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On the Weyl Matrix Balls Associated with Nondegenerate Matrix-Valued Carathéodory Functions

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The paper is aimed at a study of the limit behaviour of the normalized semi-radii of the Weyl matrix balls associated with a nondegenerate matrix-valued Carathéodory function. It turns out that the ranks of the limits of these normalized semi-radii are constant within the unit disc. This enables us a new classification of matrix-valued Carathéodory functions.

Key words: *Matrix-valued Carathéodory functions, Weyl matrix balls, limit behaviour of the semi-radii*

AMS subject classification: 30E05

0. Introduction

Inspired by SCHUR's and R. NEVANLINNA's famous papers [20] and [18], ARTEMENKO [2] and GERONIMUS [13] completely solved the interpolation problems named now after Carathéodory and Schur. Additionally, given a fixed point z_0 of the open unit disc, they observed that the set of the values at z_0 of all solutions of the interpolation problem under consideration fills a closed disc, the center and the radii of which can be explicitly expressed by the given data. In connection with boundary value problems the study of certain families of nested discs originates in WEYL's "Habilitationsschrift" [22] in which there are treated singular differential equations of second order. In the context of discrete interpolation problems, Weyl's method of nested discs was first applied by HELLINGER [14].

In their approach to matrix versions of classical interpolation problems, V.P. Potapov and his pupils worked out a natural matricial generalization of Weyl's method (see, e.g., KOVALISHINA and POTAPOV [17], KOVALISHINA [16], DUBOVOJ [5]). Hereby, discs are replaced by so-called matrix balls which had been treated in detail before by ŠMULJAN [21]. Note, that particular aspects of the matricial generalization of Weyl's method were also touched in the context of Nevanlinna-Pick interpolation by DELSARTE/GENIN/KAMP [4]. The study of matrix and operator balls in connection with

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completion and interpolation problems started with the fundamental paper [1] of ADAMJAN, AROV and KREIN on the matricial Nehari problem.

Answering to a problem posed by M.G. Krein in 1969, ORLOV [19] proved a powerful theorem which provides a matricial generalization of the known Weyl limit point - limit circle alternative. Orlov's theorem has become the basis of investigating the limit behaviour of sequences of nested matrix balls which are connected with matrix interpolation problems.

Studying the matricial versions of the interpolation problems of Carathéodory, Nevanlinna-Pick and Hamburger, KOVALISHINA [16] considered the associated families of nested matrix balls. In particular, outgoing from the limit behaviour of the right semi-radii, she introduced a classification of matrix-valued Carathéodory functions. In the framework of the matricial Schur problem, DUBOVOJ [5], [7] refined the method of Kovalishina by introducing so-called normalized left semi-radii. This led him to a more complete classification for matrix-valued Schur functions. However, there was a gap in Dubovoj's proof caused by an incorrect application of Orlov's theorem. This gap was closed in [9] (see also [7, Sections 3.11 and 5.6]).

Dubovoj's results suggest to look for a corresponding classification of matrix-valued Carathéodory functions which is based on a certain normalization of the left semi-radii in question. Our main aim in this paper is to realize these ideas. It will turn out that it is necessary to extend the known results on Weyl matrix balls associated with a given matrix-valued Carathéodory function. In particular, we have to verify explicit interrelations between the Weyl matrix balls associated with a pair $[\Omega, \check{\Omega}]$ of matrix-valued Carathéodory functions where $\check{\Omega}(z) := \Omega^*(\overline{z})$ for all $z \in \mathbb{D}$.

1. Preliminaries

Let us begin with some notations and preliminaries. Throughout this paper, let m, p and q be positive integers. We will use $\mathbb{N}_0$ and $\mathbb{C}$ to denote the set of all nonnegative integers and the set of all complex numbers, whereas $\mathbb{D}$, $\mathbb{T}$ and $\mathbb{C}_0$ stand for the open unit disc, the unit circle, and the extended complex plane:

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \quad \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}, \quad \mathbb{C}_0 := \mathbb{C} \cup \{\infty\}.$$

The symbol $0_{p\times q}$ designates the null matrix which belongs to the set $\mathbb{C}^{p\times q}$ of all $p \times q$ matrices with complex entries. The identity matrix that belongs to $\mathbb{C}^{p\times p}$ will be denoted by I_p. In cases where the size of the null matrix (respectively, the identity matrix) is clear, we will omit the indexes. If A and B are $p \times p$ Hermitian matrices, the Löwner semi-ordering $A \geqq B$ means that $A - B$ is nonnegative Hermitian. If $A - B$ is positive Hermitian, then we will write $A > B$ to indicate this fact. If $A \in \mathbb{C}^{p\times p}$, then the Hermitian matrices

$$\operatorname{Re} A := \frac{1}{2}(A + A^*) \quad \text{and} \quad \operatorname{Im} A := \frac{1}{2i}(A - A^*)$$

are called the *real part of* A and the *imaginary part of* A, respectively. If $A \in \mathbb{C}^{p\times p}$ satisfies $\operatorname{Re} A \geqq 0$, then it is readily checked that $\det(I + A) \neq 0$.

Let $\mathbb{K}_{p\times q}$ be the set of all $p \times q$ contractive matrices, i.e. the set of all $A \in \mathbb{C}^{p\times q}$ which satisfy $AA^* \leqq I$. If $M \in \mathbb{C}^{p\times q}$, $A \in \mathbb{C}^{p\times p}$ and $B \in \mathbb{C}^{q\times q}$, then the set

$$\mathfrak{K}(M; A, B) := \left\{X \in \mathbb{C}^{p\times q} : X = M + AKB,\ K \in \mathbb{K}_{p\times q}\right\}$$

is called the *(closed) matrix ball with center M, left semi-radius A and right semi-radius B*. In his nice paper [21], ŠMULJAN verified a whole collection of important properties of matrix and operator balls. In particular, he showed that the left semi-radius and the right semi-radius of any matrix ball can be chosen nonnegative Hermitian (see also [7, Corollary 1.5.3]).

A function $f : \mathbb{D} \to \mathbb{C}^{p\times q}$ is called $p \times q$ *Schur function* if f is both holomorphic and contractive in $\mathbb{D}$. We will use $\mathcal{S}_{p\times q}(\mathbb{D})$ to denote the set of all $p \times q$ Schur functions.

Lemma 1: *Let $f \in \mathcal{S}_{m\times m}(\mathbb{D})$.*

(a) There are at most m numbers $\alpha \in \mathbb{T}$ such that $\det(\alpha I + f)$ has a zero in $\mathbb{D}$.

(b) If $\alpha \in \mathbb{T}$ is such that $\det[\alpha I + f(z_0)] = 0$ for some $z_0 \in \mathbb{D}$, then $\det(\alpha I + f)$ identically vanishes in $\mathbb{D}$.

(c) If $\alpha \in \mathbb{T}$ is such that $\det[\alpha I + f(z_0)] \neq 0$ for some $z_0 \in \mathbb{D}$, then $\det(\alpha I + f)$ nowhere vanishes in $\mathbb{D}$.

Proof: Apply Lemma 2.1.6 in [7] ∎

A function $\Omega : \mathbb{D} \to \mathbb{C}^{m\times m}$ is called $m \times m$ *Carathéodory function* if Ω is holomorphic in $\mathbb{D}$ and has nonnegative Hermitian real part $\operatorname{Re}\Omega(z)$ for all $z \in \mathbb{D}$. We will write $\mathcal{C}_m(\mathbb{D})$ for the set of all $m\times m$ Carathéodory functions. There are several interesting interrelations between matricial Schur functions and matricial Carathéodory functions. For example, the following useful result holds true. For a proof we refer to [7, Propositions 2.1.2, 2.1.3 and part (f) of Lemma 1.3.12].

Proposition 1: *The following statements hold true:*

(a) Let Ω belong to $\mathcal{C}_m(\mathbb{D})$. Then $\det(I + \Omega)$ does not vanish in $\mathbb{D}$. The function $f := (I - \Omega)(I + \Omega)^{-1}$ belongs to $\mathcal{S}_{m\times m}(\mathbb{D})$ and fulfills

$$I + f = 2(I + \Omega)^{-1}. \tag{1}$$

In particular, $\det(I + f)$ has no zeros in $\mathbb{D}$. Moreover,

$$\Omega = (I - f)(I + f)^{-1} = (I + f)^{-1}(I - f) \tag{2}$$

and

$$\operatorname{rank}[\operatorname{Re}\Omega(z)] = \operatorname{rank}[I - f^*(z)f(z)] = \operatorname{rank}[I - f^*(0)f(0)]$$

for all $z \in \mathbb{D}$.

(b) Let $f \in \mathcal{S}_{m\times m}(\mathbb{D})$, and let $\eta \in \mathbb{T}$ be such that $\det[\eta I + f(z_0)] \neq 0$ for some $z_0 \in \mathbb{D}$. Then $\Omega := (\eta I - f)(\eta I + f)^{-1}$ belongs to $\mathcal{C}_m(\mathbb{D})$. Further,

$$f = \eta(I - \Omega)(I + \Omega)^{-1} = \eta(I + \Omega)^{-1}(I - \Omega).$$

For further interrelations between the function classes of Schur and Carathéodory we refer to DELSARTE/GENIN/KAMP [3] and [7, Section 2.1].

2. On the Weyl Matrix Balls Associated with a Nondegenerate Matrix-valued Schur Function

At the beginning of this section we will summarize some facts on matricial Schur sequences.

Let τ be a nonnegative integer or $\tau = \infty$. A sequence $(A_k)_{k=0}^{\tau}$ of complex $p \times q$ matrices is called *$p \times q$ Schur sequence* (respectively, *nondegenerate $p \times q$ Schur sequence*) if for every integer n with $0 \leqq n \leqq \tau$, the block Toeplitz matrix $S_n := S_n < A_0, A_1, ..., A_n >$ is contractive (resp. strictly contractive) where

$$S_n < A_0, A_1, ..., A_n > := \begin{pmatrix} A_0 & 0 & 0 & \dots & 0 \\ A_1 & A_0 & 0 & \dots & 0 \\ A_2 & A_1 & A_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_n & A_{n-1} & A_{n-2} & \dots & A_0 \end{pmatrix} . \tag{3}$$

If $(A_k)_{k=0}^{\infty}$ is a given sequence of complex $p \times q$ matrices, then the power series

$$f(z) := \sum_{k=0}^{\infty} A_k z^k , \quad z \in \mathbb{D} , \tag{4}$$

defines a $p \times q$ Schur function f if and only if $(A_k)_{k=0}^{\infty}$ is a $p \times q$ Schur sequence (see, e.g., [7, Theorem 3.1.1]). A $p \times q$ Schur function f is said to be *nondegenerate* if the sequence $(A_k)_{k=0}^{\infty}$ of its Taylor coefficients (in its Taylor series representation around the origin) is a nondegenerate $p \times q$ Schur sequence.

Now we assume that n is a nonnegative integer and that $(A_k)_{k=0}^{n}$ is a sequence of complex $p \times q$ matrices. We will use the symbol $\mathcal{S}_{p\times q}[A_0, A_1, ..., A_n]$ to denote the set of all $f \in \mathcal{S}_{p\times q}(\mathbb{D})$ for which $(A_k)_{k=0}^{n}$ is exactly the sequence of the first $n+1$ Taylor coefficients in the Taylor series representation of f around the origin. It is a well-known fact that the set $\mathcal{S}_{p\times q}[A_0, A_1, ..., A_n]$ is nonempty if and only if $(A_k)_{k=0}^{n}$ is a $p \times q$ Schur sequence (see, e.g., [7, Theorem 3.5.2]). If $(A_k)_{k=0}^{n}$ is a nondegenerate $p\times q$ Schur sequence, then $\mathcal{S}_{p\times q}[A_0, A_1, ..., A_n]$ can be parametrized by various linear fractional transformations (see, e.g., [7, Theorems 3.9.1, 3.10.1 and 5.4.3]). Moreover, in this case one can describe the set

$$\{f(z) \,:\, f \in \mathcal{S}_{p\times q}[A_0, A_1, ..., A_n]\} \tag{5}$$

for all $z \in \mathbb{D}$. To formulate this result we need some preparations.

