Zeitschrift für Analysis
und ihre Anwendungen
Vol. 12 (1993), 285 – 296

# Eigenvalue Distribution of Invariant Linear Second Order Elliptic Differential Operators with Constant Coefficients

M. BELGER

Let $\mathfrak{G}$ be a properly discontinuous group of affine transformations acting on an $n$-dimensional affine space and $P$ a $\mathfrak{G}$-invariant linear elliptic differential operator with constant coefficients. In this paper the $\mathfrak{G}$-automorphic eigenvalue problem to $P$ is solved. For the number $N(\lambda)$ of the eigenvalues which are less than or equal to the "frequency bound" $\lambda^2$ the asymptotic estimation $N(\lambda) = c_0\lambda^n + c_1\lambda^{n-1} + O(\lambda^{n-2+2/(n+1)})$ is given with $c_0$ and $c_1$ being interesting geometric invariants.

Key words: *Eigenvalue problem, eigenvalue distribution, invariant linear elliptic differential operator, lattice remainder, asymptotic estimation, principal vector*

AMS subject classification: 47 F 05, 47 A 75, 11 F 72, 11 L 07, 11 H 06, 35 P 20

## 0. Problem

Let $\mathfrak{V}$ be an $n$-dimensional vector space or later at the same time also an affine space, $\mathfrak{V}^*$ its dual; $\mathfrak{G}$ a properly discontinuous group of affine transformations acting on $\mathfrak{V}$ and having a compact fundamental domain [3]. For a $\mathfrak{G}$-invariant positive definite quadratic form $\mathfrak{P}$ on $\mathfrak{V}^*$ and for a fixed vector $\mathfrak{p} \in \mathfrak{V}^*$ we consider the differential operator

$$P[\psi] = \mathfrak{P}\Big(\frac{\partial}{\partial\mathfrak{x}} - 2\pi i\mathfrak{p}, \frac{\partial}{\partial\mathfrak{x}} - 2\pi i\mathfrak{x}\Big)[\psi], \quad \mathfrak{x} \in \mathfrak{V}, \tag{1}$$

and the assigned polynom

$$P(\mathfrak{v}) = -\mathfrak{P}(\mathfrak{v} - 2\pi\mathfrak{p}, \mathfrak{v} - 2\pi\mathfrak{p}), \quad \mathfrak{v} \in \mathfrak{V}^*. \tag{1'}$$

$\mathfrak{G}$-invariant means for $P[\ ]$ that the following relation is valid:

$$P[\psi \circ S] = P[\psi] \circ S \quad \text{for all } S \in \mathfrak{G}. \tag{1''}$$

Now look at the $\mathfrak{G}$-automorphic eigenvalue problem

$$P[\psi] + \mu\psi = 0, \quad \psi \in L_2(\mathfrak{G}). \tag{2}$$

M. Belger: Universität Leipzig, Institut für Mathematik, Augustuspl. 10, D - 04109 Leipzig

ISSN 0232-2064 / \$ 2.50 

$L_2(\mathfrak{G})$ is the Hilbert space over $\mathbb{C}$ of locally square-integrable $\mathfrak{G}$-automorphic functions. $\mathrm{spec}_{\mathfrak{G}}(P)$ denotes the eigenvalue spectrum of (2). We will investigate the eigenvalue distribution $\mathrm{dis}(\mathrm{spec}_{\mathfrak{G}}(P))$ over $\mathbb{R}^+$, where "dis" is defined by the distribution function

$$N(\lambda) = \#\{\mu \in \mathrm{spec}_{\mathfrak{G}}(P): \mu \leq \lambda^2\}. \tag{3}$$

Here sometimes $\lambda$ instead of $\lambda^2$ is taken and called in Weyl's considerations "frequency bound" [25]. To establish a good asymptotic estimation of $N(\lambda)$ we will work out the following subjects:

1. Solution of the $\mathfrak{G}$-automorphic eigenvalue problem (2).
2. Description of $N(\lambda)$ by a certain number of so-called "principal lattice vectors" in a convex domain $\lambda \cdot \mathbb{D} \subset \mathfrak{V}^*$ (see (23)/(23')).
3. Formulation of $N(\lambda)$ as a finite sum of Weyl sums.
4. Asymptotic estimation $N(\lambda) = c_0\lambda^n + c_1\lambda^{n-1} + O(\lambda^{n-2+2/(n+1)})$ with explicit calculation of $c_0$ and $c_1$ as geometric invariants. Survey of influence of fixed (fixed point-free) elements of the group $\mathfrak{G}$ on the asymptotic expression for $N(\lambda)$.

## 1. Solution of the $\mathfrak{G}$-automorphic eigenvalue problem (2)

**1.1 The orthonormal system of $\mathfrak{G}$-automorphic functions in $L_2(\mathfrak{G})$.** To introduce such a system we follow the proceeding of P. Günther in [7: § 1 and § 2].

**The lattice $\Gamma \subset \mathfrak{V}$:** We will write the affine transformation $S: \mathfrak{V} \to \mathfrak{V}$ $(S \in \mathfrak{G})$ of the $n$-dimensional affine space $\mathfrak{V}$ as a Seitzian space group symbol $S = (\sigma, \mathfrak{f})$ with $\mathfrak{x}' = S(\mathfrak{x}) = \sigma\mathfrak{x} + \mathfrak{f}$ $(\mathfrak{x}', \mathfrak{x} \in \mathfrak{V})$ as transformation formula. The components $\sigma$ and $\mathfrak{f}$ are said to be *fixed point* and *translation part* of $S$, respectively. For $R = (\rho, \mathfrak{r}) \in \mathfrak{G}$ and $S \in \mathfrak{G}$ the composition $R \circ S = (\rho\sigma, \sigma\mathfrak{r} + \mathfrak{f})$ is defined by $(R \circ S)(\mathfrak{x}) = S(R(\mathfrak{x}))$. The inverse to $S$ with respect to the identity element $E = (e, \mathfrak{o})$ $\in \mathfrak{G}$ is $S^{-1} = (\sigma^{-1}, -\sigma^{-1}\mathfrak{f})$, where $e = \mathrm{id}$ and $\mathfrak{o} \in \mathfrak{V}$ is the null vector. Now we consider the "point group" $\mathfrak{K}$ of $\mathfrak{G}$,

$$\mathfrak{K} = \{\sigma: (\sigma, \mathfrak{f}) \in \mathfrak{G} \text{ for some } \mathfrak{f} \in \mathfrak{V}\} \tag{4}$$

and the "translation group" $\mathfrak{T} \subset \mathfrak{G}$ of all translations in $\mathfrak{G}$,

$$\mathfrak{T} = \{(e, \mathfrak{t}) \in \mathfrak{G}\}. \tag{4'}$$

We know about $\mathfrak{K}$ and $\mathfrak{T}$ the following [1,3,5]: $\mathfrak{T}$ is an invariant subgroup of $\mathfrak{G}$. The factor group $\mathfrak{G}/\mathfrak{T}$ and $\mathfrak{K}$ are isomorphic and $\mathrm{ord}(\mathfrak{G}/\mathfrak{T})$ is finite. Therefore we can introduce

$$r := \mathrm{ord}(\mathfrak{G}/\mathfrak{T}) = \mathrm{ord}\,\mathfrak{K}. \tag{4''}$$

