Zeitschrift für Analysis
und ihre Anwendungen
Vol. 12 (1993), 273 - 283

On Transformations of Distribution Functions on the Unit Interval - a Generalization of the Gauß-Kuzmin-Lévy Theorem

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Let X be a random variable on [0,1) with the distribution function g_0 and h be a piecewise monotonic transformation on [0,1). Further denote the distribution function of $h(X)$ by g_1, of $h(h(X))$ by g_2, and so on. The distribution functions $g_0, g_1, g_2, \ldots$ can also be regarded as the asymptotic distribution functions of sequences $(a_n), (h(a_n)), (h(h(a_n))), \ldots$. The operator P_h defined by $g'_{n+1} = P_h g'_n$ is the so-called Frobenius-Perron operator assigned to the transformation h. We say that h preserves the distribution function k if $g_0 = k$ implies $g_1 = k$ and consequently $g_n = k$ for all n, i.e., if k' is an invariant element of the operator P_h. Then under weak additional suppositions $g_n^{(r)} = k^{(r)} + O(q^n)$ for $r = 0, 1, 2$ and for arbitrary initial distribution functions g_0, where $q < 1$ depends on h but not on g_0. This result generalizes the Gauss-Kuzmin-Lévy theorem in the metric theory of continued fractions. Particularly, piecewise linear transformations preserve the uniform distribution. In this case more precise estimates are possible under weaker suppositions.

Key words: *Piecewise monotonic transformations, Frobenius–Perron operators, invariant measures, uniform distribution of sequences, metric theory of continued fractions*
AMS subject classification: 58F11, 28D05, 11K06, 11K50

1. Introduction

Recently, Bosch [1] and Porubský, Šalát, and Strauch [14] have studied transformations h that preserve the uniform distribution of sequences. On the other hand, Goh and Schmutz [2] have discovered that high iterates of certain piecewise monotonic transformations h generate the uniform distribution even if the initial distribution is not uniform. In what follows we wish to unify both directions of research. We shall show that piecewise monotonic transformations h that preserve the uniform distribution generate also the uniform distribution in the limit if high iterates are applied to an arbitrary initial distribution function. The preserving transformation h must only satisfy weak additional assumptions. The rate of convergence is estimated to be geometric. The best estimates are possible, of course, in the case of piecewise linear transformations.

All assertions can be extended to the case of piecewise monotonic transformations h which preserve a more general distribution function k. For instance, the transformation $h(x) = \{1/x\}$ preserves the distribution function $k(x) = \log(1+x)/\log 2$. This transformation is considered in the theory of continued fractions, cf., e.g., Khintchine [10], and is the subject of the celebrated Gauss-Kuzmin-Lévy theorem which is now contained in our estimates as a special case.

The transformation h of the unit interval induces a transformation P_h of probability densities on the unit interval. This transformation is accomplished by the so-called Frobenius-Perron operator assigned to h, cf. the remark succeeding Theorem 1. The convergence of iterates of the Frobenius-Perron operator is studied by, e.g., Lasota and Yorke [12, 13], Hofbauer and Keller [3], Jabłoński, Kowalski, and Malczak [6 - 9]. Our results are especially related to those in [6 - 9] on the geometric convergence of iterates of the Frobenius-Perron operator. However, all the above mentioned authors consider only finite partitions of the unit interval [0,1) and their results

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ISSN 0232-2064 / $ 2.50

are especially not applicable to the Gauss-Kuzmin-Lévy theorem. See for example also Ishitani [5], who has considered infinite partitions in another connection. In case of finite partitions our suppositions are somewhat different from those in [6 - 9].

2. Transformations

We partition the unit interval [0,1) into a denumerable set of subintervals $[x_i, y_i)$ such that

$$[0,1) = \bigcup_{i=1}^{\infty} [x_i, y_i). \tag{1}$$

Let $h_i \in C^2[0,1)$ be given for $i = 1, 2, \ldots$. We assume the h_i increasing on [0,1) and suppose $h_i(0) = x_i, h_i(1-0) = y_i$. Then we define the piecewise monotonic transformation h by $h(x) = h_i^{-1}(x)$ on $[x_i, y_i)$.

We do not assume that $x_{i+1} = y_i$ or that $x_i = y_{i+1}$. Thus we permit an infinite number of limit points of the x_i in the unit interval. The case of $y_i = x_i$ for $i > N$ is included. Thus we consider also finite partitions of [0,1).

Furthermore we mention that some or all h_i can be decreasing with $h_i(0) = y_i, h_i(1-0) = x_i$. Then the following considerations must be modified in an obvious manner. But for the sake of transparence we suppress the explicit inclusion of this generalization.