In the following, we will work with the matrix

$$V_{nm} := (\delta_{j,n-k} I_m)_{j,k=0}^{n} , \tag{6}$$

where

$$\delta_{jk} := \begin{cases} 1 & , \ j = k \\ 0 & , \ j \neq k \end{cases} \tag{7}$$

is the Kronecker symbol. Obviously, $V_{nm}^* = V_{nm}$ and $V_{nm}^2 = I$.

Remark 1: Let $\tau \in \mathbb{N}_0$ or $\tau = \infty$, and let $(A_k)_{k=0}^{\tau}$ be a sequence of $p \times q$ complex matrices. If n is an integer with $0 \leqq n \leqq \tau$, then the matrices $S_n := S_n < A_0, A_1, ..., A_n >$ and $S_{n,*} := S_n < A_0^*, A_1^*, ..., A_n^* >$ (see (3)) fulfill the identities

$$S_{n,*} = V_{nq} S_n^* V_{np} , \; I - S_{n,*} S_{n,*}^* = V_{nq} \left(I - S_n^* S_n\right) V_{nq}$$

and

$$I - S_{n,*}^* S_{n,*} = V_{np} (I - S_n S_n^*) V_{np} .$$

Thus, $(A_k)_{k=0}^{\tau}$ is a $p \times q$ Schur sequence (respectively, nondegenerate $p \times q$ Schur sequence) if and only if $(A_k^*)_{k=0}^{\tau}$ is a $q \times p$ Schur sequence (respectively, nondegenerate $q \times p$ Schur sequence).

Suppose that $(A_k)_{k=0}^n$ is a nondegenerate $p \times q$ Schur sequence. Let $r_n : \mathbb{C} \to \mathbb{C}^{q \times q}$, $s_n : \mathbb{C} \to \mathbb{C}^{p \times q}$ and $t_n : \mathbb{C} \to \mathbb{C}^{p \times p}$ be defined by

$$r_n(z) := I_q + (1 - |z|^2)\, e_{nq}(z)\, S_n^* \left(I_{(n+1)p} - S_n S_n^*\right)^{-1} S_n e_{nq}^*(z), \tag{8}$$

$$s_n(z) := -(1 - |z|^2)\, e_{np}(z) \left(I_{(n+1)p} - S_n S_n^*\right)^{-1} S_n e_{nq}^*(z) \tag{9}$$

and

$$t_n(z) := -I_p + (1 - |z|^2)\, e_{np}(z) \left(I_{(n+1)p} - S_n S_n^*\right)^{-1} e_{np}^*(z) \tag{10}$$

where

$$e_{nm}(z) := \left(I_m,\ zI_m,\ z^2 I_m,\ \ldots,\ z^n I_m\right) . \tag{11}$$

One can verify that $r_n(z) \geqq I$ and $s_n(z)\,[r_n(z)]^{-1}\,[s_n(z)]^* \geqq t_n(z)$ hold true for all $z \in \mathbb{D}$ (see [7, part (c) of Theorem 5.5.1]). The functions $\mathcal{M}_n : \mathbb{D} \to \mathbb{C}^{p \times q}$, $\mathcal{L}_n : \mathbb{D} \to \mathbb{C}^{p \times p}$ and $\mathcal{R}_n : \mathbb{D} \to \mathbb{C}^{q \times q}$ given by

$$\mathcal{M}_n(z) := -s_n(z)\, (r_n(z))^{-1} , \tag{12}$$

$$\mathcal{L}_n(z) := s_n(z)\, (r_n(z))^{-1}\, s_n^*(z) - t_n(z) \tag{13}$$

and

$$\mathcal{R}_n(z) := (r_n(z))^{-1} \tag{14}$$

are called the *Weyl-Schur center function*, the *canonical left Weyl-Schur semi-radius function* and the *canonical right Weyl-Schur semi-radius function*, respectively, *associated with the nondegenerate $p \times q$ Schur sequence $(A_k)_{k=0}^n$*.

Lemma 2 (see [7, Lemma 5.6.2]): *Let $n \in \mathbb{N}_0$, and let $(A_k)_{k=0}^n$ be a nondegenerate $p \times q$ Schur sequence. Then $(A_k^*)_{k=0}^n$ is a nondegenerate $q \times p$ Schur sequence, and the canonical left Weyl semi-radius function $\mathcal{L}_n$ associated with $(A_k)_{k=0}^n$ and the canonical right Weyl semi-radius function $\mathcal{R}_{n,*}$ associated with $(A_k^*)_{k=0}^n$ are linked by the formula*

$$\mathcal{L}_n(z) = |z|^{2(n+1)}\, \mathcal{R}_{n,*}(\overline{z}) , \quad z \in \mathbb{D} . \tag{15}$$

In particular, $\mathcal{L}_n(0) = 0_{p \times q}$.

In view of formula (15) and the following theorem, the function $\mathcal{L}_n^\# : \mathbb{D} \to \mathbb{C}^{p \times p}$ defined by $\mathcal{L}_n^\#(z) := \mathcal{R}_{n,*}(\overline{z})$, where $\mathcal{R}_{n,*}$ is given in Lemma 2, is said to be the *canonical normalized left Weyl-Schur semi-radius function associated with $(A_k)_{k=0}^n$*.

Lemma 3: *Let $n \in \mathbb{N}_0$, and let $(A_k)_{k=0}^n$ be a nondegenerate $p \times q$ Schur sequence. Then the Weyl-Schur center function $\mathcal{M}_n$, the canonical normalized left Weyl-Schur semi-radius function $\mathcal{L}_n^\#$ and the canonical right Weyl-Schur semi-radius function $\mathcal{R}_n$ are continuous in $\mathbb{D}$. Moreover, for each $z \in \mathbb{D}$,*

$$\mathcal{M}_n(z)\, \mathcal{M}_n^*(z) < I , \tag{16}$$

$$0 < \mathcal{L}_n^\#(z) \leqq I , \tag{17}$$

$$0 < \mathcal{R}_n(z) \leqq I\,, \tag{18}$$

and

$$\det \mathcal{L}_n^\#(z) = \det \mathcal{R}_n(z)\,. \tag{19}$$

Proof: By definition, $\mathcal{M}_n$, $\mathcal{L}_n^\#$ and $\mathcal{R}_n$ are continuous. The inequalities (18) were verified in part (b) of Proposition 5.5.1 in [7], whereas (17) follows from (18) and the definition of $\mathcal{L}_n^\#$. Furthermore, (16) is a consequence of part (b) of Proposition 5.6.3 in [7] and (17). Finally, the equality (19) is clear from Lemma 5.6.3 in [7] ∎

In the following theorem, we will give an explicit expression for the so-called Weyl matrix ball associated with a nondegenerate $p \times q$ Schur sequence.

Theorem 1: *Let $n \in \mathbb{N}_0$, and let $(A_k)_{k=0}^n$ be a nondegenerate $p \times q$ Schur sequence. Further, let $\mathcal{M}_n$, $\mathcal{L}_n^\#$ and $\mathcal{R}_n$ be the Weyl-Schur center function, the canonical normalized left Weyl-Schur semi-radius function and the canonical right Weyl-Schur semi-radius function, respectively, associated with $(A_k)_{k=0}^n$. For each $z \in \mathbb{D}$, the set given in (5) coincides with the matrix ball $\mathfrak{K}\left(\mathcal{M}_n(z); |z|^{n+1}\sqrt{\mathcal{L}_n^\#(z)}, \sqrt{\mathcal{R}_n(z)}\right)$. In particular $\mathcal{M}_n(0) = A_0$.*

Proof: For $z \in \mathbb{D}\setminus\{0\}$, Theorem 1 is proved in [7, part (f) of Theorem 5.5.1]. To verify the assertion in the case $z = 0$, it remains to check that $\mathcal{M}_n(0) = A_0$, i.e. $-s_n(0) = A_0 r_n(0)$. We have $I + S_n^*(I - S_nS_n^*)^{-1}S_n = (I - S_n^*S_n)^{-1}$ and $A_0 e_{nq}(0) = e_{np}(0)S_n$. Hence,

$$\begin{aligned} A_0 r_n(0) &= A_0\, e_{nq}(0)\left[I + S_n^*(I - S_nS_n^*)^{-1}S_n\right] e_{nq}(0) \\ &= e_{np}(0)\, S_n\,(I - S_n^*S_n)^{-1}\, e_{nq}^*(0) \\ &= e_{np}(0)\,(I - S_nS_n^*)^{-1}\, S_n\, e_{nq}(0) = -s_n(0) \end{aligned}$$ ∎

Theorem 1 is in principle a consequence of general results on the image of the set of all contractive $p \times q$ matrices under a linear fractional transformation generated by a j_{pq}-contractive matrix (see DELSARTE/GENIN/KAMP [4, Appendix], DYM [8, Theorem 3.6] or [7, Theorem 1.6.3]). Observe that the authors [7, Theorem 3.9.2], [12, Part III, Theorem 16] also described the parameters of the Weyl matrix ball occurring in Theorem 1 with the aid of other quantities. Furthermore, note that the matrices $\mathcal{L}_n^\#(0)$ and $\mathcal{R}_n(0)$ occur as left and right semi-radii in the matrix ball description of the solution set of the so-called *coefficient problem* associated with the $p \times q$ Schur sequence $(A_k)_{k=0}^n$ (see, e.g., [7, Section 3.5] and [12, Part IV, Corollary 11]).

The next theorem which is taken from [7, Theorem 5.6.1] describes the limit behaviour of the sequences $(\mathcal{M}_n(z))_{n=0}^\infty$, $(\mathcal{L}_n^\#(z))_{n=0}^\infty$ and $(\mathcal{R}_n(z))_{n=0}^\infty$ associated with the sequence of Taylor coefficients of a given nondegenerate $p \times q$ Schur function.

Theorem 2: *Let f be a nondegenerate $p \times q$ Schur function, and let*

$$f(z) = \sum_{k=0}^\infty A_k z^k\,, \quad z \in \mathbb{D}\,, \tag{20}$$

be the Taylor series representation of f. For $n \in \mathbb{N}_0$*, let* $\mathcal{M}_n$*,* $\mathcal{L}_n^{\#}$ *and* $\mathcal{R}_n$ *be the Weyl-Schur center function, the canonical normalized left Weyl-Schur semi-radius function and the canonical right Weyl-Schur semi-radius function, respectively, associated with* $(A_k)_{k=0}^n$*. Then:*

(a) *For each* $z \in \mathbb{D}$, $\lim_{n\to\infty} \mathcal{M}_n(z) = f(z)$.

(b) *For each* $z \in \mathbb{D}$*, the sequences* $(\mathcal{L}_n^{\#}(z))_{n=0}^{\infty}$ *and* $(\mathcal{R}_n(z))_{n=0}^{\infty}$ *are monotonously nonincreasing and convergent. The corresponding limits* $\mathcal{L}^{\#}(z)$ *and* $\mathcal{R}(z)$ *are nonnegative Hermitian and satisfy* $\det \mathcal{L}^{\#}(z) = \det \mathcal{R}(z)$ *for all* $z \in \mathbb{D}$.

(c) *The functions* $\operatorname{rank} \mathcal{L}^{\#}$ *and* $\operatorname{rank} \mathcal{R}$ *are constant in* $\mathbb{D}$*. If* $\delta^{\#}$ *and* δ *are the corresponding values of these ranks, then* $\delta^{\#} = p$ *if and only if* $\delta = q$.

Theorem 2 leads us to the following notions.

