Eigenvalue Distribution of Invariant Linear Second Order Elliptic Differential Operators with Constant Coefficients

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1-

Let Θ be a properly discontinuous group of affine transformations acting on an n-dimensional affine space and P a G-invariant linear elliptic differential operator with constant coefficients. In this paper the 63 -automorphic eigenvalue problem to P is solved. For the number $N(\lambda)$ of the eigenvalues which are less than or equal to the "frequency bound" λ^2 the asymp-Let Θ be a properly discontinuous grous
affine space and P a Θ -invariant line
cients. In this paper the Θ -automorph
 $N(\lambda)$ of the eigenvalues which are less
totic estimation $N(\lambda) = c_0 \lambda^n + c_1 \lambda^{n-1}$
esting geom $+ O(\lambda^{n-2+2/(n+1)})$ is given with c_0 and c_1 being inter-M. BELGER

Let Θ be a properly discontinuous group of affine transformations acting

affine space and P a Θ -invariant linear elliptic differential operator v

cients. In this paper the Θ -automorphic eigenvalue pr **Example 18 and P a G-invariant linear elliptic differential operator with concients. In this paper the G-automorphic eigenvalue problem to P is solved. Followith** $N(\lambda)$ **of the eigenvalue N(\lambda) = c_0 \lambda^n + c_1 \lambda^{n-1} + O(\lambda^{n-2+2**

Key words *Eigenvalue problem eigenvalue distribution invariant linear elliptic differential operator, lattice remainder, asymptotic estimation principal vector*

AMS subject classification: 47F05, 47A75, 11F72, 11L07, 11H06, 35P20

Let $\mathfrak B$ be an n-dimensional vector space or later at the same time also an affine space, $\mathfrak B^*$ its dual; Θ a properly discontinuous group of affine transformations acting on $\mathfrak B$ and having a compact fundamental domain [3]. For a \mathfrak{G} -invariant positive definite quadratic form \mathfrak{P} on \mathfrak{B}^* and for a fixed vector $p \in \mathfrak{B}^*$ we consider the differential operator *Physical enoterine, eigenvalue enoterine, eigenvalue institution, inverting the calculation in principal vector*
 Physical detection: 47 F 05, 47 A 75, 11 F 72, 11 L 07, 11 H 06, 35 P 20
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$$
P[\psi] = \mathfrak{P}\left(\frac{\partial}{\partial \xi} - 2\pi i \mathfrak{p}, \frac{\partial}{\partial \xi} - 2\pi i \mathfrak{p}\right) [\psi], \ \mathfrak{g} \in \mathfrak{B}, \tag{1}
$$

and the assigned polynom

$$
P(\mathbf{v}) = -\mathfrak{P}(\mathbf{v} - 2\pi\mathfrak{p}, \mathfrak{v} - 2\pi\mathfrak{p}), \mathfrak{v} \in \mathfrak{B}^*.
$$
 (1)

 Θ -invariant means for $P[\]$ that the following relation is valid:

 $P[\psi \circ S] = P[\psi] \circ S$, for all $S \in \mathcal{G}$. Now look at the **@-automorphic eigenvalue problem** $\mathfrak{P}(\mathfrak{v} - 2\pi\mathfrak{p}, \mathfrak{v} - 2\pi\mathfrak{p})$, $\mathfrak{v} \in \mathfrak{B}^*$.

the means for $P[\]$ that the following relation is valid:
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 $\mathfrak{G}-\text{invariant means for } P[$] that the following relation is valid:
 $P[\psi \circ S] = P[\psi] \circ S$, for all $S \in \mathfrak{G}$.

Now look at the

$$
P[\psi] + \mu \psi = 0, \quad \psi \in L_2(\mathfrak{G}). \tag{2}
$$

M. Belger: Universität Leipzig, Institut für Mathematik, Augustuspl. 10, D - 04109 Leipzig ISSN 0232-2064. / \$ 2.50 © Heldermann Verlag Berlin Contract of the State of the State $L_2(\mathfrak{B})$ is the Hilbert space over C of locally square-integrable \mathfrak{B} -automorphic functions. spec_{$\mathfrak{cs}(P)$} denotes the eigenvalue spectrum of (2). We will investigate the eigenvalue distribution dis(spec $_{G}(P)$) over \mathbb{R}^+ , where "dis" is, defined by the distribution function *M.* BELGER
 N) is the Hilbert space over C of locally square-integrable Θ -automorphic functions.
 $\Theta(P)$ denotes the eigenvalue spectrum of (2). We will investigate the eigenvalue distri-
 N(λ) = **x**_{*µ* (*z*

$$
N(\lambda) = \mathbf{u}\{\mu \in \text{spec}_{\mathbf{G}}(P) : \mu \leq \lambda^2\}.
$$
 (3)

Here sometimes λ instead of λ^2 is taken and called in Weyl's considerations "frequency bound" [25]. To establish a good asymptotic estimation of $N(\lambda)$ we will work out the following subjects:

- 1. Solution of the \mathfrak{G} -automorphic eigenvalue problem (2).
- 2. Description of $N(\lambda)$ by a certain number of so-called "principal lattice vectors" in a convex domain $\lambda \cdot \mathbb{D} \subset \mathfrak{B}^*$ (see (23)/(23')).
- 3. Formulation of $N(\lambda)$ as a finite sum of Weyl sums.
- **4.** Asymptotic estimation $N(\lambda) = c_0 \lambda^n + c_1 \lambda^{n-1} + O(\lambda^{n-2+2/(n+1)})$ with explicit calculation of c_0 and c_1 as geometric invariants. Survey of influence of fixed (fixed point free) elements of the group Θ on the asymptotic expression for $N(\lambda)$. 2. Description of the Θ and $N(\lambda)$ by a certain number of so-called "principal lattic convex domain $\lambda \cdot D \subset \mathfrak{B}^*$ (see (23)/(23)).
3. Formulation of $N(\lambda)$ as a finite sum of Weyl sums.
4. Asymptotic estimation N

1. Solution of the **9**-automorphic eigenvalue problem (2)

1.1 The orthonormal system of \mathfrak{G} -automorphic functions in $L_2(\mathfrak{G})$. To introduce such a system we follow the proceeding of P. Günther in [7: § 1 and § 2].

The lattice $\Gamma \subset \mathfrak{B}$ **:** We will write the affine transformation $S: \mathfrak{B} \to \mathfrak{B}$ ($S \in \mathfrak{B}$) of the n-dimensional affine space $\mathfrak B$ as a Seitzian space group symbol $S = (\sigma, \mathfrak f)$ with $\mathfrak k' = S(\mathfrak k) = \sigma \mathfrak k + \mathfrak f$ $(\mathfrak k', \mathfrak k \in \mathfrak k)$ 23) as transformation formula. The components e and f are said to be *fixed point* and *transla*tion part of S, respectively. For $R = (\rho, r) \in \mathfrak{G}$ and $S \in \mathfrak{G}$ the composition $R \circ S = (\rho \circ \sigma, \sigma r + \mathfrak{f})$ is defined by $(R \circ S)(r) = S(R(r))$. The inverse to S with respect to the identity element $E = (e, 0)$ $\in \mathfrak{G}$ is $S^{-1} = (\sigma^{-1}, -\sigma^{-1})$, where $e = id$ and $\mathfrak{g} \in \mathfrak{B}$ is the null vector. Now we consider the "point" group" Ω of \mathfrak{G} , **orthonormal system of** \mathcal{D} **-automorphic functions in** $L_2(\mathcal{D})$ **.** To introduce such a system
low the proceeding of P. Guinther in [7: § 1 and § 2].
 ttice $\Gamma \subset \mathfrak{B}$ **:** We will write the affine transformation $S: \$ tice $\Gamma \subset \mathfrak{B}$: We will write

affine space \mathfrak{B} as a Seit

ransformation formula.

rt of S, respectively. For

l by $(R \circ S)(\mathfrak{x}) = S(R(\mathfrak{x}))$.
 $S^{-1} = (\sigma^{-1}, -\sigma^{-1}\mathfrak{f})$, where
 \mathfrak{B} of \mathfrak{G} ,
 $\{\sigma: (\sigma, \mathfrak$

$$
\Omega = \{ \sigma \colon (\sigma, f) \in \mathfrak{G} \text{ for some } f \in \mathfrak{B} \}
$$
 (4)

and the "translation group" $\Sigma \subset \mathfrak{G}$ of all translations in \mathfrak{G} ,

$$
\mathfrak{X} = \{ (e, t) \in \mathfrak{G} \}. \tag{4'}
$$

We know about Ω and Σ the following [1,3,5]: Σ is an invariant subgroup of Θ . The factor group $\mathfrak{G}/\mathfrak{X}$ and \mathfrak{G} are isomorphic and ord $(\mathfrak{G}/\mathfrak{X})$ is finite. Therefore we can introduce know about **Q** and **L** to \mathfrak{g} ord **Q** are isom
r = ord($\mathfrak{G}/\mathfrak{X}$) = ord **Q**.