Theorem 1 : *Let X be a random variable on $[0,1)$ with the continuous distribution function $g_0(x) = \mathcal{P}(X < x)$. Then $h(X)$ has the distribution function*

$$g_1(x) = \sum_{i=1}^{\infty} \left(g_0(h_i(x)) - g_0(x_i) \right). \tag{2}$$

Furthermore, if g_0 is absolutely continuous, $|g_0'(x)| \le C$ a.e. and if $\sum_{i=1}^{\infty} h_i'(x)$ converges a.e. in $[0,1)$ to an integrable function, then g_1 is also absolutely continuous and

$$g_1'(x) = \sum_{i=1}^{\infty} g_0'(h_i(x)) h_i'(x) \ \text{ a.e. in } [0,1). \tag{3}$$

Proof: By the theorem of total probability we have

$$\mathcal{P}(h(X) < x) = \sum_{i=1}^{\infty} \mathcal{P}(h(X) < x, x_i \le X < y_i)$$

which proves (2). Now the right-hand side of (3) converges a.e. to an integrable function, say g. Then we see by the dominated convergence theorem that

$$\int_0^x g(t)dt \ = \ \sum_{i=1}^{\infty} \int_0^x g_0'(h_i(t)) h_i'(t) dt = \sum_{i=1}^{\infty} \left(g_0(h_i(x)) - g_0(x_i) \right) = g_1(x)$$

which proves (3) ▮

Remark : By (3) the initial probability density g_0' is transformed into the probability density g_1'. This transformation defines the Frobenius-Perron operator P_h by $g_1' = P_h g_0'$.

Let $(a_k)_{k=1}^{\infty}$ be a sequence of real numbers. The sequence (a_k) is said to have the *asymptotic distribution function* g_0 if

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} 1_{[0,x)}(\{a_k\}) = g_0(x)$$

for $x \in [0,1)$, cf. Kuipers and Niederreiter [11: p.53]. Here $1_{[0,x)}$ is the indicator function of the interval $[0,x)$ and $\{a_k\}$ is the fractional part of a_k. Since only the fractional parts of a_k are of interest, we suppose in the following that $0 \le a_k < 1$.

Theorem 2: *If the sequence $(a_k)_{k=1}^{\infty}$ has the asymptotic distribution function g_0 which is assumed to be absolutely continuous, then the sequence $(h(a_k))_{k=1}^{\infty}$ has the asymptotic distribution function g_1 given by* (2).

Proof: We choose N so large that $\sum_{i=1}^{N}\left(g_0(y_i) - g_0(x_i)\right) > 1 - \epsilon$. This is possible on account of the absolute continuity of g_0. Then we have

$$\begin{aligned}\sum_{i=1}^{N}\frac{1}{K}\sum_{k=1}^{K} 1_{[x_i,h_i(x))}(a_k) &\le \frac{1}{K}\sum_{k=1}^{K} 1_{[0,x)}(h(a_k)) \\ &\le \sum_{i=1}^{N}\frac{1}{K}\sum_{k=1}^{K} 1_{[x_i,h_i(x))}(a_k) + 1 - \sum_{i=1}^{N}\frac{1}{K}\sum_{k=1}^{K} 1_{[x_i,y_i)}(a_k).\end{aligned}$$

Letting $K \to \infty$ we obtain

$$\begin{aligned}\sum_{i=1}^{N}\left(g_0(h_i(x)) - g_0(x_i)\right) &\le \liminf_{K\to\infty}\frac{1}{K}\sum_{k=1}^{K} 1_{[0,x)}(h(a_k)) \\ &\le \limsup_{K\to\infty}\frac{1}{K}\sum_{k=1}^{K} 1_{[0,x)}(h(a_k)) \le \sum_{i=1}^{N}\left(g_0(h_i(x)) - g_0(x_i)\right) + \epsilon\end{aligned}$$

which proves the theorem for $\epsilon \to 0$ ∎

Definition: We say that a transformation h *preserves the distribution function* k if $g_0 = k$ is transformed by (2) into $g_1 = k$.

The transformation h preserves the distribution function k if the probability density k' is an invariant element of the Frobenius-Perron operator P_h. Particularly, the transformation h preserves the uniform distribution if 1 is an invariant element of P_h.

Theorem 3: *The transformation h preserves the uniform distribution if and only if*

$$\sum_{i=1}^{\infty}(h_i(x) - x_i) = x \tag{4}$$

for $x \in [0,1)$. Furthermore, if

$$\sum_{i=1}^{\infty} h_i'(x) = 1 \text{ a.e. in } [0,1), \tag{5}$$

then h preserves the uniform distribution.

