$

which allows us to conclude, at least for those points of Ω_+ where $det(Sp_{12}+p_{22})$ does not vanish. Setting $\lambda=\omega$ we see that $R_S(\lambda)$ is contractive at these points. It follows that any singularity of R_S is removable, which ends the proof. ∎

Corollary 3.2. *Let $\omega\in\Omega_+$ be such that $detP(\omega)\neq 0$ and let $S\in W(\rho,P)$ be analytic in ω. Then $det(S(\omega)p_{12}(\omega)+p_{22}(\omega))\neq 0$.*

Lemma 3.3. *Let Δ be a subset of $\mathbf{D}$ having one accumulation point inside $\mathbf{D}$ and let S be a $\mathbf{C}^{n\times n}$-valued function defined on Δ and such that the kernel $(I_n-S(\lambda)S(\omega)^*)/(1-\lambda\omega^*)$ is positive on Δ. Then S has a unique extension to an analytic function inside $\mathbf{D}$. For this extension the above kernel is still positive.*

Proof: The set of functions $\frac{c}{1-\lambda\omega^*}$ where c is in $\mathbf{C}^{n\times 1}$ and ω is in Δ is dense in the Hardy space H_2^n. From the positivity of the given kernel on Δ it follows that the map T, $T\frac{c}{1-\lambda\omega^*}=\frac{S(\omega)^*}{1-\lambda\omega^*}$, is a contraction from H_2^N into itself. Its adjoint T^* is still a contraction and is defined by $(T^*f)(\lambda)=S(\lambda)f(\lambda)$ for λ in Δ and f in H_2^n, from which the claim follows. ∎

The next result is the analogue of the positivity of the matrix $\begin{pmatrix} A & B \\ B^* & D\end{pmatrix}$ defined by (3.1)-(3.3) in the present setting.

Lemma 3.4. *Let $\lambda\to R(\lambda)$ be in $W(\rho,I_n)$. Let $\omega_1,\ldots,\omega_N$ and $\nu_1,\ldots,\nu_P$ be points in Ω_+ such that $\frac{b}{a}(\omega_i)\neq\frac{b}{a}(\nu_j)$ for $i\in\{1,\ldots,N\}$, $j\in\{1,\ldots,P\}$. Then*

$$T=\begin{pmatrix} T_1 & T_2 \\ T_2^* & T_3\end{pmatrix}\geq 0,$$

where the block matrices T_1,T_2 and T_3 are defined by

$$(T_1)_{ij}=\frac{I_n-R(\omega_i)R(\omega_j)^*}{\rho_{\omega_j}(\omega_i)} \qquad (i,j\in\{1,\ldots,N\}) \quad (3.7)$$

$$(T_3)_{ij}=\frac{I_n-R(\nu_i)^*R(\nu_j)}{\rho_{\nu_i}(\nu_j)} \qquad (i,j\in\{1,\ldots,N\}) \quad (3.8)$$

$$(T_2)_{ij} = \frac{R(\omega_i) - R(\nu_j)}{b(\omega_i)a(\nu_j) - a(\omega_i)b(\nu_j)} \qquad (i \in \{1, \ldots, N\}; j \in \{1, \ldots, P\}). \tag{3.9}$$

In particular, the kernel (3.4) (with R instead of R_S) is positive in Ω_+.

Proof: Let σ be a function defined as $\sigma = b/a$. In view of (1.2), $|\sigma(\lambda)| < 1$ for $\lambda \in \Omega_+$. Since the functions a, b are both analytic in Ω_+, it follows that any singularity of σ in Ω_+ is removable. From [6] it follows that $R(\lambda) = S(\sigma(\lambda))$ where S is an analytic contraction. From this identity we obtain the nonnegativity of the matrix $T' = \begin{pmatrix} T_1' & T_2' \\ T_2'^* & T_3' \end{pmatrix}$ defined by

$$(T_1')_{ij} = \frac{I_n - R(\lambda_i)R(\lambda_j)^*}{1 - \sigma(\lambda_i)\sigma(\lambda_j)^*} \qquad (i, j \in \{1, \ldots, N\})$$

$$(T_2')_{ij} = \frac{R(\lambda_i) - R(\omega_j)}{\sigma(\lambda_i) - \sigma(\omega_j)} \qquad (i \in \{1, \ldots, N\};\ j \in \{1, \ldots, P\})$$

$$(T_3')_{ij} = \frac{I_n - R(\omega_i)^* R(\omega_j)}{1 - \sigma(\omega_i)^* \sigma(\omega_j)} \qquad (i, j \in \{1, \ldots, P\})$$

which is equivalent to the nonnegativity of the matrix T defined in Lemma 3.4 since the function a does not vanish in Ω_+. ∎

We note that the relation $R = S \circ \sigma$ was obtained in [6] using the reproducing kernel Hilbert space associated to the positive function $I_n/\rho_\omega(\lambda)$ (where $\lambda, \omega \in \Omega_+$). Now we turn to the main result of this section.

Theorem 3.5. *Let S be in the class $W(\rho, P)$ and let $\omega_i, \nu_j \in \Omega_+$ be points of analyticity of S and P^{-1} such that $\sigma(\omega_i) \neq \sigma(\nu_j)$, $i \in \{i, \ldots, N\}$, $j \in \{1, \ldots, P\}$. Then*

$$\mathbf{S} = \begin{pmatrix} S_1 & S_2 \\ S_2^* & S_3 \end{pmatrix} \geq 0, \tag{3.10}$$

where

$$(S_1)_{ij} = K_S(\omega_i, \omega_j) \quad (i, j \in \{1, \ldots, N\}) \tag{3.11}$$

$$(S_3)_{ij} = \tilde{K}_S(\nu_i, \nu_j) := -(I_n, -S(\nu_i)^*) \frac{P(\nu_i)^{-*} J P(\omega_j)^{-1}}{\rho_{\nu_i}(\nu_j)} \begin{pmatrix} I_n \\ -S(\nu_j) \end{pmatrix}$$
$$(i, j \in \{1, \ldots, P\}) \tag{3.12}$$

$$(S_2)_{ij} = (S(\omega_i), I_n) \frac{P(\omega_i) P(\nu_j)^{-1}}{a(\omega_i)b(\nu_j) - b(\omega_i)a(\nu_j)} \begin{pmatrix} I_n \\ -S(\nu_j) \end{pmatrix}$$
$$(i \in \{1, \ldots, N\}, j \in \{1, \ldots, P\}). \tag{3.13}$$

Proof: Let $P(\lambda)^{-1} = (q_{ij})_{i,j=1,2}$ be the block decomposition of P^{-1} into four $\mathbf{C}^{n\times n}$-valued blocks. It is easily checked that $\lambda \to det(q_{12}(\lambda)S(\lambda) - q_{11}(\lambda))$ is not identically vanishing and that the function R_S defined by (3.5) can be reexpressed as

$$R_S(\lambda) = -(q_{22}(\lambda)S(\lambda) - q_{21}(\lambda))(q_{12}(\lambda)S(\lambda) - q_{11}(\lambda))^{-1}. \tag{3.14}$$

To obtain (3.10) we start with the nonnegative matrix T defined by (3.7)-(3.9), with R_S instead of R. We substitute (3.6) into (3.7) and (3.14) into (3.8). In (3.9) we replace $R_S(\omega_i)$ by (3.5) and $R_S(\nu_j)$ by (3.14). Then multiplying T from the left by the matrix

$$N = \begin{pmatrix} diag(S(\omega_i)p_{12}(\omega_i) + p_{22}(\omega_i))_{i=1}^{N} & 0 \\ 0 & -diag(S(\nu_i)q_{12}(\nu_i)^* - q_{11}(\nu_i)^*)_{i=1}^{P} \end{pmatrix}$$

and from the right by N^* we obtain the required result. ∎

Corollary 3.6. *Under the assumptions of Theorem 3.5 the kernel*

$$T(\lambda,\omega) = \begin{pmatrix} \mathbf{S} & & \tilde{\psi}_1(\omega)^* \\ & & \tilde{\psi}_2(\omega)^* \\ \tilde{\psi}_1(\lambda) & \tilde{\psi}_2(\lambda) & K_S(\lambda,\omega) \end{pmatrix} \tag{3.15}$$

is positive in Ω_+, where $\mathbf{S}$ *is defined by (3.10)-(3.13) and*

$$\begin{aligned} (\tilde{\psi}_1(\lambda))_i &= (S(\lambda), I_n)\frac{P(\lambda)JP(\omega_i)^*}{\rho_{\omega_i}(\lambda)}\begin{pmatrix} S(\omega_i)^* \\ I_n \end{pmatrix} \\ (\tilde{\psi}_2(\lambda))_1 &= (S(\lambda), I_n)\frac{P(\lambda)P(\omega_i)^{-1}}{\rho_{\omega_i}(\lambda)}\begin{pmatrix} I_n \\ -S(\nu_i) \end{pmatrix}. \end{aligned}$$

Proof: It suffices to remark that the positivity of the function $T(\lambda,\omega)$ is equivalent to the nonnegativity of the $(N+P+1)nk \times (N+P+1)nk$ matrices

$$\begin{pmatrix} I_{N+P} & 0 \\ \vdots & \vdots \\ 0 & I_{nk} \end{pmatrix} \begin{pmatrix} \mathbf{S} & \begin{pmatrix} \tilde{\psi}_1(\lambda_1)^* \cdots \tilde{\psi}_1(\lambda_k)^* \\ \tilde{\psi}_2(\lambda_1)^* \cdots \tilde{\psi}_2(\lambda_k)^* \end{pmatrix} \\ \star & (K_S(\lambda_i,\lambda_j))_{i,j=1}^{k} \end{pmatrix} \begin{pmatrix} I_{N+P} & \cdots & I_{N+P} & 0 \\ 0 & \cdots & 0 & I_{nk} \end{pmatrix}$$

for every choice of points $\lambda_1, \ldots, \lambda_k \in \Omega_+$. Applying Theorem 3.5 to the points $\omega_1, \ldots, \omega_N$, $\omega_{N+1} = \lambda_1, \ldots, \omega_{N+k} = \lambda_k$ and $\nu_1, \ldots, \nu_P$ we obtain the nonnegativity of the inner matrix in this last product. ∎

Corollary 3.7. *Let S be in $W(\rho, P)$ and let R_S be defined by (3.5). Then the function $\lambda \to det(p_{11}(\lambda) - p_{12}(\lambda)R_S(\lambda))$ does not vanish identically in Ω_+ and*

$$S(\lambda) = (p_{22}(\lambda)R_S(\lambda) - p_{21}(\lambda))(p_{11}(\lambda) - p_{12}(\lambda)R_S(\lambda))^{-1}.$$

To prove the corollary note that in view of (3.14) $p_{11} - p_{12}R_S = (q_{12}S - q_{11})^{-1}$.