 $(4⁷)$

 \hat{x} has *n* generators (e, b_1) , ..., (e, b_n) with *n* linear independent translation parts b_k which are used to form the bas $\mathfrak B$ and also to form the Ω -invariant n-dimensional lattice

$$
\Gamma \coloneqq \operatorname{orb}_{\mathfrak{X}}(\mathfrak{o}) = \{ t = t^k \mathfrak{b}_k : t^k \in \mathbb{Z} \} \subset \mathfrak{B}.\tag{5}
$$

 $R = \{ \sigma : (\sigma, f) \in \mathcal{G} \}$ for some $f \in \mathcal{B} \}$

the "translation group" $\mathcal{Z} \subset \mathcal{G}$ of all translations i
 $\mathcal{Z} = \{ (e, t) \in \mathcal{G} \}$.

know about \mathcal{Q} and \mathcal{Z} the following $[1, 3, 5]$: \mathcal{Z} is $\mathcal{G}/$ The vector $a \in \mathfrak{B}$ is said to be "belonging to $\sigma \in \mathfrak{L}$ " if $(\sigma, a) \in \mathfrak{G}$. Together with a then also all vectors $a + \Gamma$ and only these are belonging to σ . So modulo Γ exactly one vector a is belonging to σ and will be denoted by $a = f$. In the coset decomposition of Θ relative to \mathfrak{X} ,

Eigenvalue Distribution of Differential Operators 287
\n
$$
\mathfrak{G} = S_1 \circ \mathfrak{X} + ... + S_r \circ \mathfrak{X}, \quad S_v = (\sigma_v, f_v)
$$
\n(6)

the elements of one of the same coset $S_v \circ \hat{\mathfrak{X}}$ have the same fixed point part σ_v but different cosets have different such parts. If (σ_1, f_1) , (σ_2, f_2) , $(\sigma_1 \circ \sigma_2, f) \in \mathfrak{B}$ it may be adventageous to think of the Frobenius congruence $B = S_1 \circ \mathfrak{X} + \mathfrak{e}$ ements of
s have different of the Frot
 $\frac{1}{2} + \frac{1}{2} = \frac{1}{2}$
and lattice ... + $S_r \circ \hat{\mathbf{\Sigma}}$, $S_v = (\sigma \circ \hat{\mathbf{S}})$

one of the same cos

erent such parts. If

benius congruence

mod Γ .
 $\Gamma^* \subset \mathfrak{B}^*$: A usually

$$
\sigma_{\mathbf{t}} \mathbf{f}_{\mathbf{t}} + \mathbf{f}_{\mathbf{t}} = \mathbf{f} \bmod \Gamma. \tag{7}
$$

The dual lattice $\Gamma^* \subset \mathfrak{B}^*$: A usually in cristallography here we turn to the dual situation. Let \mathfrak{B}^* be the dual space of linear functionals on $\mathfrak{B}, \langle \mathfrak{v}, \mathfrak{k} \rangle$ the value of $\mathfrak{v} \in \mathfrak{B}^*$ in $\mathfrak{k} \in \mathfrak{B}$. Relative to $\Gamma \subset \mathfrak{B}$, let Example 19 = $S_1 \circ \chi + ... + S_r \circ \chi$, $S_v = (O_v, I_v)$

Elements of one of the same coset $S_v \circ \chi$ have the same fixed point part G_v but different

ts have different such parts. If (o_1, f_1) , (o_2, f_2) , $(o_1 \circ o_2, f) \in \mathcal{B}$ i

$$
\Gamma^* = \{ \mathbf{u} = u_k \, \delta^k \colon u_k \in \mathbb{Z} \} \subset \mathfrak{B}^*, \quad \langle \mathbf{b}^h, \mathbf{b}_k \rangle = \delta^h_k, \tag{8}
$$

be the dual lattice in \mathfrak{B}^* . As bas \mathfrak{B}^* we use then $\{\mathfrak{b}^1, \dots, \mathfrak{b}^n\}$. Instead of $\sigma \in \mathfrak{L}$ here we need the adjoint mapping o' to **0:** *ci* is defined by \mathfrak{B} , let
 $\mathfrak{B} = \{u = u_k \delta^k : u_k \in \mathbb{Z}\} \subset \mathfrak{B}^*, \ \langle \delta^h, \delta_k \rangle = \delta^h_k,$

dual lattice in \mathfrak{B}^* . As bas \mathfrak{B}^* we use then $\{b^1, ..., b^n\}$. Instead
 $\colon \mathfrak{B}^* \to \mathfrak{B}^*$ with $\sigma^T \mathfrak{v} = \mathfrak{v} \circ \sigma$.

$$
\sigma^{\tau} \colon \mathfrak{B}^* \to \mathfrak{B}^* \quad \text{with } \sigma^{\tau} \mathfrak{v} = \mathfrak{v} \circ \sigma.
$$

adjoint mapping σ^r to σ : σ^r is defined by
 σ^r : $\mathfrak{B}^* \to \mathfrak{B}^*$ with $\sigma^r \mathfrak{v} = \mathfrak{v} \circ \sigma$.
 The pricipal classes $\mathfrak{h} \subset \Gamma^*$ **:** For a fixed lattice functional $\mathfrak{u} \in \Gamma^*$ we introduce the class

$$
\mathfrak{k} := \{ \mathfrak{u}' \in \Gamma^* : \mathfrak{u}' = \sigma^{\tau} \mathfrak{u} \text{ for all } \sigma \in \Omega \} = \{ \mathfrak{u}_1, \dots, \mathfrak{u}_n \}.
$$
 (9)

The pricipal classes $\oint C \Gamma^*$: For a fixed lattice functional $u \in \Gamma^*$ we introduce the equivalence
class
 $\mathbf{f} := \{u' \in \Gamma^* : u' = \sigma^{\mathsf{T}} u$ for all $\sigma \in \Omega\} = \{u_1, ..., u_j\}.$ (9)
Here is $I = \text{ord } \mathbf{f} \le r = \text{ord } \Omega$ as we can relative to the adjoint isotropy group to u, $\mathfrak{B}^* \to \mathfrak{B}^*$ with $\sigma^T \mathfrak{v} = \mathfrak{v} \circ \sigma$.
 icipal classes $\mathfrak{h} \subset \Gamma^*$: For a fixe
 $:= \{ u' \in \Gamma^* : u' = \sigma^T u \text{ for all } \sigma \in \Omega \}$
 $s \mid I = \text{ord } \mathfrak{k} \leq r = \text{ord } \Omega$ as we can
 $= \text{tot } \mathfrak{h} \in \Gamma^* \mathfrak{g}$: $\sigma^T u = u \$

(10)

So Γ^* is decomposed completely in a set \Re of classes $\mathfrak k$. Among these classes the so-called principal classes b play a leading part: For $\Re(u)$ we consider the character. $\chi(u, \cdot)$ with \cdot attice function

(multiply).

The by help of the post $\mathcal{L} \in \Gamma^* : \mathfrak{u}' = \sigma^T \mathfrak{u}$ for all $\sigma \in \Omega$ =

ord $\mathfrak{k} \leq r = \text{ord}\Omega$ as we can set

the adjoint isotropy group to \mathfrak{t}
 $\{\sigma \in \Omega : \sigma^T \mathfrak{u} = \mathfrak{u}\}.$

scomposed completely in a set

lasses \mathfrak{h} play

$$
\chi(u,\sigma) = \exp\{2\pi i \langle u, \{ \rangle \}, \quad (\sigma, \{ \} \in \mathfrak{G}).
$$
 (11)

In (σ, f) the vector f is well established and

$$
\chi(\mathbf{u}, \sigma) = \exp\{2\pi i \langle \mathbf{u}, \mathbf{1}\rangle\}, \quad (\sigma, \mathbf{1}) \in \Theta.
$$
\n
$$
\sigma, \mathbf{1} \text{ the vector } \mathbf{1} \text{ is well-established and}
$$
\n
$$
\varphi_{\mathbf{u}}(\mathbf{t}) = \exp\{2\pi i \langle \mathbf{u}, \mathbf{t} \rangle\}
$$
\n
$$
\mathbf{\mathfrak{X}}-\text{automorphic function on } \mathbf{\mathfrak{B}}. \text{ Therefore } \chi \text{ is correctly defined. If}
$$
\n
$$
\chi(\mathbf{u}, \sigma) = 1 \text{ for all } \sigma \in \mathbf{\mathfrak{R}}(\mathbf{u})
$$
\n
$$
\chi(\mathbf{u}, \sigma) = \mathbf{1} \text{ for all } \sigma \in \mathbf{\mathfrak{R}}(\mathbf{u})
$$
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$$
\chi(\mathbf{u}, \sigma) = \mathbf{1} \text{ for all } \sigma \in \mathbf{\mathfrak{R}}(\mathbf{u})
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$$
\chi(\mathbf{u}, \sigma) = \mathbf{1} \text{ for all } \sigma \in \mathbf{\mathfrak{R}}(\mathbf{u})
$$
\n
$$
\chi(\mathbf{u}, \sigma) = \mathbf{1} \text{ for all } \sigma \in \mathbf{\
$$

is a Σ -automorphic function on $\mathfrak B$. Therefore χ is correctly defined. If

$$
\chi(\mathbf{u},\mathbf{\sigma})=1 \text{ for all } \mathbf{\sigma} \in \mathfrak{R}(\mathbf{u})
$$
 (13)

so $\chi(u, \cdot)$ is said to be *principal character* of $\mathcal{R}(u)$ and u *principal vector* of Γ^* . Now if $u \in \mathfrak{k}$ is a principal vector, *r* contains only principal vectors and is called *principal class* b. Otherwise *^r* contains only non-principal vectors (ℓ is a non-principal class). Let $\mathfrak H$ be the set of all principal classes $\mathfrak{h} \subset \Gamma^*$.

The orthonormal system of \mathcal{D} -automorphic functions: Let $f = \{u_1, ..., u_l\} \in \mathfrak{D}$ be a principal class and rep($\mathfrak{B/B}(u_1)$ _L = { σ_1 , ... , σ_j } a system of representatives of the left coset decomposition of $\mathfrak g$ with respect to $\mathfrak R(u_1)$. Then $\mathfrak f_1, \ldots, \mathfrak f_l \in \mathfrak V$ shall be vectors belonging to $\sigma_1, \ldots, \sigma_l$, respectively, i.e. $S_v = (\sigma_v, f_v)$ for $v = 1, ..., L$.

