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The Smoothness of the Solution to a Two-Dimensional Integral Equation with Logarithmic Kernel

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We observe a two-dimensional weakly singular integral equation with logarithmic kernel. The behavior of the higher order derivatives of the solution to the equation is examined in case of bounded domain of integration with piecewise smooth boundary. Exact descriptions for the leading terms of the derivatives and estimations for the remainders are given.

Key words: ***Weakly Singular Integral Equations, Smoothness of the Solution***

AMS subject classification: 45 B 05, 45 M 05

1. Introduction. As a rule, the derivatives of a solution to a weakly singular integral equation have singularities near the boundary of the integration domain. Descriptions of the singularities or at least estimations of those are needed when effective approximate methods are constructed to solve the equation. The case of one-dimensional integral equations is analyzed in [1,4,5,7]. In [3,6] the behavior of the derivatives of the solution is examined in the case of multidimensional equations. In general, these estimations are non-improvable for the classes of kernels considered in [3,6]; typically, the singularities may occur along the whole boundary. In [2] a more special equation with logarithmically singular kernel is examined:

$$u(x) = \int_{\Omega} a(x,y) \ln|x-y|\, u(y)dy + f(x), \quad x \in \Omega \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is an open bounded set with a piecewise Lyapunov boundary and f and a are sufficiently smooth functions. It is shown that a solution to (1) and its first derivatives are continuous on the closure $\overline{\Omega}$ and the second derivatives may have logarithmical or bounded singularities only at corner points of the boundary $\partial\Omega$. Explicit formulas for the singular parts of the second derivatives are given.

In the present paper we continue the examination of a solution to (1). We describe the leading singular parts of its derivatives of an arbitrary

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order depending on the smoothness properties of a, f and $\partial\Omega$. In general, Ω may have an "inner boundary" (a part of boundary where Ω lies on both sides of $\partial\Omega$). To treat this situation formally, we use the "inner" completion Ω^* of Ω instead of the usual Euclidean closure $\overline{\Omega}$. A function $u \in C(\Omega^*)$ is piecewise continuous on $\overline{\Omega}$ and may have jumps on the inner boundary. More complete definitions are given in Section 2.

2. **Definitions and assumptions.** Let Γ be a piecewise smooth closed directed curve with the unit tangent vector $t(y)=\begin{pmatrix} t_1(y) \\ t_2(y)\end{pmatrix}$, $y\in\Gamma$. Then $\omega(y)=\begin{pmatrix}\omega_1(y)\\ \omega_2(y)\end{pmatrix}=\begin{pmatrix}-t_2(y)\\ t_1(y)\end{pmatrix}$ is the unit interior normal to Γ at the point y (if we move along the curve Γ in the positive direction, then the interior is on the left). Let $\varphi=\varphi(y)$ be the angle between the abcissa axis and the vector $t(y)$. Denote by P the matrix of rotation by $-\pi/2$ and $P_\varphi(y)$ — the matrix of rotation by $\varphi(y)$. Then

$$P=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix},\qquad P_\varphi(y)=\begin{pmatrix}\omega_2(y) & \omega_1(y)\\ -\omega_1(y) & \omega_2(y)\end{pmatrix},\qquad y\in\Gamma.$$

For a smooth curve Γ denote

$$H^{m,\mu}(\Gamma)=\left\{f\in C^m(\Gamma):\ \left|\frac{d^mf(y)}{ds^m}-\frac{d^mf(y')}{ds^m}\right|\le M|y-y'|^{\mu}\right\},\quad 0<\mu\le 1.$$

Let Ω be an open bounded set. Introduce the inner distance $d_\Omega(x^1,x^2)$ between the points $x^1,x^2\in\Omega$ as the infimimum of the lenghts of the polygonal paths in Ω which connect the points x^1 and x^2. If x^1 and x^2 belong to different connectivity components, then let $d_\Omega(x^1,x^2)=\infty$. Denote the completion of Ω with respect to d_Ω-metrics by Ω^*. Let Γ be the d_Ω-boundary of Ω^*, that is $\Gamma=\Omega^*\backslash\Omega$. Then Γ is the boundary of Ω with possibly "multiple points", which are different in d_Ω-metrics.

Assume that separation points $y^1,\dots,y^n$ divide Γ into smooth parts $\Gamma_1,\dots,\Gamma_n$ so that $\omega\in[H^{m,\mu}(\Gamma_j)]^2$, $j=1,\dots,n$. Assume that there exists a $\delta>0$ such that the normal ω is m_j times ($m_j\le m$) continuously differentiable in $\Gamma\cap S(y^j,\delta)$, where $S(y,\delta)=\{x:|x-y|<\delta\}$. If ω is discontinuous at y^j, then put $m_j=-1$. In the following we refer to these conditions as $\Gamma\in G(n,m,\mu,\{m_j\}_{j=1}^n)$. Note that $G(0,0,\mu,\emptyset)$ is the set of Lyapunov curves. Denote $m_0=\min_{j=1,\dots,n} m_j$ and, for $k=0,\pm1,\dots$,

$$R_k(\Omega^*)=R_k(\Omega^*,\{y^j,m_j\}_{j=1}^n)$$
$$=\left\{r\in C(\Omega^*\backslash\{y^1,\dots,y^n\})\ \middle|\ |r(x)|\le C\left(1+\sum_{j=1}^n\begin{cases} r\in C(y^j), & \text{if } m_j>k\\ |\ln|x-y^j||, & \text{if } m_j=k\\ |x-y^j|^{m_j-k}, & \text{if } m_j<k\end{cases}\right)\right\}$$

where $r \in C(y^j)$ means that there exists a $\delta>0$ such that r is continuous in $\Omega^* \cap S(y^j,\delta)$. Then $R_k(\Omega^*)=C(\Omega^*)$ for $k<m_0$. For example, if Ω is a rectangle, then $m_j=-1$, $j=1,2,3,4$ and $R_k(\Omega^*)=C(\overline{\Omega})$, $k\le -2$; $R_{-1}(\Omega^*)$ consists of functions which may have logarithmic singularities at the corners of Ω, and $R_k(\Omega^*)$, $k\ge 0$ includes functions with possible singularities up to the order $|x-y^j|^{-k-1}$ at the corners.