Proof: The condition (4) follows from (2) by definition. Let (5) be satisfied. Then by the monotone convergence theorem and by $h_i(0) = x_i$ we have

$$x = \int_0^x \sum_{i=1}^{\infty} h_i'(t)\,dt = \sum_{i=1}^{\infty}\int_0^x h_i'(t)dt = \sum_{i=1}^{\infty}(h_i(x) - x_i).$$

Thus (4) is satisfied ∎

Theorem 3 is related to Theorem 2.9 in Bosch [1]. Obviously (5) implies that $h_i'(x) \le 1$ on [0,1) for $i = 1, 2, \ldots$

3. Higher iterates

We define recursively the random variables X_n by $X_{n+1} = h(X_n), X_0 = X$, the sequences $(a_{k,n})$ by $a_{k,n+1} = h(a_{k,n}), a_{k,0} = a_k$, and the distribution functions g_n by

$$g_{n+1}(x) = \sum_{i=1}^{\infty} \big(g_n(h_i(x)) - g_n(h_i(0))\big). \tag{6}$$

In order to obtain g_n directly from g_0, we introduce the vector $j_n = (i_1, i_2, \ldots, i_n)$ and put

$$h(j_{n+1}; x) = h(j_n; h_{i_{n+1}}(x)),\ h(j_1; x) = h_{i_1}(x). \tag{7}$$

Corollary 4: *We have*

$$g_n(x) = \sum_{j_n} \big(g_0(h(j_n; x)) - g_0(h(j_n; 0))\big). \tag{8}$$

Proof: We apply induction on n. For $n = 1$ the equality (8) is identical with (2). From (6) and (8) we obtain

$$\begin{aligned} g_{n+1}(x) &= \sum_{i=1}^{\infty}\sum_{j_n} \big(g_0(h(j_n; h_i(x))) - g_0(h(j_n; 0))\big) \\ &\quad - \sum_{i=1}^{\infty}\sum_{j_n} \big(g_0(h(j_n; h_i(0))) - g_0(h(j_n; 0))\big) \\ &= \sum_{j_{n+1}} \big(g_0(h(j_{n+1}; x)) - g_0(h(j_{n+1}; 0))\big) \end{aligned}$$

by (7). This is (8) with n replaced by $n + 1$ ∎

Corollary 5: *If g_0 is the distribution function of X, then g_n is the distribution function of X_n. If g_0 is absolutely continuous and the asymptotic distribution function of the sequence $(a_k)_{k=1}^{\infty}$, then g_n is the asymptotic distribution function of the sequence $(a_{k,n})_{k=1}^{\infty}$.*

Corollary 6: *If the transformation h preserves the uniform distribution, then it follows that $\sum_{j_n}(h(j_n; x) - h(j_n; 0)) = x$.*

Proof: Put $g_0(x) = g_n(x) = x$ in (8) ∎

Lemma 7: *If* (5) *holds a.e. in* $[0, 1)$, *then*

$$\sum_{j_n} h'(j_n; x) = 1 \quad \text{a.e. in } [0, 1). \tag{9}$$

Furthermore, if g_0 is absolutely continuous, $|g_0'| \le C$ a.e., then g_n is absolutely continuous and

$$g_n'(x) = \sum_{j_n} h'(j_n; x) g_0'(h(j_n; x)) \quad \text{a.e. in } [0, 1). \tag{10}$$

Proof: The equality (9) will be proved by induction on n. Namely,

$$\sum_{j_{n+1}} h'(j_{n+1}; x) = \sum_{i_{n+1}} \sum_{j_n} h'(j_n; h_{i_{n+1}}(x)) h_{i_{n+1}}'(x)$$

on account of (7). Now (10) can be concluded from (8) in the same way as (3) was concluded from (2) in the proof of Theorem 1 ∎

Remark : Corollaries 4–6 and Lemma 7 can be summarized to the assertion that the transformation $h^{\circ n} = h \circ \ldots \circ h$ is given by the $h(j_n; x)$ and satisfies the same suppositions as h.

Lemma 8 : *Assume that* (5) *holds* a.e. *in* $[0,1)$. *Further let*

$$\text{a)}\ h_i'(x) \le d < 1 \qquad \text{b)}\ |h_i''(x)| \le K h_i'(x), \tag{11}$$

for $x \in [0,1)$ *and* $i = 1, 2, \ldots$. *If moreover* $g_0(x) \in C^2[0,1]$, *i.e.* , *if*

$$\text{a)}\ |g_0'(x) - 1| \le C_1 \qquad \text{b)}\ |g_0''(x)| \le C_2, \tag{12}$$

then $g_n'' \in C^2[0,1]$ *too, and*

$$\text{a)}\ |g_1''(x)| \le KC_1 + dC_2 \qquad \text{b)}\ |g_n''(x)| \le KC_1/(1-d) + d^n C_2, \quad n \ge 2. \tag{13}$$

Proof: From (7) it follows that

$$h'(j_n; x) \le d^n. \tag{14}$$

Further we show by induction on n that

$$|h''(j_n; x)| \le K h'(j_n; x)/(1-d). \tag{15}$$

Namely, from (7) we conclude that

$$\begin{aligned} |h''(j_{n+1}; x)| &= |h''(j_n; h_{i_{n+1}}(x)) h_{i_{n+1}}'^2(x) + h'(j_n, h_{i_{n+1}}(x)) h_{i_{n+1}}''(x)| \\ &\le K h'(j_n; h_{i_{n+1}}(x)) h_{i_{n+1}}'(x) d/(1-d) \ + K h'(j_n; h_{i_{n+1}}(x)) h_{i_{n+1}}'(x) \\ &\le K h'(j_{n+1}; x)/(1-d). \end{aligned}$$