$\mathfrak{T}$ has $n$ generators $(e, \mathfrak{b}_1), \ldots, (e, \mathfrak{b}_n)$ with $n$ linear independent translation parts $\mathfrak{b}_k$ which are used to form the bas $\mathfrak{V}$ and also to form the $\mathfrak{K}$-invariant $n$-dimensional lattice

$$\Gamma := \mathrm{orb}_{\mathfrak{T}}(\mathfrak{o}) = \{\mathfrak{t} = t^k\mathfrak{b}_k: t^k \in \mathbb{Z}\} \subset \mathfrak{V}. \tag{5}$$

The vector $\mathfrak{a} \in \mathfrak{V}$ is said to be "belonging to $\sigma \in \mathfrak{K}$" if $(\sigma, \mathfrak{a}) \in \mathfrak{G}$. Together with $\mathfrak{a}$ then also all vectors $\mathfrak{a} + \Gamma$ and only these are belonging to $\sigma$. So modulo $\Gamma$ exactly one vector $\mathfrak{a}$ is belonging to $\sigma$ and will be denoted by $\mathfrak{a} = \mathfrak{f}$. In the coset decomposition of $\mathfrak{G}$ relative to $\mathfrak{T}$,

$$\mathfrak{G} = S_1 \circ \mathfrak{T} + \ldots + S_r \circ \mathfrak{T}, \quad S_\nu = (\sigma_\nu, \mathfrak{f}_\nu) \tag{6}$$

the elements of one of the same coset $S_\nu \circ \mathfrak{T}$ have the same fixed point part $\sigma_\nu$ but different cosets have different such parts. If $(\sigma_1, \mathfrak{f}_1)$, $(\sigma_2, \mathfrak{f}_2)$, $(\sigma_1 \circ \sigma_2, \mathfrak{f}) \in \mathfrak{G}$ it may be adventageous to think of the Frobenius congruence

$$\sigma_1 \mathfrak{f}_2 + \mathfrak{f}_1 \equiv \mathfrak{f} \bmod \Gamma. \tag{7}$$

**The dual lattice $\Gamma^* \subset \mathfrak{B}^*$**: A usually in cristallography here we turn to the dual situation. Let $\mathfrak{B}^*$ be the dual space of linear functionals on $\mathfrak{B}$, $\langle \mathfrak{v}, \mathfrak{x} \rangle$ the value of $\mathfrak{v} \in \mathfrak{B}^*$ in $\mathfrak{x} \in \mathfrak{B}$. Relative to $\Gamma \subset \mathfrak{B}$, let

$$\Gamma^* = \{\mathfrak{u} = u_k \mathfrak{b}^k : u_k \in \mathbb{Z}\} \subset \mathfrak{B}^*, \quad \langle \mathfrak{b}^h, \mathfrak{b}_k \rangle = \delta_k^h, \tag{8}$$

be the dual lattice in $\mathfrak{B}^*$. As bas $\mathfrak{B}^*$ we use then $\{\mathfrak{b}^1, \ldots, \mathfrak{b}^n\}$. Instead of $\sigma \in \mathfrak{L}$ here we need the adjoint mapping $\sigma^\tau$ to $\sigma$: $\sigma^\tau$ is defined by

$$\sigma^\tau : \mathfrak{B}^* \to \mathfrak{B}^* \quad \text{with } \sigma^\tau \mathfrak{v} = \mathfrak{v} \circ \sigma.$$

**The pricipal classes $\mathfrak{h} \subset \Gamma^*$**: For a fixed lattice functional $\mathfrak{u} \in \Gamma^*$ we introduce the equivalence class

$$\mathfrak{k} := \{\mathfrak{u}' \in \Gamma^* : \mathfrak{u}' = \sigma^\tau \mathfrak{u} \text{ for all } \sigma \in \mathfrak{L}\} = \{\mathfrak{u}_1, \ldots, \mathfrak{u}_l\}. \tag{9}$$

Here is $l = \operatorname{ord} \mathfrak{k} \leq r = \operatorname{ord} \mathfrak{L}$ as we can see by help of the decomposition $\mathfrak{L} = \mathfrak{N}(\mathfrak{u}) \cup (\mathfrak{L} \setminus \mathfrak{N}(\mathfrak{u}))$ relative to the adjoint isotropy group to $\mathfrak{u}$,

$$\mathfrak{N}(\mathfrak{u}) = \{\sigma \in \mathfrak{L} : \sigma^\tau \mathfrak{u} = \mathfrak{u}\}. \tag{10}$$

So $\Gamma^*$ is decomposed completely in a set $\mathfrak{K}$ of classes $\mathfrak{k}$. Among these classes the so-called principal classes $\mathfrak{h}$ play a leading part: For $\mathfrak{N}(\mathfrak{u})$ we consider the character $\chi(\mathfrak{u}, \cdot)$ with

$$\chi(\mathfrak{u}, \sigma) = \exp\{2\pi i \langle \mathfrak{u}, \mathfrak{f} \rangle\}, \quad (\sigma, \mathfrak{f}) \in \mathfrak{G}. \tag{11}$$

In $(\sigma, \mathfrak{f})$ the vector $\mathfrak{f}$ is well established and

$$\varphi_{\mathfrak{u}}(\mathfrak{x}) = \exp\{2\pi i \langle \mathfrak{u}, \mathfrak{x} \rangle\} \tag{12}$$

is a $\mathfrak{T}$-automorphic function on $\mathfrak{B}$. Therefore $\chi$ is correctly defined. If

$$\chi(\mathfrak{u}, \sigma) = 1 \quad \text{for all } \sigma \in \mathfrak{N}(\mathfrak{u}) \tag{13}$$

so $\chi(\mathfrak{u}, \cdot)$ is said to be *principal character* of $\mathfrak{N}(\mathfrak{u})$ and $\mathfrak{u}$ *principal vector* of $\Gamma^*$. Now if $\mathfrak{u} \in \mathfrak{k}$ is a principal vector, $\mathfrak{k}$ contains only principal vectors and is called *principal class* $\mathfrak{h}$. Otherwise $\mathfrak{k}$ contains only non—principal vectors ($\mathfrak{k}$ is a non-principal class). Let $\mathfrak{H}$ be the set of all principal classes $\mathfrak{h} \subset \Gamma^*$.

**The orthonormal system of $\mathfrak{G}$-automorphic functions**: Let $\mathfrak{f} = \{\mathfrak{u}_1, \ldots, \mathfrak{u}_l\} \in \mathfrak{H}$ be a principal class and $\operatorname{rep}(\mathfrak{L}/\mathfrak{N}(\mathfrak{u}_1))_L = \{\sigma_1, \ldots, \sigma_l\}$ a system of representatives of the left coset decomposition of $\mathfrak{L}$ with respect to $\mathfrak{N}(\mathfrak{u}_1)$. Then $\mathfrak{f}_1, \ldots, \mathfrak{f}_l \in \mathfrak{B}$ shall be vectors belonging to $\sigma_1, \ldots, \sigma_l$, respectively, i.e. $S_\nu = (\sigma_\nu, \mathfrak{f}_\nu)$ for $\nu = 1, \ldots, l$.

**Definition:** The sum

$$\psi_{\mathfrak{h}} = \frac{1}{\sqrt{l}} \sum_{\nu=1}^{l} \varphi_{\mathfrak{u}_1} \circ S_\nu \tag{14}$$

is said to be a $\mathfrak{h}$-corresponding function on $\mathfrak{V}$.