Definition 1: Let f be a nondegenerate $p \times q$ Schur function. The functions $\mathcal{L}^{\#} : \mathbb{D} \to \mathbb{C}^{p\times p}$ and $\mathcal{R} : \mathbb{D} \to \mathbb{C}^{q\times q}$ given in Theorem 2 are called the *canonical normalized left Weyl-Schur limit semi-radius function* and the *canonical right Weyl-Schur limit semi-radius function*, respectively, *associated with f*.

Furthermore, part (c) of Theorem 2 suggests the following classification of nondegenerate $p \times q$ Schur functions.

Definition 2: Let f be a nondegenerate $p \times q$ Schur function, and let $\mathcal{L}^{\#}$ and $\mathcal{R}$ be the canonical normalized left Weyl-Schur limit semi-radius function and the canonical right Weyl-Schur limit semi-radius function, respectively, associated with f. Then f is said to have *the Schur type* $[\delta^{\#}, \delta]$ if $\operatorname{rank} \mathcal{L}^{\#}(0) = \delta^{\#}$ and $\operatorname{rank} \mathcal{R}(0) = \delta$.

Observe that, in view of Theorem 2, the classification of matrix-valued Schur functions given in Definition 2 can also be described by $\operatorname{rank} \mathcal{L}^{\#}(z_1)$ and $\operatorname{rank} \mathcal{R}(z_2)$ where z_1 and z_2 are arbitrary points which belong to $\mathbb{D}$.

Remark 2: Using a method developed by KOVALISHINA and POTAPOV [17], DUBOVOJ [3], [7, Theorem 5.6.3] showed that, for every choice of $j \in \{0, 1, ..., p-1\}$ and $k \in \{0, 1, ..., q-1\}$, there is a nondegenerate $p \times q$ Schur function of Schur type $[j, k]$, whereas part (c) of Theorem 2 yields that all the Schur types $[j, q]$, $j \in \{0, 1, ..., p-1\}$, and $[p, k]$, $k \in \{0, 1, ..., q-1\}$, are impossible. The trivial example of the constant function defined on $\mathbb{D}$ with value $0_{p\times q}$ yields $\mathcal{L}_n^{\#}(z) = I_p$ and $\mathcal{R}(z) = I_q$ for all $z \in \mathbb{D}$ and all $n \in \mathbb{N}_0$. Thus, we see that there exists a nondegenerate $p \times q$ Schur function of Schur type (p, q).

Lemma 4: *Let* $n \in \mathbb{N}_0$*, and let* $(A_k)_{k=0}^n$ *be a nondegenerate* $p \times q$ *Schur sequence. Let* $\mathcal{M}_n$*,* $\mathcal{L}_n^{\#}$ *and* $\mathcal{R}_n$ *be the Weyl-Schur center function, the canonical normalized left Weyl-Schur semi-radius function and the canonical right Weyl-Schur semi-radius function, respectively, associated with* $(A_k)_{k=0}^n$*. Then the Weyl-Schur center function* $\mathcal{M}_{n,*}$*, the canonical normalized left Weyl-Schur semi-radius function* $\mathcal{L}_{n,*}^{\#}$ *and the canonical right Weyl-Schur semi-radius function* $\mathcal{R}_{n,*}$ *associated with (the nondegenerate* $q \times p$ *Schur sequence)* $(A_k^*)_{k=0}^n$ *admit the representations*

$$\mathcal{M}_{n,*}(z) = \mathcal{M}_n^*(\overline{z}) \quad , \mathcal{L}_{n,*}^{\#}(z) = \mathcal{R}_n(\overline{z}) \quad \text{and} \quad \mathcal{R}_{n,*}(z) = \mathcal{L}_n^{\#}(\overline{z}) \qquad (21)$$

for all $z \in \mathbb{D}$.

Proof: The last identity in (21) is clear by definition. In view of $A_k = (A_k^*)^*$, $k \in \{0,1,...,n\}$, then the second one follows. It remains to check the first equation in (21). From Remark 1 we see

$$(I - S_{n,*}\, S_{n,*}^*)^{-1} = V_{nq}\,(I - S_n^*\, S_n)^{-1}\, V_{nq}$$

and

$$(I - S_{n,*}^*\, S_{n,*})^{-1} = V_{np}\,(I - S_n\, S_n^*)^{-1}\, V_{np}\,.$$

Furthermore, we have

$$z^n\, e_{np}\,(1/z) = e_{np}\,(z)\, V_{np}$$

for all $z \in \mathbb{C}\backslash\{0\}$. Using Lemma 5.6.1 and part (e) of Theorem 5.5.1 in [7], and the identity $S_n\,(I - S_n^*\, S_n)^{-1} = (I - S_n\, S_n^*)^{-1} S_n$, we get

$$\begin{aligned}
\mathcal{M}_{n,*}^*(\overline{z}) &= \Big(\left[(1-|\,\overline{z}\,|^2)e_{nq}(\overline{z})\,(I - S_{n,*}\, S_{n,*}^*)^{-1}\, S_{n,*}\, e_{np}^*(\overline{z})\right] \\
&\quad \times \left[|\,\overline{z}\,|^{2(n+1)}\; I + (1-|\,\overline{z}\,|^2) e_{np}(\overline{z})(I - S_{n,*}^*\, S_{n,*})^{-1}\, e_{np}^*(\overline{z})\right]^{-1} \Big)^* \\
&= \left[|\,z\,|^{2(n+1)}\; I + (1-|\,z\,|^2)\,|\,z\,|^{2n}\, e_{np}(1/\overline{z})(I - S_n S_n^*)^{-1} e_{np}^*(1/\overline{z})\right]^{-1} \\
&\quad \times \left[(1-|\,z\,|^2)\,|\,z\,|^{2n}\, e_{np}(1/\overline{z}) S_n\,(I - S_n^* S_n)^{-1} e_{nq}^*(1/\overline{z})\right] \\
&= \left[I + \frac{1-|\,z\,|^2}{|\,z\,|^2}\, e_{np}(1/\overline{z})\,(I - S_n S_n^*)^{-1} e_{np}^*(1/\overline{z})\right]^{-1} \\
&\quad \times \left[\frac{1-|\,z\,|^2}{|\,z\,|^2}\, e_{nq}(1/\overline{z})\, S_n^*\,(I - S_n S_n^*)^{-1} e_{np}^*(1/\overline{z})\right]^* = \mathcal{M}_n(z)
\end{aligned}$$

for all $z \in \mathbb{D}\backslash\{0\}$. Hence, the first identity in (21) is proved for each $z \in \mathbb{D}\backslash\{0\}$. In view of Lemma 3, the matrix-valued functions $\mathcal{M}_n$ and $\mathcal{M}_{n,*}$ are continuous in $\mathbb{D}$. Hence, the first equation in (21) holds true for $z = 0$ as well ■

Lemma 5: *Let f be a nondegenerate $p \times q$ Schur function, let $\mathcal{L}^\#$ and $\mathcal{R}$ be the canonical normalized left and the canonical right Weyl-Schur limit semi-radius functions, respectively, associated with f. Then $\check{f} : \mathbb{D} \to \mathbb{C}^{q\times p}$ given by*

$$\check{f}(z) := f^*(\overline{z})\,, \quad z \in \mathbb{D}\,, \tag{22}$$

is a nondegenerate $q \times p$ Schur function. If $\mathcal{L}_*^\#$ and $\mathcal{R}_*^\#$ are the canonical normalized left and the canonical right Weyl-Schur limit semi-radius functions, respectively, associated with $\check{f}$, then

$$\mathcal{L}_*^\#(z) = \mathcal{R}(\overline{z}) \quad \text{and} \quad \mathcal{R}_*(z) = \mathcal{L}_*^\#(\overline{z})$$

for all $z \in \mathbb{D}$. If f has the Schur type $(\delta^\#, \delta)$, then $\check{f}$ has the Schur type $(\delta, \delta^\#)$.

Proof: If (20) is the Taylor series representation of f, then $\check{f}(z) = \sum_{k=0}^{\infty} A_k^*\, z^k$, $z \in \mathbb{D}$, is the Taylor series representation of $\check{f}$. The Taylor series characterization of matricial Schur functions and Remark 1 show that $\check{f}$ is a nondegenerate $q \times p$ Schur function. Hence, Lemma 4 yields the assertion ■

3. Interrelations between the Weyl Matrix Balls Connected with Nondegenerate Carathéodory Sequences and Their Cayley-associated Schur Sequences

Now we will turn our attention to a more detailed study of matrix-valued Carathéodory functions. First we recall the notion of matricial Carathéodory sequences.

Let τ be a nonnegative integer or $\tau = \infty$. A sequence $(\Gamma_k)_{k=0}^{\tau}$ of complex $m \times m$ matrices is called $m \times m$ *Carathéodory sequence* (resp. *nondegenerate* $m \times m$ *Carathéodory sequence*) if for every integer n with $0 \leqq n \leqq \tau$, the block Toeplitz matrix

$$T_n := \operatorname{Re}[S_n < \Gamma_0, \Gamma_1, ..., \Gamma_n >] \tag{23}$$

(see (3)) is nonnegative Hermitian (resp. positive Hermitian). If $(\Gamma_k)_{k=0}^{\infty}$ is a given sequence of complex $m \times m$ matrices, then the power series

$$\Omega(z) := \sum_{k=0}^{\infty} \Gamma_k z^k \,, \quad z \in \mathbb{D} \,, \tag{24}$$

defines an $m \times m$ Carathéodory function if and only if $(\Gamma_k)_{k=0}^{\infty}$ is an $m \times m$ Carathéodory sequence (see, e.g., [7, Theorem 2.2.1 and 2.2.2]). An $m \times m$ Carathéodory function Ω is said to be *nondegenerate* if the sequence $(\Gamma_k)_{k=0}^{\infty}$ of its Taylor coefficients (in the Taylor series representation of Ω around the origin) is a nondegenerate $m \times m$ Carathéodory sequence.

Now we assume that n is a nonnegative integer and that $(\Gamma_k)_{k=0}^{n}$ is a sequence of complex $m \times m$ matrices. We will use the notation $\mathcal{C}_m[\Gamma_0, \Gamma_1, ..., \Gamma_n]$ to denote the set of all $\Omega \in \mathcal{C}_m(\mathbb{D})$ for which $(\Gamma_k)_{k=0}^{n}$ is exactly the sequence of the first $n+1$ Taylor coefficients in the Taylor series representation of Ω around the origin. The set $\mathcal{C}_m[\Gamma_0, \Gamma_1, ..., \Gamma_n]$ is nonempty if and only if $(\Gamma_k)_{k=0}^{n}$ is an $m \times m$ Carathéodory sequence (see, e.g., [11, Part I, Section 4]). If $(\Gamma_k)_{k=0}^{n}$ is a nondegenerate Carathéodory sequence, then $\mathcal{C}_m[\Gamma_0, \Gamma_1, ..., \Gamma_n]$ can be described by various linear fractional transformations (see, e.g., [11, Part V, Theorem 28]). Similarly as in the case of a nondegenerate matrix-valued Schur function, the set

$$\{\Omega(z) \,:\, \Omega \in \mathcal{C}_m\,[\Gamma_0, \Gamma_1, ..., \Gamma_n]\} \tag{25}$$

will turn out to be a matrix ball for all $z \in \mathbb{D}$. In order to give explicit expressions for the parameters of this matrix ball, we need some preparations.