Denote

$$\arg_j(x)=\varepsilon_\delta^j(x)\,\mathrm{Arg}(x_1-y_1^j+i(x_2-y_2^j)),$$

where $\varepsilon_\delta^j \in C^\infty(\mathbb{R}^2)$ is such that $\varepsilon_\delta^j(x)=1$ for $x\in S(y^j,\frac{\delta}{2})$, and $\varepsilon_\delta^j(x)=0$ for $x\notin S(y^j,\delta)$; δ is chosen so that the point y^j and ∞ can be connected by a continuous line lying outside of $\Omega\cap S(y^j,\delta)$. Outside of the line let the function Arg be continuous.

For a derivative D^α we define an order of differentiations by

$$D_x^{\alpha^l}=\frac{\partial}{\partial x_{i_l}}D_x^{\alpha^{l-1}},\quad l=1,\dots,k,$$

where $\alpha^0=(0,0)$, $i_1,\dots,i_k\in\{1,2\}$ and $\alpha^k=\alpha$.

3. The main result and examples. Now we can describe fully the smoothness of the solution to (1).

Theorem: *Let $\Omega\subset\mathbb{R}^2$ be an open bounded set with d_Ω-boundary $\Gamma\in G(n,m,\mu,\{m_j\}_{j=1}^n)$. Assume that $f\in C^{m+2}(\Omega^*)$, $a\in C^{m+2}(\Omega^*\times\Omega^*)$ and equation (1) has a solution in $L(\Omega)$. Then the solution $u\in C^{m_0+2}(\Omega^*)\cap C^{m+2}(\Omega^*\setminus\{y^1,\dots,y^n\})$ and, for $|\alpha|=k=2,3,\dots,m+2$,*

$$D^\alpha u(x)=\sum_{\substack{j=1\\ m_j\le k-3}}^{n} a(x,y^j)\,u(y^j)\Big[b_{11}(y^j)\Big(\frac{\partial}{\partial x_1}\Big)^{k-m_j-3}\ln|x-y^j| - b_{12}(y^j)\Big(\frac{\partial}{\partial x_1}\Big)^{k-m_j-3}\arg_j(x)\Big]+r_{k-4}(x), \tag{2}$$

where $r_{k-4}\in R_{k-4}(\Omega^*)$ and

$$B_\alpha(y^j)=\begin{pmatrix} b_{11}(y^j) & b_{12}(y^j)\\ b_{21}(y^j) & b_{22}(y^j)\end{pmatrix}=P^{\alpha_2}\frac{d^{m_j+1}}{ds_y^{m_j+1}}\big[\omega_1(y)\,P_\varphi(y)\big]\Big|_{y=y_-^j}^{y=y_+^j}P_\varphi^{m_j+1}(y^j).$$

Corollary: If the functions a, f and the d_Ω-boundary of Ω are infinitely smooth, then the solution $u\in C^\infty(\Omega^*)$.

Remark 1: Similar result is also valid for a two-dimensional integral operator with logarithmic kernel. Let a and Ω be as in the theorem and let u be such that $D^\beta u\in R_{|\beta|-3}(\Omega^*)$, $0\le|\beta|\le m+1$. Then the singularities of the function $\int_\Omega a(x,y)\ln|x-y|\,u(y)\,dy$ are described by the right-hand side of (2).

Remark 2: We may weaken the assumptions for f. Namely, if $D^\beta f \in R_{|\beta|-4}(\Omega^*)$ for $0 \le |\beta| \le m+2$, then (2) holds. If for every $0 \le |\beta| \le m+2$ the derivative $D^\beta f \in R_{|\beta|-3}(\Omega^*)$, then the singularities of the derivatives of a solution to the equation (1) are sums of those described in (2) and those of the corresponding derivatives of f.

Example 1: Let Ω be the unit square, $\Omega = (0,1) \times (0,1)$. Write out some singularities of the derivatives of the solution at the point $y^1 = (0,0)$ (at the other corners they are similar). Since

$$B_\alpha(y^1) = P^{\alpha_2}\left[\omega_1(y)\, P_\varphi(y)\right]\Big|_{y=y^1-}^{y=y^1+} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\alpha_2}(-1)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\alpha_2+1}$$

and $\arg_1(x) = \arctan \dfrac{x_2}{x}$, $x \in \Omega$, we get

$$\frac{\partial^2 u(x)}{\partial x_1^2} = -a(x, y^1)\, u(y^1) \arctan \frac{x_2}{x_1} + v(x),$$

$$\frac{\partial^2 u(u)}{\partial x_1 \partial x_2} = -a(x, y^1)\, u(y^1) \ln|x - y^1| + v_1(x),$$

$$\frac{\partial^3 u(x)}{\partial x_1^3} = a(x, y^1)\, u(y^1) \frac{x_2}{x_1^2 + x_2^2} + v_2(x)$$

and so on. Here v and v_1 are continuous in a neighbourhood of y^1 and v_2 has no more than logarithmic singularity at the point y^1.