**Remark 1:** For each $\mathfrak{v} \in \mathfrak{V}^*$ the function $\varphi_{\mathfrak{v}}$ is satisfying the relation

$$\varphi_{\mathfrak{v}} \circ S = \varphi_{\mathfrak{v}}(\mathfrak{f}) \varphi_{\sigma^\tau \mathfrak{v}} \quad \text{for all } S = (\sigma, \mathfrak{f}) \in \mathfrak{G}. \tag{15}$$

Especially for the translations $S = (e, t) \in \mathfrak{T}$ and lattice vectors $\mathfrak{v} = \mathfrak{u} \in \Gamma^*$ we see that $\varphi_{\mathfrak{u}}$ is $\mathfrak{T}$-automorphic, even $\varphi_{\mathfrak{u}} \in L_2(\mathfrak{T})$ ($L_2$-space of $\mathfrak{T}$-automorphic functions).

**Remark 2:** If $\sigma$ runs through $\mathfrak{K}$, so $\sigma^\tau \mathfrak{u}_1$ runs through $\mathfrak{h} = \{ \mathfrak{u}_1, \dots, \mathfrak{u}_l\}$ - but in general not simply ($l \leq r$). But if $\sigma$ runs only through $\text{rep}(\mathfrak{K}/\mathfrak{N}(\mathfrak{u}_1))_{\text{L}}$, so from $\mathfrak{u}_1$ every vector $\mathfrak{u}_\nu \in \mathfrak{h}$ arises exactly one time by $\mathfrak{u}_\nu = \sigma^\tau \mathfrak{u}_1$.

The $\mathfrak{h}$-corresponding functions $\psi_{\mathfrak{h}}$ are elements of $L_2(\mathfrak{G})$. As functions normed to one just the $\psi_{\mathfrak{h}}$ build a complete orthonormal system $\{\psi_{\mathfrak{h}}: \mathfrak{h} \in \mathfrak{H}\}$ in $L_2(\mathfrak{G})$ [7: §2/(2.8)].

**1.2 The $\mathfrak{G}$-automorphic eigenfunctions and spec$_{\mathfrak{G}}$ of $P$.** To prove that the $\mathfrak{h}$-corresponding functions $\psi_{\mathfrak{h}}$ are the eigenfunctions of $P$ we must investigate the action of $P$ on $\varphi_{\mathfrak{v}} \circ S$.

**Lemma 1:** *The $\mathfrak{G}$-invariant differential operator $P$ from* (1) *acts on the functions $\varphi_{\mathfrak{v}} \circ S$ from* (14) *or* (15) *according to*

$$P[\varphi_{\mathfrak{v}} \circ S] = P(2\pi\mathfrak{v}) \cdot \varphi_{\mathfrak{v}} \circ S \quad \textit{for all } S \in \mathfrak{G},\ \mathfrak{v} \in \mathfrak{V}^*. \tag{16}$$

**Proof:** The operator $P$ can be written as

$$P = P^{hk} \partial_h \partial_k - 4\pi i P^h \partial_h - 4\pi^2 P^o. \tag{17}$$

Here $P^{hk}$ are the coefficients of the quadratic form $\mathfrak{P}$ from (1'), furthermore $P^h = P^{hk} p_k$, $P^o = P^{hk} p_h p_k$, where $\mathfrak{p} = p_h \mathfrak{b}^h$ and $\partial_h = \partial/\partial x^h$, $\mathfrak{x} = x^h \mathfrak{b}_h$ - explained altogether respectively to bas$\mathfrak{V}$ or bas$\mathfrak{V}^*$. Now we apply $P$ on $\varphi_{\mathfrak{v}}$, $\mathfrak{v} = v_\nu \mathfrak{b}^\nu$: Using (12) and (8) we obtain

$$\partial_h \varphi_{\mathfrak{v}}(\mathfrak{x}) = \partial/\partial x^h (\exp 2\pi i \langle v_\nu \mathfrak{b}^\nu, x^\mu \mathfrak{b}_\mu \rangle) = \varphi_{\mathfrak{v}}(\mathfrak{x}) \cdot 2\pi i\, \partial/\partial x^h (v_\nu \cdot x^\nu) = 2\pi i v_h \varphi_{\mathfrak{v}}(\mathfrak{x})$$

$$\partial_h \partial_k \varphi_{\mathfrak{v}}(\mathfrak{x}) = (2\pi i)^2 v_h v_k \varphi_{\mathfrak{v}}(\mathfrak{x}).$$

Now (17) and after that (1') gives

$$\begin{aligned} P[\varphi_{\mathfrak{v}}] &= \left(-P^{hk}(2\pi v_h)(2\pi v_k) + 4\pi P^h (2\pi v_h) - 4\pi^2 P^o\right) \varphi_{\mathfrak{v}} \\ &= -\mathfrak{P}(2\pi\mathfrak{v} - 2\pi\mathfrak{p}, 2\pi\mathfrak{v} - 2\pi\mathfrak{p}) \varphi_{\mathfrak{v}} = P(2\pi\mathfrak{v}) \varphi_{\mathfrak{v}}. \end{aligned}$$

So (16) follows from the $\mathfrak{G}$-invariance of $P$, i.e. from (1'') ∎

If we now take into account the $\mathfrak{h}$-corresponding function $\psi_{\mathfrak{h}}$ from (14), formula (16) gives

$$P[\psi_{\mathfrak{h}}] = \frac{1}{\sqrt{l}} \sum_{\nu=1}^{l} P(2\pi \mathfrak{u})\varphi_{\mathfrak{u}} \circ S_\nu = P(2\pi\mathfrak{u})\psi_{\mathfrak{h}}, \quad \mathfrak{u} \in \mathfrak{h}. \tag{18}$$

**Definition:** If $\mathfrak{u} \in \mathfrak{k}$, we can write

$$P(2\pi\mathfrak{k}) = P(2\pi\mathfrak{u}), \tag{19}$$

(where $P(2\pi\mathfrak{k})$ can be understood as a class norm $\|\mathfrak{k}\|^2$ of $\mathfrak{k}$).

The justification for (19) comes from the $\mathfrak{L}$-automorphy of $P$ from (1′),

$$P(\sigma^T\mathfrak{v}) = P(\mathfrak{v}) \text{ for all } \sigma \in \mathfrak{L},\ \mathfrak{v} \in \mathfrak{B}^*, \tag{20}$$

and of the fact that all $\mathfrak{u} \in \mathfrak{k} = \{\mathfrak{u}_1, \dots, \mathfrak{u}_l\}$ arise e.g. from $\mathfrak{u}_1$ by means of the equivalence $\mathfrak{u} = \sigma^T\mathfrak{u}_1$, $\sigma \in \mathfrak{L}$.

**Remark 3:** If the class norms of $\mathfrak{k}_1, \mathfrak{k}_2$ are different, $P(2\pi\mathfrak{k}_1) \neq P(2\pi\mathfrak{k}_2)$, the same is always right for the classes, $\mathfrak{k}_1 \neq \mathfrak{k}_2$. But the inverse assertion is not right; if $\mathfrak{k}_1 \neq \mathfrak{k}_2$, notwithstanding may be $P(2\pi\mathfrak{k}_1) = P(2\pi\mathfrak{k}_2)$.