Lemma 3.8. *Let S be in $W(\rho, P)$, let P and P^{-1} be analytic in some neighbourhood $\mathcal{U}_\omega \subset \Omega_+$ of the point $\omega \in \Omega_+$ and let*

$$(I_n, 0)P(\lambda)JP(\lambda)^* \begin{pmatrix} I_n \\ 0 \end{pmatrix} = p_{12}(\lambda)p_{12}(\lambda)^* - p_{11}(\lambda)p_{11}(\lambda)^* \leq 0 \tag{3.16}$$

for all $\lambda \in \mathcal{U}_\omega$. Then S is analytic in $\mathcal{U}_\omega$.

Proof: Since $\det P(\lambda) \neq 0$ in $\mathcal{U}_\omega$, (3.16) forces $\det p_{11}(\lambda) \neq 0$ $(\lambda \in \mathcal{U}_\omega)$. Thus $p_{11}^{-1}(\lambda)p_{12}(\lambda)$ is a contraction for $\lambda \in \mathcal{U}_\omega$ and

$$p_{11}^{-1}(\lambda)p_{12}(\lambda) = U \begin{pmatrix} r(\lambda) & 0 \\ 0 & V \end{pmatrix} U^* \tag{3.17}$$

for some unitary matrices U and V and a strict contraction $r(\lambda)$ in $\mathcal{U}_\omega$. The invertibility of P and p_{11} implies (see [9]) $\det(p_{22}(\lambda) - p_{21}(\lambda)p_{11}^{-1}(\lambda)p_{12}(\lambda)) \neq 0$ for $\lambda \in \mathcal{U}_\omega$. Let us define $\tilde{S} = U(p_{22} - p_{21}p_{11}^{-1}p_{12})^{-1}(Sp_{11} + p_{21})U^*$. Then

$$S = (p_{22} - p_{21}p_{11}^{-1}p_{12})U^*\tilde{S}Up_{11}^{-1} - p_{21}p_{11}^{-1}. \tag{3.18}$$

Substituting (3.17), (3.18) into the inequality $K_S(\lambda, \lambda) \geq 0$ (see (1.2)) and multiplying it by the matrix $U(p_{22} - p_{21}p_{11}^{-1}p_{12})$ from the left and by its adjoint from the right we obtain

$$\begin{aligned} I_n + \tilde{S}(\lambda)\begin{pmatrix} r(\lambda) & 0 \\ 0 & V \end{pmatrix} + \begin{pmatrix} r(\lambda)^* & 0 \\ 0 & V^* \end{pmatrix}\tilde{S}(\lambda)^* \\ +\tilde{S}(\lambda)\begin{pmatrix} r(\lambda)r(\lambda)^* - I & 0 \\ 0 & 0 \end{pmatrix}\tilde{S}(\lambda)^* \geq 0 \end{aligned} \tag{3.19}$$

for $\lambda \in \mathcal{U}_\omega$. Let

$$\tilde{S} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \tag{3.20}$$

be a block decomposition of S corresponding to (3.17). Since $rr^* - I < 0, s_{11}$ and s_{21} are bounded in $\mathcal{U}_\omega$ and thus are analytic there. Using this fact and substituting (3.20) into (3.19) we obtain the boundedness of the function $\begin{pmatrix} s_{12} \\ s_{22} \end{pmatrix}(0, V) + \begin{pmatrix} 0 \\ V^* \end{pmatrix}(s_{12}^*, s_{22}^*)$ in $\mathcal{U}_\omega$. This means that $\begin{pmatrix} s_{12} \\ s_{22} \end{pmatrix}(0, V)$ has a bounded real part in $\mathcal{U}_\omega$ and, therefore, is analytic there. So, the function $\tilde{S}$ is analytic in $\mathcal{U}_\omega$, which on account of (3.17) implies the assertion of the lemma. ∎

Lemma 3.9. *Let S be in $W(\rho, P)$, let P and P^{-1} be analytic in $\omega \in \Omega_+$, let (3.13) holds for $\lambda = \omega$ and let $K_S(\omega, \omega) > 0$. Then S is analytic in ω.*

Proof: As in the proof of Lemma 3.8 we obtain that $\det p_{11}(\omega) \neq 0$ and $p_{11}^{-1}(\lambda)p_{12}(\lambda)$ is an analytic contraction in some neighbourhood $\mathcal{U}_\omega$ of ω. Let R_S be defined by (3.5). By

Lemma 3.1 the strict positivity of $K_S(\omega,\omega)$ implies that R_S is a strict contraction in $\mathcal{U}_\omega$. Then $det(p_{11}(\lambda)-p_{12}(\lambda)R_S(\lambda)) \neq 0$ for $\lambda \in \mathcal{U}_\omega$ and by Corollary 3.7 S is analytic in Ω. ∎

In the previous lemma, the condition $K_S(\omega,\omega) > 0$ cannot be relaxed to $K_S(\omega,\omega) \geq 0$, as we now show on an example.

Example 3.10. *Let $\rho_z(\lambda) = 1-\lambda z^*$, $\Omega_+ = \mathbf{D}$, $P(\lambda) = \begin{pmatrix} \frac{1}{2} & \lambda \\ 0 & 1 \end{pmatrix}$, $\omega = \frac{1}{2} \in \Omega_+$. Then the function $S(\lambda) = 2(1-2\lambda)^{-1}$ belongs to $W(\rho,P)$ since $K_S(\lambda,\lambda) = 0 \quad (\lambda \in \Omega_+)$. We have $(I_n,0)P(\frac{1}{2})JP(\frac{1}{2})^*\begin{pmatrix} I_n \\ 0 \end{pmatrix} = 0$, but S has a pole in $\lambda = \frac{1}{2}$.*

In conclusion we show the nonemptiness of $W(\rho,P)$.

Lemma 3.11. *Let A, B be $n \times n$ matrices such that $rank\begin{pmatrix} A \\ B \end{pmatrix} = n$. Then there exists a contractive matrix $S \in \mathbf{C}^{n\times n}$ such that $(SB+A)$ is invertible.*

Proof: Let us suppose that $det(SB+A) = 0$ for every contraction in $\mathbf{C}^{n\times n}$. Then the function $\lambda \to det(\lambda B + A)$ is identically equal to zero, and the pencil $\lambda B + A$ is singular. By a theorem of Kronecker on the canonical form of singular pencils [12], there exist nonsingular matrices P and Q such that

$$PAQ = \begin{pmatrix} diag\begin{pmatrix} 0 & 0 \\ 0 & L_{\mu_i} \end{pmatrix}_{i=1,\dots,n-r} & 0 & 0 \\ 0 & diag\begin{pmatrix} L_{\varepsilon_i} & 0 \\ 0 & L^*_{\eta_i} \end{pmatrix}_{i=1,\dots,T} & 0 \\ 0 & 0 & A_0 \end{pmatrix},$$

$$PBQ = \begin{pmatrix} diag\begin{pmatrix} 0 & 0 \\ 0 & M_{\mu_i} \end{pmatrix}_{i=1,\dots,n-r} & 0 & 0 \\ 0 & diag\begin{pmatrix} M_{\varepsilon_i} & 0 \\ 0 & M^*_{\eta_i} \end{pmatrix}_{i=1,\dots,T} & 0 \\ 0 & 0 & B_0 \end{pmatrix},$$

where L_k and M_k are $k \times (k+1)$-matrices defined by

$$L_k = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \ddots & \ddots & & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}, \qquad M_k = \begin{pmatrix} 1 & 0 & & \cdots & 0 \\ 0 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

and $r = rank(A,B)$. The indices μ_i, ε_i and η_i are chosen to be of increasing order and $det(A_0 + \lambda B_0) \not\equiv 0$.

Let V_i $(i = 1,\dots,n-r)$ denote the $(\mu_i+1)\times(\mu_i+1)$-matrix with all entries equal to zero, at the exception of the first entry of the second column which is equal to $v_i \in \mathbf{C}$. Then the matrix $\begin{pmatrix} 0 & 0 \\ 0 & L^\bullet_{\mu_i} \end{pmatrix} + V_i\begin{pmatrix} 0 & 0 \\ 0 & M^\bullet_{\mu_i} \end{pmatrix} = \begin{pmatrix} v_i & 0 \\ 0 & I_{\mu_i} \end{pmatrix}$ is not singular for $v_i \neq 0$.