Example 2: Let the boundary of Ω be such that $y_2 = |y_1|^3$ in a neighbourhood of the point $y^1 = (0,0)$. Then,

$$\omega(y) = \frac{1}{\sqrt{1 + 9y_1^4}}\begin{pmatrix} -3y_1^2 \operatorname{sgn} y_1 \\ 1 \end{pmatrix}.$$

The function ω is continuously differentiable, but its second derivative is discontinuous at the point y^1. Therefore $m_1 = 1$ and the solution to the corresponding integral equation has continuous derivatives up to order 3 in the neighbourhood of y^1. Since

$$B_\alpha(y^1) = P^{\alpha_2} \frac{d^2}{ds_y^2}\left[\omega_1(y)\, P_\varphi(y)\right]\Big|_{y=y^1_-}^{y=y^1_+} P_\varphi^2(y^1)$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\alpha_2}(-12)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^2 = -12\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\alpha_2}$$

and $\arg_1(x) = \operatorname{arccot} \dfrac{x_1}{x_2}$, $x \in \Omega$, we get

$$\frac{\partial^4 u(x)}{\partial x_1 \partial x_2^3} = -12\, a(x, y^1)\, u(y^1) \operatorname{arccot} \frac{x_1}{x_2} + v(x),$$

$$\frac{\partial^4 u(x)}{\partial x_1^4} = \frac{\partial^4 u(x)}{\partial x_2^4} = 12\, a(x, y^1)\, u(y^1) \ln|x - y^1| + v_1(x),$$

$$\frac{\partial^5 u(x)}{\partial x_2^5} = 12\, a(x, y^1)\, u(y^1) \frac{x_2}{x_1^2 + x_2^2} + v_2(x),$$

where v, v_1 and v_2 are similar to the previous example.

Example 3: Let y^1 be a cuspidal point of Γ. Then $\omega_1(y^1_+) = -\omega_1(y^1_-)$ and

$$\omega_1(y)\, P_\varphi(y)\Big|_{y=y^1_-}^{y=y^1_+} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore the leading terms of the singularities will disappear and the second derivatives of the solution have no singularities at y^1. The third derivatives have no more than logarithmic singularity and the derivatives of order k $(k>3)$ may have singularities up to power $k-3$ at the point y^1.

4. The lemmas. In this section we present some auxiliary results we need for proving the theorem.

Lemma 1 (see [1]): *Let Γ_0, a part of the d_Ω-boundary of Ω, be a Lyapunov curve with beginning and end points y^1 and y^2, respectively. Let the function $g \in C(\Omega^* \times \Gamma_0)$ be such that $g(x,\cdot) \in H^{0,\mu}(\Gamma_0)$. Then*

$$\int_{\Gamma_0} g(x,y)\frac{x_l - y_l}{|x-y|^2}\, ds_y = \sum_{j=1}^{2}(-1)^j g(x,y^j)\Big[\omega_l(y^j)\arg_j(x) + (-1)^l \omega_{3-l}(y^j)\ln|x-y^j|\Big] + v(x),$$

where $v \in C(\Omega^*)$.

Denote for $x,y \in \mathbb{R}^2$, $x \neq y$

$$q(x,y) = \begin{pmatrix} (x_1-y_1)/|x-y|^2 \\ (x_2-y_2)/|x-y|^2 \end{pmatrix}.$$

Note that

$$\frac{\partial}{\partial x_2} q(x,y) = P\frac{\partial}{\partial x_1} q(x,y) \quad \text{and} \quad D_x^\alpha q(x,y) = P^{\alpha_2}\Big(\frac{\partial}{\partial x_1}\Big)^{|\alpha|} q(x,y),$$

and, for $y \in \Gamma$,

$$\frac{\partial}{\partial s_y} q(x,y) = \frac{\partial}{\partial t_y} q(x,y) = \omega_2(y)\frac{\partial}{\partial y_1} q(x,y) - \omega_1(y)\frac{\partial}{\partial y_2} q(x,y)$$

$$= -\Big(\omega_2(y)\frac{\partial}{\partial x_1} q(x,y) - \omega_1(y)\, P\frac{\partial}{\partial x_1} q(x,y)\Big) = -P_{-\varphi}(y)\frac{\partial}{\partial x_1} q(x,y).$$

Therefore, for $(x,y) \in \mathbb{R}^2 \times \Gamma$,

$$\frac{\partial}{\partial x_1} q(x,y) = -P_\varphi(y)\frac{\partial}{\partial s_y} q(x,y) \quad \text{and} \quad D_x^\alpha q(x,y) = P^{\alpha_2}\Big[-P_\varphi(y)\frac{\partial}{\partial s_y}\Big]^{|\alpha|} q(x,y).$$

Define also the operator $\mathcal{D}$ by

$$\mathcal{D}f(x,y) = \frac{\partial}{\partial s_y}\Big[f(x,y)P_\varphi(y)\Big], \quad x \in \Omega,\ y \in \Gamma,$$

where f may be a scalar or a 2×2 matrix function.

Lemma 2: *Let Ω be an open bounded set with d_Ω-boundary $\Gamma \in G(n,m,\mu,\{m_j\}_{j=1}^n)$ and $0 \le k \le m$, $p > -m_0$. Assume that $g \in C^{k+1}(\Omega^* \times \Omega^*)$ and $D^\beta v \in R_{|\beta|-p-1}(\Omega^*)$, $0 \le |\beta| \le k$. Then*

$$\int_\Gamma g(x,y)\, v(y)\, D_x^\alpha \frac{x_i - y_i}{|x-y|^2}\, \omega_{i_0}(y)\, ds_y$$

$$= \begin{cases} r_{k-p}(x), & \text{if } p \le 1, \\ \sum\limits_{\substack{j=1 \\ m_j<k}} g(x,y^j)v(y)\Big[b_{i1}(y^j)\Big(\frac{\partial}{\partial x_1}\Big)^{k-m_j-1} \ln|x-y^j| - & \\ \qquad - b_{i2}(y^j)\Big(\frac{\partial}{\partial x_1}\Big)^{k-m_j-1} \arg_j(x)\Big] + r_{k-2}(x), & \text{if } p \ge 2, \end{cases}$$

where $|\alpha| = k$, $i, i_0 = 1,2$, $r_i \in R_i(\Omega^*)$ and

$$B_\alpha(y^j) = P^{\alpha_2} \frac{d^{m_j+1}}{ds_y^{m_j+1}} \Big[\omega_{i_0}(y)\, P_\varphi(y)\Big]\Big|_{y=y^j_-}^{y=y^j_+} P_\varphi^{m_j+1}(y^j).$$

