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Integral and Boundary Value Problems for Nonlinear Systems of Composite Type

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Using the Schauder fixed point theorem we establish the solvability of an initial-boundary value problem for a nonlinear first order system of composite type. The procedure depends on first establishing a priori estimates for the solutions. This investigation generalizes the results of [1, 3, 4].

Key words: *Nonlinear composite systems, Schauders fixed point theorem, a priori estimates*
AMS subject classification: 35F20, 35F30

§1. Proper formulation of the initial and boundary value problem

In an earlier work Gilbert and Schneider [4] considered linear, first order systems of composite type. In the present work, we consider nonlinear first order systems of composite type

$$\begin{aligned} &w_{\bar z} = F(z,w,w_z,s),\ w=(w_1,\dots,w_n)',\ F=(F_1,\dots,F_n)', \\ &F = Q^1 w_z + Q^2 \bar w_{\bar z} + A^1 w + A^2 \bar w + A^3 s + A^4 \\ &Q^j = (Q^j_{k\ell})\ (j=1,2),\ A^j=(A^j_{k\ell})\ (j=1,2,3),\ A^4=(A^4_{11},\dots,A^4_{n1})' \end{aligned} \tag{1.1}$$

$$\begin{aligned} &s_y = G(z,w,s),\ s=(s_1,\dots,s_n)',\ G=(G_1,\dots,G_n)' \\ &G = B^1 w + B^2 \bar w + B^3 s + B^4,\ B^j=(B^j_{k\ell})\ (j=1,2,3),\ B^4=(B^4_{11},\dots,B_{n1})' \end{aligned} \tag{1.2}$$

in a bounded simply connected domain D, where

$$\begin{aligned} &Q^j_{k\ell} := Q^j_{k\ell}(z,w,w_z,s)\ (j=1,2),\ A^j_{k\ell} = A^j_{k\ell}(z,w,s)\quad (j=1,2,3) \\ &B^j_{k\ell} := B^j_{k\ell}(z,w,s)\ (j=1,\dots,4;\ \ell\le k,\ \ell\le n) \end{aligned}$$

and w_k, $Q^j_{k\ell}, A^j_{k\ell}, B^j_{k\ell}$ $(j=1,2)$, $A^4_{k\ell}$ are complex-valued functions, $B^2_{k\ell} = \overline{B^1_{k\ell}}$, s_k, $A^3_{k\ell}$, $B^j_{k\ell}$ $(j=3,4)$ are real-valued functions. For convenience, it may be assumed that D is the unit disk, and $\gamma = \{z : |z|=1,\ y\le 0\}$ is the lower boundary of D.

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We suppose that the system (1.1) and (1.2) satisfies **Condition C** below:

(C_1) $Q^j_{k\ell}(z,w,U,s)$ $(j=1,2)$, $A^j_{k\ell}(z,w,s)$ $(j=1,\ldots,4)$ are measurable in $z \in D$ for all continuous vectors $w(z), s(z)$ and all measurable vectors $U(z) = (U_1(z),\ldots,U_n(z))'$ on $\bar{D}$, and satisfy $(1 \le k,\ \ell \le n)$

$$L_p[A^j_{k\ell}(z,w(z),\ s(z)),\bar{D}] \le k_0\ (j=1,2,4),\ \text{but}\ L[A^3_{k\ell},\bar{D}] \le \varepsilon \qquad L_p[A^j_{k\ell},\bar{D}] \le \varepsilon\ (j=1,2;\ k<\ell) \tag{1.3}$$

where $p\ (>2)$, k_0, ε are positive constants.

(C_2) The foregoing functions are continuous in $w_k(z) \in \mathbb{C}$ (the complex plane) and $s_k(z) \in \mathbb{R}$ (the real axis) for almost every point $z \in D$ and $U_k(z) \in \mathbb{C}$, $k=1,\ldots,n$.

(C_3) The complex system (1.1) satisfies the uniform ellipticity condition

$$|F_k(z,w,U^1,s) - F_k(z,w,U^2,s)| \le \sum_{\ell=1}^{n} q_{k\ell}\,|U^1_\ell - U^2_\ell|$$

$$\sum_{\ell=1}^{n} q_{k\ell} \le q_k < \frac{1}{n}, k=1,\ldots,n \tag{1.4}$$

for almost every point $z \in D$ and w_k, U^1_k, $U^2_k \in \mathbb{C}$, $s_k \in \mathbb{R}$ $(k=1,\ldots,n)$, in which $q_{k\ell}$, q_k are positive constants and $q_{k\ell} \le \varepsilon$, $k<\ell$.