Let U_i denote the $(\varepsilon_i + \eta_i + 1) \times (\varepsilon_i + \eta_i + 1)$-matrix with all entries equal to zero, at the exception of the $(\varepsilon_i + 1)$-th entry of the first column which is equal to $u_i \in \mathbb{C}$. Then the matrix $\begin{pmatrix} L_{\varepsilon_i} & 0 \\ 0 & L_{\eta_i}^* \end{pmatrix} + U_i \begin{pmatrix} M_{\varepsilon_i} & 0 \\ 0 & M_{\eta_i}^* \end{pmatrix}$ is not singular for $u_i \neq 0$. We chose $u_1, \ldots, u_T, v_1, \ldots, v_{n-r}$ and λ_0 such that $det(A_0 + \lambda_0 B_0) \neq 0$ and such that

$$S = P^{-1} \begin{pmatrix} diag(V_i) & 0 & 0 \\ 0 & diag(U_i) & 0 \\ 0 & 0 & \lambda_0 I \end{pmatrix} P$$

is a contraction. Then

$$SB + A = P^{-1} \left\{ \begin{pmatrix} diag(V_i) & 0 & 0 \\ 0 & diag(U_i) & 0 \\ 0 & 0 & \lambda_o I \end{pmatrix} PBQ + PAQ \right\} Q^{-1}$$

is not singular, which concludes the proof. ■

Corollary 3.12. *Let A, B be $n \times n$ matrices such that $rank(A, B) = n$. Then there exists a contractive matrix $S \in \mathbb{C}^{n \times n}$ such that $(BS + A)$ is invertible.*

Corollary 3.13. *The class $\Omega(\rho, P)$ is nonempty.*

Proof: Let $P = (p_{ij})$ be the decomposition of P into four $\mathbb{C}^{n \times n}$-valued functions . By Corollary 3.7, it suffices to prove that, for some contractive matrix R, the function $\lambda \to det(p_{11}(\lambda) - p_{12}(\lambda) R_S(\lambda))$ does not vanish identically. This, in turn follows from Corollary 3.12 since $rank(p_{11}(\lambda), p_{12}(\lambda)) = n$ for all points λ where $det P(\lambda) \neq 0$.

4 The interpolation problems $IP(\rho, P)$

The interpolation problems $IP(\rho, P)$ were introduced in Definition 1.4. In this section we present a necessary and sufficient condition (Theorem 4.1) in terms of positive kernels for a $\mathbb{C}^{n \times n}$-valued function S to be a solution of $IP(\rho, P)$. As a corollary we obtain a necessary condition (the nonnegativity of the matrix K defined in (4.2)-(4.4)) in terms of the data for the problem to have a solution.

Theorem 4.1. *Let S be a $\mathbb{C}^{n \times n}$-valued meromorphic function analytic in ω_i $(i = 1, \ldots, N)$. Then S is a solution of the interpolation problem $IP(\rho, P)$ if and only if the function*

$$K(z, w) = \begin{pmatrix} K_1 & K_2 & \psi_1(\omega)^* \\ K_2^* & K_3 & \psi_2(\omega)^* \\ \psi_1(\lambda) & \psi_2(\lambda) & K_S(\lambda, \omega) \end{pmatrix} \quad (4.1)$$

is positive in Ω_+. *Here the function* K_S *is defined by (1.2), the block–matrices* K_1, K_2, K_3 *are defined by*

$$(K_1)_{ij} = (c_i, h_i)\frac{P(\omega_i)JP(\omega_j)^*}{\rho_{\omega_j}(w_i)}\begin{pmatrix} c_j^* \\ h_j^* \end{pmatrix} \tag{4.2}$$

$$(K_2)_{ij} = \begin{cases} (c_i, h_i)\frac{P(\omega_i)P(\omega_j)^{-1}}{a(\omega_i)b(\omega_j)-b((\omega_i)(a(\omega_j)}\begin{pmatrix} f_j^* \\ -d_j^* \end{pmatrix} & (i \neq j) \\ \frac{\gamma_i+(c_i,h_i)P'(\omega_i)P(\omega_i)^{-1}\begin{pmatrix} f_i^* \\ -d_i^* \end{pmatrix}}{b(\omega_i)a'(\omega_i)-a(\omega_i)b'(\omega_i)} & (i = j) \end{cases} \tag{4.3}$$

$$(K_3)_{ij} = -(f_i, -d_i)\frac{P(\omega_i)^{-*}P(\omega_j)^{-1}}{\rho_{\omega_i}(w_j)}\begin{pmatrix} f_j^* \\ -d_j^* \end{pmatrix} \quad (i, j = 1, \ldots, N) \tag{4.4}$$

and the matrix–valued functions ψ_1, ψ_2 *are defined by*

$$(\psi_1(\lambda))_i = (c_i, h_i)\frac{P(\omega_i)JP(\lambda)^*}{\rho_{\omega_i}(\lambda)^*}\begin{pmatrix} S(\lambda)^* \\ I_n \end{pmatrix}$$

$$(\psi_2(\lambda))_i = (f_i, -d_i)\frac{P(\omega_i)^{-*}P(\lambda)^*}{(a(\lambda)b(\omega_i) - a(\omega_i)b(\lambda))^*}\begin{pmatrix} S(\lambda)^* \\ I_n \end{pmatrix} \quad (i = 1, \ldots, N).$$

Proof: We first suppose that the function S belongs to $W(\rho, P)$ and is a solution of the interpolation problem $IP(\rho, P)$. We consider the kernel defined in (3.15) with N points $\omega_1, \ldots, \omega_N$ where ω_i are the interpolation points and with N points $\nu_1, \ldots, \nu_N$ in Ω_+ with $\sigma(\nu_i) \neq \sigma(\omega_j)$, $i, j \in \{1, \ldots, N\}$. Multiplying $T(\lambda, \omega)$ from the left by

$$\phi = \begin{pmatrix} diag(h_i) & 0 & 0 \\ 0 & diag(f_i) & 0 \\ 0 & 0 & I_n \end{pmatrix}$$

and from the right by ϕ^* and letting ν_i tend to ω_i $(i = 1, \ldots, N)$, we obtain (4.1) by Lemma 1.1. Conversely, let S be a $\mathbf{C}^{n \times n}$–valued function analytic at the points $\omega_i, i = 1, \ldots, N$ and for which (4.1) holds. Then in particular the function $K(\lambda, \omega)$ is positive in Ω_+ and the function S belongs to $W(\rho, P)$.

The nonnegativity of the matrix–valued function (4.1) implies that, for $\lambda = \omega$,

$$\begin{pmatrix} (K_1)_{ii} & (K_2)_{ii} & (\psi_1(\lambda))_i \\ (K_2^*)_{ii} & (K_3)_{ii} & (\psi_2(\lambda))_i \\ ((\psi_1(\lambda))_i^* & (\psi_2(\lambda))_i^* & K_S(\lambda, \lambda) \end{pmatrix} \geq 0. \tag{4.5}$$

Therefore, the function ψ_2 is bounded in compact neighbourhoods of the points ω_i. Therefore, $f_i S(\omega_i)^* = d_i$ $(i = 1, \ldots, N)$ so that S satisifes the first interpolation conditions in (1.3). The other interpolation conditions are obtained as follows: we set in (4.5) $\lambda = \omega_i$ and obtain

$$\begin{pmatrix} c_i & h_i \\ S(\lambda_i) & I_n \end{pmatrix} P(\lambda_i)JP(\lambda_i)^* \begin{pmatrix} c_i & h_i \\ S(\lambda_i) & I_n \end{pmatrix}^* \geq 0.$$

From this we obtain that the matrix

$$M = \begin{pmatrix} c_i & h_i \\ S(\lambda_i) & I_n \end{pmatrix} P(\lambda_i) \in \mathbb{C}^{(n+r_i)\times 2n}$$

is J-nonnegative. Thus, the rank of M is less or equal to n. Since $det P(\lambda_i) \neq 0$, $rank M = n$. Since the rank of the matrix $(S(\lambda_i), I_n)$ is n, there exists a matrix $g \in \mathbb{C}^{r_i \times n}$ such that $(I_{r_i}, g)\begin{pmatrix} c_i & h_i \\ S(\lambda_i) & I_n \end{pmatrix} = 0$. Hence, $h_i = -g$ and $c_i = h_i S(\lambda_i)$ $(i = 1, \dots, N)$ which form the second set of interpolation conditions.

To obtain the last set, multiply (4.5) by the matrix

$$N = \begin{pmatrix} 0 & I_{s_i} & 0 \\ I_{r_i} & 0 & -h_i \end{pmatrix} \in \mathbb{C}^{(r_i+s_i)\times(r_i+s_i+n)}$$

from the left and by N^* from the right, and let λ go to λ_i.

Taking into account the two first set of interpolation conditions and that

$$\lim_{\lambda\to\omega_i} \frac{h_i(S(\lambda), I_n)P(\lambda)P(\omega_i)^{-1}\begin{pmatrix} f_i^* \\ -d_i^* \end{pmatrix}}{b(\omega_i)a(\lambda) - a(\omega_i)b(\lambda)} = \frac{(c_i, h_i)P'(\omega_i)P(\omega_i)^{-1}\begin{pmatrix} f_i^* \\ -d_i^* \end{pmatrix} + h_i S'(\omega_i) f_i^*}{b(\omega_i)a'(\omega_i) - a(\omega_i)b'(\omega_i)}$$

we obtain the inequality

$$\begin{pmatrix} (K_3)_{ii} & \frac{\gamma_i^* - f_i S'(\omega_i^*)h_i^*}{(b(\omega_i)a'(\omega_i) - a(\omega_i)b'(\omega_i))^*} \\ \frac{\gamma_i - h_i S'(\omega_i) f_i^*}{b(\omega_i)a'(\omega_i) - a(\omega_i)b'(\omega_i)} & 0 \end{pmatrix} \geq 0$$

from which $\gamma_i = h_i S'(\lambda_i) f_i^*$ $(i = 1, \dots, N)$ follows. ∎

It follows from the preceding analysis that the nonnegativity of the matrix K is a necessary condition to ensure that $IP(\rho, P)$ is solvable.