Definition: The sum

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\n
$$
\psi_{\mathfrak{h}} = \frac{1}{\sqrt{I}} \sum_{\nu=1}^{I} \varphi_{\mathfrak{u}_1} \circ S_{\nu}
$$
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$$
\psi_{\mathfrak{h}} = \frac{1}{\sqrt{I}} \sum_{\nu=1}^{I} \varphi_{\mathfrak{u}_1} \circ S_{\nu}
$$
\n
$$
\text{Remark 1: For each } \mathfrak{v} \in \mathfrak{B}^* \text{ the function } \varphi_{\mathfrak{v}} \text{ is satisfying the relation}
$$
\n
$$
\varphi_{\mathfrak{v}} \circ S = \varphi_{\mathfrak{v}}(\mathfrak{f}) \varphi_{\sigma^{\mathfrak{v}} \mathfrak{v}} \text{ for all } S = (\sigma, \mathfrak{f}) \in \mathfrak{G}. \tag{15}
$$
\n
$$
\text{scaling for the translations } S = (e, t) \in \mathfrak{X} \text{ and lattice vectors } \mathfrak{v} = \mathfrak{u} \in \Gamma^* \text{ we see that } \varphi_{\mathfrak{v}} \text{ is } \mathfrak{X} - \mathfrak{v} \text{ is } \mathfrak{v} \text{ is } \mathfrak{X
$$

is said to be a \mathfrak{h} -corresponding function on $\mathfrak{B}.$

Remark 1: For each $v \in \mathfrak{B}^*$ the function φ_v is satisfying the relation

$$
\varphi_{\mathbf{D}} \circ S = \varphi_{\mathbf{D}}(\mathfrak{f}) \varphi_{\mathbf{D}^{\mathsf{T}} \mathbf{D}} \quad \text{for all } S = (\sigma, \mathfrak{f}) \in \mathfrak{G}. \tag{15}
$$

Especially for the translations $S = (e, t) \in \mathbb{Z}$ and lattice vectors $v = u \in \Gamma^*$ we see that φ_u is \mathbb{Z} automorphic, even $\varphi_{\mathbf{u}} \in L_2(\mathfrak{X})$ (L_2 -space of \mathfrak{X} -automorphic functions).

Remark 2: If σ runs through Ω , so $\sigma^{\nu}u_1$ runs through $\mathfrak{h} = \{u_1, \ldots, u_l\}$ - but in general not simply $(I \le r)$. But if σ runs only through rep $(\mathcal{B}/\mathcal{R}(u_1))_L$, so, from u_1 every vector $u_1 \in \mathfrak{h}$ arises exactly one time by $u_{1} = \sigma^{T} u_{1}$.

The $\mathfrak b$ -corresponding functions $\psi_{\mathfrak b}$ are elements of $L_2(\mathfrak G)$. As functions normed to one just the $\psi_{\mathbf{b}}$ build a complete orthonormal system $\{\psi_{\mathbf{b}}: \mathbf{b} \in \mathbf{b}\}\$ in $L_2(\mathbf{b})$ [7: §2/(2.8)]. The \mathfrak{b} -corresponding functions $\psi_{\mathbf{b}}$ are elements of $L_2(\mathfrak{B})$. As functions normed to the $\psi_{\mathbf{b}}$ build a complete orthonormal system $\{\psi_{\mathbf{b}}: \mathfrak{b} \in \mathfrak{H}\}$ in $L_2(\mathfrak{B})$ [7: §2/(2.8)].
1.2

1.2 The \bigcirc **-automorphic eigenfunctions and spec_o of** *P***. To prove that the h-corresponding**

Lemma 1: *The G-invariant differential operator P from (1) acts on the functions* $\varphi_{\mathbf{b}} \circ S$ *from (14) or* (15) *according to* ents of L_2 :
 \vdots $\theta \in \mathfrak{H}$ in
 $\text{ec}_{\mathfrak{B}}$ of *P*.
 \vdots investiga
 $\text{erator } P$ from
 \mathfrak{B}^* . **p** p [(14) or (15) a
 p [φ _{**p**} o S] = P
 Proof: The open
 $p = p^{hk} \partial_h \partial_k$ **orphic eigenfunctions and spe**
 e Θ -invariant differential ope
 coording to
 $(2\pi\mathfrak{v}) \varphi_{\mathfrak{v}} \circ S$ *for all* $S \in \Theta$, $\mathfrak{v} \in$
 erator *P* can be written as
 $-4\pi i P^h \partial_h - 4\pi^2 P^o$.
 coefficients of t

$$
P[\varphi_{\mathbf{v}} \circ S] = P(2\pi \mathbf{v}) \cdot \varphi_{\mathbf{v}} \circ S \quad \text{for all } S \in \mathfrak{B}, \mathbf{v} \in \mathfrak{B}^*.
$$
 (16)

Proof: The operator Pcan be written as

$$
P = P^{hk}\partial_h\partial_k - 4\pi i P^h\partial_h - 4\pi^2 P^0. \tag{17}
$$

Here *P*^{hk} are the coefficients of the quadratic form \mathfrak{P} from (1'), furthermore $P^h = P^h k p_k$, $P^o =$ *phk_{phpk}*, where $\hat{p} = p_h \hat{p}^h$ and $\partial_h = \partial/\partial x^h$, $\hat{r} = x^h \hat{p}_h - \hat{r}$ explained altogether respectively to bas 93. Now we apply P on φ_p , $\varphi = v_p \hat{p}^v$: Using (12) and (8) we obtain *i*(14) or (15) according to
 P[φ _B o *S*] = *P*($2\pi\upsilon$) φ _B o *S* for all *S* $\in \Theta$, $\upsilon \in \mathfrak{B}^*$.
 Proof: The operator *P* can be written as
 P = $P^{hk}\partial_h\partial_k - 4\pi i P^h\partial_h - 4\pi^2 P^0$.
 Phk are the

$$
\partial_h \varphi_{\mathbf{v}}(\mathbf{r}) = \partial / \partial x^h(\exp 2\pi i \langle v, \mathbf{v}^v, x^{\mu} \mathbf{v}_u \rangle) = \varphi_{\mathbf{v}}(\mathbf{r}) \cdot 2\pi i \partial / \partial x^h(v, x^{\nu}) = 2\pi i v_h \varphi_{\mathbf{v}}(\mathbf{r})
$$

$$
\partial_h \partial_k \varphi_{\mathbf{D}}(\mathbf{r}) = (2\pi \mathbf{i})^2 v_h v_k \varphi_{\mathbf{D}}(\mathbf{r}).
$$

Now (17) and after that $(1')$ gives

$$
P[\varphi_{\mathbf{v}}] = (-P^{hk}(2\pi v_h)(2\pi v_k) + 4\pi P^h(2\pi v_h) - 4\pi^2 P^o)\varphi_{\mathbf{v}}
$$

 $= -\mathfrak{P}(2\pi\mathfrak{v}-2\pi\mathfrak{p}, 2\pi\mathfrak{v}-2\pi\mathfrak{p})\varphi_{\mathfrak{w}} = P(2\pi\mathfrak{v})\varphi_{\mathfrak{w}}.$

So (16) follows from the \mathfrak{G} -invariance of *P*, i.e. from (1ⁿ)

If we now take into account the h-corresponding function $\psi_{\mathbf{b}}$ from (14), formula (16) gives

$$
P[\psi_{\mathfrak{h}}] = \frac{1}{\sqrt{I}} \sum_{v=1}^{I} P(2\pi u) \varphi_{\mathfrak{u}} \circ S_{v} = P(2\pi u) \psi_{\mathfrak{h}}, \quad \mathfrak{u} \in \mathfrak{h}.
$$
 (18)
Definition: If $\mathfrak{u} \in \mathfrak{k}$, we can write

$$
P(2\pi \mathfrak{k}) = P(2\pi \mathfrak{u}),
$$
 (19)
are $P(2\pi \mathfrak{k})$ can be understood as a class norm $\|\mathfrak{k}\|^2$ of \mathfrak{k}).

Definition: If *u* € f, we can write

(19) $= P(2\pi u)$, (19)
 (19)
 (19) can be understood as a class norm $||\mathbf{f}||^2$ of \mathbf{f}).
 (19)
 (19)
 (19) can be understood as a class norm $||\mathbf{f}||^2$ of \mathbf{f}).
 (20)
 (20)
 (20)

act that all $u \in$

(where $P(2\pi\mathbf{t})$ can be understood as a class norm $\|\mathbf{t}\|^2$ of \mathbf{t}).

The justification for (19) comes from the Ω -automorphy of P from (1'),

$$
P(\sigma^{\mathsf{T}}\mathbf{v}) = P(\mathbf{v}) \text{ for all } \sigma \in \mathfrak{L}, \mathbf{v} \in \mathfrak{B}^*,
$$
 (20)

and of the fact that all $u \in \mathfrak{k} = \{u_1, \ldots, u_l\}$ arise e.g. from u_1 by means of the equivalence $u =$ *o*^τu₁, σεΩ.

Remark 3: If the class norms of f_1 , f_2 are different, $P(2\pi f_1)$ $\neq P(2\pi f_2)$, the same is always **Remark 3:** If the class norms of f_1, f_2 are different, $P(2\pi f_1) \neq P(2\pi f_2)$, the same is always right for the classes, $f_1 \neq f_2$. But the inverse assertion is not right; if $f_1 \neq f_2$, notwithstanding may be $P(2\pi\ell_1) = P(2\pi\ell_2)$.