Proof: Instead of $D_x^\alpha\big((x_i - y_i)/|x-y|^2\big)$ we may consider $(\partial/\partial x_1)^k q(x,y)$. Examine the integral over some Γ_j with beginning and end points y^1 and y^2, respectively. Fix a point $x^0 \in \Omega$. Let δ_l, $l = 1,2$ be such that $\Gamma_j \cap S(y^l, \delta)$ consists of only one curve for every $\delta < \delta_l$ and let $\varepsilon_l = \min\{\delta_l, |x^0 - y^l|/2, |y^1 - y^2|/2\}$. Denote $\Gamma_j^l = \Gamma_j \cap S(y^l, \varepsilon_l)$, $l = 1,2$, and $\tilde{\Gamma}_j = \Gamma_j \setminus (\Gamma_j^1 \cup \Gamma_j^2)$. Then we can divide the integral over Γ_j into integrals over Γ_j^1, Γ_j^2 and $\tilde{\Gamma}_j$. Let the end points of Γ_j^1 be y^1 and $\tilde{y}^1$. First let us consider the case when $k > p + m_1$. Then we may integrate the integral over Γ_j^1 by parts $p + m_1 + 1$ times:

$$\int_{\Gamma_j^1} g(x,y)\, v(y) \Big(\frac{\partial}{\partial x_1}\Big)^k q(x,y)\, \omega_{i_0}(y)\, ds_y$$

$$= -\sum_{l=0}^{p+m_1} \mathcal{D}^l\big[g(x,y)\, v(y)\, \omega_{i_0}(y)\big] P_\varphi(y) \Big(\frac{\partial}{\partial x_1}\Big)^{k-l-1} q(x,y)\Big|_{y=y^1}^{y=\tilde{y}^1} \tag{3}$$

$$+ \int_{\Gamma_j^1} \mathcal{D}^{p+m_1+1}\big[g(x,y)\, v(y)\, \omega_{i_0}(y)\big]\Big(\frac{\partial}{\partial x_1}\Big)^{k-p-m_1-1} q(x,y)\, ds_y.$$

Since

$$\Big|\Big(\frac{\partial}{\partial x_1}\Big)^{k-p-m_1-1} q(x^0,y)\Big| \le \frac{c}{|x^0 - y|^{k-p-m_1}} \le \frac{c}{\varepsilon_1^{k-p-m_1}} \le \frac{c}{|x^0 - y^1|^{k-p-m_1}}, \quad y \in \Gamma_j^1$$

and the derivative of order $(p + m_1 + 1)$ of the function v has at most logarithmic singularity at the point y^1, we may estimate the absolute value of the last integral as follows:

$$\frac{c}{|x^0 - y^1|^{k-p-m_1}} \int_{\Gamma_j^1} (|\ln|y - y^1|| + 1)\, ds_y \le \frac{c}{|x^0 - y^1|^{k-p-m_1}}.$$

Examine the integral over $\tilde{\Gamma}_j$ integrating it k times by parts:

$$\int_{\tilde{\Gamma}_j^1} g(x,y)\,v(y)\Big(\frac{\partial}{\partial x_1}\Big)^k q(x,y)\omega_{l_0}(y)\,ds_y$$
$$= -\sum_{l=0}^{k-1} \mathcal{D}^l\big[g(x,y)\,v(y)\,\omega_{l_0}(y)\big]P_\varphi(y)\Big(\frac{\partial}{\partial x_1}\Big)^{k-l-1} q(x,y)\Big|_{y=\tilde{y}^1}^{y=\tilde{y}^2}$$
$$+\int_{\tilde{\Gamma}_j} \mathcal{D}^k\big[g(x,y)\,v(y)\,\omega_{l_0}(y)\big]q(x,y)\,ds_y.$$

From Lemma 1 we get that the last integral can be estimated at the point x^0 as follows:

$$C\sum_{j=1}^{2}\big|\mathcal{D}^k\big[g(x^0,y)\,v(y)\,\omega_{l_0}(y)\big]\big|\big(1+\big|\ln|x^0-\tilde{y}^j|\big|\big).$$

In a neighbourhood of $\tilde{y}^1$ we get the upper bound

$$C\big(1+\big|\ln|x^0-\tilde{y}^1|\big|\big)\left\{\begin{array}{ll}\big(1+\big|\ln|y-y^1|\big|\big), & k=m_1+p+1\\ \dfrac{1}{|y-y^1|^{k-m_1-p-1}}, & k>m_1+p+1\end{array}\right\}\le\frac{C}{|x^0-y^1|^{k-p-m_1}}.$$

Adding the integrals over Γ_j^1, $\tilde{\Gamma}_j$ and Γ_j^2, the addends at $\tilde{y}^1$ with $l=0,1,\dots,p+m_1$ will reduce. If $p+m_1<k$, then the remaining addends where $l=p+m_1+1,\dots,k$ are estimable by

$$\frac{C}{|x^0-y^1|^{k-l}}\left\{\begin{array}{ll}\big(1+\big|\ln|x^0-y^1|\big|\big), & l=m_1+p+1\\ \dfrac{1}{|x^0-y^1|^{l-m_1-p-1}}, & l>m_1+p+1\end{array}\right\}\le\frac{C}{|x^0-y^1|^{k-p-m_1}}.$$

Now let us consider the singularity at the point y^1. We got the addends which include y^1 when calculating the integral over Γ_j^1 (see (3)). Since y^1 is the beginning point of Γ_j and also the end point of some $\Gamma_{j'}$, addends similar to these we have got arise but with the opposite signs. As the function v is at least $p+m_1$ times and the functions ω and P_φ are at least m_1 times continuously differentiable at the point y^1 (if they are smoother, then there exists y^j, which is different from y^1 in d_Ω-metrics, but the same in $\mathbb{R}^2$-metrics, where the functions ω and P_φ are exactly m_1 times continuously differentiable), the addends where $l=0,\dots,\min\{m_1, m_1+p\}$ will reduce. If $p>0$, then the addend where $l=m_1+1$ is the following:

$$g(x,y)\,v(y)\,\frac{d^{m_1+1}}{ds_y^{m_1+1}}\big[\omega_{l_0}(y)\,P_\varphi(y)\big]P_\varphi^{m_1+1}(y)\Big(\frac{\partial}{\partial x_1}\Big)^{k-m_1-2}q(x,y)\Big|_{y=y^1_-}^{y=y^1_+}.$$

Multiplying it by P^{α_2} and taking the l-th component of the vector, we get the leading term appearing in the assertion of the Lemma. Remaining terms include as singularities only $\big(\frac{\partial}{\partial x_1}\big)^l q(x,y)$, $l=0,\dots,k-m_1-2$ and therefore are estimable by $C/|x^0-y^1|^{k-m_1-2}$. Since previous estimations include $C/|x^0-y^1|^{k-p-m_1}$, in case $p=1$ we must count the leading term into the remainder, too.