The matrix K will be called the *informative block-matrix* associated to the problem $IP(\rho, P)$.

5 Solution of $IP(\rho, P)$: the nondegenerate case

In this section we suppose that the informative block matrix K (defined in the previous section) is strictly positive and describe the set of all solutions under this hypothesis. We begin with some lemmas. Then we introduce the notion of P-positive pairs and finally state and prove the main result, Theorem 5.1. The next lemma is taken from [5].

Lemma 5.1. *Let A and B be in $\mathbb{C}^{N\times N}$ and M be in $\mathbb{C}^{n\times N}$ such that:*

(i) for some point μ in Ω_0, $det(a(\mu)A - b(\mu)B) \neq 0$;

(ii) the columns of $F(\lambda) = M(a(\lambda)A - b(\lambda)B)^{-1}$ are linearly independent as functions of λ.

Let furthermore $K \in \mathbf{C}^{N\times N}$ be an invertible Hermitian solution of the equation

$$A^*KA - B^*KB = M^*JM. \tag{5.1}$$

Then the $\mathbf{C}^{2n\times 2n}$-valued function

$$\psi(\lambda) = I_{2n} - \rho_\mu(\lambda)F(\lambda)K^{-1}F(\mu)^*J \tag{5.2}$$

is J-unitary in Ω_0 and

$$J - \psi(\lambda)J\psi(\omega)^* = \rho_\omega(\lambda)F(\lambda)K^{-1}F(\omega)^*, \tag{5.3}$$

where λ and ω are points of analyticity of ψ. We note that there always exists a point μ in Ω_0 for which $a(\mu) \neq 0$ (see [6]).

A formula for ψ^{-1} is presented in the following lemma. We first need some notation and introduce the functions

$$\Gamma(\lambda) = (a(\lambda)A - b(\lambda)B)^{-1}, \tag{5.4}$$

$$G(\lambda) = (a(\lambda)B^* - b(\lambda)A^*)^{-1}. \tag{5.5}$$

Lemma 5.2. *The function θ defined by*

$$\theta(\lambda) = I_{2n} + \rho_\mu(\lambda)MG(\mu)^*K^{-1}G(\lambda)M^*J \tag{5.6}$$

is the inverse of the function ψ defined in (5.2). Furthermore, for λ and ω points of analyticity of θ,

$$\theta(\lambda)J\theta(\omega)^* - J = \rho_\omega(\lambda)MG(\mu)^*K^{-1}G(\lambda)G(\mu)^{-1}KG(\mu)^{-*}G(\omega)^*K^{-1}G(\mu)M^*; \tag{5.7}$$

$$\theta(\omega)^*J\theta(\lambda) - J = \rho_\omega(\lambda)JMG(\omega)^*K^{-1}G(\lambda)M^*J. \tag{5.8}$$

Proof: The proof is computational. From (5.1), (5.7) we obtain

$$\theta(\lambda)J\theta(\omega)^* - J = MG(\mu)^*K^{-1}G(\lambda)\mathcal{L}(\lambda,\omega)G(\omega)^*K^{-1}G(\mu)G^*, \tag{5.9}$$

where

$$\begin{aligned}\mathcal{L}(\lambda,\omega) = \; & \rho_\mu(\lambda)(a(\mu)B^* - b(\mu)A^*)^*K(a(\omega)^*B - b(\omega)^*A)\\ & +\rho_\mu(\omega)^*(a(\lambda)B^* - b(\lambda)A^*)K(a(\mu)^*B - b(\mu)^*A)\\ & +\rho_\mu(\lambda)\rho_\mu(\omega)^*(A^*KA - B^*KB),\end{aligned}$$

which can be rewritten as

$$\mathcal{L}(\lambda,\omega) = \rho_\omega(\lambda)(a(\mu)B^* - b(\mu)A^*)K(a(\mu)^*B - b(\mu)^*A). \tag{5.10}$$

Substituting (5.10) into (5.9) we obtain (5.7). Equality (5.8) is proved mainly in the same way.

We now turn to the proof that $\theta(\lambda)\psi(\lambda) = I_{2n}$. We have from (5.2) and (5.6)

$$\theta(\lambda)\psi(\lambda) = I_{2n} + \rho_\mu(\lambda)MG(\mu)^*K^{-1}G(\lambda)\mathcal{N}(\lambda)\Gamma(\lambda)K^{-1}\Gamma(\mu)^*M^*J, \qquad (5.11)$$

where

$$\mathcal{N}(\lambda) = \Gamma(\mu)^{-*}K\Gamma^{-1}(\lambda) - G^{-1}(\lambda)KG(\mu)^{-*} - \rho_\mu(\lambda)M^*JM.$$

Substituting (1.1), (5.1), (5.4) and (5.5) into the last equality we obtain

$$\begin{aligned}\mathcal{N}(\lambda) = \; & (a(\mu)A - b(\mu)B)^*K(a(\lambda)A - b(\lambda)B) \\ & -(a(\lambda)B^* - b(\lambda)A^*)K(a(\mu)^*B - b(\mu)^*A) \\ & -(a(\lambda)a(\mu)^* - b(\lambda)b(\mu)^*)(A^*KA - B^*KB) = 0,\end{aligned}$$

which both with (5.11) implies $\theta(\lambda)\psi(\lambda) = I_{2n}$. ∎

We will apply the above lemmas to the following set of matrices:

$$A = \begin{pmatrix} diag(a(\omega_i)^*I_{r_i})_{i=1,\dots,N} & 0 \\ 0 & diag(b(\omega_i)I_{s_i})_{i=1,\dots,N} \end{pmatrix} \qquad (5.12)$$

$$B = \begin{pmatrix} diag(b(\omega_i)^*I_{r_i})_{i=1,\dots,N} & 0 \\ 0 & diag(a(\omega_i)I_{s_i})_{i=1,\dots,N} \end{pmatrix} \qquad (5.13)$$

$$M^* = \begin{pmatrix} ((c_i, h_i)P(\omega_i)J)_{i=1,\dots,N} \\ ((f_i, -d_i)P(\omega_i)^{-*})_{i=1,\dots,N} \end{pmatrix}. \qquad (5.14)$$

It is then easily verified that the informative block matrix K is a solution of (5.1).

We now turn to the notion of P-positive pairs.

Definition 5.3. *A pair $\{p(\lambda), q(\lambda)\}$ of $\mathbb{C}^{n\times n}$-valued functions meromorphic in Ω is called P-positive if*

(i) $det(p(\lambda)p(\lambda)^* + q(\lambda)q(\lambda)^*) \not\equiv 0$ in Ω (nondegeneracy of the pair)

(ii) the function

$$K_{p,q}(\lambda,\omega) = (p(\lambda), q(\lambda))\frac{P(\lambda)JP(\omega)^*}{\rho_\omega(\lambda)}\begin{pmatrix} p(\omega)^* \\ q(\omega)^* \end{pmatrix} \qquad (5.15)$$

is positive on the set of analyticity of the pair (p, q).

We will denote by $\mathcal{P}(\rho, P)$ the class of all P-positive pairs. In $\mathcal{P}(\rho, P)$ we introduce an equivalence relation as follows: a pair $\{p, q\}$ is, by definition, equivalent to a pair $\{p_1, q_1\}$ if there exists a $\mathbb{C}^{n\times n}$-valued function X, meromorphic and with nonidentically vanishing determinant in Ω and such that $(p_1(\lambda), q_1(\lambda)) = X(\lambda)(p(\lambda), q(\lambda))$. In the next sequence of lemmas we study the set $\mathcal{P}(\rho, P)$ in more detail.

Lemma 5.4. *There exists a one-to-one correspondence between the classes of equivalence of P-positive pairs and the set of $\mathbf{C}^{n\times n}$-valued analytic contractions in Ω_+ from the class $W(\rho, I_n)$. Namely:*

(i) Every pair $\{u, v\}$ in $\mathcal{P}(\rho, P)$ is equivalent to some pair $\{p, q\}$ of the form

$$(p, q) = (R, I_n)P^{-1}, \tag{5.16}$$

where R belongs to $W(\rho, I_n)$.

(ii) For every R as in (i) the pair defined by (5.16) is P-positive.

Proof: Let $\{u, v\}$ be a P-positive pair. Therefore, at those points λ in Ω_+ where u and v are analytic, we have

$$(u(\lambda), v(\lambda))\frac{P(\lambda)JP(\lambda)^*}{\rho_\lambda(\lambda)}\begin{pmatrix} u(\lambda)^* \\ v(\lambda)^* \end{pmatrix} \geq 0$$

and hence

$$\varphi_1(\lambda)\varphi_1(\lambda)^* \leq \varphi_2(\lambda)\varphi_2(\lambda)^*, \tag{5.17}$$

where we have set

$$\varphi_1(\lambda) = u(\lambda)p_{11}(\lambda) + v(\lambda)p_{21}(\lambda), \quad \varphi_2(\lambda) = u(\lambda)p_{12}(\lambda) + v(\lambda)p_{22}(\lambda).$$