From (15) we derive

$$|\log h'(j_n; x) - \log h'(j_n; y)| \le |x - y| |h''(j_n; \xi)| / h'(j_n; \xi) \le K/(1-d),$$

$$h'(j_n; x)/h'(j_n; y) \le L \tag{16}$$

for arbitrary $x, y \in [0,1)$. Now we see that (9) holds uniformly in x. By termwise differentiation we get

$$\sum_{j_n} h''(j_n; x) = 0, \ \ g_n''(x) = \sum_{j_n} h''(j_n; x)(g_0'(h(j_n; x)) - 1) + \sum_{j_n} h'^2(j_n; x) g_0''(h(j_n; x))$$

also uniformly in [0,1) such that (13) follows ∎

4. Asymptotic estimates

In the following we wish to show the geometric convergence of the g_n to the uniform distribution or to a more general distribution function k which is preserved under the piecewise monotonic transformation h given by the h_i.

Theorem 9 : *Let* $g_0 \in C^2[0,1]$ *and assume that* (5) *and* (11) *and are satisfied. Then there exists a real* $q < 1$ *such that*

$$\text{a)}\ g_n'(x) = 1 + O(q^n) \qquad \text{b)}\ g_n''(x) = O(q^n). \tag{17}$$

Proof: From (16) we get $\Delta_{j_n} = h(j_n;1) - h(j_n;0) = h'(j_n;\xi_{j_n}) \le Lh'(j_n;x)$. By Lemma 7 we have

$$\begin{aligned}
g_n'(x) - 1 &= \sum_{j_n} h'(j_n;x)\big(g_0'(h(j_n;x)) - 1\big) \\
&= \sum_{j_n} \big(h'(j_n;x) - \Delta_{j_n}/L\big)\big(g_0'(h(j_n;x)) - 1\big) \\
&\quad + \frac{1}{L}\sum_{j_n}\left(\Delta_{j_n}\big(g_0'(h(j_n;x)) - 1\big) - \int_{h(j_n;0)}^{h(j_n;1)} \big(g_0'(x) - 1\big)dx\right)
\end{aligned}$$

where the equalities

$$\sum_{j_n} \Delta_{j_n} = 1, \ \sum_{j_n} \int_{h(j_n;0)}^{h(j_n;1)} \big(g_0'(x) - 1\big)dx = \int_0^1 \big(g_0'(x) - 1\big)dx = 0$$

are used. The first one can be proved by induction on n whereas the second one is a consequence of the first one. Since $g_0 \in C^2[0,1]$, condition (12) is fulfilled. Thus we obtain

$$\begin{aligned}
|g_n'(x) - 1| &\le \sum_{j_n} \big(h'(j_n;x) - \Delta_{j_n}/L\big)C_1 + \left|\frac{1}{L}\sum_{j_n} \Delta_{j_n}\big(g_0'(h(j_n;x)) - g_0'(h(j_n;\tau_{j_n}))\big)\right| \\
&\le (1 - 1/L)C_1 + \left|\frac{1}{L}\sum_{j_n} \Delta_{j_n}(x - \tau_{j_n})g_0''(h(j_n;\sigma_{j_n}))h'(j_n;\sigma_{j_n})\right| \\
&\le (1 - 1/L)C_1 + d^n C_2/L,
\end{aligned}$$

by (14). On the other hand, from Lemma 8 we get $|g_n''(x)| \le KC_1/(1-d) + d^n C_2$. Now we choose l so large that $d^l \le min\big((1-d)/6K, 1/6\big)$. Moreover we suppose $K > 0$ and enlarge C_1 or C_2 such that $C_2 = 3KC_1/(1-d)$. Then

$$|g_l'(x) - 1| \le (1 - 1/2L)C_1 \quad \text{and} \quad |g_l''(x)| \le 3K(1 - 1/2L)C_1/(1-d).$$

Starting with g_k instead of g_0 we arrive at

$$|g_{ml+k}'(x) - 1| \le (1 - 1/2L)^m C \quad \text{and} \quad |g_{ml+k}''(x)| \le 3K(1 - 1/2L)^m C/(1-d)$$

for $0 \le k < n, m = 0, 1, \ldots$, and $C = \sup_{x\in[0,1], 0\le k<l} \big(|g_k'(x) - 1|, |g_k''(x)|(1-d)/3K\big)$ ∎

Note that $q = (1 - 1/2L)^{1/l}$ depends on h but not on g_0. Only C depends on g_0. If $|h'(x)| \le M$ for $x \in [0,1]$, then $h_i'(x) \ge 1/M$, and the supposition (11)/b) will be satisfied in many cases because $h_i \in C^2[0,1)$. However, $|h'(x)| \le M$ can only hold for finite partitions of the unit interval. The supposition (11)/a) will also be satisfied in many cases since $h_i'(x) \le 1$ is implied by (5). Thus the condition (5) is the most significant supposition for Theorem 9, it ensures that h preserves the uniform distribution.