Suppose that $(\Gamma_k)_{k=0}^{n}$ is a nondegenerate $m \times m$ Carathéodory sequence. Then the matrices Γ_0, $\mathfrak{S}_n := S_n < \Gamma_0, \Gamma_1, ..., \Gamma_n >$ and $T_n := \operatorname{Re} \mathfrak{S}_n$ are nonsingular. Set

$$z_n := (\Gamma_n, \Gamma_{n-1}, ..., \Gamma_1)\,, \quad y_n := (\Gamma_1^*, \Gamma_2^*, ..., \Gamma_n^*)^*\,,$$

$$l_n := \begin{cases} \operatorname{Re}\Gamma_0 & , \ n=0 \\ \operatorname{Re}\Gamma_0 - \frac{1}{4} z_n T_{n-1}^{-1} z_n^* & , \ n>0 \end{cases} ,$$

$$r_n := \begin{cases} \operatorname{Re}\Gamma_0 & , \ n=0 \\ \operatorname{Re}\Gamma_0 - \frac{1}{4} y_n^* T_{n-1}^{-1} y_n & , \ n>0 \end{cases} ,$$

and

$$\mathfrak{T}_n := (\mathfrak{S}_n^{-1})^* T_n \mathfrak{S}_n^{-1}\,.$$

Lemma 28 in [11, Part V] shows that $l_n \geqq 0$. Furthermore, we define the matrix polynomials η_n, ζ_n, η'_n and ζ'_n by

$$\eta_n(z) := e_{nm}(z)\, T_n^{-1}\, e^*_{nm}(0)\,, \quad \zeta_n(z) := \varepsilon^*_{nm}(0)\, T_n^{-1}\, \varepsilon_{nm}(z)\,, \tag{26}$$

$$\eta'_n(z) := e_{nm}(z)\, \mathfrak{T}_n^{-1}\, e^*_{nm}(0)\,, \quad \zeta'_n(z) := \varepsilon^*_{nm}(0)\, \mathfrak{T}_n^{-1}\, \varepsilon_{nm}(z)\,, \tag{27}$$

$z \in \mathbb{C}$, where $e_{nm} : \mathbb{C} \to \mathbb{C}^{m\times(n+1)m}$ and $\varepsilon_{nm} : \mathbb{C} \to \mathbb{C}^{(n+1)m\times m}$ are given by (11) and

$$\varepsilon_{nm}(z) := (\overline{z}^n I_m,\ \overline{z}^{n-1} I_m, \ldots,\ \overline{z} I_m, I_m)^*\,; \quad z \in \mathbb{C}\,. \tag{28}$$

If $x_0, x_1, \ldots, x_n$ are complex $p \times q$ matrices, and if the $p \times q$ matrix polynomial X is given by

$$X(z) := \sum_{k=0}^{n} x_k\, z^k\,, \quad z \in \mathbb{C}\,,$$

then the reciprocal matrix polynomial $\tilde{X}$ of X with respect to the unit circle $\mathbf{T}$ and the formal degree n is defined by

$$\tilde{X}(z) := \sum_{k=0}^{n} x^*_{n-k}\, z^k\,, \quad z \in \mathbb{C}\,.$$

In this sense, let $\tilde{\eta}_n$ (respectively, $\tilde{\eta}'_n, \tilde{\zeta}_n, \tilde{\zeta}'_n$) be the reciprocal matrix polynomial of η_n (respectively, $\eta'_n, \zeta_n, \zeta'_n$) with respect to $\mathbf{T}$ and the formal degree n.

One can check that

$$P(z) := \zeta^*_n(z)\, l_n\, \zeta_n(z) - |\, z\,|^2\, (\tilde{\eta}_n(z))^*\, r_n\, \tilde{\eta}_n(z) \tag{29}$$

and

$$Q(z) := \eta_n(z)\, r_n\, \eta^*_n(z) - |\, z\,|^2\, \tilde{\zeta}_n(z)\, l_n \left(\tilde{\zeta}_n(z)\right)^* \tag{30}$$

are positive Hermitian for all $z \in \mathbb{D}$ (see [11, Part V, Theorem 29]). The functions $\mathfrak{M}_n : \mathbb{D} \to \mathbb{C}^{m\times m}$, $\mathfrak{L}^\#_n : \mathbb{D} \to \mathbb{C}^{m\times m}$ and $\mathfrak{R}_n : \mathbb{D} \to \mathbb{C}^{m\times m}$ given by

$$\begin{aligned} \mathfrak{M}_n(z) &:= \left[\eta'_n(z)\, (\Gamma_0^{-1})^*\, r_n\, \eta^*_n(z) + |\, z\,|^2\, \tilde{\zeta}'_n(z) \Gamma_0^{-1}\, l_n \left(\tilde{\zeta}_n(z)\right)^*\right] [Q(z)]^{-1}\,, \\ \mathfrak{L}^\#_n(z) &:= [P(z)]^{-1} \quad \text{and} \quad \mathfrak{R}_n(z) := [Q(z)]^{-1} \end{aligned}$$

are called the *Weyl-Carathéodory center function*, the *canonical normalized left Weyl-Carathéodory semi-radius function* and the *canonical right Weyl-Carathéodory semi-radius function*, respectively, *associated with the nondegenerate* $m \times m$ *Carathéodory sequence* $(\Gamma_k)_{k=0}^n$.

Lemma 6: *Let* $n \in \mathbb{N}_0$, *and let* $(\Gamma_k)_{k=0}^n$ *be a nondegenerate* $m \times m$ *Carathéodory sequence. Then the Weyl-Carathéodory center function* $\mathfrak{M}_n$, *the canonical normalized left Weyl-Carathéodory semi-radius function* $\mathfrak{L}^\#_n$ *and the canonical right Weyl-Carathéodory semi-radius function* $\mathfrak{R}_n$ *associated with* $(\Gamma_k)_{k=0}^n$ *are continuous. Moreover, for each* $z \in \mathbb{D}$, $\det \mathfrak{L}^\#_n(z) = \det \mathfrak{R}_n(z)$,

$$\operatorname{Re} \mathfrak{M}_n(z) \geqq 0\,, \quad \mathfrak{L}^\#_n(z) > 0\,, \quad \textit{and} \quad \mathfrak{R}_n(z) > 0\,. \tag{31}$$

Proof: From their definition, the continuity of $\mathfrak{M}_n$, $\mathfrak{L}_n^\#$ and $\mathfrak{R}_n$ is clear. Corollary 7 in [11, Part IV] yields the determinant identity. Theorem 1.6.3 in [7], part (d) of Proposition 12 in [11, Part V] and the proof of Theorem 29 in [11, Part V] show that both $\mathfrak{L}_n^\#(z)$ and $\mathfrak{R}_n(z)$ are positive Hermitian for all $z \in \mathbb{D}$. The combination of Theorems 28 and 29 in [11, Part V] provides finally the first inequality in (31) ∎

The following theorem, which is taken from [11, Part V, Theorem 29], gives now the announced explicit matrix ball description of the set given in (25).

Theorem 3: *Let $n \in \mathbb{N}_0$, and let $(\Gamma_k)_{k=0}^n$ be a nondegenerate $m \times m$ Carathéodory sequence. Further, let $\mathfrak{M}_n$, $\mathfrak{L}_n^\#$ and $\mathfrak{R}_n$ be the Weyl-Carathéodory center function, the canonical normalized left Weyl-Carathéodory semi-radius function and the canonical right Weyl-Carathéodory semi-radius function associated with $(\Gamma_k)_{k=0}^n$. For each $z \in \mathbb{D}$, the set given in (25) coincides with the matrix ball*

$$\mathfrak{K}\left(\mathfrak{M}_n(z)\,;\, |\, z\,|^{n+1}\sqrt{2\,\mathfrak{L}_n^\#(z)}\,,\,\sqrt{2\,\mathfrak{R}_n(z)}\right)\,. \tag{32}$$

In particular, $\mathfrak{M}_n(0) = \Gamma_0$.

Note that the function $\mathfrak{L}_n : \mathbb{D} \to \mathbb{C}^{m\times m}$ defined by $\mathfrak{L}_n(z) := |\, z\,|^{2(n+1)}\, \mathfrak{L}_n^\#(z)$, $z \in \mathbb{D}$, is called the *canonical left Weyl-Carathéodory semi-radius function associated with the nondegenerate $m \times m$ Carathéodory sequence $(\Gamma_k)_{k=0}^n$*, whereas the matrix ball given in (32) is called the *Weyl matrix ball associated with $(\Gamma_k)_{k=0}^n$ at the point $z \in \mathbb{D}$*. Observe that KOVALISHINA [16, Formulas (52) and (53)] and the authors [11, Part IV, Theorem 27] expressed the parameters of the Weyl matrix ball (32) in other terms. Furthermore, note that the matrices $\mathfrak{L}_n^\#(0)$ and $\mathfrak{R}_n(0)$ occur as left and right radius of the matrix ball which describes the solution set of the so-called coefficient problem associated with the $m \times m$ Carathéodory sequence $(\Gamma_k)_{k=0}^n$ (see, e.g., [7, Section 3.4] and [11, Part V]).

Remark 3: Let $\tau \in \mathbb{N}_0$ or $\tau = \infty$, and let $(\Gamma_k)_{k=0}^\tau$ be a sequence of $m \times m$ complex matrices. If n is an integer with $0 \leqq n \leqq \tau$, and if V_{nm} is given by (6) and (7), then $\mathfrak{S}_{n,*} := S_n < \Gamma_0^*, \Gamma_1^*, ..., \Gamma_n^* >$ and $T_{n,*} := \operatorname{Re} \mathfrak{S}_{n,*}$ satisfy

$$\mathfrak{S}_{n,*} = V_{nm}\, \mathfrak{S}_n^*\, V_{nm} \tag{33}$$

and

$$T_{n,*} = V_{nm}\, T_n\, V_{nm}\,. \tag{34}$$

Thus, $(\Gamma_k)_{k=0}^\tau$ is an $m \times m$ Carathéodory sequence (respectively, a nondegenerate $m \times m$ Carathéodory sequence) if and only if $(\Gamma_k^*)_{k=0}^\tau$ is an $m \times m$ Carathéodory sequence (respectively, a nondegenerate $m \times m$ Carathéodory sequence).

Lemma 7: *Let $n \in \mathbb{N}_0$, and let $(\Gamma_k)_{k=0}^n$ be a nondegenerate $m \times m$ Carathéodory sequence. Let $\mathfrak{M}_n$, $\mathfrak{L}_n^\#$ and $\mathfrak{R}_n$ be the Weyl-Carathéodory center function, the canonical normalized left and the canonical right Weyl-Carathéodory Semi-radius functions, respectively, associated with $(\Gamma_k)_{k=0}^n$. Then the Weyl-Carathéodory center function $\mathfrak{M}_{n,*}$, the canonical normalized left and the canonical right Weyl-Carathéodory semi-radius functions $\mathfrak{L}_{n,*}^\#$ and $\mathfrak{R}_{n,*}$, respectively, associated with (the nondegenerate $m \times m$ Carathéodory sequence) $(\Gamma_k^*)_{k=0}^n$ admit the representations*

$$\mathfrak{M}_{n,*}(z) = \mathfrak{M}_n^*(\overline{z})\,,\; \mathfrak{L}_{n,*}^\#(z) = \mathfrak{R}_n(\overline{z}) \quad \textit{and} \qquad \mathfrak{R}_{n,*}(z) = \mathfrak{L}_n^\#(\overline{z})$$

for all $z \in \mathbb{D}$.