(C_4) $B^j_{k\ell}(z,w,s)$ $(j=1,\ldots,4)$, $G_k(z,w,s)$ $(1 \le k,\ell \le n)$ are continuous in $z \in D$ for all Hölder continuous functions w^j_k, $s^j_k \in C_\beta(\bar{D})$ $(j=1,2; k=1,\ldots,n)$, and satisfy

$$\begin{aligned} &C_\beta[B^j_{k\ell}(z,w^1,s^1),\bar{D}] \le k_0\ (j=1,\ldots,4)\\ &w^j = (w^j_1,\ldots,w^j_n)',\ s^j = (s^j_1,\ldots,s^j_n)'\ (j=1,2)\\ &G(z,w^1,s^1) - G(z,w^2,s^2) = \tilde{B}^1(w^1-w^2) + \tilde{B}^2\,(\bar{w}^1-\bar{w}^2) + \tilde{B}^3\,(s^1-s^2) \end{aligned} \tag{1.5}$$

where $\tilde{B}^j = (\tilde{B}^j_{k\ell})$, $\tilde{B}^j_{k\ell} \in C_\beta(\bar{D})$ $(j=1,\ldots,4;\ 1 \le k,\ \ell \le n, (0<\beta<1)$ is real.

We discuss a proper initial–boundary value problem (**Problem A**) for the system (1.1) and (1.2), with the following initial and boundary conditions $(\Gamma := \partial D)$:

$$\begin{aligned} &\mathrm{Re}\,[\overline{\lambda(z)}\,w(z)] = P(z) + H(z),\ \lambda(z) = (\lambda_{k\ell}(z))\\ &P(z) = (P_1(z),\ldots,P_n(z))', z \in \Gamma \end{aligned} \tag{1.6}$$

$$a(z)s(z) = Q(z),\ a(z) = (a_{k\ell}(z)),\ Q(z) = (Q_1(z),\ldots,Q_n(z))',\ z \in \gamma \tag{1.7}$$

where $\lambda_{k\ell}, P_k$ are Hölder continuous functions, and $|\lambda_{kk}(z)| = 1$, $|a_{kk}(t)| = 1$, and P_k, Q_k satisfy

$$\begin{aligned} &C_\alpha[\lambda_{k\ell}(z),\Gamma] \le k_1, C_\alpha[P_k(z),\Gamma] \le k_1, C_\alpha[a_{k\ell}(z),\gamma] \le k_1\quad (1 \le k,\ell \le n)\\ &C_\alpha[\lambda_{k\ell}(z),\Gamma] \le \varepsilon,\ aC_\alpha[a_{k\ell}(z),\gamma] \le \varepsilon\ (1 \le k < \ell \le n),\ C_\alpha[Q_k(z),\ \gamma] \le k_2\ (1 \le k \le n) \end{aligned} \tag{1.8}$$

in which $a\ (> \frac{1}{2})$, $k_1, k_2\ (\geq 0)$ are constants. Moreover, for $s \in \Gamma$ and $1 \leq k \leq n$,

$$H(z) = \begin{pmatrix} H_1(z) \\ \vdots \\ H_n(z) \end{pmatrix} \tag{1.9}$$

$$H_k(z) = \begin{cases} 0, & \text{if } \kappa_k = \frac{1}{2\pi}\Delta_\Gamma \arg \lambda_{kk}(z) \geq 0 \\ H_{k0} + \operatorname{Re} \sum\limits_{m=1}^{-\kappa_k - 1} (H_{km}^+ + iH_{km}^-)\, z^m, & \text{if } \kappa_k < 0,\ 1 \leq k \leq n \end{cases}$$

where $H_{k0}, H_{km}^{\pm}$ $(m = 1, \ldots, -\kappa_k - 1)$ are unknown real constants to be determined appropriately. Moreover, if $\kappa_k \leq 0$, we assume that the solution w of (1.1) satisfies the point condition $(1 \leq k \leq n)$

$$\operatorname{Im}[\overline{\lambda(a_j)}\, w(a_j)] = b^j,\ b^j = (b_1^j, \ldots, b_n^j)',\ j \in \{j\} = \{1, \ldots, 2\kappa_k + 1\} \tag{1.10}$$

where a_j are distinct points on Γ, and b_k^j are real constants with the condition $|b_k^j| \leq k_3$, and $k_3\ (\geq 0)$ is a constant.