We remark that the supposition $g_0, h_i \in C^2[0,1]$ can be replaced by a Lipschitz condition for g_0' and h_i'.

Now we assume that the transformation h preserves the distribution function $k = k(x)$, where $k \in C^2[0,1]$ and

$$k'(x) \ge \delta > 0 \tag{18}$$

for $x \in [0,1]$. Then by (2)

$$k(x) = \sum_{i=1}^{\infty} \big(k(h_i(x)) - k(x_i)\big). \tag{19}$$

We introduce the transformation $h^* = k \circ h \circ k^{-1}$ by $h_i^*(x) = k(h_i(k^{-1}(x)))$. In view of (19) h_i^* fulfils (4), i.e., h^* preserves the uniform distribution. Since $h = k^{-1} \circ h^* \circ k$, we have $h^{\circ n} = k^{-1} \circ (h^*)^{\circ n} \circ k$. The function k transforms the initial distribution function g_0 into $g_0^* = g_0 \circ k^{-1}$. On account of (18), the distribution function g_0^* fulfils the suppositions of Theorem 9 in the same way as g_0. The distribution function g_0^* is transformed by $(h^*)^{\circ n}$ into g_n^*. For g_n^* the estimates (17) hold true. Finally k^{-1} transforms into $g_n = g_n^* \circ k$.

Corollary 10 : *Let the transformation h preserve the distribution function $k \in C^2[0,1]$ with (18). Assume that $h^* = k \circ h \circ k^{-1}$ satisfies the suppositions of Theorem 9. Furthermore let $g_0 \in C^2[0,1]$. Then*

$$\text{a)}\ g_n(x) = k(x) + O(q^n) \quad \text{b)}\ g_n'(x) = k'(x) + O(q^n) \quad \text{c)}\ g_n''(x) = k''(x) + O(q^n) \tag{20}$$

uniformly for $x \in [0,1]$.

Notice that the last estimate in (20) implies both others since $\int_0^1 g_n'(x)dx = \int_0^1 k'(x)dx = 1$, $g_n(0) = k(0) = 0$.

Corollary 10 is related to results by Jabłoński, Kowalski, and Malczak [7, 8]. But these authors consider only finite partitions of the unit interval. Moreover, in [8] they assume the h_i to be convex, and in [7] they suppose that the h_i satisfy certain complicated conditions which are quite different from our suppositions. On the other hand, in [7, 8] the initial probability density g_0' is required only to be of bounded variation.

Theorem 11 : *Let the transformation h preserve the distribution function $k \in C^2[0,1)$ with (18). Assume that $h^* = k \circ h \circ k^{-1}$ satisfies the suppositions (5) and (11). Furthermore, let $g_0 \in C^2[0,1]$. Then (20) holds with $q \le K/2 + d$.*

Proof: Let g_0^* satisfy (12)/b). Then we conclude that

$$|g_0^{*\prime}(x) - 1| = \left|\int_0^1 (g_0^{*\prime}(x) - g_0^{*\prime}(t))dt\right| \le \int_0^1 |g_0^{*\prime}(x) - g_0^{*\prime}(t)|dt \le \int_0^1 |x-t|C_2 dt \le C_2/2,$$

and from (13)/a) that $|g_1^{*\prime\prime}(x)| \le KC_2/2 + dC_2 = (K/2+d)C_2$. Now (20) can be obtained by iterating the given estimates ∎

We remark that more precisely $|g_n^{*\prime\prime}(x)| \le C_2 q^n$ and $|g_n^{*\prime}(x) - 1| \le C_2 q^n/2$ are derived in the proof of Theorem 11. Further we obtain

$$|g_n^*(x) - x| \le \min(x, 1-x)C_2 q^n/2 \le x(1-x)C_2 q^n.$$

In all these estimates C_2 is an upper bound for the absolute value of the second derivative of $g_0^* = g_0 \circ k^{-1}$. For g_n we obtain more precisely the estimates

$$\begin{aligned}
&\text{a)} \quad |g_n(x) - k(x)| \le k(x)(1-k(x))C_2 q^n \\
&\text{b)} \quad |g_n'(x) - k'(x)| \le k'(x)C_2 q^n/2 \\
&\text{c)} \quad |g_n''(x) - k''(x)| \le \big(|k''(x)|/2 + k'^2(x)\big)C_2 q^n.
\end{aligned} \tag{21}$$

Theorem 11 is an improvement of a result by Jabłoński and Malczak [6, 9] who have essentially estimated q by $K + d$, again only in the case of finite partitions and for g_0' of bounded variation.