We have proved Lemma 2 in the neighbourhood of y^1 when $k>p+m_1$. If $k\le p+m_1$, then the proof is similar, only we do not need to cut off the neighbourhood of y^1 but can integrate by parts k times over the whole curve. If $k\le m_1$, then all singularities will reduce and the remainder is continuous in the neighbourhood of y^1; if $k>m_1$, then we get the leading term and the remainder analogously to the previous case. In the neighbourhoods of the other singular points the proof is similar■

Lemma 3 ([3]): *Let $\Omega\subset\mathbb{R}^2$ be an open bounded set with piecewise smooth d_Ω-boundary Γ. Assume that $K\in C^k((\Omega\times\Omega)\setminus\{x=y\})$ and there exists $\nu<2$ such that*

$$|D_x^\alpha K_\beta(x,y)|\le C\begin{cases}1+|\ln|x-y||\ , & \nu+|\alpha|=0,\\ 1+|x-y|^{-\nu-\alpha}\ , & \nu+|\alpha|\ne 0,\end{cases}\qquad |\alpha|+|\beta|\le k,$$

where

$$K_\beta(x,y) = \Big(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial y_1}\Big)^{\beta_1}\Big(\frac{\partial}{\partial x_2}+\frac{\partial}{\partial y_2}\Big)^{\beta_2}K(x,y).$$

Then for $u\in C^k(\Omega^*)$ the function $\int_\Omega K(x,y)u(y)dy$ is k times continuously differentiable in Ω and

$$D^\alpha\int_\Omega K(x,y)\,u(y)\,dy=\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\int_\Omega K_{\alpha-\beta}(x,y)\,D^\beta u(y)\,dy$$
$$+\sum_{l=0}^{k-1}\sum_{\beta\le\alpha^l}\binom{\alpha^l}{\beta}\int_\Gamma D_x^{\alpha-\alpha^{l+1}}K_{\alpha^l-\beta}(x,y)\,D^\beta u(y)\omega_{i_{l+1}}(y)\,ds_y,\quad x\in\Omega\ ,\ |\alpha|=k.$$

Fix a point $x^0\in\Omega$. Denote

$$\delta_p=1/2\min_{\substack{j=1,\dots,n\\ m_j\le m_0+p}}|x^0-y^j|,\quad p=0,1,\dots\qquad\text{and}\quad \Omega_p=\Omega\cap S(x^0,\delta_p).$$

Let

$$\Delta_p=\Big(\frac{1}{\delta_0^p}+\frac{1}{\delta_1^{p-1}}+\dots+\frac{1}{\delta_{p-1}}+|\ln\delta_p|+1\Big).$$

Note that

$$\Delta_p\frac{1}{\delta_l^j}\le C\begin{cases}\Delta_{p+j}\ , & l<p,\\ \Delta_{p+j+1}, & l=p,\\ \Delta_{l+j}\ , & l>p\end{cases}$$

and, for $p=0,1,\dots$,

$$R_{m_0+p}(\Omega^*)=\big\{r\in C(\Omega^*\setminus\{y^1,\dots,y^n\}:\ r\in C(y^j)\ \text{if}\ m_j\ge k,\ r(x^0)\le C\Delta_p\big\},$$

where C is independent of $x^0\in\Omega$.

Lemma 4: Assume that $k(x,y) = g(x,y)\ln|x-y| + g_1(x,y)\frac{x_1-y_1}{|x-y|^2}$, *where* $g, g_1 \in C^{p+1}(\Omega^*\times\Omega^*)$. *Let* $v\in C(\Omega^*)$ *be such that* $D^\beta v \in R_{m+|\beta|-1}(\Omega^*)$ *for each* $|\beta|\le p$. *Then*

$$D^\alpha \int_\Omega k(x,y)\, v(y)\, dy \in R_{p+m_0}(\Omega^*), \qquad |\alpha| = p+1.$$

Proof: Let the point x^0 and $\delta_0,\dots,\delta_p$ be fixed as above. Then

$$\int_\Omega k(x,y)\,v(y)\,dy = \int_{\Omega\setminus\Omega_0} k(x,y)\,v(y)\,dy + \int_{\Omega_0\setminus\Omega_1} k(x,y)\,v(y)\,dy + \dots$$

$$+ \int_{\Omega_{p-1}\setminus\Omega_p} k(x,y)\,v(y)\,dy + \int_{\Omega_p} k(x,y)\,v(y)\,dy.$$

Differentiate the integrals at $x\in\Omega_p$ and after that estimate them at the point x_0. We can differentiate the first integral under the integral sign:

$$\left|D^\alpha \int_{\Omega\setminus\Omega_0} k(x^0,y)\,v(y)\,dy\right| \le C\int_{\Omega\setminus\Omega_0}\left|D_x^\alpha k(x^0,y)\right| dy$$

$$\le C\int_{\Omega\setminus\Omega_0}\frac{dy}{|x^0-y|^{|\alpha|+1}} \le C\int_{\delta_0}^{\operatorname{diam}\Omega} 2\pi r\frac{dr}{r^{p+2}} \le C\Delta_p.$$