We claim that the determinant of ρ_2 is not identically vanishing in Ω_+. Indeed, let λ_0 be a point of analyticity of ρ_2 in Ω_+ where $\varphi_2(\lambda_0) = 0$. Without loss of generality we may suppose that $detP(\lambda_0) \neq 0$. Since $det\varphi_2(\lambda_0) = 0$, there exists a vector $h \in \mathbf{C}^{1\times n}$ such that $h\varphi_2(\lambda_0) = 0$. From (5.17) we have $h\varphi_1(\lambda_0) = 0$ and thus $h(u(\lambda_0), v(\lambda_0))P(\lambda_0) = 0$, so that $hu(\lambda_0) = hv(\lambda_0) = 0$. By property (i) in Definition 5.1, λ belongs to a set of isolated points without accumulation points inside Ω_+. From (5.17), the meromorphic function $R(\lambda) = \varphi_2^{-1}(\lambda)\varphi_1(\lambda)$ is a contraction and thus is analytic in Ω_+. Moreover, $(u(\lambda), v(\lambda)) = \varphi_2(\lambda)(R(\lambda), I_n)P^{-1}(\lambda)$ and hence, $K_{u,v}(\lambda, \omega) = \varphi_2(\lambda)K_R(\lambda, \omega)\varphi_2(\omega)^*$. Since $\det\varphi_2(\lambda) \not\equiv 0$ Lemma 1.1 insures that the pair $\{u, v\}$ belongs to $\mathcal{P}(\rho, P)$ if and only if R belongs to $W(\rho, I_n)$. ∎

Definition 5.5. *An equivalence class of P-positive pairs is called proper if for all pairs $\{p(\lambda), q(\lambda)\}$ in the class, det q is not identically vanishing.*

Lemma 5.6. *Every equivalence class in $\mathcal{P}(\rho, P)$ is proper if*

$$(0, I_n)P(\lambda)^{-*}JP(\lambda)^{-1}\begin{pmatrix} 0 \\ I_n \end{pmatrix} > 0 \quad (\lambda \in \Omega_+) \tag{5.18}$$

or, equivalently, if

$$(I_n, 0)P(\lambda)JP(\lambda)^*\begin{pmatrix} I_n \\ 0 \end{pmatrix} < 0 \quad (\lambda \in \Omega_+). \tag{5.19}$$

Proof: Let $\{p, q\}$ be a pair in $\mathcal{P}(\rho, P)$. Then, up to a multiplicative factor, $\{p, q\}$ is of the form (5.16). Using the block decomposition $P^{-1} = (q_{ij})$ of the function P^{-1} into four $\mathbb{C}^{n\times n}$-valued blocks we obtain $q(\lambda) = R(\lambda)q_{12}(\lambda) + q_{22}(\lambda)$. From (5.18) we have $q_{12}(\lambda)^* q_{12}(\lambda) \leq q_{22}(\lambda)^* q_{22}(\lambda)$, and the nondegeneracy of P forces the determinant of q_{22} to be not identically vanishing. Thus, $q_{12}(\lambda)q_{22}(\lambda)^{-1}$ is a strict contraction for λ in Ω_+ where P^{-1} is analytic. Hence $det\, q(\lambda) = det((I_n + R(\lambda)q_{12}(\lambda)q_{22}(\lambda)^{-1})q_{22}(\lambda))$ is not identically vanishing in Ω_+. The equivalence of (5.18) and (5.19) follows from the $n \times n$-block decompositions of P and P^{-1} and the evident equality $(I_n, 0)P(\lambda)P(\lambda)^{-1}\begin{pmatrix} 0 \\ I_n \end{pmatrix} = 0$. ∎

The following subclass of $\mathcal{P}(\rho, P)$ will be of interest.

Definition 5.7. *A pair $\{p, q\}$ in $\mathcal{P}(\rho, P)$ will be in the class $\partial\mathcal{P}(\rho, P)$ if*

$$(p(\lambda), q(\lambda))P(\lambda)JP(\lambda)^*\begin{pmatrix} p(\lambda)^* \\ q(\lambda)^* \end{pmatrix} = 0, \quad \lambda \in \Omega_+.$$

We now state the analogue of Lemma 5.4 for the subclass $\partial P(\rho, P)$.

Lemma 5.8. *There exists a one-to-one correspondence between classes of equivalence of $\partial\mathcal{P}(\rho, P)$ and the set of unitary $n \times n$-matrices. Namely: a pair $\{p, q\}$ is in $\partial\mathcal{P}(\rho, P)$ if and only if it can be written as*

$$(p(\lambda), q(\lambda)) = X(\lambda)(R, I_n)P^{-1}(\lambda) \quad \textit{for} \quad \lambda \in \Omega_+,$$

where the function X is $\mathbb{C}^{n\times n}$-valued and meromorphic in Ω_+, with nonidentically vanishing determinant, and R is a unitary element in $\mathbb{C}^{n\times n}$.

The proof of this lemma goes along the lines of the proof of Lemma 5.4 and will be omitted.

With these lemmas out of the way we turn to the main result of this section.

Theorem 5.9. *Under the hypothesis that the informative block matrix K is strictly positive, the solutions of the problem $IP(\rho, P)$ are described as follows: let θ be the matrix-valued function defined in (5.7) and let*

$$\mathcal{A}(\lambda) = P(\lambda)\theta(\lambda)P(\lambda)^{-1}. \tag{5.20}$$

Let $\mathcal{A} = (a_{ij})$ and $P = (p_{ij})$ be the decompositions of $\mathcal{A}$ and P into four $\mathbf{C}^{n\times n}$-valued functions and let $\mathcal{P}^\circ(\rho, P)$ be the set of all P-positive pairs $\{p,q\}$ such that

$$det\left((p(\lambda),q(\lambda))\mathcal{A}(\lambda)\begin{pmatrix} p_{12}(\lambda) \\ p_{22}(\lambda)\end{pmatrix}\right)^{-1}(p(\lambda)a_{12}(\lambda)+q(\lambda)a_{22}(\lambda)) \not\equiv 0. \tag{5.21}$$

Then the solutions of $IP(\rho, P)$ are parametrized by the linear fractional transformation

$$S(\lambda) = (p(\lambda)a_{12}(\lambda)+q(\lambda)a_{22}(\lambda))^{-1}(p(\lambda)a_{11}(\lambda)+q(\lambda)a_{21}(\lambda)) \tag{5.22}$$

when $\{p,q\}$ varies in $\mathcal{P}^\circ(\rho, P)$. More precisely:

(i) Every solution S of $IP(\rho, P)$ is of the form (5.22) for some pair $\{p,q\}$ in $\mathcal{P}^\circ(\rho, P)$.

(ii) Conversely, for every pair $\{p,q\}$ in $\mathcal{P}^\circ(\rho, P)$ the function S defined by (5.22) is in $W(\rho, P)$ and is a solution of $IP(\rho, P)$.

(iii) Two different pairs $\{p,q\}$ and $\{p_1,q_1\}$ lead to the same S in (5.22) if and only if they are equivalent.

Proof: By Theorem 4.1, the $\mathbf{C}^{n\times n}$-valued function S analytic in ω_i $(i = 1,\ldots,N)$ is a solution of $IP(\rho, P)$ if and only if the kernel $K(\lambda,\omega)$ defined by (4.1) is nonnegative. By Lemma 1.2, this is equivalent to the nonnegativity of the function

$$K_S(\lambda,\omega) - (\psi_1(\lambda),\psi_2(\lambda))K^{-1}\begin{pmatrix}\psi_1(\omega)^* \\ \psi_2(\omega)^*\end{pmatrix} \geq 0,$$

which can be rewritten as

$$(S(\lambda), I_n)P(\lambda)\left\{\frac{J}{\rho_\omega(\lambda)} - M\Gamma(\lambda)K^{-1}\Gamma(\omega)^*M^*\right\}P(\omega)^*\begin{pmatrix}S(\omega)^* \\ I_n\end{pmatrix} \geq 0, \tag{5.23}$$

where Γ and M are defined via (5.4), (5.12)–(5.14). Using (5.3) and Lemma 5.2, we rewrite inequality (5.23) as

$$(S(\lambda), I_n)\frac{P(\lambda)\theta(\lambda)^{-1}J\theta(\omega)^{-*}P(\omega)^*}{\rho_\omega(\lambda)}\begin{pmatrix}S(\omega)^* \\ I_n\end{pmatrix} \geq 0$$

or, equivalently, as

$$(S(\lambda), I_n)\frac{\mathcal{A}(\lambda)^{-1}P(\lambda)JP(\omega)^*\mathcal{A}(\omega)^{-*}}{\rho_\omega(\lambda)}\begin{pmatrix}S(\omega)^* \\ I_n\end{pmatrix} \geq 0. \tag{5.24}$$

Let $\{p,q\}$ be defined by $(p,q) = (S,I_n)\mathcal{A}^{-1}$. It follows from (5.23) that $\{p,q\}$ is a P-positive pair. Furthermore,

$$pa_{12}+qa_{22} = I_n, \quad pa_{11}+qa_{21} = S$$

and in view of Corollary 3.2

$$det\left((p,q)\mathcal{A}\begin{pmatrix}p_{12} \\ p_{22}\end{pmatrix}\right)^{-1}(pa_{12}+qa_{22}) = det(sp_{12}+p_{22}) \neq 0$$

in ω_i $(i = 1, \ldots, N)$.Therefore, S admits a representation (5.22).