Proof: Let $l_{0,*} := \operatorname{Re}\Gamma_0^*$ and $r_{0,*} := \operatorname{Re}\Gamma_0^*$. If $n \in \mathbb{N}$, then let $z_{n,*} := (\Gamma_n^*, \Gamma_{n-1}^*, ..., \Gamma_1^*)$, $y_{n,*} := (\Gamma_1, \Gamma_2, ..., \Gamma_n)^*$, $l_{n,*} := \operatorname{Re}\Gamma_0^* - \frac{1}{4} z_{n,*} T_{n-1,*}^{-1} z_{n,*}^*$, $r_{n,*} := \operatorname{Re}\Gamma_0^* - \frac{1}{4} y_{n,*}^* T_{n-1}^{-1} y_{n,*}$. Furthermore, let $\mathfrak{T}_{n,*} := (\mathfrak{S}_{n,*}^{-1})^* T_{n,*} \mathfrak{S}_{n,*}^{-1}$, and let the matrix polynomials $\eta_{n,*}, \zeta_{n,*}, \eta_{n,*}'$ and $\zeta_{n,*}'$ be defined by

$$\eta_{n,*}(z) := e_{nm}(z)\, T_{n,*}^{-1}\, e_{nm}^*(0)\ ,\ \zeta_{n,*}(z) := \varepsilon_{nm}^*(0)\, T_{n,*}^{-1}\, \varepsilon_{nm}(z)\ ,$$

$$\eta_{n,*}'(z) := e_{nm}(z)\, \mathfrak{T}_{n,*}^{-1}\, e_{nm}^*(0)\ ,\ \zeta_{n,*}(z) := \varepsilon_{nm}^*(0)\, \mathfrak{T}_{n,*}^{-1}\, \varepsilon_{nm}(z)\ .$$

Obviously, $z_{n,*} = y_n^* V_{n-1,m}$ and $y_{n,*} = V_{n-1,m} z_n^*$. In view of (34) and $\operatorname{Re}\Gamma_0^* = \operatorname{Re}\Gamma_0$, then $l_{n,*} = r_{n,*}$ and $r_{n,*} = l_{n,*}$. Using the identities (34) and $e_{nm}(z) = \varepsilon_{nm}^*(\overline{z}) V_{nm}$, $z \in \mathbb{C}$, we obtain $\eta_{n,*}(z) = \zeta_n^*(\overline{z})$, $\zeta_{n,*}(z) = \eta_n^*(\overline{z})$ for all $z \in \mathbb{C}$. From (33) and (34) it follows $\mathfrak{T}_{n,*} = V_{nm} \mathfrak{T}_n V_{nm}$, and hence $\eta_{n,*}'(z) = [\zeta_n'(\overline{z})]^*$, $\zeta_{n,*}'(z) = [\eta_n'(\overline{z})]^*$ for all $z \in \mathbb{C}$. Thus, we have

$$\begin{aligned} \mathfrak{L}_{n,*}^{\#}(z) &= \left[\eta_n(\overline{z})\, r_n\, \eta_n^*(\overline{z}) - |\, z\,|^2\ \tilde{\zeta}_n(\overline{z})\, l_n \left[\tilde{\zeta}_n(\overline{z})\right]^*\right]^{-1} = \mathfrak{R}_n(\overline{z})\ ,\\ \mathfrak{R}_{n,*}(z) &= \left[\zeta_n^*(\overline{z})\, l_n\, \zeta_n(\overline{z}) - |\, z\,|^2\ [\tilde{\eta}_n(\overline{z})]^*\, r_n\, \tilde{\eta}_n(\overline{z})\right]^{-1} = \mathfrak{L}_n^{\#}(\overline{z})\ , \end{aligned}$$

and, in view of Theorem 29 in [11, Part V],

$$\mathfrak{M}_{n,*}(z) = \left([\zeta_n'(\overline{z})]^*\ \Gamma_0^{-1}\, l_n\, \zeta_n(\overline{z}) + |\, z\,|^2\ [\tilde{\eta}_n'(z)]^*\ (\Gamma_0^{-1})^*\, r_n\, \tilde{\eta}_n(\overline{z})\right) \left[\mathfrak{L}_n^{\#}(\overline{z}\right]^{-1} = \mathfrak{M}_n^*(\overline{z})$$

for all $z \in \mathbb{D}$ ∎

Let $\tau \in \mathbb{N}_0$ or $\tau = \infty$. Further, let $(B_k)_{k=0}^{\tau}$ be a sequence of $m \times m$ complex matrices with $\det(I + B_0) \neq 0$. By virtue of part (d) of Lemma 1.1.21 in [7], there is a unique sequence $(C_k)_{k=0}^{\tau}$ such that

$$S_n < C_0, C_1, ..., C_n > = (I - S_n < B_0, B_1, ..., B_n >)\ (I + S_n < B_0, B_1, ..., B_n >)^{-1}$$

for all integers n with $0 \leqq n \leqq \tau$. This sequence $(C_k)_{k=0}^{\tau}$ is called the *Cayley transform of* $(B_k)_{k=0}^{\tau}$.

Proposition 2: *Let* $\tau \in \mathbb{N}_0$ *or* $\tau = \infty$. *Then:*

(a) Let $(\Gamma_k)_{k=0}^{\tau}$ *be an* $m \times m$ *Carathéodory sequence. Then* $\det(I + \Gamma_0) \neq 0$ *and the Cayley transform* $(A_k)_{k=0}^{\tau}$ *of* $(\Gamma_k)_{k=0}^{\tau}$ *is an* $m \times m$ *Schur sequence. If* $(\Gamma_k)_{k=0}^{\tau}$ *is nondegenerate, then* $(A_k)_{k=0}^{\tau}$ *is nondegenerate as well.*

(b) Let $(A_k)_{k=0}^{\tau}$ *be an* $m \times m$ *Schur sequence. If* $\det((I + A_0) \neq 0$, *then the Cayley transform* $(\Gamma_k)_{k=0}^{\tau}$ *of* $(A_k)_{k=0}^{\tau}$ *is an* $m \times m$ *Carathéodory sequence. If* $(A_k)_{k=0}^{\tau}$ *is nondegenerate, then* $\det(I + A_0) \neq 0$ *and* $(\Gamma_k)_{k=0}^{\tau}$ *is nondegenerate as well.*

Proof: Use part (b) of Lemma 1.1.13 and Lemma 1.3.12 in [7] ∎

Lemma 8: *Let* $\tau \in \mathbb{N}_0$ *or* $\tau = \infty$. *Let* $(B_k)_{k=0}^{\tau}$ *be a sequence of* $m \times m$ *complex matrices with* $\det(I + B_0) \neq 0$, *and let* $(C_k)_{k=0}^{\tau}$ *be the Cayley transform of* $(B_k)_{k=0}^{\tau}$. *Then:*

(a) $\det(I + C_0) \neq 0$.

(b) $(B_k)_{k=0}^{\tau}$ *is the Cayley transform of* $(C_k)_{k=0}^{\tau}$.

(c) $(C_k^*)_{k=0}^{\tau}$ *is the Cayley transform of* $(B_k^*)_{k=0}^{\tau}$.

Proof: The application of parts (c) and (f) of Lemma 1.3.12 in [7] provides parts (a) and (b) of the assertion. It remains to check part (c). Let n be an integer with $0 \leqq n \leqq \tau$. If V_{nm} is given by (6) and (7), then

$$
\begin{aligned}
& S_n < C_0^*, C_1^*, \dots, C_n^* > \\
= \; & V_{nm}\,(S_n < C_0, C_1, \dots, C_n >)^*\,V_{nm} \\
= \; & V_{nm}\,[I + (S_n < B_0, B_1, \dots, B_n >)^*]^{-1}\,[I - (S_n < B_0, B_1, \dots, B_n >)^*]\,V_{nm} \\
= \; & [I + V_{nm}\,(S_n < B_0, B_1, \dots, B_n >)^*\,V_{nm}]^{-1}\,[I - V_{nm}\,(S_n < B_0, B_1, \dots, B_n >)^*\,V_{nm}] \\
= \; & (I + S_n < B_0^*, B_1^*, \dots, B_n^* >)^{-1}\,(I - S_n < B_0^*, B_1^*, \dots, B_n^* >) \\
= \; & (I - S_n < B_0^*, B_1^*, \dots, B_n^* >)\,(I + S_n < B_0^*, B_1^*, \dots, B_n^* >)^{-1}\,.
\end{aligned}
$$

The proof is complete ■

From Proposition 1 we know that the Cayley transform of a matricial Carathéodory function is a matricial Schur function. Now we can see that the property of being nondegenerate will be preserved.

Lemma 9: *The following statements hold true:*

(a) *Let* Ω *be a nondegenerate* $m \times m$ *Carathéodory function. Then the Cayley transform* $f := (I - \Omega)(I + \Omega)^{-1}$ *of* Ω *is a nondegenerate* $m \times m$ *Schur function.*

(b) *Let* f *be a nondegenerate* $m \times m$ *Schur function. Then* $\det(I + f)$ *does not vanish in* $\mathbb{D}$ *and the Cayley transform* $f := (I - f)(I + f)^{-1}$ *is a nondegenerate* $m \times m$ *Carathéodory function.*

Proof: In view of the above mentioned characterization of matrix-valued Schur and Carathéodory functions, the application of Proposition 2 yields the assertion ■

The next considerations are aimed at an explicit description of the interrelations between the parameters of the Weyl matrix balls associated with a nondegenerate $m \times m$ Carathéodory sequence $(\Gamma_k)_{k=0}^{n}$ and with that nondegenerate $m \times m$ Schur sequence which is the Cayley transform of $(\Gamma_k)_{k=0}^{n}$. In particular, we will need some facts from J-theory.

Let J be a $q \times q$ signature matrix, i.e., J belongs to $\mathbb{C}^{q \times q}$ and satisfies $J^* = J$ and $J^2 = I$. A $q \times q$ matrix A is said to be *J-contractive* (respectively, *J-unitary*) if $A^* J A \leqq J$ (respectively, $A^* J A = J$). A $q \times q$ matrix-valued function B which is meromorphic in the extended complex plane $\mathbb{C}_0 := \mathbb{C} \cup \{\infty\}$ is called *J-elementary factor* if the following three conditions are satisfied:

(i) B has exactly one pole $z_0 \in \mathbb{C}_0$.

(ii) For each $z \in \mathbb{D} \setminus \{z_0\}$, the matrix $B(z)$ is J-contractive.

(iii) For each $z \in \mathbb{T}\setminus\{z_0\}$, the matrix $B(z)$ is J-unitary.

In the following, we will mainly consider the $2m \times 2m$ signature matrix

$$j_{mm} := \operatorname{diag}(I_m, -I_m) . \tag{35}$$

The next result will turn out to play a key role for introducing a Dubovoj-like classification in the Carathéodory class $\mathcal{C}_m(\mathbb{D})$.

Proposition 3: *Let $n \in \mathbb{N}_0$, and let $(\Gamma_k)_{k=0}^n$ be a nondegenerate $m \times m$ Carathéodory sequence. Assume that $\mathfrak{M}_n$, $\mathfrak{L}_n^\#$ and $\mathfrak{R}_n$ are the Weyl-Carathéodory center function, the canonical normalized left and the canonical right Weyl-Carathéodory semi-radius functions, respectively, associated with $(\Gamma_k)_{k=0}^n$. Let $(A_k)_{k=0}^n$ be the Cayley transform of $(\Gamma_k)_{k=0}^n$. Further, let $\mathcal{M}_n$, $\mathcal{L}_n^\#$ and $\mathcal{R}_n$ be the Weyl-Schur center function, the canonical normalized left and the canonical right Weyl-Schur semi-radius functions, respectively, associated with (the nondegenerate $m \times m$ Schur sequence) $(A_k)_{k=0}^n$. For each $z \in \mathbb{D}$, then*

$$\begin{aligned}\mathfrak{M}_n(z) &= \left([I - \mathcal{M}_n(z)]\,[\mathcal{R}_n(z)]^{-1}\,[I + \mathcal{M}_n(z)] + |\, z \,|^{2(n+1)}\, \mathcal{L}_n^\#(z)\right) \\ &\quad \times\left([I + \mathcal{M}_n(z)]\,[\mathcal{R}_n(z)]^{-1}\,[I + \mathcal{M}_n(z)]^* - |\, z \,|^{2(n+1)}\, \mathcal{L}_n^\#(z)\right)^{-1} ,\end{aligned} \tag{36}$$

$$\mathfrak{L}_n^\#(z) = \left([I + \mathcal{M}_n(z)]^*\,\left[\mathcal{L}_n^\#(z)\right]^{-1}[I + \mathcal{M}_n(z)] - |\, z \,|^{2(n+1)}\, \mathcal{R}_n(z)\right)^{-1} , \tag{37}$$

and

$$\mathfrak{R}_n(z) = \left([I + \mathcal{M}_n(z)]\,[\mathcal{R}_n(z)]^{-1}\,[I + \mathcal{M}_n(z)]^* - |\, z \,|^{2(n+1)}\, \mathcal{L}_n^\#(z)\right)^{-1} . \tag{38}$$