**Theorem 1:** *To each principal class $\mathfrak{h} \in \mathfrak{H}$ we can assign exactly one eigenvalue $\mu = \mu_{\mathfrak{h}}$ of the $\mathfrak{G}$-automorphic eigenvalue problem* (2), *namely*

$$\mu_{\mathfrak{h}} = -P(2\pi\mathfrak{h}) \tag{21}$$

*with*

$$m_{\mathfrak{G}}(\mu_{\mathfrak{h}}) = \operatorname{card}\{\mathfrak{h}' \in \mathfrak{H}: P(2\pi\mathfrak{h}') = P(2\pi\mathfrak{h})\} \tag{22}$$

*as multiplicity; thereby the $\mathfrak{h}$-corresponding function $\psi_{\mathfrak{h}}$ belongs to $\mu_{\mathfrak{h}}$ as the eigenfunction. The set* $\operatorname{spec}_{\mathfrak{G}}(P) = \{\mu_{\mathfrak{h}}: \mathfrak{h} \in \mathfrak{H}\}$ *is the complete $\mathfrak{G}$-automorphic eigenvalue spectrum of the $\mathfrak{G}$-invariant differential operator $P$ from* (1).

**Proof:** The correspondence $\mathfrak{h} \to \psi_{\mathfrak{h}}$ from(14), and (18), prove the first part of the theorem. The completeness of $\operatorname{spec}_{\mathfrak{G}}(P)$ follows from the completeness of the orthonormal system $\{\psi_{\mathfrak{h}}: \mathfrak{h} \in \mathfrak{H}\}$ of $L_2(\mathfrak{G})$. Let $\psi = \sum c_{\mathfrak{h}}\psi_{\mathfrak{h}}$ (summation over $\mathfrak{h} \in \mathfrak{H}$) be an arbitrary $\mathfrak{G}$-automorphic eigenfunction of $P$ to the eigenvalue $\mu \neq \mu_{\mathfrak{h}}$ for all $\mathfrak{h} \in \mathfrak{H}$. Then from (2), (18), (19) and (21) for each $\mathfrak{h} \in \mathfrak{H}$ there follows $c_{\mathfrak{h}}(\mu_{\mathfrak{h}} - \mu) = 0$. Consequently there would be $c_{\mathfrak{h}} = 0$ and therefore $\psi = 0$ which is a contradiction ∎

## 2. $N(\lambda)$ as the number of principal classes $\mathfrak{h}$ contained in a certain convex domain $\lambda \cdot \mathbb{D} \subset \mathfrak{B}^*$

The operator $P$ has the following geometric appearance.

**Definition:** The domains in $\mathfrak{B}^*$

$$\mathbb{D} = \{\mathfrak{v} \in \mathfrak{B}^*: -P(\mathfrak{v} + 2\pi\mathfrak{p}) \le (1/2\pi)^2\} \tag{23}$$

$$\lambda\cdot\mathbb{D} = \{\mathfrak{v} \in \mathfrak{B}^*: -P(\mathfrak{v} + 2\pi\mathfrak{p}) \le (\lambda/2\pi)^2\} \tag{23'}$$

$$\mathfrak{p} + \lambda\cdot\mathbb{D} = \{\mathfrak{v} \in \mathfrak{B}^*: -P(2\pi\mathfrak{v}) \le \lambda^2\} \tag{23''}$$

in this order are said to be *gauge domain*, *homothetical expansion* of $\mathbb{D}$ with $\lambda > 0$ as factor, *parallel translated domain* by the vector $\mathfrak{p} \in \mathfrak{B}^*$ (from (1)).

The $\mathfrak{G}$-invariance of $P$ means for these domains

**Lemma 2:** *The gauge domain* $\mathbb{D}$ *and so also all its homothetical expansions* $\lambda\cdot\mathbb{D}$ *are* $\mathfrak{L}$-*invariant. Therefore for an equivalence class* $\mathfrak{k} \in \mathfrak{K}$ *there is valid*

$$\textit{either } \mathfrak{k} \subset (\mathfrak{p} + \lambda\cdot\mathbb{D}) \ \textit{ or } \ \mathfrak{k} \cap (\mathfrak{p} + \lambda\cdot\mathbb{D}) = \emptyset. \tag{24}$$

Now if we look at $N(\lambda)$ from (3) and $\mu_{\mathfrak{h}}$ from (21) we could ask for the geometric locus containing all $\mathfrak{h}$ with $\mu_{\mathfrak{h}} \le \lambda^2$. The formulas (21), (19), (1'), (23') and (24) yield

**Proposition 1:** *The number of eigenvalues* $\mu_{\mathfrak{h}} \le \lambda^2$ *is given by*

$$N(\lambda) = \mathrm{card}\{\mathfrak{h} \in \mathfrak{H}: \mathfrak{h} \subset (\mathfrak{p} + \lambda\cdot\mathbb{D})\}. \tag{25}$$

## 3. $N(\lambda)$ as a finite sum of Weyl sums

**3.1 A proposition of P. Günther.** Let

$$\mathfrak{B}^*(\sigma) = \ker(\sigma^\tau - \mathrm{id}) \quad \text{and} \quad \Gamma^*(\sigma) = \Gamma^* \cap \mathfrak{B}^*)\sigma) \tag{26}$$

be the eigenspace to the eigenvalue 1 of $\sigma^\tau$ and the $\mathbb{Z}$-module of all lattice functionals of $\mathfrak{B}^*(\sigma)$, respectively (look at (8)). According to [7: Proposition 2.2], for a function $f: \mathfrak{B}^* \to \mathbb{C}$ it is valid

$$\sum_{\mathfrak{h}\in\mathfrak{H}} \frac{1}{\mathrm{card}\,\mathfrak{h}} \sum_{\mathfrak{u}\in\mathfrak{h}} f(\mathfrak{u}) = \frac{1}{r} \sum_{\sigma\in\mathfrak{L}} W(\sigma) \tag{27}$$

so far as

$$W(\sigma) := \sum_{\mathfrak{u}\in\Gamma^*(\sigma)} \chi(\mathfrak{u},\sigma) f(\mathfrak{u}) \tag{28}$$

is absolutely convergent for all $\sigma \in \mathfrak{L}$.

**3.2 The characteristic function $\chi_\lambda$ of $\lambda\cdot\mathbb{D}$.** Let $\chi$ be the characteristic function of $\mathbb{D}$ and $\chi_\lambda$ that of $\lambda\cdot\mathbb{D}$. From the definition of $\chi_\lambda$ and the $\mathfrak{L}$-invariance of $\lambda\cdot\mathbb{D}$ (Lemma 2) you can easily see

**Lemma 3:** *For* $\mathfrak{v} \in \mathfrak{B}^*$ *we have*

$$\chi_\lambda(\mathfrak{v}) = \chi(\tfrac{1}{\lambda}\cdot\mathfrak{v}) \quad \textit{for all } \lambda > 0 \tag{29}$$

$$\chi_\lambda(\sigma^\tau \mathfrak{v}) = \chi_\lambda(\mathfrak{v}) \quad \textit{for all } \sigma \in \mathfrak{L}, \tag{29'}$$

*i.e.* $\chi_\lambda$ *is* $\mathfrak{L}$*-automorphic on* $\mathfrak{V}^*$.