Theorem 1: To each principal class $\mathfrak{h} \in \mathfrak{H}$ we can assign exactly one eigenvalue $\mu \circ \mu_{\mathfrak{h}}$ of *the 3-automorphic eigen Value problem (2), namely*

 $\mu_{\mathbf{b}} = -P(2\pi\mathbf{b})$ (21)

with

$$
m_{\mathfrak{S}}(\mu_{\mathfrak{h}})=\text{card}\left\{\mathfrak{h}'\in \mathfrak{H}\colon P(2\pi\mathfrak{h}')=P(2\pi\mathfrak{h})\right\}
$$

*as multiplicity; thereby the h-corresponding function ψ_b belongs to μ_b as the eigenfunction.
The set spec_®(P) = {μ_b: b ∈ \$}} is the complete [®]-automorphic eigenvalue spectrum of the 3i-invariant differential operator P from (1).*

Proof: The correspondence $\mathfrak{h} \to \psi_{\mathfrak{h}}$ from(14), and (18), prove the first part of the theorem. The completeness of spec $_{c}(\mathbf{P})$ follows from the completeness of the orthonormal system ${\{\psi_{\mathbf{B}}\}}$: $\mathfrak{b} \in \mathfrak{H}$ of $L_2(\mathfrak{G})$. Let $\psi = \sum c_{\mathbf{b}} \psi_{\mathbf{b}}$ (summation over $\mathfrak{b} \in \mathfrak{H}$) be an arbitrary \mathfrak{G} -automorphic eigenfunction of P to the eigenvalue $\mu \neq \mu_b$ for all $\beta \in \mathcal{S}_b$. Then from (2), (18), (19) and (21) for each $\mathbf{b} \in \mathcal{S}$ there follows $c_{\mathbf{b}}(\mu_{\mathbf{b}} - \mu) = 0$. Consequently there would be $c_{\mathbf{b}} = 0$ and therefore $\psi =$ 0 which is a contradiction \blacksquare ID is $L_2(\Theta)$. Let $\psi = \sum c_\beta \psi_\beta$ (summation over $\theta \in \mathfrak{H}$) be an arbitrary Θ -automorphic ei-
unction of P to the eigenvalue $\mu + \mu_\beta$ for all $\theta \in \mathfrak{H}$. Then from (2), (18), (19) and (21) for
 $\theta \in \mathfrak{H}$ t

2. $N(\lambda)$ as the number of principal classes b contained in a certain convex domain $\lambda \cdot D \subset \mathfrak{B}^*$

The operator *P* has the following geometric appearance.

Definition: The domains in \mathfrak{B}^*

(22)

M. BELGER
\n
$$
\lambda \cdot \mathbb{D} = \{ \mathbf{v} \in \mathfrak{B}^* : -P(\mathbf{v} + 2\pi \mathbf{p}) \le (\lambda/2\pi)^2 \}
$$
\n
$$
\mathbf{p} + \lambda \cdot \mathbb{D} = \{ \mathbf{v} \in \mathfrak{B}^* : -P(2\pi \mathbf{v}) \le \lambda^2 \}
$$
\n(23')
\nis order are said to be gauge domain, homothetical expansion of \mathbb{D} with $\lambda > 0$ as factor,
\n*llel translated domain* by the vector $\mathbf{p} \in \mathfrak{B}^*$ (from (1)).
\nThe \mathfrak{B} -invariance of *P* means for these domains
\n**Lemma 2:** The gauge domain \mathbb{D} and so also all its homothetical expansions $\lambda \cdot \mathbb{D}$ are \mathbb{R} -
\n*rain*. Therefore for an equivalence class $\mathbf{f} \in \mathfrak{R}$ there is valid
\neither $\mathbf{f} \subset (\mathbf{p} + \lambda \cdot \mathbb{D})$ or $\mathbf{f} \cap (\mathbf{p} + \lambda \cdot \mathbb{D}) = \mathbb{D}$.
\n(24)
\nNow if we look at $N(\lambda)$ from (3) and $\mu_{\mathbf{p}}$ from (21) we could ask for the geometric locus
\naining all \mathbf{b} with $\mu_{\mathbf{b}} \le \lambda^2$. The formulas (21), (19), (1'), (23') and (24) yield

in this order are said to be *gauge domain, homothetical expansion* of D with $\lambda > 0$ as factor, *parallel translated domain* by the vector $\mathfrak{p} \in \mathfrak{B}^*$ (from (1)).

The Θ -invariance of *P* means for these domains

Lemma 2: The gauge domain $\mathbb D$ and so also all its homothetical expansions $\lambda \cdot \mathbb D$ are $\mathbb Q$ *invariant. Therefore for an equivalence class F E S there is valid* The \mathcal{G} -invariance of *P* means for these domain
Lemma 2: The gauge domain \mathbb{D} and so also air
tiant. Therefore for an equivalence class $\mathbf{\hat{t}} \in \mathcal{R}$
either $\mathbf{\hat{t}} \in (\mathfrak{p} + \lambda \cdot \mathbb{D})$ or $\mathbf{\hat{t}} \cap (\mathfr$

Now if we look at $N(\lambda)$ from (3) and μ_b from (21) we could ask for the geometric locus containing all \mathfrak{h} with $\mu_{\mathbf{b}} \leq \lambda^2$. The formulas (21), (19), (1'), (23') and (24) yield

Proposition 1: The number of eigenvalues $\mu_h \leq \lambda^2$ *is given by*

3. $N(\lambda)$ as a finite sum of Weyl sums

3.1 A proposition of P. GUnther. Let

$$
\mathfrak{B}^{*}(\sigma) = \ker(\sigma^{\top} - id) \quad \text{and} \quad \Gamma^{*}(\sigma) = \Gamma^{*} \cap \mathfrak{B}^{*}(\sigma)
$$
 (26)

all b with $\mu_{\beta} \le \lambda^2$. The formulas (21), (19), (1), (2)

ition 1: The number of eigenvalues $\mu_{\beta} \le \lambda^2$ is give

card {b \in \$: b \subset (b + λ ·lD)}.

a finite sum of Weyl sums

sition of P. Günther. Let

= ker be the eigenspace to the eigenvalue 1 of σ^T and the Z-module of all lattice functionals of \mathfrak{B}^{*} (o), respectively. (look at (8)). According to [7: Proposition 2.2], for a function $f: \mathfrak{B}^{*} \rightarrow \mathbb{C}$ it is valid **proposition of P.**
 $3^*(\sigma) = \ker(\sigma^{\tau} - i)$
 e eigenspace to
 b, respectively (id id
 c
 d cardb $\frac{1}{\mu \epsilon b}$
 d cardb $\frac{1}{\mu \epsilon b}$ *f***(a)** and $\Gamma^*(\sigma) = \Gamma^* \cap \mathfrak{B}^* (\sigma)$.

the eigenvalue 1 of σ^{\dagger} and the eigenvalue 1 of σ^{\dagger} and the eigenvalue 1 of σ^{\dagger} and the $f(u) = \frac{1}{r} \sum_{\sigma \in \mathfrak{B}} W(\sigma)$. 3. $N(\lambda)$ as a finite sum of Weyl sums

3.1 A proposition of P. Günther. Let
 $\mathfrak{B}^*(\sigma) = \ker(\sigma^\tau - id)$ and $\Gamma^*(\sigma) = \Gamma^* \cap \mathfrak{B}^* \cap \sigma$

be the eigenspace to the eigenvalue 1 of σ^τ and the Z-module of al
 $\mathfrak{B}^*(\$ **P. Günther.** Let

id) and $\Gamma^*(\sigma) = \Gamma^* \cap \mathfrak{B}^* (\sigma)$

o the eigenvalue 1 of σ^{τ} and the Z-modu

look at (8)). According to [7: Proposition 2
 $\int_{\sigma}^{\tau} f(u) = \frac{1}{r} \sum_{\sigma \in \mathfrak{B}} W(\sigma)$.

(u, σ) $f(u)$

gent for al

$$
\sum_{\mathbf{b}\in\mathfrak{D}}\frac{1}{\operatorname{card}\mathbf{b}}\sum_{\mathbf{u}\in\mathbf{b}}f(\mathbf{u})=\frac{1}{r}\sum_{\sigma\in\mathbf{B}}W(\sigma)
$$
 (27)

$$
\sum_{\mathbf{b} \in \mathfrak{D}} \frac{1}{\operatorname{card} \mathfrak{b}} \sum_{\mathbf{u} \in \mathbf{b}} f(\mathbf{u}) = \frac{1}{r} \sum_{\sigma \in \mathbf{B}} W(\sigma)
$$
(27)
ur as

$$
W(\sigma) := \sum_{\mathbf{u} \in \Gamma^*(\sigma)} \chi(\mathbf{u}, \sigma) f(\mathbf{u})
$$
(28)

is absolutely convergent for all **0 6** 2.

3.2 The characteristic function χ_{λ} **of** λ **. D.** Let. χ be the characteristic function of D and χ is absolutely convergent for all $\sigma \in \Omega$.

3.2 The characteristic function χ_{λ} of λ . D. Let χ be the characteristic function of D and χ_{λ} that of $\lambda \cdot D$. From the definition of χ_{λ} and the Ω -in 25 T(d), respectively (look at (8)). According to [7: Proposition 2.2], for a function $f:$

is valid
 $\sum_{\theta \in \mathcal{S}} \frac{1}{\text{card}b} \sum_{u \in b} f(u) = \frac{1}{r} \sum_{\theta \in \mathcal{S}} W(\theta)$

so far as
 $W(\theta) = \sum_{u \in \mathbb{I}^m(\theta)} \chi(u, \theta) f(u)$

is a Lemma 3: For $v \in \mathbb{R}^m$ we have
 $\mathcal{U}(\sigma) := \sum_{u \in \Gamma^m(\sigma)} \chi(u, \sigma) f(u)$

solutely convergent for all $\sigma \in \mathbb{R}$.

The characteristic function χ_{λ} of $\lambda \cdot D$. Let χ be the characteristic function of of $\lambda \cdot D$. F $f(u, \sigma) f(u)$
gent for all $\sigma \in \Omega$.
it cfunction χ_{λ} of $\lambda \cdot D$.
he definition of χ_{λ} and the definition of χ_{λ} and the δ or all $\lambda > 0$

(25)

(29)

Eigenvalue Distribution of Differential Operators

\n
$$
\chi_{\lambda}(\sigma^{\intercal}v) = \chi_{\lambda}(v) \text{ for all } \sigma \in \Omega,
$$
\n
$$
\chi_{\lambda} \text{ is } \Omega \text{-automorphic on } \mathfrak{B}^*.
$$
\n(29')

i.e. χ_{λ} *is* Ω *-automorphic on* \mathfrak{B}^* .