As $v\in C^k(\Omega^*_{k-1})$, $k=1,2,\dots,p$, we can differentiate the integrals over $\Omega_{k-1}\setminus\Omega_k$ k times by Lemma 3. We obtain that

$$\left|D^\alpha \int_{\Omega_{k-1}\setminus\Omega_k} k(x,y)\,v(y)\,dy\right|$$

$$= D^{\alpha-\alpha^k}\Bigg[\sum_{\beta\le\alpha^k}\binom{\alpha^k}{\beta}\int_{\Omega_{k-1}\setminus\Omega_k} k_{\alpha^k-\beta}(x,y)\,D^\beta v(y)\,dy$$

$$+\sum_{l=0}^{k-1}\sum_{\beta\le\alpha^l}\binom{\alpha^l}{\beta}\int_{\partial(\Omega_{k-1}\setminus\Omega_k)} D_x^{\alpha^k-\alpha^{l+1}} k_{\alpha^l-\beta}(x,y)\,D^\beta v(y)\,\omega_{i_{l+1}}(y)\,ds_y\Bigg].$$

Estimate the integrals in the first sum:

$$\left|\int_{\Omega_{k-1}\setminus\Omega_k} D_x^{\alpha-\alpha^k} k_{\alpha^k-\beta}(x^0,y)\,D^\beta v(y)\,dy\right|$$

$$\le C\Delta_{k-1}\int_{\Omega_{k-1}\setminus\Omega_k}\frac{dy}{|x^0-y|^{p+2-k}} \le C\Delta_{k-1}\begin{Bmatrix}(1+|\ln\delta_k|), & k=p\\ \frac{1}{\delta_k^{p-k}}, & k<p\end{Bmatrix} \le C\Delta_p.$$

We can differentiate the integral over Ω_p $p+1$ times by Lemma 3. Thus

$$D^\alpha\int_{\Omega_p} k(x,y)\,v(y)\,dy = \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\int_{\Omega_p} k_{\alpha-\beta}(x,y)\,D^\beta v(y)\,dy$$

$$+\sum_{l=0}^{p}\sum_{\beta\le\alpha^l}\binom{\alpha^l}{\beta}\int_{\partial\Omega_p} D_x^{\alpha-\alpha^{l+1}} k_{\alpha^l-\beta}(x,y)\,D^\beta v(y)\,\omega_{i_{l+1}}(y)\,ds_y.$$

Note that all double integrals are continuous on $\Omega^* \cap S(x^0,\delta_p)$.

Now let us add all the boundary integrals arisen within the proof:

$$\sum_{l=0}^{p} \sum_{\beta\le\alpha^l} \binom{\alpha^l}{\beta} \int_{\partial\Omega_l} D^{\alpha-\alpha^{l+1}} k_{\alpha^l-\beta}(x,y)\, D^\beta v(y)\omega_{l+1}(y)\, ds_y .$$

Observe the integral over $\partial\Omega_l$. We may present it in the form

$$\sum_{k=0}^{p-l+1} \int_{\partial\Omega_l} h_k(x,y)\, D_x^{\nu^k} \ln|x-y|\, D^\beta v(y)\, \omega_{l+1}(y)\, ds_y ,$$

where $h_k \in C^{|\beta|+k}(\Omega^*\times\Omega^*)$, $|\nu^k| = k$ and $|\beta| \le l$, hence $D^\beta v \in C^{l+1-|\beta|}(\Omega^*)$. For $k \le l+1-|\beta|$ we may integrate by parts $k-1$ times to obtain

$$\int_{\partial\Omega_l} h_k(x,y)\Big(\frac{\partial}{\partial x_1}\Big)^{k-1} q(x,y)\, D^\beta v(y)\,\omega_{l+1}(y)\, ds_y$$

$$= -\sum_{k_1=0}^{k-2} \sum_{y^0\in Y_l} \mathcal{D}^{k_1}\big[h_k(x,y)\, D^\beta v(y)\,\omega_{l+1}(y)\big] P_\varphi(y)\Big(\frac{\partial}{\partial x_1}\Big)^{k-k_1-2} q(x,y)\Big|_{y=y^0_+}^{y=y^0_-}$$

$$+ \int_{\partial\Omega_l} \mathcal{D}^{k-1}\big[h_k(x,y)\, D^\beta v(y)\,\omega_{l+1}(y)\big] q(x,y)\, ds_y$$

where Y_l is the set of angular points of $\partial\Omega_l$ (the set of intersection points of Γ and the circle with center x^0 and radius δ_l). Consider the last integral. On $\partial\Omega_l \cap \partial\Omega$ we may use Lemma 1. On the remaining part of $\partial\Omega_l$ we may estimate the components of $q(x^0,y)$ by $1/\delta_l$. As $\Big(\frac{\partial}{\partial s_y}\Big)^{k_1} D^\beta v \in R_{m_0+|\beta|+k_1-1}(\Omega^*)$, the absolute value of the integral is estimable at the point x^0 by

$$C\Bigg[\sum_{k_1=0}^{k-2} \Delta_{|\beta|+k_1-1}\frac{1}{\delta_l^{k-k_1-1}} + \Delta_{|\beta|+k-2}\Big(|\ln\delta_l| + \frac{1}{\delta_l}\,2\pi\delta_l\Big)\Bigg] \le C\Delta_{l+k-1} \le C\Delta_p .$$

For $k > l+2-|\beta|$, integrating by parts we must take into account the singularities which the derivatives of order $k-1$ of $D^\beta v$ may have at the points $y^j \in \Omega_l$ where $m_j < k+m_0-|\beta|-1$. In that case we have to cut out small neighbourhoods of these points $\tilde\Gamma_j = \Gamma \cap S(y^j,\varepsilon_j)$, where $\varepsilon_j \le |x^0-y^j|/2$ is chosen so that the circle $S(y^j,\varepsilon_j)$ would not contain other singular points of the boundary except the ones which are the same in $\mathbb{R}^2$-metrics. Outside the neighbourhoods we may integrate by parts $k-1$ times. In addition to the addends similar to those we have got in case $k \le l-2-|\beta|$, we then obtain

$$\sum_{k_1=0}^{k-2} \sum_{y^0\in\tilde Y} \pm\mathcal{D}^{k_1}\big[h_k(x,y)\, D^\beta v(y)\,\omega_{l+1}(y)\big] P_\varphi(y)\Big(\frac{\partial}{\partial x_1}\Big)^{k-k_1-2} q(x,y)\Big|_{y=y^0}, \quad (4)$$

where $\tilde Y$ is the set of the end points of the neighbourhoods and the sign +(-) is chosen when the point y^0 is one of the beginning (end) points of the domain of integration.