Let conversely $\{p, q\}$ be in $\mathcal{P}^\circ(\rho, P)$ and let S be defined by (5.22). Then

$$(S(\lambda), I_n) = (p(\lambda)a_{12}(\lambda) + q(\lambda)a_{22}(\lambda))^{-1}(p(\lambda), q(\lambda))\mathcal{A}(\lambda)$$

and the P-positivity of the pair $\{p, q\}$ is equivalent to (5.24). Let us introduce a pair $\{u, v\}$ by

$$(u(\lambda), v(\lambda)) = (p(\lambda), q(\lambda))\mathcal{A}(\lambda). \tag{5.25}$$

Since $det\mathcal{A}(\lambda) \not\equiv 0$ and $\{p, q\}$ is nondegenerate, it follows from (5.24) that $\{u, v\}$ is nondegenerate as well. Since $\{p, q\} \in \mathcal{P}(\rho, P)$ and $\theta(\lambda)J\theta(\lambda)^* \geq J$ for all $\lambda \in \Omega_+$ (see (5.7)), then, in view of (5.20) and (5.25) (and with some abuse of notation)

$$(u, v)\frac{PJP^*}{\rho}\begin{pmatrix} u^* \\ v^* \end{pmatrix} = (p, q)P\frac{\theta J\theta^*}{\rho}P^*\begin{pmatrix} p^* \\ q^* \end{pmatrix} \geq (p, q)\frac{PJP^*}{\rho}\begin{pmatrix} p^* \\ q^* \end{pmatrix} \geq 0 \tag{5.26}$$

and, hence, $\{u, v\}$ belongs to $\mathcal{P}(\rho, P)$. According to Lemma 5.4

$$(u, v) = X(R, I_n)P^{-1} \tag{5.27}$$

for some $\mathbf{C}^{n\times n}$-valued meromorphic function X $(detX(\lambda) \not\equiv 0)$ and $\mathbf{C}^{n\times n}$-valued analytic contraction $R \in W(\rho, I_n)$. Since $detP(\omega_i) \neq 0$, it follows from (5.27) that the functions $X^{-1}u$ and $X^{-1}v$ are analytic in ω_i. By (5.27) $X = up_{12} + vp_{22}$, and using (5.25) we can rewrite (5.21) as

$$detX^{-1}(\lambda)v(\lambda) \neq 0 \quad for\ \lambda = \omega_i\ (i = 1, \ldots, N). \tag{5.28}$$

Comparing (5.22) and (5.25) we obtain that the function $S = v^{-1}u = (X^{-1}v)^{-1}(X^{-1}u)$ is analytic in ω_i $(i = 1, \ldots, N)$. We have shown that S defined by (5.22) is analytic in ω_i and satisfies (5.24). According to Theorem 4.1, S is a solution of $IP(\rho, P)$.

The proof of (iii) is quite straightforward and will be omitted. ∎

The matrix-valued function $\mathcal{A}(\lambda)$ given by (5.25) will be called *the resolvent matrix* of the problem.

We note that the set $\mathcal{P}^\circ(\rho, P)$ introduced in the above theorem consists of all pairs $\{p, q\} \in \mathcal{P}(\rho, P)$ which lead under the linear fractional transformation (5.22) to a function S analytic in the interpolating points ω_i $(i = 1, \ldots, N)$.

The set $\mathcal{P}^\circ(\rho, P)$ may be empty; then the strict positivity of the informative matrix K does not ensure the solvability of the problem $IP(\rho, P)$ as we now illustrate on an example.

Example 5.10. *Let $\omega \in \Omega_+$ and $c \in \mathbf{C}^{1\times n}$ be such that*

$$(c,0)P(\omega)JP(\omega)^* \begin{pmatrix} c^* \\ 0 \end{pmatrix} > 0 \tag{5.29}$$

and let h be the zero vector of $\mathbf{C}^{1\times n}$. Then the tangential problem with the interpolation data ω, c, h (that is, to find all functions $S \in W(\rho, P)$ analytic at ω and such that $hS(w) = c$) has no solutions although its informative matrix coincide with (5.29) and is strictly positive.

If the inequality (3.13) holds for all λ in some neighborhood $\mathcal{U}_{\omega_i}$ of each interpolating point ω_i, it follows from Lemma 3.9 and the previous discussion that the set $\mathcal{P}^\circ(\rho, P)$ can be defined as the set of all pairs $\{p,q\} \in \mathcal{P}(\rho, P)$ such that

$$det(p(\lambda)a_{12}(\lambda) + q(\lambda)a_{22}(\lambda)) \neq 0. \tag{5.30}$$

A particular case of interest is considered in the next lemma.

Lemma 5.11. *Let the interpolating point ω_i be such that*

$$(I_n,0)P(\omega_i)JP(\omega_i)^* \begin{pmatrix} I_n \\ 0 \end{pmatrix} < 0 \quad (i = 1, \ldots, N). \tag{5.31}$$

Then $\mathcal{P}^\circ(\rho, P) = \mathcal{P}(\rho, P)$.

Proof: Let $\{p,q\}$ be in $\mathcal{P}(\rho, P)$ and let $\{u,v\} \in \mathcal{P}(\rho, P)$ be defined by (5.25). Let $\omega \in \Omega_+$ be a point in a sufficiently small neighbourhood of ω_1 such that p, q, P and P^{-1} are analytic in ω, $rank(p(\omega), q(\omega)) = n$ and

$$(I_n,0)P(\omega)JP(\omega)^* \begin{pmatrix} I_n \\ 0 \end{pmatrix} < 0 \tag{5.32}$$

(such a point exists in view of (5.31) and the nondegeneracy of $\{p,q\}$). Let us suppose $detv(\omega) = 0$ and let $h \in \mathbf{C}^{1\times n}$ be a nonzero vector for which

$$hv(\omega) = 0. \tag{5.33}$$

Since $det\mathcal{A}(\omega) \neq 0$ (in view of (5.2), (5.6),(5.20), $\mathcal{A}$ and $\mathcal{A}^{-1}$ have singularities only at ω_i, $i = 1, \ldots, N$ and at the singular points of P and P^{-1}), the pair $\{u,v\}$ is nondegenerate in ω and on account of (5.33), $hu(\omega) \neq 0$. Substituting (5.33) into (5.26) and using (5.19) we obtain

$$0 > hu(\omega)(I,0)\frac{P(\omega)JP(\omega)^*}{\rho_\omega(\omega)} \begin{pmatrix} I \\ 0 \end{pmatrix} u(\omega)^*h^* \geq 0.$$

The last contradiction implies $detv(\omega) \neq 0$ which leads to (5.30). In view of Lemma 5.6 $\{p,q\} \in \mathcal{P}^\circ(\rho, P)$. ∎

Lemma 5.12. *Let (3.13) hold for all λ in some neighbourhood of each interpolating point ω_i and let a pair $\{p,q\} \in \mathcal{P}(\rho, P)$ be strictly positive:*

$$(p(\lambda)q(\lambda))\frac{P(\lambda)JP(\lambda)^*}{\rho_\lambda(\lambda)}\begin{pmatrix} p(\lambda)^* \\ q(\lambda)^* \end{pmatrix} > 0 \quad (\lambda \in \Omega_+). \tag{5.34}$$

Then $\{p,q\} \in \mathcal{P}^\circ(\rho, P)$.

Proof: As above we choose a point $\omega \in \mathcal{U}_{\omega_1}$ such that p, q, P, P^{-1} are analytic in ω, $rank(p(\omega), q(\omega)) = n$ and (3.13) holds for $\lambda = \omega$. We define a pair $\{u, v\}$ by (5.25) and suppose that (5.33) is valid for some nonzero $h \in \mathbf{C}^{1\times n}$. Substituting (5.33) into (5.26) and using (3.13), (5.34) we obtain

$$0 \geq (hu(\omega), 0)P(\omega)JP(\omega)^*(hu(\omega), 0)^* \geq hK_{p,q}(\omega, \omega)h^* > 0.$$

The last contradiction forces $det v(\omega) \neq 0$, and so $\{p,q\} \in \mathcal{P}^\circ(\rho, P)$. ∎

Corollary 5.13. *Let K be strictly positive and let (3.13) hold for all λ in some neighbourhood of each point ω_i $(i = 1, \ldots, N)$. Then the problem $IP(\rho, P)$ has a solution.*

6 Solution of $IP((\rho, P))$: the degenerate case

We now consider the case where $K \geq 0$. The problem $IP(\rho, P)$ is still solvable and the solutions will also be described in terms of a linear fractional transformation. Let $l = rankK$ and let $e_{i_1}, \ldots, e_{i_l}$ be vectors from the canonical basis of $\mathbf{C}^{1\times N}$ such that

$$Lin\{e_{i_j},\ j \in \{1, \ldots, l\}\} \cap KerK = \{0\}, \tag{6.1}$$

where Lin stands for linear span and $KerK = \{\ c \in \mathbf{C}^{1\times N} \mid cK = 0\}$.

Let Q be the element of $C^{l\times N}$ defined by

$$Q = \begin{pmatrix} e_{i_1} \\ \vdots \\ e_{i_l} \end{pmatrix}. \tag{6.2}$$

Then $QKQ^* > 0$ and, in particular, the rank of QKQ^* is l, the rank of K. Thus, inequality (4.5) is equivalent to the positivity of the function

$$K(z, w) = \begin{pmatrix} QKQ^* & Q\Gamma(\omega)^*M^*P(\omega)^*\begin{pmatrix} S(\omega)^* \\ I_n \end{pmatrix} \\ (S(\lambda), I_n)P(\lambda)M\Gamma(\lambda)Q^* & K_S(\lambda, \omega) \end{pmatrix} \tag{6.3}$$

together with the condition

$$(S(\lambda), I_n)P(\lambda)M\Gamma(\lambda)P_{KerK} = 0, \tag{6.4}$$

where P_{KerK} is the orthogonal projection onto $KerK$.

Theorem 6.1. *Let $K \geq 0$, $l = rankK$ and let Q be as in (6.2). Let μ be a point in Ω_0, where $|a(\mu)| \neq 0$, and let $\mathcal{A}(\lambda) = P(\lambda)\theta(\lambda)P(\lambda)^{-1}$, where*

$$\theta(\lambda) = I_{2n} + \rho_\mu(\lambda)MG(\mu)^*Q^*(QKQ^*)^{-1}QG(\lambda)M^*J \tag{6.5}$$

(with Γ, G, M being defined via (5.4), (5.5), (5.14)).