Now regard χ_{λ} as a partial function on $-\mathfrak{p} + \Gamma^* \subset \mathfrak{B}^*$. Then χ_{λ} is a class function depending only on the equivalence classes - \mathfrak{p} + \mathfrak{k} of the lattice - \mathfrak{p} + Γ^\ast for all \mathfrak{k} ϵ

Now regard
$$
\chi_{\lambda}
$$
 as a partial function on $-\mathfrak{p} + \Gamma^* \subset \mathfrak{B}^*$. Then χ_{λ} is a class function depending on the equivalence classes $-\mathfrak{p} + \mathfrak{k}$ of the lattice $-\mathfrak{p} + \Gamma^*$ for all $\mathfrak{k} \in \mathfrak{R}$:

$$
\chi_{\lambda}(-\mathfrak{p} + \mathfrak{k}) = \begin{cases} 1 & \text{if } \mathfrak{k} \subset \mathfrak{p} + \lambda \cdot \mathbb{D} \\ 0 & \text{if } \mathfrak{k} \not\subset \mathfrak{p} + \lambda \cdot \mathbb{D} \end{cases}
$$
(30)

Eigenval
 λ (b) for all $\sigma \in \Omega$,

comorphic on \mathfrak{B}^* .
 λ as a partial function on -p

equivalence classes -p + \mathfrak{k} of t

= $\begin{cases} 1 & \text{if } \mathfrak{k} \subset \mathfrak{p} + \lambda \cdot \mathbb{D} \\ 0 & \text{if } \mathfrak{k} \not\subset \mathfrak{p} + \lambda \cdot \mathbb{D} \end{cases$ (see also (24)). Now we set going proposition (27)/(28) choosing $f(u)$ in accordance with $f(u)$ $\pm \chi_3(-\mathfrak{p} + \mathfrak{u}) = \chi_3(-\mathfrak{p} + \mathfrak{h})$ for $\mathfrak{u} \in \mathfrak{h} \in \mathfrak{H}$. Then

$$
\chi_{\lambda}(\sigma^{\intercal}b) = \chi_{\lambda}(b) \text{ for all } \sigma \in \Omega, \qquad (29)
$$
\n
$$
\chi_{\lambda} \text{ is } \Omega \text{-automorphic on } \mathfrak{B}^*.
$$
\nNow regard χ_{λ} as a partial function on $-b + \Gamma^* \subset \mathfrak{B}^*$. Then χ_{λ} is a class function depending on the equivalence classes $-b + \ell$ of the lattice $-b + \Gamma^*$ for all $\ell \in \mathfrak{B}$:\n
$$
\chi_{\lambda}(-b + \ell) = \begin{cases} 1 & \text{if } \ell \subset p + \lambda \cdot \mathbb{D} \\ 0 & \text{if } \ell \not\subset p + \lambda \cdot \mathbb{D} \end{cases}
$$
\nalso (24)). Now we set going proposition (27)/(28) choosing $f(u)$ in accordance with $f(u)$:\n
$$
(\lambda(-b + u) = \chi_{\lambda}(-b + b) \text{ for } u \in b \in \mathfrak{D}. \text{ Then}
$$
\n
$$
W(\sigma) := \sum_{u \in \Gamma^*(\sigma) \cap (\mu + \lambda \cdot \mathbb{D})} \chi(u, \sigma)
$$
\nfinite and so an absolutely convergent series. Because of (30) and (25) the left-hand side\n
$$
N(\lambda) = \frac{1}{r} \sum_{\sigma \in \mathcal{B}} W(\sigma).
$$
\n(32)\nSplitting of $N(\lambda)$ into isodimensional summands. Le be\n
$$
n(\sigma) := \dim \mathfrak{B}^*(\sigma)
$$
\n(33)

is a finite and so an absolutely convergent series. Because of (30) and (25) the left-hand side of (27) is equal to $N(\lambda)$ so that

$$
N(\lambda) = \frac{1}{r} \sum_{\sigma \in \mathbf{S}} W(\sigma). \tag{32}
$$

$\overline{3.3}$ Splitting of $N(\lambda)$ into isodimensional summands. Le be

$$
n(\sigma) \coloneqq \dim \mathfrak{B}^*(\sigma) \tag{33}
$$

and

$$
n(\sigma) := \dim \mathfrak{B}(\sigma)
$$
\n
$$
\mathfrak{B}_m := \{ \sigma \in \mathfrak{B} : n(\sigma) = m \}, \quad m = 0, 1, \dots, n.
$$
\n(34)

For $\sigma \in \mathcal{Q}_m$ the Z-module $\Gamma^*(\sigma)$ from (26) has *m* linearly independent generators. Now (32) can be dissected according to

Proposition 2: $N(\lambda)$ *is the sum of isodimensional summands:*

Splitting of
$$
N(\lambda)
$$
 into isodimensional summands. Le be
\n $n(\sigma) := \dim \mathfrak{B}^*(\sigma)$ (33)
\n $\mathfrak{B}_m := \{\sigma \in \mathfrak{B} : n(\sigma) = m\}, m = 0, 1, ..., n.$ (34)
\n $\sigma \in \mathfrak{B}_m$ the Z-module $\Gamma^*(\sigma)$ from (26) has *m* linearly independent generators. Now (32) can
\nassociated according to
\n**Proposition 2:** $N(\lambda)$ is the sum of isodimensional summands:
\n $N(\lambda) = \frac{1}{r} \sum_{m=0}^n \sum_{\sigma \in \mathfrak{B}_m} W(\sigma)$ (35)
\n $\text{re } W(\sigma)$ with $(\sigma, \mathfrak{f}) \in \mathfrak{B}$ are the Weyl sums (31)/(11), or for a specific purpose formulated,

where W(o) with (o,) *£ 0 are the Weyl sums* (31)/(11), *or for a specific purpose formulated,*

$$
\mathfrak{L}_{m} \coloneqq \{ \sigma \in \mathfrak{L} : n(\sigma) = m \}, m = 0, 1, ..., n. \tag{34}
$$
\nFor $\sigma \in \mathfrak{L}_{m}$ the Z-module $\Gamma^{*}(\sigma)$ from (26) has *m* linearly independent generators. Now (32) can be dissected according to\n
\n**Proposition 2:** $N(\lambda)$ is the sum of isodimensional summands:\n
$$
N(\lambda) = \frac{1}{r} \sum_{m=0}^{n} \sum_{\sigma \in \mathfrak{L}_{m}} W(\sigma) \tag{35}
$$
\nwhere $W(\sigma)$ with $(\sigma, f) \in \mathfrak{B}$ are the Weyl sums (31)/(11), or for a specific purpose formulated,\n
$$
W(\sigma) = \sum_{\mu = 0} \sum_{\sigma \in \mathfrak{R}_{m}} \exp\{2\pi i \langle \mu, f \rangle\}.
$$
\n
$$
= P(\pi \mu) \leq \lambda^{2}
$$
\nThe special kind of summation in (36) in comparison with that of (31) follows from (23'').\n
\n**Definition:** In (35) the summand with $m = n$ is said to be principal part and that with $m = n - 1$ secondary part of $N(\lambda)$.\n
\n**Remark 4:** All the other summands of $N(\lambda)$ with $m \leq n - 2$ will be proved subordinate and get into the remainder during the asymptotic estimation of $N(\lambda)$ in Subsections 4.2/4.3 (see (49)).

The special kind of summation in (36) in comparison with that of (31) follows from $(23'')$.

Definition: In (35) the summand with $m = n$ is said to be *principal part* and that with $m =$

Remark 4: All the other summands of $N(\lambda)$ with $m \leq n-2$ will be proved subordinate and get into the remainder during the asymptotic estimation of $N(\lambda)$ in Subsections 4.2/4.3 (see (49)).

4. The asymptotic estimation of $N(\lambda)$

4.1 Formulation of the Weyl sum *W(d)* **in coordinates relative to bas** r*(d). Let be

Formula 10.1 For the **Weyl sum**
$$
W(\sigma)
$$
 in coordinates relative to $\text{bas } \Gamma^*(\sigma)$. Let be
\n $\text{bas } \Gamma^*(\sigma) := \{c^1(\sigma), ..., c^m(\sigma)\}, \text{ bas } \Gamma^* = \{b^1, ..., b^n\},$ \n
$$
c^0(\sigma) = c^0(\sigma)b^h, c^0(\sigma) \in \mathbb{Z} \quad (h = 1, ..., n; v = 1, ..., m).
$$
\n(37)
\n $\text{auss of } \tau^0(\sigma) \in \Gamma^*(\sigma) \text{ there is } (\sigma^T - \text{id})\tau^0(\sigma) = 0. \text{ Therefore } c^0(\sigma) \text{ for each } v \text{ is a solution of}$
\nsystem of linear equations $(\sigma^T - \text{id})\tau^0(\sigma) = 0$ $(j = 1, ..., n)$ and naturally $\sigma^Tb^i = \sigma^i_jb^j$.
\n**Agreement:** Latin indices run through 1, ..., n and Greek indices through 1, ..., m
\n\n in the exception of $\sigma \in \mathbb{R}$.
\nFor $\mu \in \Gamma^*(\sigma)$ and for $\beta \in \mathbb{S}^*(\sigma)$ as the invariant vector from (1') we write
\n $\mu = u'_\nu c^0(\sigma) = u'_\nu c^0_\mu(\sigma)b^h = u_h b^h$ and $\beta = \rho'_\nu c^0(\sigma) = \rho'_\nu c^0_h(\sigma)b^h = \rho_h b^h$. (38)
\nIn we have

Because of $\tau^{\vee}(0) \in \Gamma^*(0)$ there is $(\sigma^{\tau} - id)\tau^{\vee}(0) = 0$. Therefore $c^{\vee}(0)$ for each v is a solution of the system of linear equations $(\sigma_i^i - \delta_i^j)c_i^{\vee}(\sigma) = 0$ $(j = 1, ..., n)$ and naturally $\sigma^{\intercal}b^i = \sigma_i^j b^j$.