Now regard $\chi_\lambda$ as a partial function on $-\mathfrak{p} + \Gamma^* \subset \mathfrak{V}^*$. Then $\chi_\lambda$ is a class function depending only on the equivalence classes $-\mathfrak{p} + \mathfrak{k}$ of the lattice $-\mathfrak{p} + \Gamma^*$ for all $\mathfrak{k} \in \mathfrak{K}$:

$$\chi_\lambda(-\mathfrak{p} + \mathfrak{k}) = \begin{cases} 1 & \text{if } \mathfrak{k} \subset \mathfrak{p} + \lambda\cdot\mathbb{D} \\ 0 & \text{if } \mathfrak{k} \not\subset \mathfrak{p} + \lambda\cdot\mathbb{D} \end{cases} \tag{30}$$

(see also (24)). Now we set going proposition (27)/(28) choosing $f(\mathfrak{u})$ in accordance with $f(\mathfrak{u}) := \chi_\lambda(-\mathfrak{p} + \mathfrak{u}) = \chi_\lambda(-\mathfrak{p} + \mathfrak{h})$ for $\mathfrak{u} \in \mathfrak{h} \in \mathfrak{H}$. Then

$$W(\sigma) := \sum_{\mathfrak{u}\in\Gamma^*(\sigma)\cap(\mathfrak{p}+\lambda\cdot\mathbb{D})} \chi(\mathfrak{u}, \sigma) \tag{31}$$

is a finite and so an absolutely convergent series. Because of (30) and (25) the left-hand side of (27) is equal to $N(\lambda)$ so that

$$N(\lambda) = \frac{1}{r}\sum_{\sigma\in\mathfrak{L}} W(\sigma). \tag{32}$$

**3.3 Splitting of $N(\lambda)$ into isodimensional summands.** Le be

$$n(\sigma) := \dim \mathfrak{V}^*(\sigma) \tag{33}$$

and

$$\mathfrak{L}_m := \{\sigma \in \mathfrak{L}: n(\sigma) = m\}, \quad m = 0, 1, \ldots, n. \tag{34}$$

For $\sigma \in \mathfrak{L}_m$ the Z-module $\Gamma^*(\sigma)$ from (26) has $m$ linearly independent generators. Now (32) can be dissected according to

**Proposition 2:** $N(\lambda)$ *is the sum of isodimensional summands*:

$$N(\lambda) = \frac{1}{r}\sum_{m=0}^{n} \sum_{\sigma\in\mathfrak{L}_m} W(\sigma) \tag{35}$$

*where* $W(\sigma)$ *with* $(\sigma, \mathfrak{f}) \in \mathfrak{G}$ *are the Weyl sums* (31)/(11), *or for a specific purpose formulated,*

$$W(\sigma) = \sum_{\substack{\mathfrak{u} \equiv \mathfrak{p} \bmod \Gamma^*(\sigma) \\ -P(2\pi\mathfrak{u}) \leq \lambda^2}} \exp\{2\pi i\langle \mathfrak{u}, \mathfrak{f}\rangle\}. \tag{36}$$

The special kind of summation in (36) in comparison with that of (31) follows from (23'').

**Definition:** In (35) the summand with $m = n$ is said to be *principal part* and that with $m = n - 1$ *secondary part* of $N(\lambda)$.

**Remark 4:** All the other summands of $N(\lambda)$ with $m \leq n-2$ will be proved subordinate and get into the remainder during the asymptotic estimation of $N(\lambda)$ in Subsections 4.2/4.3 (see (49)).

## 4. The asymptotic estimation of $N(\lambda)$

**4.1 Formulation of the Weyl sum $W(\sigma)$ in coordinates relative to bas $\Gamma^*(\sigma)$.** Let be

$$\begin{aligned} &\operatorname{bas}\Gamma^*(\sigma) := \{\mathfrak{c}^1(\sigma),\dots,\mathfrak{c}^m(\sigma)\}, \quad \operatorname{bas}\Gamma^* = \{\mathfrak{b}^1,\dots,\mathfrak{b}^n\},\\ &\mathfrak{c}^\nu(\sigma) = c_h^\nu(\sigma)\mathfrak{b}^h, \quad c_h^\nu(\sigma)\in\mathbb{Z} \quad (h=1,\dots,n;\ \nu=1,\dots,m). \end{aligned} \tag{37}$$

Because of $\tau^\nu(\sigma)\in\Gamma^*(\sigma)$ there is $(\sigma^T - \mathrm{id})\tau^\nu(\sigma) = 0$. Therefore $c_i^\nu(\sigma)$ for each $\nu$ is a solution of the system of linear equations $(\sigma_j^i - \delta_j^i)c_i^\nu(\sigma) = 0$ $(j = 1,\dots,n)$ and naturally $\sigma^T\mathfrak{b}^i = \sigma_j^i\mathfrak{b}^j$.

Agreement: Latin indices run through $1,\dots,n$ and Greek indices through $1,\dots,m$ - only with the exception of $\sigma\in\mathfrak{K}$.

For $\mathfrak{u}\in\Gamma^*(\sigma)$ and for $\mathfrak{p}\in\mathfrak{B}^*(\sigma)$ as the invariant vector from (1′) we write

$$\mathfrak{u} = u'_\nu\mathfrak{c}^\nu(\sigma) = u'_\nu c_h^\nu(\sigma)\mathfrak{b}^h = u_h\mathfrak{b}^h \quad\text{and}\quad \mathfrak{p} = p'_\nu\mathfrak{c}^\nu(\sigma) = p'_\nu c_h^\nu(\sigma)\mathfrak{b}^h = p_h\mathfrak{b}^h. \tag{38}$$

Then we have

$$\langle\mathfrak{u},\mathfrak{f}\rangle = u'_\nu s^\nu(\sigma) \quad\text{with}\quad s^\nu(\sigma) = \langle\mathfrak{c}^\nu(\sigma),\mathfrak{f}\rangle. \tag{39}$$

Now looking at (17) we introduce the symmetric $m\times m$-matrix $(P^{\nu\mu}(\sigma))$ with

$$P^{\nu\mu}(\sigma) = P^{hk}c_h^\nu(\sigma)c_k^\mu(\sigma), \quad \Delta(\sigma) := \det(P^{\nu\mu}(\sigma)). \tag{40}$$

By (38) this makes possible to write $P$ in form of

$$-P(2\pi\mathfrak{u}) = (2\pi)^2P^{\nu\mu}(\sigma)w'_\nu w'_\mu, \quad w'_\nu = u'_\nu - p'_\nu. \tag{41}$$

Therefore Proposition 2 in coordinates relative to bas $\Gamma^*(\sigma)$ can be formulated as

**Proposition 3:** *$N(\lambda)$ (so as in Proposition 2) is the sum of the Weyl sums*

$$W(\sigma) = e^{2\pi i p'_\nu s^\nu(\sigma)} \sum_{\substack{w'_\nu \equiv -p_\nu \bmod(1)\\ P^{\nu\mu}(\sigma)w'_\nu w'_\mu = (\lambda/2\pi)^2}} e^{2\pi i w'_\nu s^\nu(\sigma)}. \tag{42}$$

**Remark 5:** For $\sigma = e$ ($e$ - identity in $\mathfrak{K}$) we obtain

$$\begin{aligned} &n(e) = n,\ \mathfrak{K}_n = \{e\},\ \mathfrak{B}^*(e) = \mathfrak{B}^*,\ \Gamma^*(e) = \Gamma^*,\ \mathfrak{c}^\nu(e) = \mathfrak{b}^\nu\\ &c_h^\nu(e) = \delta_h^\nu,\ u'_\nu = u_\nu,\ p'_\nu = p_\nu,\ P^{\nu\mu}(e) = P^{\nu\mu},\ \Delta(e) = \det(P^{\nu\mu}). \end{aligned} \tag{43}$$

**4.2 Landau's estimation of lattice remainder applied to the Weyl sum $W(\sigma)$.** In (42) we have the sum of the unimodular weights $\exp\{2\pi i w'_\nu s^\nu(\sigma)\}$ which load the lattice functionals $\mathfrak{w}\in\Gamma^*(\sigma)$ within the $m = n(\sigma)$-dimensional ellipsoid $(\mathfrak{p} + \lambda\cdot\mathbb{D}) \cap \mathfrak{B}^*(\sigma)$. The estimation of such a sum $W(\sigma)$ is a classical problem which was worked out above all by E. Landau ([14: Chapter I/(7) and (10)] and [19]). As we know this leads to the result

$$W(\sigma) = \frac{\delta_\sigma}{2^m\sqrt{\pi}^m\sqrt{\Delta(\sigma)}\,\Gamma\left(\frac{m+2}{2}\right)}\lambda^m + O\left(\lambda^{m-2+\frac{2}{m+1}}\right) \tag{44}$$

$$\delta_\sigma = 1 \text{ if } s^\nu(\sigma) \in \mathbb{Z} \quad \text{and} \quad \delta_\sigma = 0 \text{ otherwise.} \tag{45}$$

**Definition:** $\delta_\sigma$ will be called *Landau's $\delta$-symbol* which is assigned to $\sigma$ (see Proposition 4).