Then the formula (5.26) gives a parametrization of all the solutions of $IP(\rho, P)$ when the parameter $\{p, q\}$ varies in $\mathcal{P}^\circ(\rho, P)$ and is of the form

$$(p(\lambda), q(\lambda)) = \begin{pmatrix} \hat{R}(\lambda) & 0 & I & 0 \\ 0 & R & 0 & I \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} P(\lambda)^{-1}, \tag{6.6}$$

where $U \in \mathbf{C}^{n\times n}$ and $R \in \mathbf{C}^{\nu\times\nu}$ are unitary matrices and $\hat{R}$ is a $\mathbf{C}^{(n-\mu)\times(n-\nu)}$–valued analytic function in Ω_+ with contractive values, and

$$\nu = rank\ M\Gamma(\mu)P_{KerK}. \tag{6.7}$$

Proof: The proof is divided into the following three steps.

Step 1. *The solutions of inequality (6.3) are parametrized by (5.25) with $\mathcal{A}(\lambda) = P(\lambda)\theta(\lambda)P(\lambda)^{-1}$ and θ as in (6.5).*

Step 2. *A pair $\{p, q\}$ in $\mathcal{P}^\circ(\rho, P)$ is a parameter leading to a solution if and only if*

$$(p(\lambda), q(\lambda))P(\lambda)M\Gamma(\mu)P_{KerK} \equiv 0. \tag{6.8}$$

Step 3. *A pair $\{p, q\}$ is a parameter leading to a solution if and only if it is of the form (6.6).*

Proof of Step 1. The matrices A, B defined in (5.12), (5.13) are diagonal. Thus

$$AQ^* = Q^*QAQ^*, \quad BQ^* = Q^*QBQ^* \tag{6.9}$$

and, taking into account (5.4), (5.5),

$$\Gamma(\lambda)Q^* = Q^*Q\Gamma(\lambda)Q^*, \quad QG(\lambda) = QG(\lambda)Q^*Q. \tag{6.10}$$

From (5.1) and (6.9) we obtain

$$(QA^*Q^*)(QKQ^*)(QAQ^*) - (QB^*Q^*)(QKQ^*)(QBQ^*) = QM^*JMQ^*.$$

We can therefore apply Lemma 5.2 and the function θ defined in (6.5) is J–unitary on Ω_0 and J–expansive in Ω_+. Moreover,

$$J - \theta(\lambda)^{-1}J\theta(\omega)^{-*} = \rho_\omega(\lambda)MQ^*Q\Gamma(\lambda)Q^*(QKQ^*)^{-1}Q\Gamma(\omega)^*Q^*QM.$$

Using (6.10) this last identity is rewritten as

$$J-\theta(\lambda)^{-1}J\theta(\omega)^{-*}=\rho_\omega(\lambda)M\Gamma(\lambda)Q^*(QKQ^*)^{-1}Q\Gamma(\omega)^*M^*. \tag{6.11}$$

Similary, from (5.7) we obtain

$$\theta(\lambda)J\theta(\omega)^*-J=\rho_\omega(\lambda)DQKQ^*D^*, \tag{6.12}$$

where

$$D=MG(\mu)^*Q^*(QKQ^*)^{-1}QG(\lambda)G^{-1}(\mu)Q^*. \tag{6.13}$$

Since $QKQ^*>0$, then (6.3) is equivalent to the positivity of the kernel:

$$(S(\lambda),I_n)P(\lambda)\left\{\frac{I}{\rho_\omega(\lambda)}-M\Gamma(\lambda)Q^*(QKQ^*)^{-1}Q\Gamma(\omega)^*M^*\right\}P(\omega)^*\begin{pmatrix}S(\omega)^*\\ I_n\end{pmatrix}\geq 0.$$

In view of (6.11) the last inequality can be rewritten as

$$(S(\lambda),I_n)P(\lambda)\frac{\theta(\lambda)^{-1}J\theta(\omega)^{-*}}{\rho_\omega(\lambda)}P(\omega)^*\begin{pmatrix}S(\omega)^*\\ I_n\end{pmatrix}\geq 0,$$

or equivalently

$$(S(\lambda),I_n)P(\lambda)\frac{\theta(\lambda)^{-1}P(\lambda)JP(\omega)^*\theta(\omega)^{-*}}{\rho_\omega(\lambda)}P(\lambda)^*\begin{pmatrix}S(\omega)^*\\ I_n\end{pmatrix}\geq 0. \tag{6.14}$$

From the proof of Theorem 5.1, all solutions to (6.12) are parametrized by a linear fractional transformation (5.22) with parameter $\{p,q\}$ from $\mathcal{P}^\circ(\rho,P)$.

Proof of Step 2. If S is of the form (5.22), we have

$$(S(\lambda),I_n)=(p(\lambda)a_{12}(\lambda)+q(\lambda)a_{22}(\lambda))^{-1}(p(\lambda),q(\lambda))\mathcal{A}(\lambda). \tag{6.15}$$

Substituting (6.15) into (6.4) we obtain

$$(p(\lambda),q(\lambda))\mathcal{A}(\lambda)P(\lambda)M\Gamma(\lambda)P_{KerK}=0. \tag{6.16}$$

Using (6.5) and the identity

$$a(\lambda)G(\lambda)M^*JM\Gamma(\lambda)=G(\lambda)A^*K-KB\Gamma(\lambda)$$

we obtain

$$\begin{aligned}&\mathcal{A}(\lambda)P(\lambda)M\Gamma(\lambda)\\ &=P(\lambda)(I+\rho_\mu(\lambda)MG(\mu)^*Q^*(QKQ^*)^{-1}QG(\lambda)M^*J)M\Gamma(\lambda)\\ &=P(\lambda)M\Gamma(\lambda)+\tfrac{\rho_\mu(\lambda)}{a(\lambda)}P(\lambda)MG(\mu)^*Q^*(QKQ^*)^{-1}Q(G(\lambda)A^*K-KB\Gamma(\lambda)).\end{aligned} \tag{6.17}$$

Substituting (6.17) into (6.16) we get

$$(p(\lambda),q(\lambda))P(\lambda)M\left(I+\frac{\rho_\mu(\lambda)}{a(\lambda)}G(\mu)^*Q^*(QKQ^*)^{-1}QKB\right)\Gamma(\lambda)P_{KerK}=0, \tag{6.18}$$

which can be rewritten as

$$\frac{a(\mu)}{a(\lambda)}(p(\lambda),q(\lambda))P(\lambda)MG(\mu)^*\left(I+\frac{\rho_\mu(\lambda)}{a(\mu)}(I-Q^*(QKQ^*)^{-1}QKB\Gamma(\lambda))\right)P_{KerK}=0. \tag{6.19}$$

Since $a(\mu)\neq 0$ and

$$I-Q^*(QKQ^*)^{-1}QK=P_{KerK}(I-Q^*(QKQ^*)^{-1}QK),$$

the equation (6.19) is equivalent to the vanishing of

$$(p(\lambda),q(\lambda))P(\lambda)MG(\mu)^*P_{KerK}\left(I+\frac{\rho_\mu(\lambda)}{a(\mu)}(I-Q^*(QKQ^*)^{-1}QKB\Gamma(\lambda)P_{KerK})\right).$$

Since the matrix

$$I+\frac{\rho_\mu(\lambda)}{a(\mu)}\left(I-Q^*(QKQ^*)^{-1}QKB\Gamma(\lambda)\right)P_{KerK}$$

is invertible and

$$a(\mu)^*G(\mu)^*=b(\mu)\Gamma(\mu), \tag{6.20}$$

we obtain (6.8).

Proof of Step 3. We need first to prove a lemma.

Lemma 6.2. *Let $\{p,q\}$ be in $\mathcal{P}(\rho,P)$ and f,g be in $\mathbf{C}^{n\times r}$, $\lambda_0\in\Omega_+$, $detP(\lambda_0)\neq 0$. Suppose that*

$$f^*f=g^*g \tag{6.21}$$

and

$$(p(\lambda_0),q(\lambda_0))P(\lambda_0)\begin{pmatrix} f\\ g\end{pmatrix}=0. \tag{6.22}$$

Then, up to equivalence,

$$(p(\lambda),q(\lambda))=\begin{pmatrix}\hat{R}(\lambda) & 0 & I & 0\\ 0 & R & 0 & I\end{pmatrix}\begin{pmatrix} U & 0\\ 0 & U\end{pmatrix}P(\lambda)^{-1}, \tag{6.23}$$

where $U\in\mathbf{C}^{n\times n}$, $R\in C^{\nu\times\nu}$ are unitary matrices, $\hat{R}(\lambda)$ is a $\mathbf{C}^{(n-\nu)\times(n-\nu)}$–valued analytic contraction in Ω_+ and $\nu=rank\begin{pmatrix} f\\ g\end{pmatrix}=rankf$.