Agreement: Latin indices run through 1, ..., *n* and Greek indices through 1, ..., *m -* only with the exception of *o € 2.*

For $u \in \Gamma^*(\sigma)$ and for $p \in \mathfrak{B}^*(\sigma)$ as the invariant vector from (1') we write

have of τ^ν(σ) ε l^ν(σ) there is (σ'-id)τ^ν(σ) = 0. Therefore c^ν(σ) for each v is a solution of system of linear equations (σⁱ - δⁱ)c^ν(σ) = 0 (j = 1, ..., n) and naturally σ^τbⁱ = σⁱb^j.

\nAgreement: Latin indices run through 1, ..., n and Greek indices through 1, ..., m
\nly with the exception of σ ε ℜ.

\nFor u ε Γ[∗](σ) and for ρ ε Ψ[∗](σ) as the invariant vector from (1') we write

\nu = u_ν'c^ν(σ) = u_ν'c^ν(σ)b^h = u_hb^h

\nand
$$
\mathbf{p} = \rho_{\nu}c\mathbf{v}^{\nu}(\sigma) = \rho_{\nu}c\rho_{\nu}(\sigma)bh = \rho_{h}bh
$$
.

\n(38)

\nin we have

\n $\langle \mathbf{u}, \{\rangle = u_{\nu}^{\prime} s^{\nu}(\sigma) \text{ with } s^{\nu}(\sigma) = \langle \epsilon^{\nu}(\sigma), \{\rangle.$

\nlooking at (17) we introduce the symmetric $m \times m$ -matrix $(P^{\nu\mu}(\sigma))$ with

\n $P^{\nu\mu}(\sigma) = P^{hk}c^{\nu}_{h}(\sigma)c^{\mu}_{k}(\sigma), \Delta(\sigma) := \text{det}(P^{\nu\mu}(\sigma)).$

\n(40)

\n38) this makes possible to write P in form of

\n $-P(2\pi u) = (2\pi)^{2}P^{\nu\mu}(\sigma)w_{\nu}^{\prime}w_{\mu}^{\prime}, w_{\nu}^{\prime} = u_{\nu}^{\prime} - p_{\nu}^{\prime}.$

\n(41)

\nbefore Proposition 2 in coordinates relative to bas^τ(σ) can be formulated as

Then we. have

$$
\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}_v' \mathbf{s}^{\mathbf{v}}(\mathbf{0}) \quad \text{with} \quad \mathbf{s}^{\mathbf{v}}(\mathbf{0}) = \langle \mathbf{c}^{\mathbf{v}}(\mathbf{0}), \mathbf{v} \rangle. \tag{39}
$$

Now looking at (17) we introduce the symmetric $m \times m$ -matrix $(P^{\nu \mu}(\sigma))$ with

$$
\langle \mu, \{\rangle = \mu'_{\nu} s^{\nu}(\sigma) \quad \text{with} \quad s^{\nu}(\sigma) = \langle c^{\nu}(\sigma), \{\rangle}.
$$
\n(39)\nlooking at (17) we introduce the symmetric $m \times m$ -matrix $(P^{\nu} \mu(\sigma))$ with

\n
$$
P^{\nu} \mu(\sigma) = P^{hk} c_h^{\nu}(\sigma) c_k^{\mu}(\sigma), \quad \Delta(\sigma) := \det(P^{\nu} \mu(\sigma)).
$$
\n(40)

By(38) this makes possible to write *P* in form of

$$
P(2\pi u) = (2\pi)^2 P^{\nu\mu}(\sigma) w'_{\nu} w'_{\mu}, \quad w'_{\nu} = u'_{\nu} - p'_{\nu}.
$$
 (41)

Proposition 3: $N(\lambda)$ (so as in Proposition 2) is the sum of the Weyl sums

Now looking at (17) we introduce the symmetric
$$
m \times m
$$
-matrix $(P^{\nu\mu}(\sigma))$ with
\n $P^{\nu\mu}(\sigma) = P^{hk}c_h^{\nu}(\sigma)c_k^{\mu}(\sigma), \Delta(\sigma) = \det(P^{\nu\mu}(\sigma)).$ (40)
\nBy (38) this makes possible to write P in form of
\n $-P(2\pi u) = (2\pi)^2 P^{\nu\mu}(\sigma)w_v'w_\mu', w_v' = u_v' - p_v'.$ (41)
\nTherefore Proposition 2 in coordinates relative to $\text{bas} \Gamma^*(\sigma)$ can be formulated as
\n**Proposition 3:** $N(\lambda)$ (*so as in Proposition 2*) is the sum of the Weyl sums
\n $W(\sigma) = e^{2\pi i p_v' s^{\nu}(\sigma)} \sum_{\substack{w_v' = -p_v \bmod(1) \\ P^{\nu\mu}(\sigma)w_v'w_\mu' = (\lambda/2\pi)^2}} e^{2\pi i w_v' s^{\nu}(\sigma)}.$ (42)

Remark 5: For $\sigma = e$ (e - identity in Ω) we obtain

$$
n(e) = n, \quad \Omega_n = \{e\}, \quad \mathfrak{B}^*(e) = \mathfrak{B}^*, \quad \Gamma^*(e) = \Gamma^*, \quad e^{\vee}(e) = \mathfrak{b}^{\vee}
$$
\n
$$
c_h^{\vee}(e) = \delta_h^{\vee}, \quad u_v' = u_v, \quad p_v' = p_v, \quad P^{\vee}\mu(e) = P^{\vee}\mu, \quad \Delta(e) = \det(P^{\vee}\mu). \tag{43}
$$

4.2 Landau's estimation of lattice remainder applied to the Weyl sum W(ø). In (42) we have the sum of the unimodular weights $exp{2\pi i w'_0 s^v(\sigma)}$ which load the lattice functionals $m \in \Gamma^*(\sigma)$ within the $m = n(\sigma)$ -dimensional ellipsoid $(p + \lambda \cdot D)$ o $\mathfrak{B}^*(\sigma)$. The estimation of such a sum $W(\sigma)$ is a classical problem which was worked out above all by E. Landau ([14: Chapter $I/(7)$ and (10)] and [19]). As we know this leads to the result $P^{\nu\mu}(\sigma)w_{\nu}^{\prime}w_{\mu}^{\prime}=(\lambda/2\pi)^{2}$

Remark 5: For $\sigma = e(e - i \text{dentity in } \Omega)$ we obtain
 $n(e) = n$, $\Omega_{n} = \{e\}$, $\mathfrak{B}^{*}(e) = \mathfrak{B}^{*}$, $\Gamma^{*}(e) = \Gamma^{*}$, $e^{\nu}(e) = \mathfrak{b}^{\nu}$
 $c_{n}^{\nu}(e) = \delta_{n}^{\nu}$, $u_{\nu}^{\prime} = u_{\nu}$ *W*_y = $\frac{1}{P}V_{\mu}(\sigma)W_{\nu}V_{\mu}^{'} = (\lambda/2\pi)^2$
 Remark 5: For $\sigma = e(e^{-i\phi})W_{\mu}^{'} = (\lambda/2\pi)^2$
 Remark 5: For $\sigma = e(e^{-i\phi})W_{\mu}^{'} = (\lambda/2\pi)^2$
 Remark 5: For $\sigma = e(e^{-i\phi})W_{\mu}$ is a contribution of $\sigma^*W_{\mu}^{'} = \sigma^*W_{\mu}^{'} =$

$$
W(\sigma) = \frac{\delta_{\sigma}}{2^m \sqrt{\pi}^m \sqrt{\Delta(\sigma)} \Gamma(\frac{m+2}{2})} \lambda^m + O\left(\lambda^{m-2+\frac{2}{m+1}}\right)
$$
(44)

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\n
$$
\delta_{\sigma} = 1 \text{ if } s^{\nu}(\sigma) \in \mathbb{Z} \text{ and } \delta_{\sigma} = 0 \text{ otherwise.}
$$
\n(45)
\n**Definition:**
$$
\delta_{\sigma}
$$
 will be called Landau's
$$
\delta
$$
-symbol which is assigned to
$$
\sigma
$$
 (see Proposition 4).

Definition: δ_{σ} will be called *Landau's* δ -*symbol* which is assigned to σ (see Proposition 4).

4.3 N(A) and the *m* **- dimensional volumes** $\mathbf{v}_0 = \mathbf{v}_0 + \mathbf{v}_0$ **. (45)

Definition:** δ_o will be called *Landau's* δ -*symbol* which is assigned to σ (see Proposition 4).
 4.3 N(λ **) and the** *m* - dimensio **Definition:** δ_{σ} will be called *Landau's* δ -*symbol* which is assigned to σ (see Proposition 4).
4.3 $N(\lambda)$ **and the** m - **dimensional volumes** $vol_{m}(\lambda \cdot D \cap \mathfrak{B}^{m}(\sigma))$. Let be $\sigma \in \mathfrak{L}_{m}$ and $m = n(\sigma)$ damental domain of "). So we can introduce the m-dimensional volume of $\mathbb D$ \cap $\mathfrak{B}^*(\sigma)$, **Definition:** δ_{σ} will be ca
 N(λ **) and the m** - dimer
 $\mathfrak{B}^*(\sigma)$ be equipped with

ental domain of "). So we

vol_m($\mathbb{D} \cap \mathfrak{B}^*(\sigma)$) = $\mathbb{D} \cap$ **Solutional volumes volumes volumes volumes**
a measure μ_{σ}^* of the can introduce the
 $\int_{\mathfrak{B}} d\mu^*(\mathfrak{v}) = \int_{\mathfrak{D}} d\mu$ *don of Differential Operators* 293
 (45)
 do. differential Operators 293
 *do. d***₀.** *do.**do* mensional volumes vol_m($\lambda \cdot D$ o \mathbb{S}^*
ith a measure μ_o^* of the normalizat
b we can introduce the m-dimension
 $\int_{D \cap \mathfrak{B}^*} d\mu^*(v) = \int_{D \cap \mathfrak{B}^*} d\sigma / \int_{\mathfrak{B}(\Gamma^*(\sigma))} d\sigma$

$$
\text{vol}_m(\mathbb{D} \cap \mathfrak{B}^*(\sigma)) = \int_{\mathbb{D} \cap \mathfrak{B}^*(\sigma)} d\mu^*(\mathfrak{v}) = \int_{\mathbb{D} \cap \mathfrak{B}^*(\sigma)} d\mathfrak{v} \int_{\mathfrak{B}(\Gamma^*(\sigma))} d\mathfrak{v}.
$$
 (46)

Remark 6: In an affine space \mathfrak{B}^* the affine volume $\int_C dv$ is a relative invariant of weight -1. The quotient of two such volumes, so as in (46), is an absolute invariant.