**4.3 $N(\lambda)$ and the $m$-dimensional volumes $\mathrm{vol}_m(\lambda\cdot\mathbb{D} \cap \mathfrak{V}^*(\sigma))$.** Let be $\sigma \in \mathfrak{L}_m$ and $m = n(\sigma)$. Let $\mathfrak{V}^*(\sigma)$ be equipped with a measure $\mu_\sigma^*$ of the normalization $\mu_\sigma^*(\mathfrak{F}(\Gamma^*(\sigma))) = 1$ ($\mathfrak{F}(\cdot)$ - "fundamental domain of"). So we can introduce the $m$-dimensional volume of $\mathbb{D} \cap \mathfrak{V}^*(\sigma)$,

$$\mathrm{vol}_m(\mathbb{D} \cap \mathfrak{V}^*(\sigma)) = \int_{\mathbb{D}\cap\mathfrak{V}^*(\sigma)} d\mu^*(\mathfrak{v}) = \int_{\mathbb{D}\cap\mathfrak{V}^*(\sigma)} d\mathfrak{v} \Big/ \int_{\mathfrak{F}(\Gamma^*(\sigma))} d\mathfrak{v}. \tag{46}$$

**Remark 6:** In an affine space $\mathfrak{V}^*$ the affine volume $\int_G d\mathfrak{v}$ is a relative invariant of weight -1. The quotient of two such volumes, so as in (46), is an absolute invariant.

In the case that $\mathfrak{V}$ and $\mathfrak{V}^*$ are Euclidean spaces, and so especially $\mathfrak{V}^*(\sigma)$ is an Euclidean space with the metric fundamental tensor $g^{\nu\mu}(\sigma)$, $g(\sigma) = \det(g^{\nu\mu}(\sigma))$, we define as usual

$$\mathrm{vol}_m(\mathbb{D} \cap \mathfrak{V}^*(\sigma)) = \int_{\mathbb{D}\cap\mathfrak{V}^*(\sigma)} \sqrt{g(\sigma)}\, d\mu^*(\sigma) \quad \text{with} \quad \mathrm{vol}_m(\mathfrak{F}(\Gamma^*(\sigma))) = 1. \tag{47}$$

If $W(\sigma)$ from(44) is belonging to a group element $\sigma \in \mathfrak{L}_m$ with $\delta_\sigma = 1$, the factor before $\lambda$ in (44) is the volume of an $m$-dimensional ellipsoid, namely of

$$\lambda\cdot\mathbb{D} \cap \mathfrak{V}^*(\sigma) = \left\{\mathfrak{v} = v'_\nu \mathfrak{c}^\nu(\sigma) \colon P^{\nu\mu}(\sigma) v'_\nu v'_\mu \le \left(\tfrac{\lambda}{2\pi}\right)^2\right\}. \tag{48}$$

Therefore $W(\sigma)$ from (44) has the form

$$W(\sigma) = \delta_\sigma \cdot \mathrm{vol}_m(\mathbb{D} \cap \mathfrak{V}^*(\sigma))\lambda^m + O\left(\lambda^{m-2+\frac{2}{m+1}}\right). \tag{49}$$

Here the order of the remainder term in Proposition 2 (resp. Proposition 3) allows to carry out the summation for $m = n$ (yielding then the principal part of $N(\lambda)$) and only just for $m = n - 1$ (producing the secondary part). Now we ascertain that $m = n(\sigma) = n$ is true only for $\sigma = e$ and we have $\mathbb{D} \cap \mathfrak{V}^*(e) = \mathbb{D}$ (see also Remark 5). Because the null vector $\mathfrak{t} = \mathfrak{o} \in \mathfrak{V}$ is belonging to $\sigma = e$ we get $s^\nu(e) = \langle \mathfrak{c}^\nu(e), \mathfrak{o}\rangle = 0 \in \mathbb{Z}$ and hence $\delta_e = 1$. We lodge all summnads of $N(\lambda)$ for $m \le n - 2$ in (35) (Proposition 2) in $O(\lambda^{n-2+2/(n+1)})$. So Proposition 2 can be explained now as

**Theorem 2:** *The eigenvalue number $N(\lambda)$ is satisfying the estimation*

$$N(\lambda) = \frac{1}{r}\mathrm{vol}_n(\mathbb{D})\lambda^n + \frac{1}{r}\sum_{\sigma\in\mathfrak{L}_{n-1}} \mathrm{vol}_{n-1}(\mathbb{D} \cap \mathfrak{V}^*(\sigma))\delta_\sigma \cdot \lambda^{n-1} + O(\lambda^{n-2+2/(n+1)}) \tag{50}$$

*where Landau's symbol $\delta_\sigma$ is to be taken from Proposition* 4.

**Remark 7:** With regard to Remark 6 the assertion (50) of Theorem 2 can be understood also as a result of affine spectral geometry.

**4.4 Landau's $\delta$-symbol and the influence of the fixed elements from $\mathfrak{G}$ on $N(\lambda)$.** The decomposition $\mathfrak{V} = \mathfrak{V}(\sigma) \oplus \mathfrak{V}^\perp(\sigma)$ of the vector space $\mathfrak{V}$ into the subcpaces

$$\mathfrak{B}(\sigma) = \ker(\sigma - \mathrm{id}) \quad \text{and} \quad \mathfrak{B}^{\perp}(\sigma) = \mathrm{im}(\sigma - \mathrm{id}) \tag{51}$$

and the sublattices

$$\Gamma(\sigma) = \Gamma \cap \mathfrak{B}(\sigma) \quad \text{and} \quad \Gamma^{\perp}(\sigma) = \Gamma \cap \mathfrak{B}^{\perp}(\sigma) \tag{52}$$

with $n(\sigma) = \dim \mathfrak{B}(\sigma) = \dim \Gamma(\sigma)$ makes possible to formulate the following fixed point properties.

**Lemma 4:** *The affine transformation* $(\sigma, \mathfrak{f}) \in \mathfrak{G}$ *acting on* $\mathfrak{B}$ *has a fixed point* $\mathfrak{x}_o \in \mathfrak{B}$ *if and only if* $\mathfrak{f} \in \mathfrak{B}^{\perp}(\sigma)$.