Proof: Using Lemma 5.3 we obtain that, up to equivalence,

$$(p(\lambda),q(\lambda))=(R(\lambda),I_n)P(\lambda)^{-1}, \tag{6.24}$$

where $R(\lambda)$ is a $\mathbf{C}^{n\times n}$–valued analytic contraction in Ω_+. Substituting (6.22) into (6.24) we obtain $R(\lambda_0)f=-g$. In view of (6.21) $R(\lambda_0)$ acts isometrically on the set $Ranf=\{h\in\mathbf{C}^n\mid h=fy \text{ for some } y\in\mathbf{C}^{1\times r}\}$. Therefore $R(\lambda)f=-g$ for all λ in Ω_+. We note that the dimension of $Ranf$ is equal to $rankf=\nu$ and, in view of (6.21), $rankf=rank\begin{pmatrix} f\\ g\end{pmatrix}$. Therefore $R(\lambda)$ admits a representation

$$R(\lambda)=U^*\begin{pmatrix}\hat{R}(\lambda) & 0\\ 0 & R\end{pmatrix}U \tag{6.25}$$

with unitary matrices $U \in \mathbb{C}^{n\times n}$, $R \in \mathbb{C}^{\nu\times\nu}$ and $\hat{R}(\lambda)$ being a $\mathbb{C}^{(n-\nu)\times(n-\nu)}$-valued analytic contraction in Ω_+. Substituting (6.25) into (6.24) we obtain (6.23) which ends the proof of the lemma. ∎

To finish the proof of Step 3 we note that, in view of (5.1), (5.4),

$$\begin{aligned}
&\Gamma(\mu)^* M^* J M \Gamma(\mu)\\
&= \Gamma(\mu)^*(A^*KA - B^*KB)\Gamma(\mu)\\
&= \tfrac{1}{a(\mu)a(\mu)^*}(I + b(\mu)^*\Gamma(\mu)^*B^*)K(I + b(\mu)B\Gamma(\mu)) - \Gamma(\mu)^*B^*KB\Gamma(\mu)\\
&= \tfrac{1}{a(\mu)a(\mu)^*}(K + b(\mu)^*\Gamma(\mu)^*B^*K + b(\mu)KB\Gamma(\mu)),
\end{aligned}$$

which implies

$$P_{KerK}\Gamma(\mu)^* M^* J M \Gamma(\mu) P_{KerK} = 0. \tag{6.26}$$

Both (6.26) and (6.4) mean that the matrix

$$\begin{pmatrix} f \\ g \end{pmatrix} = M\Gamma(\mu)P_{KerK} \tag{6.27}$$

satisfies the conditions of Lemma 6.2. An application of Lemma 6.2 to a matrix $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)$ of the form (6.27) leads to (6.6). ∎

Corollary 6.3. *Let $P(\lambda) = \left(\begin{smallmatrix} p_{11}(\lambda) & p_{12}(\lambda) \\ p_{21}(\lambda) & p_{22}(\lambda) \end{smallmatrix}\right)$, where $p_{ik}(\lambda) = \pi_{ik}(\lambda)I_n$ ($\pi_{ik}: \Omega \to \mathbb{C}$). Then the parameters $\{p, q\}$ in (5.21) are of the form*

$$p(\lambda) = \begin{pmatrix} \hat{p}(\lambda) & 0 \\ 0 & p_0(\lambda) \end{pmatrix} U, \quad q(\lambda) = \begin{pmatrix} \hat{q}(\lambda) & 0 \\ 0 & q_0(\lambda) \end{pmatrix} U,$$

with unitary $U \in \mathbb{C}^{n\times n}$, a fixed pair $\{p_0, q_0\} \in \partial\mathcal{P}(\rho, P)$ and the parameter $\{\hat{p}, \hat{q}\}$ varying in $\mathcal{P}_{n-\nu}(\rho, P)$.

Let, moreover, $P(\lambda)$ satisfy the condition (5.18). Then all the solutions of $IP(\rho, P)$ are parametrized by the linear fractional transformation

$$S(\lambda) = (\sigma(\lambda)a_{12}(\lambda) + a_{22}(\lambda))^{-1}(\sigma(\lambda)a_{11}(\lambda) + a_{21}(\lambda))$$

with the resolvent matrix $\mathcal{A}(\lambda)$ defined as in Theorem 6.1 and parameters $\sigma(\lambda)$ of the form

$$\sigma(\lambda) = \begin{pmatrix} \hat{\sigma}(\lambda) & 0 \\ 0 & \sigma_0(\lambda) \end{pmatrix} U,$$

where $U \in \mathbb{C}^{n\times n}$ is a unitary matrix, σ_0 is a fixed $\mathbb{C}^{\nu\times\nu}$-valued function of the class $\partial W_\nu(\rho, P)$ and where the parameter $\hat{\sigma}$ varies in $W_{n-\nu}(\rho, P)$.

Note that Lemma 5.14 still holds for $K \geq 0$. As a corollary we obtain

Lemma 6.4. *Let the informative matrix K be nonnegative and let the interpolating points ω_i satisfy (5.31). Then the problem $IP(\rho, P)$ has a solution.*

On the other hand,the result of Corollary 5.13 cannot be extended to the relaxed requirement $K \geq 0$ since all pairs $\{p,q\}$ of the form (6.6) are not strictly positive.

Lemma 6.5. *Let K be nonnegative and let (3.13) hold for all λ in some neighbourhood of each interpolating point ω_i $(i = 1, \dots, N)$. Then a pair $\{p,q\} \in \mathcal{P}(\rho, P)$ leads under the linear fractional transformation (5.22) to a solution S of $IP(\rho, P)$ if and only if $\{p,q\}$ satisfies (6.8) and*

$$det \left((p(\lambda), 0)P(\lambda)MM^*P(\lambda)^* \begin{pmatrix} p(\lambda)^* \\ 0 \end{pmatrix} + q(\lambda)q(\lambda)^* \right) \not\equiv 0 \tag{6.28}$$

(or equivalently, rank $((p(\lambda),0)P(\lambda)M, \ q(\lambda)) = n$ at almost all points $\lambda \in \Omega_+$).

Proof: Let $\{p,q\} \in \mathcal{P}(\rho, P)$ satisfies (6.8), (6.28) and let $\{p,q\}$ be defined by (5.25). According to remark before Lemma 5.11, it sufficies to prove that $det \ v(\lambda) \not\equiv 0$. Let $\lambda \in \Omega_+$ be a point in a sufficiently small neighbourhood $\mathcal{U}_{\omega_1}$ of ω_1 such that p, q, P, P^{-1} are analytic in gw,

$$rank((p(\omega), 0)P(\omega)M, q(\omega)) = n \tag{6.29}$$

and

$$(I_n, 0)P(\omega)JP(\omega)^* \begin{pmatrix} I_n \\ 0 \end{pmatrix} \leq 0. \tag{6.30}$$

Let us suppose that $det \ v(\omega) = 0$ and (5.32) holds for some nonzero vector $h \in \mathbf{C}^{1\times n}$. In view of (5.20), (5.25), (5.33), (6.12) and (6.30),

$$\begin{aligned} 0 &\geq (hu(\omega), 0)P(\omega)JP(\omega)^*(hu(\omega), 0)^* \\ &= h(p(\omega), q(\omega))P(\omega)JP(\omega)^*(p(\omega), q(\omega))^*h^* + \\ &+ \rho_\omega(\omega)h(p(\omega), q(\omega))P(\omega)DQKQ^*D^*P(\omega)^*(p(\omega), q(\omega))^*h^*. \end{aligned}$$

Since both terms in the right-hand side of the last equality are nonnegative and $QKQ^* > 0$, then $h(p(\omega), q(\omega))P(\omega)D = 0$. Substituting (6.13) into the last equality and taking into account the nondegeneracy of matrices $QG(\lambda)Q^*$, $QG^{-1}(\mu)Q^*$ and QKQ^{-1} we obtain $h(p(\omega), q(\omega))P(\omega)MG(\mu)^*Q^* = 0$, which in view of (6.20) is equivalent to

$$h(p(\omega), q(\omega))P(\omega)M\Gamma(\mu)^*Q^* = 0. \tag{6.31}$$

Using (6.1) , (6.2) we obtain from (6.8) and (6.31) $h(p(\omega), q(\omega))P(\omega)M\Gamma(\mu) = 0$. Since $\Gamma(\mu)$ is nondegenerate, the last equality implies

$$h(p(\omega), q(\omega))P(\omega)M = 0. \tag{6.32}$$

Substituting (5.20) and (6.5) into (5.25) and taking into account (6.32) we obtain

$$
\begin{aligned}
&h(u(\omega), v(\omega)) \\
&= h(p(\omega), q(\omega))P(\omega)\{I + \rho_\mu(\lambda)MG(\mu)^*Q^*(QKQ^*)^{-1}QG(\omega)M^*J\}P^{-1}(\omega) \qquad (6.33) \\
&= h(p(\omega), q(\omega)).
\end{aligned}
$$

In view of (5.33) $hq(\omega) = 0$ which both with (6.32) contradicts (6.29). Therefore, $det\ v(\omega) \neq 0$ and according to Theorem 6.1 and the remark before Lemma 5.11, $\{p,q\}$ leads under (5.22) to a solution S of the problem $IP(\rho, P)$.

Let conversely $\{p,q\}$ be a parameter in (5.22) which leads to a solution of $IP(\rho, P)$ and let $\{u,v\}$ be defined as in (5.25). According to Theorem 6.1 (Step 2) $\{p,q\}$ satisfies (6.8). In view of (5.28),

$$det\ v(\lambda) \not\equiv 0. \qquad (6.34)$$

Let $\omega \in \mathcal{U}_{\omega_1}$ be an arbitrary point of the analyticity of p, q, P and P^{-1} and let us suppose that $h((p(\omega), 0)P(\omega)M, q(\omega)) = 0$ for some nonzero vector $h \in \mathbf{C}^{1\times n}$. By (6.33), $hv(\omega) = 0$ which, in view of the arbitrariness of ω, contradicts (6.34). This shows that $\{p,q\}$ satisfies (6.28) and ends the proof . ∎

Corollary 6.6. *Let K be nonnegative, let (3.13) hold for all λ in some neighbourhood of each interpolating point ω_i and let*

$$rank(I,0)P(\lambda)M = n \qquad (6.35)$$

for almost all $\lambda \in \Omega_+$ (or, in particular, $rankM = 2n$). Then

(i) $\mathcal{P}^\circ(\rho, P) = \mathcal{P}(\rho, P)$;

(ii) the problem $IP(\rho, P)$ is solvable.

Proof: (i) follows by Lemma 5.6 from (6.28) which under assumption (6.35) is equivalent to the nondegeneracy of $\{p,q\}$. The second assertion follows from (i) since the set of pairs $\{p,q\}$ of the form (6.6) is not empty. ∎

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