Example 12 : We consider the transformation $h(x) = \{1/x\}$ which plays an important role in the theory of continued fractions. It can be given by $h_i(x) = 1/(x+i)$, $x_i = 1/i$, $y_i = 1/(1+i)$ and leads to the recurrence

$$g_{n+1}(x) = \sum_{i=1}^{\infty} \left(g_n\left(\frac{1}{i}\right) - g_n\left(\frac{1}{x+i}\right)\right).$$

Since the h_i are decreasing, the sign here is opposite to that in (6). The transformation h preserves the distribution function $k(x) = \log(1+x)/\log 2$, i.e., the transformation h^* given by

$$h_i^*(x) = \frac{1}{\log 2}\big(\log(i+t) - \log(i-1+t)\big), \quad t = e^{x \log 2},$$

preserves the uniform distribution. We have

$$h_i^{*\prime}(x) = -t/(i+t)(i-1+t),$$
$$h_i^{*\prime\prime}(x)/h_i^{*\prime}(x) = \log 2 \cdot \big(i(i-1) - t^2\big)/(i+t)(i-1+t).$$

The supposition (5) is satisfied in the modified form $\sum h_i^{*\prime}(x) = -1$, observing that the h_i^* are decreasing. The supposition (11)/a) holds with $d = 1/2$, (11)/b) holds with $K = \log 2$. The supposition (18) for k holds with $\delta = 1/2 \log 2$. Thus (21) is valid for $q \leq (\log 2 + 1)/2 < 0.85$ if $g_0 \in C^2[0,1]$. For $g_0(x) = x$ we get $C_2 = \sup_{x \in [0,1]} \left(k^{-1}(x)\right)'' < 1$.

Note that *the arguments leading to the estimates* (21) *are very simple and quite elementary.* The application of Corollary 10 is possible but supplies a larger value for q. Moreover, the reasoning being the basis of Corollary 10 is more complicated.

The estimate (21)/a) is an improved version of Kuzmin's celebrated theorem, cf., e.g., Khintchine [10: p. 76]. Kuzmin has given the remainder $O\left(q^{\sqrt{n}}\right)$. The improved remainder in (21)/a) is due to Lévy who found it independently of Kuzmin and only a bit later. Many years later the constant q was improved by Szüsz [15] to $q < 0.4$, and again later Wirsing [16] has found that $q < 0.3037$ and that this estimate is not improvable. For a survey on the history of the problem cf. also Iosifescu [4].

5. Piecewise linear transformations

In the following we consider transformations h for which

$$h_i(x) = x_i + (y_i - x_i)x. \tag{22}$$

We assume $x_{i+1} = y_i$ for all i, i.e., the x_i can accumulate at most in the right end of the unit interval. All transformations h defined by (22) preserve the uniform distribution. If h is given by (22), then we can relax the suppositions on g_0.

Theorem 13 : *Let g_0 be concave on $[0,1)$ and assume that $g_0'(0)$ and $g_0'(1)$ exist. Then we have the non-uniform estimate*

$$|g_1(x) - x| \leq qx(1-x)\big(g_0'(0) - g_0'(1)\big), \quad \text{where } q = max_i(|x_i - x_{i+1}|). \tag{23}$$

Proof: We estimate

$$S = g_1(x) - x = \sum_{i=1}^{\infty} \Big(g_0(h_i(x)) - g_0(x_i) - x\big(g_0(x_{i+1}) - g_0(x_i)\big)\Big).$$

For $i > N$ we replace x_i by 1. Moreover we choose an integer m_1 and replace x by r/m_1 and x_i by r_i/m_1 for $i \leq N$, where r and r_i are integers and where the r_i are increasing. Then S changes over to S_{m_1} where $|S - S_{m_1}| \leq \epsilon$. This is possible if we choose first N and then m_1 large enough. We set $m = m_1^2$ and write $S_{m_1} = \sum_{n=1}^{m} a_n b_n$, where $b_n = g_0(n/m) - g_0((n-1)/m)$ and where

$$a_n = \begin{cases} 1 - r/m_1 & for \quad r_i m_1 < n \leq r_i m_1 + (r_{i+1} - r_i)r, \\ -r/m_1 & for \quad r_i m_1 + (r_{i+1} - r_i)r < n \leq r_{i+1} m_1. \end{cases}$$

We put $A_n = \sum_{k=1}^{n} a_k$ and get

$$A_{r_i m_1} = 0, A_m = 0 \quad \text{as well as} \quad 0 \leq A_n \leq R = r(1 - r/m_1)max_i(r_{i+1} - r_i).$$

In $S_{m_1} = \sum a_n b_n$ we apply summation by parts and obtain

$$\begin{aligned} S_{m_1} &= \sum_{n=1}^{m-1} A_n(b_n - b_{n+1}), \\ |S_{m_1}| &\leq R \sum_{n=1}^{m-1} (b_n - b_{n+1}) \leq R(b_1 - b_m) \\ &= R\big(g_0(1/m) - g_0(0) - g_0(1) + g_0((m-1)/m)\big) \quad \to qx(1-x)\big(g_0'(0) - g_0'(1)\big) \end{aligned}$$

as $m_1 \to \infty$. This proves our assertion as $\epsilon \to 0$ ▮

Corollary 14: *Under the suppositions of Theorem 13 we have*

$$|g_n(x) - x| \leq q^n x(1-x)\big(g_0'(0) - g_0'(1)\big). \tag{24}$$

Proof: If q is the maximal length of the intervals for h, then q^n is the maximal length of the intervals for $h^{\circ n}$ ▮