In the case that $\mathfrak B$ and $\mathfrak B^*$ are Euclidean spaces, and so especially $\mathfrak B^*(\sigma)$ is an Euclidean space with the metric fundamental tensor $g^{\nu\mu}(\sigma)$, $g(\sigma)$ = det $(g^{\nu\mu}(\sigma))$, we define as usual

B (d) be equipped with a measure
$$
\mu_G
$$
 of the normalization $\mu_G(\mathfrak{F}(I \cap \sigma)) = I(\mathfrak{F}(I \cap \sigma))$
\n $\text{vol}_m(D \cap \mathfrak{B}^*(\sigma)) = \int d\mu^*(\sigma) = \int d\sigma \int \int d\sigma$. (46)
\n $\text{Den } \mathfrak{B}^*(\sigma) = \int d\mu^*(\sigma) = \int d\sigma \int \int d\sigma$. (46)
\n**Remark 6:** In an affine space \mathfrak{B}^* the affine volume $\int_G d\sigma$ is a relative invariant of weight
\nthe quotient of two such volumes, so as in (46), is an absolute invariant.
\nIn the case that \mathfrak{B} and \mathfrak{B}^* are Euclidean spaces, and so especially $\mathfrak{B}^*(\sigma)$ is an Euclidean
\ne with the metric fundamental tensor $g^{\vee}\mu(\sigma)$, $g(\sigma) = \det(g^{\vee}\mu(\sigma))$, we define as usual
\n $\text{vol}_m(D \cap \mathfrak{B}^*(\sigma)) = \int \int \int g(\sigma) d\mu^*(\sigma) \quad \text{with} \quad \text{vol}_m(\mathfrak{F}(\Gamma^*(\sigma))) = 1.$ (47)
\n $\text{Do } \mathfrak{B}^*(\sigma)$
\n $\text{Vol}(G)$ from (44) is belonging to a group element $\sigma \in \mathfrak{B}_m$ with $\delta_{\sigma} = 1$, the factor before λ in
\nis the volume of an *m*-dimensional ellipsoid, namely of
\n $\lambda \cdot \mathfrak{D} \cap \mathfrak{B}^*(\sigma) = \{\mathfrak{v} \cdot \mathfrak{v}^*(\sigma) : P^{\vee}\mu(\sigma) \nu_{\mathfrak{v}}^* \nu_{\mathfrak{t}}^* \leq (\frac{\lambda}{2\pi})^2\}$. (48)
\nefore

If $W(\sigma)$ from(44) is belonging to a group element $\sigma \in \mathcal{Q}_m$ with $\delta_{\sigma} = 1$, the factor before λ in (44) is the volume of an m -dimensional ellipsoid, namely of

$$
\lambda \cdot \mathbb{D} \cdot \mathfrak{B}^*(\sigma) = \left\{ \mathfrak{v} = v'_{\mathfrak{v}} \mathfrak{e}^{\mathfrak{v}}(\sigma) \colon P^{\mathfrak{v}\mu}(\sigma) v'_{\mathfrak{v}} v'_{\mu} \leq \left(\frac{\lambda}{2\pi}\right)^2 \right\}. \tag{48}
$$

Therefore $W(\sigma)$ from (44) has the form

$$
W(\sigma) = \delta_{\sigma} \cdot \text{vol}_{m}(\mathbb{D} \cap \mathfrak{B}^{*}(\sigma)) \lambda^{m} + O\left(\lambda^{m-2+\frac{2}{m+1}}\right). \tag{49}
$$

In the case that $\mathfrak B$ and $\mathfrak B^*$ are Euclidean spaces, and so espe
 with the metric fundamental tensor $g^VW(\sigma)$, $g(\sigma) = \det(g^VW(\sigma))$
 $\text{vol}_m(\mathbb{D} \cap \mathfrak B^*(\sigma)) = \int_{\mathbb{D} \cap \mathfrak B^*(\sigma)} f(g(\sigma) d\mu^*(\sigma) \text{ with } \text{vol}_m(\mathfrak{F}(\Gamma^$ Here the order of the remainder term in Proposition 2 (resp. Proposition 3) allows to carry out the summation for $m = n$ (yielding then the principal part of $N(\lambda)$) and only just for $m = n - 1$ (producing the secondary part). Now we ascertain that $m = n(\sigma) = n$ is true only for $\sigma = e$ and we have $\mathbb{D} \cap \mathfrak{B}^*(e) = \mathbb{D}$ (see also Remark 5). Because the null vector $t = 0 \in \mathfrak{B}$ is belonging to $\sigma = e$ *we get s* $v(e) = \langle e^v(e), \rho \rangle = 0$ \in Z and hence $\delta_e = 1$. We lodge all summnads of $N(\lambda)$ for *m* $\leq n-2$ in (35) (Proposition 2) in $O(\lambda^{n-2+2/(n+1)})$. So Proposition 2 can be explained now as summation for $m = n$ (yielding then the principal part of N
ducing the secondary part). Now we ascertain that $m = n(\sigma)$
ave $D \cap \mathfrak{B}^*(e) = D$ (see also Remark 5). Because the null v
 $w e get s^v(e) = \langle e^v(e), e \rangle = 0 \in \mathbb{Z}$ and hen 3) allows to carry
only just for $m = 1$
rue only for $\sigma = e$
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Now we ascertain that $m = n(\sigma) = n$ is t
o Remark 5). Because the null vector $t =$
 $0 \in \mathbb{Z}$ and hence $\delta_{\sigma} = 1$. We lodge all sun
 $O(\lambda^{n$ *W*(σ) = $\delta_{\sigma} \cdot \text{vol}_{m}(\mathbb{D} \cap \mathfrak{B}^{*}(\sigma))\lambda^{m} + O(\lambda^{m-2+\frac{m}{m+1}})$.

Here the order of the remainder term in Proposition 2 (resp. Proposition 3) allows to carry

the summation for $m = n$ (yielding then the principal

Theorem 2: The eigenvalue number $N(\lambda)$ is satisfying the estimation

$$
N(\lambda) = \frac{1}{r} \operatorname{vol}_n(\mathbb{D}) \lambda^n + \frac{1}{r} \sum_{\sigma \in \mathbb{S}_{n-1}} \operatorname{vol}_{n-1}(\mathbb{D} \cap \mathfrak{B}^*(\sigma)) \delta_{\sigma} \lambda^{n-1} + O(\lambda^{n-2+2/(n+1)})
$$
(50)

Remark 7: With regard to Remark 6 the assertion (50) of Theorem 2 can be understood also as a result of affine spectral geometry.

4.4 Landau's δ -symbol and the influence of the fixed elements from \mathcal{B} on $N(\lambda)$. The decomalso as a result of affine spectral geometry.
 4.4 Landau's δ **-symbol and the influence of the fixed elements from** \mathcal{D} **on** $N(\lambda)$ **. The decomposition** $\mathcal{B} = \mathfrak{B}(\sigma) \oplus \mathfrak{B}^1(\sigma)$ **of the vector space** \mathcal{B} **i**

4 M. BELGER
\n
$$
\mathfrak{B}(\sigma) = \ker(\sigma - id) \quad \text{and} \quad \mathfrak{B}^1(\sigma) = \text{im}(\sigma - id)
$$
\n1 the sublattices
\n
$$
\mathfrak{D}(\sigma) = \mathfrak{D}(\sigma) \quad \text{and} \quad \mathfrak{D}^1(\sigma) = \mathfrak{D}(\sigma) \quad \text{(51)}
$$

and the sublattices

M. BELGER
\n
$$
\mathfrak{B}(\sigma) = \ker(\sigma - id) \quad \text{and} \quad \mathfrak{B}^1(\sigma) = \text{im}(\sigma - id)
$$
\n(51)

\nthe sublattices

\n
$$
\Gamma(\sigma) = \Gamma \cap \mathfrak{B}(\sigma) \quad \text{and} \quad \Gamma^1(\sigma) = \Gamma \cap \mathfrak{B}^1(\sigma)
$$
\n(52)

\n
$$
\Gamma(\sigma) = \dim \mathfrak{B}(\sigma) = \dim \Gamma(\sigma) \text{ makes possible to formulate the following fixed point proper-}
$$

and $\mathfrak{B}^{\perp}(\sigma) = \text{im}(\sigma - \text{id})$ (51)

and $\Gamma^{\perp}(\sigma) = \Gamma \cap \mathfrak{B}^{\perp}(\sigma)$ (52)
 $\Gamma(\sigma)$ makes possible to formulate the following fixed point properwith $n(\sigma) = \dim \mathfrak{B}(\sigma) = \dim \Gamma(\sigma)$ makes possible to formulate the following fixed point properties.