Now let us consider the integral over $\tilde{\Gamma}_j$. In the neighbourhood of y^j the function $D^\beta v(y)$ is $m_j-m_0-|\beta|$ times continuously differentiable. Therefore we can integrate by parts $m_j-m_0-|\beta|$ times. Adding the result to the addends of sum (4) where $y^0\in\tilde{\Gamma}_j$ we get

$$\sum_{k_1=m_j-m_0-|\beta|}^{k-2}\ \sum_{y^0\in\tilde{Y}\cap\tilde{\Gamma}_j} \pm\mathcal{D}^{k_1}\left[h_k(x,y)D^\beta v(y)\omega_{l_{l+1}}(y)\right]P_\varphi(y)\left(\frac{\partial}{\partial x_1}\right)^{k-k_1-2}q(x,y)\Big|_{y=y^0}$$

$$+\int_{\tilde{\Gamma}_j}\mathcal{D}^{m_j-m_0-|\beta|}\left[h_k(x,y)\,D^\beta v(y)\omega_{l_{l+1}}(y)\right]\left(\frac{\partial}{\partial x_1}\right)^{k-1-(m_j-m_0-|\beta|)}q(x,y)\,ds_y.$$

Since the distance from the point x^0 to the part of the boundary $\tilde{\Gamma}_j$ is at least $\delta_{m_j-m_0}/2$, the last expression is estimable at the point x^0 as follows:

$$C\sum_{k_1=m_j-m_0-|\beta|}^{k-2}\Delta_{|\beta|+k_1-1}\frac{1}{\delta^{k-k_1-1}_{m_j-m_0}}+\Delta_{m_j-m_0-1}\frac{1}{\delta^{k-(m_j-m_0-|\beta|)}_{m_j-m_0}}\int_{\tilde{\Gamma}_j}ds_y$$

$$\le C\Delta_{k+|\beta|-1}\le C\Delta_p.$$

Since all boundary integrals are also continuous in $\Omega^*\cap S(x^0,\delta_p)$, the lemma is proved■

5. The proof of the theorem. Let the assumptions of the theorem be fulfilled. At first we prove that then $u\in C^{m_0+2}(\Omega^*)$. Differentiating both sides of the equation (1) with respect to x_l and denoting

$$K(x,y)=\frac{\partial a(x,y)}{\partial x_l}\ln|x-y|+a(x,y)\frac{x_l-y_l}{|x-y|^2}$$

we obtain

$$\frac{\partial u(x)}{\partial x_l}=\int_\Omega K(x,y)\,u(y)\,dy+\frac{\partial f(x)}{\partial x_l}. \tag{5}$$

Further we use mathematical induction. We know that $u\in C^1(\Omega^*)$. We must show that if $u\in C^k(\Omega^*)$ and $k\le m_0+1$, then $u\in C^{k+1}(\Omega^*)$. The function K satisfies the assumptions of Lemma 3 for $\nu=1$. If $u\in C^k(\Omega^*)$, then we may differentiate formula (5) by Lemma 3. Therefore, for $|\alpha|=k$, $x\in\Omega$,

$$D^\alpha\frac{\partial u(x)}{\partial x_l}=\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\int_\Omega K_{\alpha-\beta}(x,y)\,D^\beta u(y)\,dy$$

$$+\sum_{l=0}^{k-1}\sum_{\beta\le\alpha^l}\binom{\alpha^l}{\beta}\int_\Gamma D_x^{\alpha-\alpha^{l+1}}K_{\alpha^l-\beta}(x,y)\,D^\beta u(y)\omega_{l_{l+1}}(y)\,ds_y+D^\alpha\frac{\partial f(x)}{\partial x_l}. \tag{6}$$

In the first sum all addends are continuous on Ω^* as weakly singular integrals. In the second sum the integrand is the following:

$$\sum_{\gamma \le \alpha-\alpha^{l+1}} \binom{\alpha-\alpha^{l+1}}{\gamma} \Big[D_x^{\alpha-\alpha^{l+1}-\gamma} \frac{\partial a_{\alpha^l-\beta}(x,y)}{\partial x_l} D_x^{\gamma} \ln|x-y|\, D^{\beta}u(y)\, \omega_{l_{l+1}}(y)$$

$$+ D_x^{\alpha-\alpha^{l+1}-\gamma} a_{\alpha^l-\beta}(x,y)\, D_x^{\gamma} \frac{x_l-y_l}{|x-y|^2} D^{\beta}u(y)\, \omega_{l_{l+1}}(y) \Big].$$

Since $a \in C^{m+2}(\Omega^*)$, the function $D_x^{\alpha-\alpha^{l+1}-\gamma} \partial a_{\alpha^l-\beta}/\partial x_l$ is at least $|\gamma|+1$ times continuously differentiable. According to the induction assumption $D^{\beta}u \in C^{k-|\beta|}(\Omega^*) \subset C^{|\gamma|+1}(\Omega^*)$. Then by Lemma 2, where $k=|\gamma|-1$ and p may be arbitrarily large, the integral from the first sum is continuous on Ω^*. Analogously in the second sum the coefficient of $D_x^{\gamma}(x_l-y_l)/|x-y|^2$ belongs to $C^{|\gamma|+1}(\Omega^*)$ and therefore by Lemma 2 the integral from the second sum is also continuous on Ω^*. Consequently $u \in C^{k+1}(\Omega^*)$.