We remark that (24) can be generalized if g_0 is concave for $b_{2i} \leq x \leq b_{2i+1}$ and convex for $b_{2i+1} \leq x \leq b_{2i+2}, i = 1, 2, \ldots, N$. Then

$$|g_n(x) - x| \leq q^n x(1-x) \sum_{i=1}^{N} \gamma_i,$$

where

$$\gamma_i = g_0'(b_{2i} + 0) - g_0'(b_{2i+1} - 0) - g_0'(b_{2i+1} + 0) + g_0'(b_{2i+2} - 0).$$

In this connection $b_i = b_{i+1}$ is allowed. Corollary 14 generalizes Theorem 3 in Goh and Schmutz [2]. We remark further that the assertions of Theorem 13 and Corollary 14 are valid if the x_i have a finite number of limit points in the unit interval or if the number of limit points is arbitrary and g_0 is absolutely continuous.

Again we consider $h = k^{-1} \circ h^* \circ k$, where h^* is defined by (22), i.e.,

$$h_i(x) = k^{-1}\big(x_i + (y_i - x_i)k(x)\big). \tag{25}$$

Corollary 15: *Let the transformation h be given by (25), let g_0 fulfil the suppositions of Theorem 13. Further assume that k is convex and that $k'(0), k'(1) > 0$. Then*

$$|g_n(x) - k(x)| \leq q^n k(x)\big(1 - k(x)\big)\big(g_0'(0)/k'(0) - g_0'(1)/k'(1)\big). \tag{26}$$

Proof: By supposition k^{-1} is concave and consequently $g_0^* = g_0 \circ k^{-1}$ is concave, too. Furthermore, $g_0^{*\prime}(0) = g_0'(0)/k'(0)$ and $g_0^{*\prime}(1) = g_0'(1)/k'(1)$. It follows that $|g_n^*(x) - x| \leq q^n x(1-x)\big(g_0'(0)/k'(0) - g_0'(1)/k'(1)\big)$ which implies (26) ∎

Corollary 15 generalizes and improves Theorem 1 in Goh and Schmutz [2].

Example 16: We consider the functions h and k of Example 12. But we replace $(h^*)^{\circ n}$ by a piecewise linear function H_n^* with the same partition of [0,1) as $(h^*)^{\circ n}$, i.e.,

$$H_n^*(x) = \begin{cases} \big(x - h^*(j_n;0)\big)/\Delta_{j_n} & \text{for} \quad h^*(j_n;0) \leq x < h^*(j_n;1), \\ \big(x - h^*(j_n;1)\big)/\Delta_{j_n} & \text{for} \quad h^*(j_n;1) \leq x < h^*(j_n;0), \end{cases}$$

where $\Delta_{j_n} = |h^*(j_n;1) - h^*(j_n;0)|$. Note that H_n^* is different from $(H_1^*)^n$. We have

$$h^*(j_n;x) = \frac{1}{\log 2}\log\left(1 + \frac{p_n + p_{n-1}(t-1)}{q_n + q_{n-1}(t-1)}\right),$$

cf. Khintchine [10: p. 79]. Here p_n and q_n are the numerator and the denominator, respectively, of the n-th convergent $p_n/q_n = [i_1, i_2, \ldots, i_n]$ formed with the partial quotients $i_1, i_2, \ldots, i_n$. It follows that

$$log2 \cdot \Delta_{j_n} = \left|\log\left(1 + \frac{p_n + p_{n-1}}{q_n + q_{n-1}}\right) - \log\left(1 + \frac{p_n}{q_n}\right)\right| = \left|\log\left(1 + \frac{(-1)^n}{(q_n + p_n)(q_n + q_{n-1})}\right)\right|.$$

The maximum of Δ_{j_n} is achieved for $j_n = (1, \ldots, 1)$. In this case we have $p_n = f_n, q_n = f_{n+1}$, where $f_{n+2} = f_{n+1} + f_n$, $f_1 = f_2 = 1$ is the sequence of Fibonacci numbers. It is well-known that $f_n \sim \big((1+\sqrt{5})/2\big)^n/\sqrt{5}$ as $n \to \infty$. We put $H_n = k^{-1} \circ H_n^* \circ k$ and obtain $|H_n(x) - k(x)| < Cq^n$, for $q = 2/\big(3 + \sqrt{5}\big) \leq 0.3820$.

In the following last two theorems let h be given by (22), but $x_{i+1} = y_i$ for all i is not needed, i.e., the x_i can have an arbitrary number of limit points in the unit interval.