Lemma 4: The affine transformation $(\sigma, f) \in \mathfrak{B}$ acting on \mathfrak{B} has a fixed point $\mathfrak{e}_0 \in \mathfrak{B}$ if and *only if* $f \in \mathfrak{B}^{\perp}(\sigma)$.

Proof: From (σ, f) _{$\mathbf{f}_0 = \mathbf{f}_0$ there follows $(\sigma - id)\mathbf{f}_0 = -f$, i.e. $-f \in \mathbb{B}^1(\sigma)$ and so also $f \in \mathbb{B}^1(\sigma)$.} Inversely, for $f \in \mathfrak{B}^1(\sigma)$ there is also $-f \in \mathfrak{B}^1(\sigma)$ and so by (51) there is a vector $\mathfrak{r}_o \in \mathfrak{B}$ with - $=(\sigma - id)\xi_0$, that is $(\sigma, f)\xi_0 = \xi_0$.

Corollary: Assume $(\sigma, f) \in \mathcal{G}$ has a fixes point in \mathcal{B} . Then $(\sigma, f + t) \in \mathcal{G}$ has a fixes point in \mathfrak{B} *if and only if* $\mathfrak{t} \in \Gamma^1(\sigma)$.

Proof: From $(\sigma, f) \mathfrak{k}_0 = \mathfrak{k}_0$ there follows $(\sigma - id) \mathfrak{k}_0 = -f$, i.e. $-rsely$, for $f \in \mathfrak{B}^{\perp}(0)$ there is also $-f \in \mathfrak{B}^{\perp}(0)$ and so by (51)
 $-i d) \mathfrak{k}_0$, that is $(\sigma, f) \mathfrak{k}_0 = \mathfrak{k}_0 \blacksquare$
Corollary: Assu **Proof:** let be $f \in \mathbb{S}^1(\sigma)$ (Lemma 4), that is $f = \sigma \xi_0 - \xi_0$, $\xi_0 \in \mathbb{S}$. a) Assume $(\sigma, f + t)\xi_1 = \xi_1$, ξ_1 $\epsilon \mathfrak{B}$, so there is true that $\sigma(\mathfrak{e}_0 + \mathfrak{e}_1) - (\mathfrak{e}_0 + \mathfrak{e}_1) = -t \epsilon \mathfrak{B}^{\perp}(\sigma)$ and then $t \epsilon \mathfrak{B}^{\perp}(\sigma)$. Because (σ, f) and $(0, f + t)$ are in \mathfrak{B} , by (7) there follows that $t \in \Gamma$ and then by (52) $t \in \Gamma^1(\sigma)$. b) Vice versa from t $\epsilon \Gamma^1(\sigma)$ there follows $t \epsilon \mathfrak{B}^1(\sigma)$, and under the assumption $f \epsilon \mathfrak{B}^1(\sigma)$ we obtain -t, -f $\epsilon \mathfrak{B}^1(\sigma)$, i.e. $-t = \sigma_{\xi_2} - \xi_2$ and $-f = \sigma_{\xi_3} - \xi_3$ ($\xi_2, \xi_3 \in \mathcal{B}$). So there is true that $\sigma(\xi_2 + \xi_3) + t + f = (\sigma, f + t)(\xi_2 + t)$ \mathbf{g}_3) = \mathbf{g}_2 + \mathbf{g}_3 **.**

Proposition 4: Let be $(\sigma, f) \in \mathcal{D}$. Then $\delta_{\sigma} = 1$ is true if and only if there is a lattice functio*ñal* $t_0 \in \Gamma$ *wih the property that* $(\sigma, f + t_0)$ has a fixed point $t_0 \in \mathfrak{B}$, *i.e. that* $f + t_0 \in \mathfrak{B}^1(\sigma)$ *.*

Proof: We have to take into consideration that $\langle v, \mathbf{r} \rangle = 0$ if $v \in \mathfrak{B}^*(\sigma)$ and $\mathbf{r} \in \mathfrak{B}^1(\sigma)$ (see (26) and (51); for understanding use dual bases in $\mathfrak{B} = \mathfrak{B}(\sigma) \oplus \mathfrak{B}^4(\sigma)$ and $\mathfrak{B}^* = \mathfrak{B}^*(\sigma) \oplus \mathfrak{B}^{*+}(\sigma))$.

a) Assume $\delta_{\sigma} = 1$ for a fixed $\sigma \in \mathcal{Q}$, i.e. $s^{\nu}(\sigma) = \langle c^{\nu}(\sigma), \rho \rangle \in \mathbb{Z}$ for all $\nu = 1, ..., m$ (see (45), (39) and (37)). Then for an arbitrary $u = u'_0 c^{\nu}(\sigma) \in \Gamma^*(\sigma)$ there is true that $\langle u, f \rangle \in \mathbb{Z}$. If we now **Proof:** We have to take into consideration that $\langle v, \xi \rangle = 0$ if $v \in \mathbb{R}^*(\sigma)$ and $\xi \in \mathbb{R}^1(\sigma)$ (see
(26) and (51); for understanding use dual bases in $\mathfrak{B} = \mathfrak{B}(\sigma) \oplus \mathfrak{B}^1(\sigma)$ and $\mathfrak{B}^* = \mathfrak{B}^*($ **a)** Assume $\delta_{\sigma} = 1$ for a fixed $\sigma \in \mathcal{R}$, i.e. $s^{\vee}(\sigma) = \langle c^{\vee}(\sigma), f \rangle \in \mathcal{Z}$ for all $\nu = 1, ..., m$ (39) and (37)). Then for an arbitrary $\mu = u'_{\sigma} c^{\vee}(\sigma) \in \Gamma^{*}(\sigma)$ there is true that $\langle u, f \rangle \in \mathcal{Z}$. I
dec $-f + b \in \Gamma$ and then $f + t_0 = f_2 + r \in \mathfrak{B}^{\perp}(0)$.

b) Conversely, let there exists a $t_0 \in \Gamma$ with $f + t_0 \in \mathfrak{B}^{\perp}(0)$; we prove that $s^{\vee}(0) \in \mathbb{Z}$ for all ν = 1,..., *m*, i.e. δ_{σ} = 1. We write $s'(\sigma) = \langle c'(\sigma), f \rangle = \langle c'(\sigma), f + t_{\sigma} \rangle - \langle c'(\sigma), t_{\sigma} \rangle$. Here $\langle c''(\sigma), t_{\sigma} \rangle$ ϵ Z because of t_0 ϵ Γ , $c^{\vee}(\sigma)$ ϵ $\Gamma^*(\sigma)$ and so $c^{\vee}(\sigma)$ ϵ Γ^* . Now using the introductory remark of the proof we find $\langle c^{\nu}(\sigma), (+t_{o}) \rangle = 0$ because $(+t_{o} \in \mathfrak{B}^{+}(\sigma), c^{\nu}(\sigma) \in \Gamma^{*}(\sigma))$ and so $c^{\nu}(\sigma) \in \mathfrak{B}^{*}(\sigma)$. Summariting we get $s^{\vee}(\sigma) \in \mathbb{Z}$

4.5 Survey of the influence of fixed (fixed point - free) elements of group 0 on the asymptotic expression for $N(\lambda)$ **.** If we ask for the intrinsic reason of the appearance of the pricipal term $c_0 \lambda^n$ and the secondary term $c_1 \lambda^{n-1}$ in $N(\lambda) = c_0 \lambda^n + c_1 \lambda^{n-1} + O(\lambda^{n-2+2/(n+1)})$ we can answer (Proposition 4):

(i) For $\sigma \in \mathcal{R}_m$ the fixed elements $(\sigma, f + t_o) \in \mathcal{B}$ produce in (49) resp. (50) the volume terms vol $_m^{}(\mathbb{D} \, \cap \, \mathfrak{B}^{\boldsymbol *}(o))\cdot\lambda^m$ whereas fixed point-free elements from \mathfrak{G} make contributions only to the remainder term $O(\lambda^{m-2+2/(m+1)})$. So we have the following knowledge: Eigenvalue Distribution of Differential Operators 295

(i) For $\sigma \in \mathbb{R}_m$ the fixed elements $(\sigma, f + t_o) \in \mathcal{G}$ produce in (49) resp. (50) the volume

s vol_m($\mathbb{D} \cap \mathfrak{B}^*(\sigma)$) λ^m whereas fixed point-free el

(ii) The identity $(e, \mathfrak{g}) \in \mathfrak{G}$ produces the principal part of $N(\lambda)$ (because $\delta_e = 1$, $e \in \mathfrak{L}_n$).

part of $N(\lambda)$.

Concluding remark: The theory developed above can be applied e.g. for crystallographic groups, especially for the 230 space groups. For short it is recommendable to investigate an *ⁿ* (iii) The fixed elements $(\sigma, f + t_0) \in \mathfrak{B}$, $\sigma \in \mathfrak{L}_{n-1}$, produces the summands of the secondary
part of $N(\lambda)$.
Concluding remark: The theory developed above can be applied e.g. for crystallographic
groups, esp (ii) The identity $(e, 0) \in \mathfrak{B}$ produces the principal part of $N(\lambda)$ (because $\delta_e = 1$, $e \in \mathfrak{B}_n$).

(iii) The fixed elements $(\sigma, \mathfrak{f} + t_0) \in \mathfrak{B}$, $\sigma \in \mathfrak{B}_{n-1}$, produces the summands of the secondary
 n = 2 demonstrate a considerable improvement if we turn from $N(\lambda) \sim c_0 \lambda^n$ to $N(\lambda) \sim c_0 \lambda^n$ $+c_1\lambda^{n-1}$ (see the Dissertation B of the author: *Zur asymptotischen Verteilung der Eigenwerte €3- invarianter linearer elliptischer Differentialoperatoren mit konstanten Koeffizienten.* Universität Leipzig *1989).*

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