Proof: Proposition 2 shows that $(A_k)_{k=0}^n$ is a nondegenerate Schur sequence. Using (3) we set $S_n := S_n < A_0, A_1, \ldots A_n >$ and $H_n := (S_n, I)^*(I - S_n S_n^*)^{-1}(S_n, I)$. In view of Theorem 4.4.1 in [7], the function $B_n : \mathbb{C}_0\setminus\{0\} \to \mathbb{C}^{2m\times 2m}$ defined by

$$B_n(z) :=$$

$$= \begin{cases} I - \frac{1-z}{z} j_{mm} \cdot \operatorname{diag}(e_{nm}(1), e_{nm}(1)) \cdot H_n \cdot \operatorname{diag}(e_{nm}^*(1/\overline{z}), e_{nm}(1/\overline{z})) , & z \in \mathbb{C}\setminus\{0\}, \\ I + J_{mm} \cdot \operatorname{diag}(e_{nm}(1), e_{nm}(1)) \cdot H_n \cdot \operatorname{diag}(e_{nm}^*(0), e_{nm}^*(0)) , & z = \infty \end{cases}$$

where $e_{nm} : \mathbb{C} \to \mathbb{C}^{m\times(n+1)m}$ is given by (11), is a full-rank j_{mm}-elementary factor with pole of order $n+1$ at $z = 0$ which satisfies $B_n(0) = I$. Part (a) of Theorem 5.5.1 in [7] yields $\det B(z) \neq 0$ for all $z \in \mathbb{C}\setminus\{0\}$. From Proposition 9 in [10] we know then that $B_n^\square := \varepsilon_{n+1} B_n$, where $\varepsilon_{n+1} : \mathbb{C} \to \mathbb{C}$ is defined by $\varepsilon_{n+1}(z) := z^{n+1}$, is a full-rank j_{mm}-elementary factor with pole of order $n+1$ at $z = \infty$ which satisfies $B_n^\square(1) = I$. Using Theorem 5.5.1 in [7], Proposition 5.5.1 in [7] and (13), we obtain

$$\begin{aligned}[B_n^\square(z)]^{-1}\, j_{mm}\, \left([B_n^\square(z)]^{-1}\right)^* &= \frac{1}{|\, z \,|^{2(n+1)}} \begin{pmatrix} I & 0 \\ -\mathcal{M}_n(z) & I \end{pmatrix} \\ \times \begin{pmatrix} (\mathcal{R}_n(z))^{-1}, & 0 \\ 0 & , -|\, z \,|^{2(n+1)}\, \mathcal{L}_n^\#(z) \end{pmatrix} &\begin{pmatrix} I & 0 \\ -\mathcal{M}_n(z) & I \end{pmatrix}^*\end{aligned} \tag{39}$$

holds true for all $z \in \mathbb{D}\backslash\{0\}$. If E is a strictly contractive $m \times m$ matrix, then the so-called *Halmos extension*

$$H(E) := \begin{pmatrix} \sqrt{I-EE^*}^{-1} & E\sqrt{I-E^*E}^{-1} \\ E^*\sqrt{I-EE^*}^{-1} & \sqrt{I-E^*E}^{-1} \end{pmatrix}$$

of E is obviously j_{mm}-unitary. Furthermore, if v and w are unitary $m \times m$ matrices, then $\operatorname{diag}(v,w)$ is clearly j_{mm}-unitary. The matrix

$$U_{mm} := \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$

satisfies $U^*_{mm} j_{mm} U_{mm} = U_{mm} j_{mm} U^*_{mm} = -j_{mm}$. The product of j_{mm}-unitary matrices is j_{mm}-unitary as well. Thus, if $E_1, E_2, ..., E_n$ are strictly contractive $m \times m$ matrices, and if v and w are unitary $m \times m$ matrices, then the product

$$U^*_{mm} \cdot H(E_1) \cdot H(E_2) \cdot ... \cdot H(E_n) \cdot U_{mm} \cdot \operatorname{diag}(v,w)$$

is j_{mm}-unitary. Every j_{mm}-unitary matrix is nonsingular and its inverse matrix is j_{mm}-unitary as well (see, e.g., [7, Part (c) of Lemma 1.3.15]). Hence, we see from Corollary 20 in [6], Proposition 20 in [12, Part IV] and part (a) of Proposition 12 in [11, Part V] that there is a j_{mm}-unitary matrix X_n such that the function $\mathcal{E}_n : \mathbb{C} \to \mathbb{C}^{2m\times 2m}$ defined by

$$\mathcal{E}_n(z) := \frac{1}{\sqrt{2}} \begin{pmatrix} -z\sqrt{r_n}\,\Gamma_0^{-1}\,\tilde{\eta}'_n(z) & , & z\sqrt{r_n}\,\tilde{\eta}_n(z) \\ \sqrt{l_n}\,\left(\Gamma_0^{-1}\right)^*\,\zeta'_n(z) & , & \sqrt{l_n}\,\zeta_n(z) \end{pmatrix}$$

admits the representation

$$\mathcal{E}_n(z) = z^{n+1}\, X_n\, B_n(z)\, C_m$$

for all $z \in \mathbb{C}\backslash\{0\}$, where

$$C_m := \frac{1}{\sqrt{2}} \begin{pmatrix} -I_m & I_m \\ I_m & I_m \end{pmatrix}.$$

Part (c) of Proposition 12 in [11, Part V] shows $\det \mathcal{E}_n(z) \neq 0$ for all $z \in \mathbb{C}\backslash\{0\}$. Set $\mathfrak{W}_n := \mathcal{E}_n^{-1} j_{mm} (\mathcal{E}_n^{-1})^*$. Using $C_m^2 = I$ we get

$$\begin{aligned} \mathfrak{W}_n(z) &= \frac{1}{|z|^{2(n+1)}} C_m \left[B_n(z)\right]^{-1} j_{mm} \left(\left[B_n(z)\right]^{-1}\right)^* C_m \\ &= C_m \left[B_n^{\square}(z)\right]^{-1} j_{mm} \left(\left[B_n^{\square}(z)\right]^{-1}\right)^* C_m \end{aligned} \tag{40}$$

for all $z \in \mathbb{C}\backslash\{0\}$. On the other hand, part (d) of Proposition 12 in [11, Part V] provides

$$\left[\mathcal{E}_n(z)\right]^{-1} = \frac{1}{\sqrt{2}\, z^{n+1}} \begin{pmatrix} -\eta_n(z)\sqrt{r_n} & , & z\,\tilde{\zeta}_n(z)\sqrt{l_n} \\ \eta'_n(z)\,\left(\Gamma_0^{-1}\right)^*\,\sqrt{r_n} & , & z\,\tilde{\zeta}'_n(z)\,\Gamma_0^{-1}\,\sqrt{l_n} \end{pmatrix}$$

and, consequently,

$$\mathfrak{W}_n(z) = \frac{1}{2\,|z|^{2(n+1)}} \begin{pmatrix} W_{11;n}(z) & , & W_{12;n}(z) \\ W_{21;n}(z) & , & W_{22;n}(z) \end{pmatrix}$$

for all $z \in \mathbb{C}\setminus\{0\}$, where the functions $W_{11;n} : \mathbb{C} \to \mathbb{C}^{m\times m}$, $W_{21;n} : \mathbb{C} \to \mathbb{C}^{m\times m}$, $W_{12;n} : \mathbb{C} \to \mathbb{C}^{m\times m}$ and $W_{22;n} : \mathbb{C} \to \mathbb{C}^{m\times m}$ are given by

$$\begin{aligned} W_{11;n}(z) &:= \eta_n(z)\, r_n\, \eta_n^*(z) - |z|^2\, \tilde{\zeta}_n(z)\, l_n \left(\tilde{\zeta}_n(z)\right)^*, \\ W_{21;n}(z) &:= -\eta_n'(z)\left(\Gamma_0^{-1}\right)^* r_n\, \eta_n^*(z) - |z|^2\, \tilde{\zeta}_n'(z)\,\Gamma_0^{-1} l_n \left(\tilde{\zeta}_n(z)\right)^*, \\ W_{12;n}(z) &:= W_{21;n}^*(z) \end{aligned}$$

and

$$W_{22;n}(z) := \eta_n'(z)\left(\Gamma_0^{-1}\right)^* r_n\,\Gamma_0^{-1}\left(\eta_n'(z)\right)^* - |z|^2\, \tilde{\zeta}_n'(z)\,\Gamma_0^{-1} l_n \left(\Gamma_0^{-1}\right)^*\left(\tilde{\zeta}_n{}'(z)\right)^*.$$

For all $z \in \mathbb{D}$, we get obviously

$$W_{11;n}(z) = \left(\mathfrak{R}_n(z)\right)^{-1} \quad\text{and}\quad W_{21;n}(z) = -\mathfrak{M}_n(z)\left(\mathfrak{R}_n(z)\right)^{-1}. \tag{41}$$

By virtue of Theorem 1.6.3 in [7], for all $z \in \mathbb{C}\setminus\{0\}$, the matrix

$$Y_n(z) := \frac{1}{2\,|z|^{2(n+1)}}\left[W_{21;n}(z)\left[W_{11;n}(z)\right]^{-1} W_{12;n}(z) - W_{22;n}(z)\right]$$

is positive Hermitian, and its inverse matrix coincides with the $m\times m$ block in the right upper corner of $\left(-\mathfrak{W}_n(z)\right)^{-1} = -E_n^*(z)\, j_{mm}\, E_n(z)$. Hence,

$$Y_n(z) = 2\left(\zeta_n^*(z)\, l_n\, \zeta_n(z) - |z|^2\left[\tilde{\eta}_n(z)\right]^* r_n\, \tilde{\eta}_n(z)\right)^{-1} = 2\,\mathfrak{L}_n^{\#}(z)$$

for all $z \in \mathbb{D}\setminus\{0\}$. From Lemma 1.1.7 in [7] then we obtain

$$\begin{aligned} \mathfrak{W}_n(z) &= \frac{1}{|z|^{2(n+1)}}\begin{pmatrix} I & 0 \\ -\mathfrak{M}_n(z) & I \end{pmatrix} \\ &\times \begin{pmatrix} \frac{1}{2}\left(\mathfrak{R}_n(z)\right)^{-1}, & 0 \\ 0, & -2\,|z|^{2(n+1)}\,\mathfrak{L}_n^{\#}(z) \end{pmatrix}\begin{pmatrix} I & 0 \\ -\mathfrak{M}_n(z) & I \end{pmatrix}^* \end{aligned} \tag{42}$$

for each $z \in \mathbb{D}\setminus\{0\}$. Comparing (39), (40) and (42) it follows

$$\begin{aligned} &\begin{pmatrix} -I & I \\ I & I \end{pmatrix}\begin{pmatrix} I & 0 \\ -\mathcal{M}_n(z) & I \end{pmatrix} \\ &\times \begin{pmatrix} [\mathcal{R}_n(z)]^{-1}, & 0 \\ 0, & -|z|^{2(n+1)}\,\mathcal{L}_n^{\#}(z) \end{pmatrix}\begin{pmatrix} I & 0 \\ -\mathcal{M}_n(z) & I \end{pmatrix}^*\begin{pmatrix} -I & I \\ I & I \end{pmatrix} \\ &= 2\begin{pmatrix} I & O \\ -\mathfrak{M}_n(z) & I \end{pmatrix}\begin{pmatrix} \frac{1}{2}[\mathfrak{R}_n(z)]^{-1}, & 0 \\ 0, & -2\,|z|^{2(n+1)}\,\mathfrak{L}_n^{\#}(z) \end{pmatrix}\begin{pmatrix} I & 0 \\ -\mathfrak{M}_n(z) & I \end{pmatrix}^* \end{aligned}$$