Theorem 17: *If $g_0 \in C^2[0,1)$, then $|g_n'(x) - 1| \leq q^n \sup_{x\in[0,1)} |g_0'(x)|$.*

Proof: We have

$$g_1'(x) - 1 = \sum_{i=1}^{\infty}\left(h_i'(x)g_0'(h_i(x)) - \int_{x_i}^{y_i} g_0'(x)dx\right) = \sum_{i=1}^{\infty}(y_i - x_i)\big(g_0'(h_i(x)) - g_0'(\xi_i)\big)$$

a.e. and consequently $|g_1'(x) - 1| \leq q \sup_{x\in[0,1)} |g_0''(x)|$. Now the assertion follows by considering $h^{\circ n}$ instead of h ∎

Theorem 18: *If $g_0 \in C^k[0,1], k \geq 2$, then $|g_n^{(k)}(x)| < q^{(k-1)n} \sup_{x\in[0,1)} |g_0^{(k)}(x)|$.*

Proof: We differentiate (2) and obtain

$$g_1^{(k)}(x) = \sum_{i=1}^{\infty}(h_i'(x))^k g_0^{(k)}(h_i(x)) \quad \text{and} \quad |g_1^{(k)}(x)| \leq q^{k-1} \sup_{x\in[0,1)} |g_0^{(k)}(x)| \sum_{i=1}^{\infty} h_i'(x) \tag{27}$$

which implies the assertion ∎

Example 19: Let $g_0(x) = x + \beta x(1-x)$, where $-1 \leq \beta \leq 1$. Then $g_0''(x) = -2\beta$ and therefore $g_1''(x) = -2\beta q$, $q = \sum_{i=1}^{\infty}(h_i'(x))^2 = \sum_{i=1}^{\infty}(y_i - x_i)^2 < 1$ in view of (27). We obtain $g_n''(x) = -2\beta q^n$ which implies $g_n(x) = x + \beta q^n x(1-x)$.

Acknowledgement: The author is very indebted to L. Heinrich for the references to [3–5,12] and for the proof of (16) which was formerly an additional supposition in Theorem 9. Furthermore, the author thanks one referee for many useful hints.

REFERENCES

[1] Bosch, W.: *Functions that preserve the uniform distribution of sequences.* Trans. Amer. Math. Soc. **307** (1988), 143 - 152.

[2] Goh, W. M. Y. and E. Schmutz: *Universal asymptotic distribution functions mod 1.* Lect. Notes Pure Appl. Math. **124** (1990), 619 - 626.

[3] Hofbauer, F. and G. Keller: *Ergodic properties of invariant measures for piecewise monotonic transformations.* Math. Z. **180** (1982), 119 - 140.

[4] Iosifescu, M.: *A survey of the metric theory of continued fractions, fifty years after Doeblin's 1940 paper.* In: Probability theory and mathematical statistics (Proc. V. Vilnius Conf. Vilnius/Lith. 1989, eds.: B. Grigelonis, Yu. V. Prokhorov, V.V. Sazonov, V. Statulevičius). Vol. **1** (1990), 550 - 572.

[5] Ishitani, H.: *A central limit theorem of mixed type for a class of 1-dimensional transformations.* Hiroshima Math. J. **16** (1986), 161 - 188.

[6] Jabłoński, M.: *On the convergence of iterates of the Frobenius-Perron operator for expanding mappings.* Ann. Pol. Math. **48** (1988), 127 - 130.

[7] Jabłoński, M., Kowalski, Z. S., and J. Malczak: *The rate of convergence of iterates of the Frobenius-Perron operator for Lasota-Yorke transformations.* Univ. Iagell. Acta Math. **25** (1985), 189 - 193.

[8] Jabłoński, M. and J. Malczak: *The rate of convergence of iterates of the Frobenius-Perron operator for piecewise convex transformations.* Bull. Polish Acad. Sci. Math. **31** (1983), 249 - 254.

[9] Jabłoński, M. and J. Malczak: *The rate of convergence of the Frobenius-Perron operator.* Colloq. Math. **49** (1984), 67 - 72.

[10] Khintchine, A.: *Kettenbrüche.* Leipzig: B.G. Teubner Vertragsgesellschaft 1956.

[11] Kuipers, L. and H. Niederreiter: *Uniform distribution of sequences.* New York: Wiley 1974.

[12] Lasota, A. and J. A. Yorke: *On the existence of invariant measures for piecewise monotonic transformations.* Trans. Amer. Math. Soc. **186** (1973), 481 - 488.

[13] Lasota, A. and J. A. Yorke: *Exact dynamical systems and the Frobenius-Perron operator.* Trans. Amer. Math. Soc. **273** (1982), 375 - 384.

[14] Porubský, S., Šalát, T., and O. Strauch: *Transformations that preserve uniform distribution.* Acta Arith. **49** (1988), 459 - 479.

[15] Szüsz, P.: *Über einen Kusminschen Satz.* Acta Math. Acad. Sci. Hungary **12** (1961), 447 - 453.

[16] Wirsing, E.: *On the theorem of Gauss-Kusmin-Lévy and a Frobenius-type theorem for function spaces.* Acta Arith. **24** (1974), 507 - 528.

Received 09.01.1992