Since $u \in C^{m_0+2}(\Omega^*)$ we can differentiate formula (6) by Lemma 3 once more. We get the formula similar to (6) where $k=m_0+2$. The first sum is continuous on Ω^* again. If we write out the integrand in the second sum, we shall get that all the addends except the one where $l=0$ and $\gamma=\alpha-\alpha^1$ are continuous on Ω^*. From Lemma 2 where $k=|\gamma|=m_0+1$ we obtain the fact that the exception has a logarithmic singularity at those points y^j where $m_j=m_0$:

$$\int_{\Gamma} a(x,y)\, D_x^{\alpha-\alpha^1} \frac{x_l-y_l}{|x-y|^2}\, u(y)\, \omega_{l_1}(y)\, ds_y$$

$$= \sum_{\substack{j=1 \\ m_j=m_0}}^{n} a(x,y)\, u(y) \big[b_{l1}(y) \ln|x-y^j| + b_{l2}(y^j) \arg_j(x) \big] + r_{m_0-1}(x), \quad r_{m_0-1} \in C(\Omega^*),$$

$$B(y^j) = P^{\alpha_2-\alpha_2^1} \frac{d^{m_0+1}}{ds_y^{m_0+1}} \big[\omega_{l_1}(y)\, P_{\varphi}(y) \big] \Big|_{y=y^j_-}^{y=y^j_+} P_{\varphi}^{m_0+1}(y^j).$$

As $\omega_2 P_{\varphi} = I + P\omega_1 P_{\varphi}$ (therefore we may substitute $P^{l_1-1}\omega_1 P_{\varphi}$ for $\omega_{l_1} P_{\varphi}$) and $b_{ij}=b'_{ij}$, $i,j=1,2$, where $B'(y^j)=P^{i-1}B(y^j)$, we have got the assertion of the theorem for $k=m_0+3$.

Now let us use mathematical induction again. Suppose that the theorem is valid for some $k=m_0+3+p$ ($p \ge 0$, $m_0+p+4 \le m+2$). Show that then the theorem is true for derivatives of order $k+1$. Examine formula (6) where $k=m_0+2$. Denote $k(x,y)=K_{\alpha-\beta}(x,y)$, $v(y)=D^{\beta}u(y)$. These functions satisfy the conditions of Lemma 4 and therefore the derivative of order $p+1$ from the first sum in formula (6) belongs to $R_{p+m}(\Omega^*)=R_{k-3}(\Omega^*)$. In boundary integrals we can differentiate under the integral sign:

$$D^{\nu}\int_{\Gamma} D_x^{\alpha-\alpha^{l+1}} K_{\alpha^l-\beta}(x,y)\, D^{\beta}u(y)\omega_{l_{l+1}}(y)\, ds_y$$

$$=\sum_{\gamma\le\nu+\alpha-\alpha^{l+1}} \binom{\nu+\alpha-\alpha^{l+1}}{\gamma} \int_{\Gamma}\Big[D_x^{\nu+\alpha-\alpha^{l+1}-\gamma}\, \frac{\partial a_{\alpha^l-\beta}(x,y)}{\partial x_l}\, D_x^{\gamma}\ln|x-y|$$

$$+D_x^{\nu+\alpha-\alpha^{l+1}-\gamma}\, a_{\alpha^l-\beta}(x,y)\, D_x^{\gamma}\, \frac{x_l-y_l}{|x-y|^2}\Big] D^{\beta}u(y)\omega_{l_{l+1}}(y)\, ds_y,\quad |\nu|=p+1.$$

Examine the coefficient at $D_x^{\gamma}\ln|x-y|$. The factor which includes the derivatives of the function a is at least $|\gamma|+1$ times continuously differentiable. Since according to the induction hypothesis $D^{\beta+\eta}u\in R_{|\beta|+|\eta|-3}(\Omega^*)$ if $|\eta|\le p+2$, by Lemma 2 where $k=|\gamma|-1$ and $p=2-|\beta|$, we get that for $|\beta|=0$ the integral belongs to $R_{|\gamma|-2}(\Omega^*)\subset R_{p+m}(\Omega^*)=R_{k-3}(\Omega^*)$ and for $|\beta|>0$ to $R_{|\gamma|+|\beta|-3}(\Omega^*)\subset R_{p+m-1}(\Omega^*)=R_{k-4}(\Omega^*)$.

Estimating the addend which includes $D_x^{\gamma}(x_l-y_l)/|x-y|^2$, we use Lemma 2, where $k=|\gamma|$ and $p=2-|\beta|$. Similarly to the previous part of proof we get the result that for $|\beta|>0$ corresponding integrals belong to $R_{k-3}(\Omega^*)$. If $|\beta|=0$, then the integral belongs to $R_{|\gamma|-1}(\Omega^*)$. If $|\gamma|<p+m_0+2$, then $R_{|\gamma|-1}(\Omega^*)\subset R_{k-3}(\Omega^*)$. If $|\gamma|=p+m_0+2$, then $\gamma=\nu+\alpha-\alpha^1$ and the addend is the following:

$$\int_{\Gamma} a(x,y)\, D_x^{\nu-\alpha-\alpha^1}\, \frac{x_l-y_l}{|x-y|^2}\, u(y)\,\omega_{l_1}(y)\, ds_y.$$

After using Lemma 2 we obtain the fact that the integral is equal to the expression

$$\sum_{\substack{j=1\\ m_j<k-1}}^{n} a(x,y^j)u(y^j)\Big[b_{11}(y^j)\Big(\frac{\partial}{\partial x_1}\Big)^{k-m_j-2}\ln|x-y^j|$$

$$-\, b_{12}(y^j)\Big(\frac{\partial}{\partial x_1}\Big)^{k-m_j-2}\arg_j(x)\Big]+r_{k-3}(x),$$

where $r_{k-3}\in R_{k-3}(\Omega^*)$ and

$$B(y^j)=P^{(\nu+\alpha)_2+l-1}\,\frac{d^{m_j+1}}{ds_y^{m_j+1}}\Big[\omega_1(y)\,P_{\varphi}(y)\Big]\Big|_{y=y^j_-}^{y=y^j_+}\, P_{\varphi}^{m_j+1}(y^j).$$

We have got the assertion of the theorem for $|\alpha|=k+1$ ■

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