**Proof:** From $(\sigma,\mathfrak{f})\mathfrak{x}_o = \mathfrak{x}_o$ there follows $(\sigma - \mathrm{id})\mathfrak{x}_o = -\mathfrak{f}$, i.e. $-\mathfrak{f} \in \mathfrak{B}^{\perp}(\sigma)$ and so also $\mathfrak{f} \in \mathfrak{B}^{\perp}(\sigma)$. Inversely, for $\mathfrak{f} \in \mathfrak{B}^{\perp}(\sigma)$ there is also $-\mathfrak{f} \in \mathfrak{B}^{\perp}(\sigma)$ and so by (51) there is a vector $\mathfrak{x}_o \in \mathfrak{B}$ with $-\mathfrak{f} = (\sigma - \mathrm{id})\mathfrak{x}_o$, that is $(\sigma,\mathfrak{f})\mathfrak{x}_o = \mathfrak{x}_o$ ∎

**Corollary:** *Assume* $(\sigma,\mathfrak{f}) \in \mathfrak{G}$ *has a fixes point in* $\mathfrak{B}$. *Then* $(\sigma, \mathfrak{f} + \mathfrak{t}) \in \mathfrak{G}$ *has a fixes point in* $\mathfrak{B}$ *if and only if* $\mathfrak{t} \in \Gamma^{\perp}(\sigma)$.

**Proof:** let be $\mathfrak{f} \in \mathfrak{B}^{\perp}(\sigma)$ (Lemma 4), that is $\mathfrak{f} = \sigma\mathfrak{x}_o - \mathfrak{x}_o$, $\mathfrak{x}_o \in \mathfrak{B}$. **a)** Assume $(\sigma, \mathfrak{f} + \mathfrak{t})\mathfrak{x}_1 = \mathfrak{x}_1$, $\mathfrak{x}_1 \in \mathfrak{B}$, so there is true that $\sigma(\mathfrak{x}_o + \mathfrak{x}_1) - (\mathfrak{x}_o + \mathfrak{x}_1) = -\mathfrak{t} \in \mathfrak{B}^{\perp}(\sigma)$ and then $\mathfrak{t} \in \mathfrak{B}^{\perp}(\sigma)$. Because $(\sigma,\mathfrak{f})$ and $(\sigma, \mathfrak{f} + \mathfrak{t})$ are in $\mathfrak{G}$, by (7) there follows that $\mathfrak{t} \in \Gamma$ and then by (52) $\mathfrak{t} \in \Gamma^{\perp}(\sigma)$. **b)** Vice versa from $\mathfrak{t} \in \Gamma^{\perp}(\sigma)$ there follows $\mathfrak{t} \in \mathfrak{B}^{\perp}(\sigma)$, and under the assumption $\mathfrak{f} \in \mathfrak{B}^{\perp}(\sigma)$ we obtain $-\mathfrak{t}, -\mathfrak{f} \in \mathfrak{B}^{\perp}(\sigma)$, i.e. $-\mathfrak{t} = \sigma\mathfrak{x}_2 - \mathfrak{x}_2$ and $-\mathfrak{f} = \sigma\mathfrak{x}_3 - \mathfrak{x}_3$ ($\mathfrak{x}_2, \mathfrak{x}_3 \in \mathfrak{B}$). So there is true that $\sigma(\mathfrak{x}_2 + \mathfrak{x}_3) + \mathfrak{t} + \mathfrak{f} = (\sigma, \mathfrak{f} + \mathfrak{t})(\mathfrak{x}_2 + \mathfrak{x}_3) = \mathfrak{x}_2 + \mathfrak{x}_3$ ∎

**Proposition 4:** *Let be* $(\sigma,\mathfrak{f}) \in \mathfrak{G}$. *Then* $\delta_\sigma = 1$ *is true if and only if there is a lattice functional* $\mathfrak{t}_o \in \Gamma$ *wih the property that* $(\sigma, \mathfrak{f} + \mathfrak{t}_o)$ has a fixed point $\mathfrak{x}_o \in \mathfrak{B}$, *i.e. that* $\mathfrak{f} + \mathfrak{t}_o \in \mathfrak{B}^{\perp}(\sigma)$.

**Proof:** We have to take into consideration that $\langle \mathfrak{v}; \mathfrak{x}\rangle = 0$ if $\mathfrak{v} \in \mathfrak{B}^*(\sigma)$ and $\mathfrak{x} \in \mathfrak{B}^{\perp}(\sigma)$ (see (26) and (51); for understanding use dual bases in $\mathfrak{B} = \mathfrak{B}(\sigma) \oplus \mathfrak{B}^{\perp}(\sigma)$ and $\mathfrak{B}^* = \mathfrak{B}^*(\sigma) \oplus \mathfrak{B}^{*\perp}(\sigma)$).

**a)** Assume $\delta_\sigma = 1$ for a fixed $\sigma \in \mathfrak{L}$, i.e. $s^{\nu}(\sigma) = \langle \mathfrak{c}^{\nu}(\sigma), \mathfrak{f}\rangle \in \mathbb{Z}$ for all $\nu = 1, \dots, m$ (see (45), (39) and (37)). Then for an arbitrary $\mathfrak{u} = u_{\nu}' \mathfrak{c}^{\nu}(\sigma) \in \Gamma^*(\sigma)$ there is true that $\langle \mathfrak{u}, \mathfrak{f}\rangle \in \mathbb{Z}$. If we now decompose $\mathfrak{f} = \mathfrak{f}_1 + \mathfrak{f}_2$ into $\mathfrak{f}_1 \in \mathfrak{B}(\sigma)$ and $\mathfrak{f}_2 \in \mathfrak{B}^{\perp}(\sigma)$ we obtain $\langle \mathfrak{u}, \mathfrak{f}_2\rangle = 0$ because $\mathfrak{u} \in \mathfrak{B}^*(\sigma)$. Then we have $\langle \mathfrak{u}, \mathfrak{f}\rangle = \langle \mathfrak{u}, \mathfrak{f}_1\rangle \in \mathbb{Z}$ and therefore $\mathfrak{f}_1 \in \Gamma(\sigma)$. For each $\mathfrak{r} \in \Gamma^{\perp}(\sigma)$ there is $\mathfrak{t}_o := -\mathfrak{f} + \mathfrak{v} \in \Gamma$ and then $\mathfrak{f} + \mathfrak{t}_o = \mathfrak{f}_2 + \mathfrak{r} \in \mathfrak{B}^{\perp}(\sigma)$.

**b)** Conversely, let there exists a $\mathfrak{t}_o \in \Gamma$ with $\mathfrak{f} + \mathfrak{t}_o \in \mathfrak{B}^{\perp}(\sigma)$; we prove that $s^{\nu}(\sigma) \in \mathbb{Z}$ for all $\nu = 1, \dots, m$, i.e. $\delta_\sigma = 1$. We write $s^{\nu}(\sigma) = \langle \mathfrak{c}^{\nu}(\sigma), \mathfrak{f}\rangle = \langle \mathfrak{c}^{\nu}(\sigma), \mathfrak{f} + \mathfrak{t}_o\rangle - \langle \mathfrak{c}^{\nu}(\sigma), \mathfrak{t}_o\rangle$. Here $\langle \mathfrak{c}^{\nu}(\sigma), \mathfrak{t}_o\rangle \in \mathbb{Z}$ because of $\mathfrak{t}_o \in \Gamma$, $\mathfrak{c}^{\nu}(\sigma) \in \Gamma^*(\sigma)$ and so $\mathfrak{c}^{\nu}(\sigma) \in \Gamma^*$. Now using the introductory remark of the proof we find $\langle \mathfrak{c}^{\nu}(\sigma), \mathfrak{f} + \mathfrak{t}_o\rangle = 0$ because $\mathfrak{f} + \mathfrak{t}_o \in \mathfrak{B}^{\perp}(\sigma)$, $\mathfrak{c}^{\nu}(\sigma) \in \Gamma^*(\sigma)$ and so $\mathfrak{c}^{\nu}(\sigma) \in \mathfrak{B}^*(\sigma)$. Summariting we get $s^{\nu}(\sigma) \in \mathbb{Z}$ ∎