for all $z \in \mathbb{D}\setminus\{0\}$. This implies

$$\begin{aligned} &\begin{pmatrix} [I+\mathcal{M}_n(z)]\,[\mathcal{R}_n(z)]^{-1}\,[I+\mathcal{M}_n(z)]^* - |z|^{2(n+1)}\,\mathcal{L}_n^{\#}(z), & -[I+\mathcal{M}_n(z)]\,[\mathcal{R}_n(z)]^{-1}\,[I-\mathcal{M}_n(z)]^* - |z|^{2(n+1)}\,\mathcal{L}_n^{\#}(z) \\ -[I-\mathcal{M}_n(z)]\,[\mathcal{R}_n(z)]^{-1}\,[I+\mathcal{M}_n(z)] - |z|^{2(n+1)}\,\mathcal{L}_n^{\#}(z), & [I-\mathcal{M}_n(z)]\,[\mathcal{R}_n(z)]^{-1}\,[I-\mathcal{M}_n(z)]^* - |z|^{2(n+1)}\,\mathcal{L}_n^{\#}(z) \end{pmatrix} \\ &= \begin{pmatrix} [\mathfrak{R}_n(z)]^{-1}, & -[\mathfrak{R}_n(z)]^{-1}\,[\mathfrak{M}_n(z)]^* \\ -\mathfrak{M}_n(z)\,[\mathfrak{R}_n(z)]^{-1}, & \mathfrak{M}_n(z)\,[\mathfrak{R}_n(z)]^{-1}\,[\mathfrak{M}_n(z)]^* - 4\,|z|^{2(n+1)}\,\mathfrak{L}_n^{\#}(z) \end{pmatrix} \end{aligned}$$

and therefore (38) and (36) for each $z \in \mathbb{D}\setminus\{0\}$. From Remark 3 we know that $(\Gamma_k^*)_{k=0}^n$ is a nondegenerate $m \times m$ Carathéodory sequence. Lemma 8 shows that $(A_k^*)_{k=0}^n$ is the Cayley transform of $(\Gamma_k^*)_{k=0}^n$. Thus, we see from Proposition 2 that $(A_k^*)_{k=0}^n$ is a nondegenerate $m \times m$ Schur sequence. Using Lemma 7, Lemma 4, and the notations given there, we then obtain from (38) that

$$\begin{aligned}
\mathfrak{L}_n^{\#}(z) &= \mathfrak{R}_{n,*}(\overline{z}) \\
&= \Big([I+\mathcal{M}_{n,*}(\overline{z})]\,[\mathcal{R}_{n,*}(\overline{z})]^{-1}\,[I+\mathcal{M}_{n,*}(\overline{z})]^* - |\,z\,|^{2(n+1)}\,\mathcal{L}_{n,*}^{\#}(\overline{z})\Big)^{-1} \\
&= \Big([I+\mathcal{M}_n(z)]^*\,[\mathcal{L}_n^{\#}(z)]^{-1}\,[I+\mathcal{M}_n(z)] - |\,z\,|^{2(n+1)}\,\mathcal{R}_n(z)\Big)^{-1}
\end{aligned}$$

holds true for all $z \in \mathbb{D}\setminus\{0\}$. Finally, from Lemmas 3 and 6 we see that identities (36), (37) and (38) are satisfied for $z = 0$ as well ∎

We want to draw the attention of the reader to the interesting fact that the equation (37) shows that (with exception of the point $z = 0$) the canonical normalized left Weyl-Carathéodory semi-radius function $\mathfrak{L}_n^{\#}$ depends explicitly as well on the canonical normalized left Weyl-Schur semi-radius function $\mathcal{L}_n^{\#}$ as on the canonical right Weyl-Schur semi-radius function $\mathcal{R}_n$. The formula (38) can be interpreted similarly.

4. Limit Behaviour of the Weyl Matrix Balls Associated with a Nondegenerate Matrix-valued Carathéodory Function

Our approach to the study of the limit behaviour of the parameters of the Weyl matrix balls associated with a given nondegenerate matrix-valued Carathéodory function Ω is based on the use of the Cayley transform $f := (I-\Omega)(I+\Omega)^{-1}$ of Ω. This enables us to use DUBOVOJ's results [5], [7, Section 5.6] on the limit behaviour of the parameters of the Weyl matrix balls associated with a given nondegenerate matrix-valued Schur function.

Parts of the following theorem go back to KOVALISHINA [16]. It is the analogue to Theorem 2 which handles the case of matrix-valued Schur functions.

Theorem 4: *Let Ω be a nondegenerate $m \times m$ Carathéodory function, and let*

$$\Omega(z) = \sum_{k=0}^{\infty} \Gamma_k z^k\,, \quad z \in \mathbb{D}\,, \tag{43}$$

be the Taylor series representation of Ω. For $n \in \mathbb{N}_0$, let $\mathfrak{M}_n$, $\mathfrak{L}_n^{\#}$, $\mathfrak{L}_n$ and $\mathfrak{R}_n$ be the Weyl-Carathéodory center function, the canonical normalized left, the canonical left and the canonical right Weyl-Carathéodory semi-radius functions, respectively, associated with $(\Gamma_k)_{k=0}^n$. Then:

(a) For each $z \in \mathbb{D}$, $\lim_{n\to\infty} \mathfrak{M}_n(z) = \Omega(z)$.

(b) For each $z \in \mathbb{D}$, the sequences $(\mathfrak{L}_n^{\#}(z))_{n=0}^{\infty}$ and $(\mathfrak{R}_n(z))_{n=0}^{\infty}$ are monotonously nonincreasing and convergent. The corresponding limits $\mathfrak{L}^{\#}(z)$ and $\mathfrak{R}(z)$ are nonnegative Hermitian for all $z \in \mathbb{D}$. Moreover, $\lim_{n\to\infty} \mathfrak{L}_n(z) = 0$ for each $z \in \mathbb{D}$.

Proof: According to Lemma 9, $f := (I - \Omega)(I + \Omega)^{-1}$ is a nondegenerate $m \times m$ Schur function for which $\det(I + f)$ does not vanish in $\mathbb{D}$. Let (20) be the Taylor series representation of f. In the following, we will use the notations given in Proposition 3 and the corresponding proof. Let $n \in \mathbb{N}_0$. From Theorem 4.4.3 in [7] we see that there is a full-rank j_{mm}-elementary factor b_n with pole of order one at $z = 0$ such that $b_n(1) = I$ and $B_{n+1} = b_n B_n$. Hence, part (a) of Theorem 5.5.1 in [7] and Lemma 1.3.15 in [7] provide

$$\begin{aligned} &[B_{n+1}(z)]^{-1} j_{mm} \left([B_{n+1}(z)]^{-1}\right)^* - [B_n(z)]^{-1} j_{mm} \left([B_n(z)]^{-1}\right)^* \\ = \; &[B_n(z)]^{-1} \left([b_n(z)]^{-1} j_{mm} \left([b_n(z)]^{-1}\right)^* - j_{mm}\right) \left([B_n(z)]^{-1}\right)^* \geqq 0 \end{aligned}$$

for all $z \in \mathbb{D}\setminus\{0\}$. In view of (40), we obtain then

$$2 \mid z \mid^{2(n+2)} \mathfrak{W}_{n+1}(z) - 2 \mid z \mid^{2(n+1)} \mathfrak{W}_n(z) \geqq 0$$

for all $z \in \mathbb{D}\setminus\{0\}$. Using the first identity in (41) and Lemma 6 we get

$$[\mathfrak{R}_{n+1}(z)]^{-1} = W_{11;n+1}(z) \geqq W_{11;n}(z) = [\mathfrak{R}_n(z)]^{-1} > 0$$

and, consequently,

$$\mathfrak{R}_n(z) \geqq \mathfrak{R}_{n+1}(z) > 0 \tag{44}$$

for all $z \in \mathbb{D}\setminus\{0\}$. We know from Lemma 6 that $\mathfrak{R}_{n+1}$ and $\mathfrak{R}_n$ are continuous. Thus, (44) also holds true for $z = 0$. For every choice of z in $\mathbb{D}$, we obtain then from Lemma 7 that

$$\mathfrak{L}^{\#}_{n+1}(z) \geqq \mathfrak{L}^{\#}_n(z) > 0$$

is satisfied. Hence, for each $z \in \mathbb{D}$, the sequences $(\mathfrak{L}^{\#}_n(z))_{n=0}^{\infty}$ and $(\mathfrak{R}_n(z))_{n=0}^{\infty}$ are monotonously nonincreasing and converge to some $m \times m$ nonnegative Hermitian matrices $\mathfrak{L}^{\#}(z)$ and $\mathfrak{R}(z)$, respectively. Now let $z \in \mathbb{D}$. Then we can conclude

$$\lim_{n\to\infty} \mathfrak{L}_n(z) = \lim_{n\to\infty} \mid z \mid^{n+1} \mathfrak{L}^{\#}_n(z) = 0_{m\times m} . \tag{45}$$

Theorem 3 yields that the matrix $\Omega(z)$ belongs to

$$\mathfrak{K}(z) := \bigcap_{n=0}^{\infty} \mathfrak{K}_n(z)$$

where

$$\mathfrak{K}_n(z) := \mathfrak{K}\left(\mathfrak{M}_n(z); \mid z \mid^{n+1} \sqrt{2\,\mathfrak{L}^{\#}_n(z)}, \sqrt{2\mathfrak{R}_n(z)}\right) , \quad n \in \mathbb{N}_0 ,$$

and that $\mathfrak{K}_{n+1}(z) \subseteq \mathfrak{K}_n(z)$ for all $n \in \mathbb{N}_0$. A theorem due to ŠMULJAN [21] (see also [7, Theorem 1.5.3]) shows then that the sequence $(\mathfrak{M}_n(z))_{n=0}^{\infty}$ converges to some complex $m \times m$ matrix $\mathfrak{M}(z)$ and, in view of (45), that $\mathfrak{K}(z)$ coincides with the matrix ball $\mathfrak{K}(\mathfrak{M}(z); 0_{m\times m}, \mathfrak{R}(z)) = \{\mathfrak{M}(z)\}$. This implies finally $\Omega(z) = \mathfrak{M}(z)$ ∎

Theorem 4 leads us to the following notions.

Definition 3: Let Ω be a nondegenerate $m \times m$ Carathéodory function. Then the functions $\mathfrak{L}^{\#} : \mathbb{D} \to \mathbb{C}^{m\times m}$ and $\mathfrak{R} : \mathbb{D} \to \mathbb{C}^{m\times m}$ given in part (b) of Theorem 4 are called the *canonical normalized left* and the *canonical right Weyl-Carathéodory limit semi-radius functions*, respectively, *associated with* Ω.

Lemma 10: *Let Ω be a nondegenerate $m \times m$ Carathéodory function. Let $\mathfrak{L}^{\#}$ and $\mathfrak{R}$ be the canonical normalized left and the canonical right Weyl-Carathéodory limit semi-radius functions, respectively, associated with Ω. Then $\check{\Omega} : \mathbb{D} \to \mathbb{C}^{m \times m}$ given by*

$$\check{\Omega}(z) := \Omega^*(\overline{z}), \quad z \in \mathbb{D}, \tag{46}$$

is a nondegenerate $m \times m$ Carathéodory function. If $\mathfrak{L}_*^{\#}$ and $\mathfrak{R}_*$ are the canonical normalized left and the canonical right Weyl-Carathéodory limit semi-radius functions, respectively, associated with $\check{\Omega}$, then

$$\mathfrak{L}_*^{\#}(z) = \mathfrak{R}(\overline{z}) \quad \textit{and} \qquad \mathfrak{R}_*(z) = \mathfrak{L}^{\#}(\overline{z})$$

for all $z \in \mathbb{D}$.