**4.5 Survey of the influence of fixed (fixed point - free) elements of group $\mathfrak{G}$ on the asymptotic expression for $N(\lambda)$.** If we ask for the intrinsic reason of the appearance of the pricipal term $c_o\lambda^n$ and the secondary term $c_1\lambda^{n-1}$ in $N(\lambda) = c_o\lambda^n + c_1\lambda^{n-1} + O(\lambda^{n-2+2/(n+1)})$ we can answer (Proposition 4):

(i) For $\sigma \in \mathfrak{L}_m$ the fixed elements $(\sigma, \mathfrak{f} + t_0) \in \mathfrak{G}$ produce in (49) resp. (50) the volume terms $\mathrm{vol}_m(\mathbb{D} \cap \mathfrak{B}^*(\sigma)) \cdot \lambda^m$ whereas fixed point-free elements from $\mathfrak{G}$ make contributions only to the remainder term $O(\lambda^{m-2+2/(m+1)})$. So we have the following knowledge:

(ii) The identity $(e, \mathfrak{o}) \in \mathfrak{G}$ produces the principal part of $N(\lambda)$ (because $\delta_e = 1$, $e \in \mathfrak{L}_n$).

(iii) The fixed elements $(\sigma, \mathfrak{f} + t_0) \in \mathfrak{G}$, $\sigma \in \mathfrak{L}_{n-1}$, produces the summands of the secondary part of $N(\lambda)$.

**Concluding remark:** The theory developed above can be applied e.g. for crystallographic groups, especially for the 230 space groups. For short it is recommendable to investigate an $n = 2$-dimensional group, e.g. $\mathfrak{G} = \Delta^2_{p31m}$ acting on $\mathfrak{B} = \mathbb{E}^2$ and having $P = c(\partial_1^2 + \partial_1\partial_2 + \partial_2^2)$ ($\partial_i = \partial/\partial x^i$) as the $\mathfrak{G}$-invariant operators for all $c > 0$. The 10 possible exmples for $\mathfrak{G}$ in the case $n = 2$ demonstrate a considerable improvement if we turn from $N(\lambda) \sim c_0\lambda^n$ to $N(\lambda) \sim c_0\lambda^n + c_1\lambda^{n-1}$ (see the Dissertation B of the author: *Zur asymptotischen Verteilung der Eigenwerte $\mathfrak{G}$-invarianter linearer elliptischer Differentialoperatoren mit konstanten Koeffizienten*. Universität Leipzig 1989).

## REFERENCES

[1] BELGER, M. and L. EHRENBERG: *Theorie und Anwendung der Symmetriegruppen* (Minöl: Vol. 23). Thun - Frankfurt/Main: Verlag Harri Deutsch 1981.

[2] BERARD, P.H.: *Spectres et groups cristallographiques* I: *Domain Euclidiens*. Inv. math. **58** (198), 179 - 199.

[3] BIEBERBACH, L.: *Über die Bewegunggruppen der Euklidischen Räume*. Math. Ann. **70** (1911), 297 - 336 and **72** (1912), 400 - 412.

[4] BRÜNING, J.: *Zur Abschätzung der Spektralfunktion elliptischer Operatoren*. Math. Z. **137** (1974), 75 - 85.

[5] BURKHARDT, J.J.: *Die Bewegungsgruppen der Kristallographie*. Basel - Stuttgart: Birkhäuser Verlag 1966.

[6] COURANT, R.: *Über die Eigenwerte bei denDifferentialgleichungen der Mathematischen Physik*. Math. Z. **7** (1920), 1 - 57.

[7] GÜNTHER, P.: *The Poisson formula for Euclidean space groups and some of its applications* I. Z. Anal. Anw. **1** (1982)1, 12 - 23.

[8] GÜNTHER, P.: *Über eine Funktionsgleichung für den Gitterrest*. Math. Nachr. **76** (1977), 5 - 27.

[9] GÜNTHER, P.: *Gitterpunktprobleme in symmetrischen Riemannschen Räumen vom Rang* 1. Math. Nachr. **9** (1980), 5 - 27.

[10] HELGASON, S.: *Groups and Geometric Analysis*. Orlando: Academic Press 1984.

[11] HÖRMANDER, L.: *The spectral function of an elliptic operator*. Acta Math. **121** (1968), 139 - 218.

[12] IVRII, V.YA.: *The Weyl asymptotic formula for the Laplace-Beltrami operator in Riemannian polyhedra and in domains with canonical boundary singularities*. Soviet Math. Dokl. **33** (1986), 596 - 599.

[13] KOBAYASHI, S. and K. NOMIZU: *Foundations of Differential Geometry*, Vol. I. New York: Intersci. Publ. 1963.

[14] LANDAU, E.: *Ausgewählte Abhandlungen zur Gitterpunktlehre*. Berlin: Dt. Verlag Wiss. 1962.

[15] LASUTKIN, W.F. and D.YA. TERMAN: *On the estimation of the remainder in the formular of H. Weyl* (in Russian). Funkt. Anal. Pril. **15** (1982), 81 - 82.

[16] LAX, P.D. and R.S. PHILLIPS: *The asymptotic distribution of lattice points in Euclidean and non-Euclidean spaces*. J. Funct. Anal. **46** (1982), 280 - 350.

[17] MELROSE, R.: *Weyl's conjecture for manifolds with concave boundary.* In: Proc. Symp. Pure Math. Providence, R.I.: Amer. Math. Soc. 1980, pp. 257 - 274.

[18] MÜLLER, W.: *Zetafunktion in Geometrie und Analysis.* In: Euler-Festband (Festakt und Wissenschaftliche Konferenz aus Anlaß des 200. Todestages von Leonhard Euler, 15./16. Sept. 1983 in Berlin; ed.: W. Engel). Abh. Akad. Wiss. DDR, Abt. Naturwiss. - Technik 1985, 1N, pp. 57 - 63.

[19] NOVAK, B.: *A remark on the theory of lattice points in ellipsoids* II. Comm. Math. Univ. Carolinae **9** (1968), 547 - 561.

[20] SAFAROV, YU. G.: *Asymptotical spectrum of pseudo-differential operators with periodical bicharacteristics* (in Russian). Rep. Sci. Sem. LOMI (Leningrad) **152** (1986), 94 - 104.

[21] WASILIEV, D. G.: *Asymptotical spectrum of boundary problems* (in Russian). Trudy Moscov Math. Obšč. **49** (1986), 167 - 237.

[22] WEYL, H.: *Über die asymptotische Verteilung der Eigenwerte.* Göttinger Nachr. (1911), 110 - 117.

[23] WEYL, H.: *Über das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit Anwendungen auf die Theorie der Hohlraumstrahlung).* Math. Ann. **71** (1912), 441 - 479.

[24] WEYL, H.: *Über das Spektrum der Hohlraumstrahlung.* J. reine angew. Math. **141** (1912), 163 - 181.

[25] WEYL, H.: *Das asymptotische Verteilungsgesetz der Eigenwerte eines beliebig gestalteten elastischen Körpers.* Rend. Circ. Mat. Palermo **39** (1915), 1 - 50.

[26] WINTGEN, G.: *Zur Darstellungstheorie der Raumgruppen.* Math. Ann. **118** (1941/43), 195 - 215.

[27] WOLF, J. A.: *Spaces of Constant Curvature.* New York: McGraw Hill 1967.

Received 20.02.1992; in revised form 27.07.1992