Proof: The function $\check{\Omega}$ has the Taylor series representation $\check{\Omega}(z) = \sum_{k=0}^{\infty} \Gamma_k^* z^k$, $z \in \mathbb{D}$. According to Remark 3, $\check{\Omega}$ is nondegenerate. Thus, the application of Lemma 7 completes the proof ■

Now we are able to formulate the first main result of this paper.

Theorem 5: *Suppose that Ω is a nondegenerate $m \times m$ Carathéodory function. Let $\mathfrak{L}^{\#}$ and $\mathfrak{R}$ be the canonical normalized left and the canonical right Weyl-Carathéodory limit semi-radius functions, associated with Ω. Further, let $\mathcal{L}^{\#}$ and $\mathcal{R}$ be the canonical normalized left and the canonical right Weyl-Schur limit semi-radius functions, respectively, associated with (the nondegenerate $m \times m$ Schur function) $f := (I - \Omega)(I + \Omega)^{-1}$. Then:*

(a) For each $z \in \mathbb{D}$,

$$0 \leqq \mathfrak{L}^{\#}(z) = [I + f(z)]^{-1} \mathcal{L}^{\#}(z) \left([I + f(z)]^{-1}\right)^* \leqq \operatorname{Re}\Omega(z) \tag{47}$$

and

$$0 \leqq \mathfrak{R}(z) = \left([I + f(z)]^{-1}\right)^* \mathcal{R}(z) [I + f(z)]^{-1} \leqq \operatorname{Re}\Omega(z). \tag{48}$$

(b) For each $z \in \mathbb{D}$,

$$\det \mathfrak{L}^{\#}(z) = \det \mathfrak{R}(z), \tag{49}$$

$$\operatorname{rank} \mathfrak{L}^{\#}(z) = \operatorname{rank} \mathcal{L}(z) = \operatorname{rank} \mathcal{L}(0), \tag{50}$$

and

$$\operatorname{rank} \mathfrak{R}(z) = \operatorname{rank} \mathcal{R}(z) = \operatorname{rank} \mathcal{R}(0). \tag{51}$$

In particular, $\operatorname{rank} \mathfrak{L}^{\#}(0) = m$ *if and only if* $\operatorname{rank} \mathfrak{R}(0) = m$.

Proof: By virtue of Lemma 9, f is a nondegenerate $m \times m$ Schur function, and $\det(I + f)$ nowhere vanishes in $\mathbb{D}$. Let (43) and (18) be the Taylor series representations of Ω and f. From part (d) of Lemma 1.1.21 in [7] we know that, for each $n \in \mathbb{N}_0$, $(A_k)_{k=0}^n$ is the Cayley transform of $(\Gamma_k)_{k=0}^n$. In the following, we will use the notations given above. Assume that $n \in \mathbb{N}_0$ and $z \in \mathbb{D}$. From Lemma 3 we see that (14) and (16) hold true. In view of Proposition 3 we obtain then

$$[I + \mathcal{M}_n(z)]^{-1} \left([\mathfrak{R}_n(z)]^{-1} + |z|^{2(n+1)} \mathcal{L}_n^{\#}(z)\right) \left([I + \mathcal{M}_n(z)]^{-1}\right)^* = [\mathcal{R}_n(z)]^{-1} \geqq I$$

and hence

$$\mathcal{R}_n(z) = [I + \mathcal{M}_n(z)]^* H_n(z) [I + \mathcal{M}_n(z)] \tag{52}$$

where

$$H_n(z) := \left([\mathfrak{R}_n(z)]^{-1} + |z|^{2(n+1)} \mathcal{L}_n^{\#}(z)\right)^{-1}.$$

Lemmas 3 and 6 show that $H_n(z)$ is positive Hermitian. Obviously,

$$H_n(z) = \mathfrak{R}_n(z) \left[I + |z|^{2(n+1)} \mathcal{L}_n^{\#}(z) \mathfrak{R}_n(z)\right]^{-1}.$$

Theorem 2 and part (b) provide

$$\lim_{n\to\infty} \left[I + |z|^{2(n+1)} \mathcal{L}_n^{\#}(z) \mathfrak{R}_n(z)\right] = I + \left(\lim_{n\to\infty} |z|^{2(n+1)}\right) \mathcal{L}^{\#}(z) \mathfrak{R}(z) = I$$

and therefore $\lim_{n\to\infty} H_n(z) = \mathfrak{R}(z)$. Thus, part (a) of Theorem 2 and (52) imply

$$\mathfrak{L}^{\#} = (I + f)^{-1} \mathcal{L}^{\#} \left[(I + f)^{-1}\right]^*. \tag{53}$$

Lemma 5 yields that $\check{f}$ given by (22) is a nondegenerate $m \times m$ Schur function, whereas Lemma 10 shows that $\check{\Omega}$ defined by (46) is a nondegenerate $m \times m$ Carathéodory function. Lemma 2.1.11 in [7] provides $\check{f} = (I - \check{\Omega})(I + \check{\Omega})^{-1}$. From Lemmas 10 and 5, and identity (53) we then get

$$\begin{aligned} \mathfrak{R}(z) = \mathfrak{L}_*^{\#} &= \left[I + \check{f}(\bar{z})\right]^{-1} \mathcal{L}_*^{\#}(\bar{z}) \left(\left[I + \check{f}(\bar{z})\right]^{-1}\right)^* \\ &= \left([I + f(z)]^{-1}\right)^* \mathcal{R}(z) [I + f(z)]^{-1} \end{aligned} \tag{54}$$

for all $z \in \mathbb{D}$. From Proposition 5.6.3 in [7] we know that

$$\mathcal{L}_n^{\#}(z) \leqq I - \mathcal{M}_n(z) \mathcal{M}_n^*(z) \quad \text{and} \quad \mathcal{R}_n(z) \leqq I - \mathcal{M}_n^*(z) \mathcal{M}_n(z)$$

hold for all $n \in \mathbb{N}_0$ and all $z \in \mathbb{D}$. In view of part (a) of Theorem 2, letting $n \to \infty$ we get

$$\mathcal{L}^{\#}(z) \leqq I - f(z) f^*(z) \quad \text{and} \quad \mathcal{R}(z) \leqq I - f^*(z) f(z)$$

for all $z \in \mathbb{D}$. Applying Theorem 4, (53), (54) and parts (f), (g) of Lemma 1.3.12 in [7] we obtain then

$$0 \leqq \mathfrak{L}^{\#}(z) \leqq [I + f(z)]^{-1} [I - f(z) f^*(z)] \left([I + f(z)]^{-1}\right)^* = \operatorname{Re}\Omega(z)$$

and

$$0 \leqq \mathfrak{R}(z) \leqq \left([I + f(z)]^{-1}\right)^* [I - f^*(z) f(z)] [I + f(z)]^{-1} = \operatorname{Re}\Omega(z)$$

for all $z \in \mathbb{D}$. Using the equalities in (47) and (48), we get from parts (b) and (c) of Theorem 2 that (49), (50) and (51) are satisfied for all $z \in \mathbb{D}$. The formula (49) yields then that $\operatorname{rank} \mathfrak{L}^{\#}(0) = m$ if and only if $\operatorname{rank} \mathfrak{R}(0) = m$ ∎

Theorem 5 should be compared with Proposition 3. In contrast to a finite stage n, the Weyl-Carathéodory limit semi-radius function $\mathfrak{L}^{\#}$ (respectively $\mathfrak{R}$) only depends from the corresponding Weyl-Schur limit semi-radius function $\mathcal{L}^{\#}$ (respectively $\mathcal{R}$) and not on both of them.

KOVALISHINA [16] recognized that $\operatorname{rank}\mathfrak{R}$ is a constant function in $\mathbb{D}$. However, her proof contained a gap connected with an incorrect application of Orlov's Theorem.

Part (b) of Theorem 5 suggests the following classification of nondegenerate $m \times m$ Carathéodory functions.

Definition 4: Let Ω be a nondegenerate $m \times m$ Carathéodory function, and let $\mathfrak{L}$ and $\mathfrak{R}$ be the canonical normalized left and the canonical right Weyl-Carathéodory limit semi-radius functions, respectively, associated with Ω. Then Ω is called to be of *Carathéodory-type* $[\beta^{\#}, \beta]$ if $\operatorname{rank}\mathfrak{L}^{\#}(0) = \beta^{\#}$ and $\operatorname{rank}\mathfrak{R}(0) = \beta$.

Observe that, in view of Theorem 5, the classification of matrix-valued Carathéodory functions given in Definition 4 can also be described by $\operatorname{rank}\mathfrak{L}^{\#}(z_1)$ and $\operatorname{rank}\mathfrak{R}(z_2)$ where z_1 and z_2 are arbitrary points which belong to $\mathbb{D}$.

Remark 4: Suppose that Ω is a nondegenerate $m \times m$ Carathéodory function which has the Carathéodory-type $[\beta^{\#}, \beta]$. Then Lemma 10 shows that $\check{\Omega}$ given by (46) is a nondegenerate $m \times m$ Carathéodory function of Carathéodory-type $[\beta, \beta^{\#}]$.

Lemma 11: *Let Ω be a nondegenerate $m \times m$ Carathéodory function, and let $f := (I-\Omega)(I+\Omega)^{-1}$. If Ω has the Carathéodory-type $[\beta^{\#}, \beta]$, then the nondegenerate $m \times m$ Schur function f has the Schur type $[\beta^{\#}, \beta]$.*

Proof: Apply Theorem 5 ■

Now we are going to state our second main result.

Theorem 6: *For every choice of j and k in $\{0, 1, ..., m-1\}$, there exists a nondegenerate $m \times m$ Carathéodory function Ω of Carathéodory-type $[j, k]$. Furthermore, there is a nondegenerate $m \times m$ Carathéodory function of type $[m, m]$. If $k \in \{0, 1, ..., m-1\}$, then there is no nondegenerate $m \times m$ Carathéodory function which has the Carathéodory-type $[m, k]$ or $[k, m]$.*

Proof: Let $f := (I-\Omega)(I+\Omega)^{-1}$. Then we see from Lemma 1.3.12 in [7] that $\Omega = (I-f)(I+f)^{-1}$ holds true. Combining Remark 2 and Theorem 5 we obtain immediately the assertion ■

Note that the proof of the corresponding result in the case of matrix-valued Schur functions (see [7, Theorem 5.6.1]) is essentially based on a nice construction going back to KOVALISHINA and POTAPOV [17].

The following result is taken from [11, Part V, Lemma 26 and Remark 30].

Proposition 4: *If Ω is a nondegenerate $m \times m$ Carathéodory function, then* $\det \Omega$ *nowhere vanishes in $\mathbb{D}$, and Ω^{-1} is a nondegenerate $m \times m$ Carathéodory function as well.*

Theorem 7: *Let Ω be a nondegenerate $m \times m$ Carathéodory function of Carathéodory-type $[\beta^{\#}, \beta]$. Then Ω^{-1} is a nondegenerate $m \times m$ Carathéodory function of the same Carathéodory-type $[\beta^{\#}, \beta]$.*

Proof: Use the formulas (12) and (13) stated in Lemma 27 in [13, Part V] and the equations $\mathfrak{L}_n^{\#}(0) = l_n$ and $\mathfrak{R}_n(0) = r_n$, $n \in \mathbb{N}_0$, which were proved in [13, Part V, p. 295] ■

By virtue of the matricial version of the F.Riesz-Herglotz Theorem (see, e.g., [7, Theorem 2.2.2]), every matrix-valued Carathéodory function is connected to a unique nonnegative Hermitian Borel measure F on the unit circle $\mathbb{T}$. According to a famous theorem due to KOLMOGOROV [15], every such measure F can be conceived as nonstochastic spectral measure of an appropriately chosen multivariate stationary sequence. Having this in mind it can be shown that the semi-radii of the Weyl matrix balls associated with a given nondegenerate Carathéodory sequence admit a clear interpretation in the context of prediction theory for multivariate stationary sequences. We will discuss this in detail somewhere else.

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