In the following, we first give á priori estimates of solutions for Problem A. Afterwards, we prove the solvability of Problem A by using the Schauder fixed-point theorem.

Under some further restrictions, we can discuss the uniqueness of the solution for *Problem A*. The results in this paper are generalizations of the results in [3] and [1].

§2. À priori estimate of solutions for the initial-boundary value problem

First of all, we consider *Problem A* for the system (1.1), (1.2) with $n = 1$:

$$\begin{aligned} w_{1\bar{z}} &= F_1(z, w_1, w_{1z}, s_1) \\ F_1 &= Q_{11}^1 w_{1z} + Q_{11}^2 \bar{w}_{1\bar{z}} + A_{11}^1 w_1 + A_{11}^2 \bar{w}_1 + A_{11}^3 s_1 + A_{11}^4 \end{aligned} \tag{2.1}$$

$$s_{1y} = G_1(z, w_1, s_1),\ G_1 = B_{11}^1 w_1 + B_{11}^2 \bar{w}_1 + B_{11}^3 s_1 + B_{11}^4\,, \tag{2.2}$$

the corresponding initial and boundary condition is as follows:

$$\operatorname{Re}[\overline{\lambda_{11}(z)}\, w_1(z)] = P_1(z) + H_1(z),\ z \in \Gamma \tag{2.3}$$

$$a_{11}(z)\, s_1(z) = Q_1(z),\ z \in \gamma \tag{2.4}$$

$$\operatorname{Im}[\overline{\lambda_{11}(a_j)}\, w_1(a_j)] = b_1^j,\ j \in \{j\} = \{1, \ldots, \kappa_1 + 1\}\,. \tag{2.5}$$

According to the method used in reference [1], we can give an à priori estimate of solutions for the above problem, namely

Lemma 2.1: *Suppose that the system (2.1), (2.2) satisfies Condition C. Then the solution w_1 of the initial-boundary value problem (2.1)-(2.5) satisfies the estimate*

$$L_{p_0}[|w_{1\bar{z}}| + |w_{1z}|, \bar{D}] + C[w_1, \bar{D}] \leq M_1 \left(\sum_{j=1}^{3} k_j + k_0 \right) \doteq M_1 k^* \tag{2.6}$$

$$C_\beta[s,\bar{D}] + C[s_y,\bar{D}] \le M_2 k^\star \tag{2.7}$$

where p_0 $(2 < p_0 \le p), \beta = \min(\alpha, 1-2/p_0)$ *are non-negative constants,* $M_j = M_j(q_{11}, p_0, k', \beta, \kappa_1)$ $(j = 1,2)$ *are real constants only depending on* $q_{11}, p_0, k', \beta, \kappa_1$, $k' := (k_0, k_1, k_2, k_3)$.

Theorem 2.2: *Let the system (1.1), (1.2) satisfy Condition C and let* ε *in Condition C and (1.8) be small enough. Then the solution* $[w,s]$ *of Problem A for (1.1), (1.2) satisfies the estimate*

$$\begin{aligned} X &= C_\beta[w,\bar{D}] + L_{p_0}[|w_{\bar{z}}| + |w_z|, D] \\ &= \sum_{k=1}^{n} \{C_\beta[w_k,\bar{D}] + L_{p_0}[|w_{k\bar{z}}| + |w_{kz}|, \bar{D}]\} \le M_3 \end{aligned} \tag{2.8}$$

$$\begin{aligned} Y &= C^\star_\beta[S,\bar{D}] = C_\beta[S,\bar{D}] + C[S_y,\bar{D}] \\ &= \sum_{k=1}^{n} \{C_\beta[s_k,\bar{D}] + C\,[s_{ky},\bar{D}]\} \le M_4 \end{aligned} \tag{2.9}$$

where $M_j = M_j(q_0, p_0, k', \beta, \kappa)$ $(j = 3,4)$, $q_0 = (q_{k\ell})$, $\kappa = (\kappa_1, \ldots, \kappa_n)$, $k' = (k_0, k_1, k_2, k_3)$, p_0 *and* β *are constants as stated in Lemma 2.1.*

Proof: Let the solution $[w,s]$ be inserted into the system (1.1), (1.2), the initial and boundary condition (1.6), (1.7), (1.10). It is clear that $[w_k, s_k]$ is a solution of the composite type system $(k = 1, \ldots, n)$

$$\begin{aligned} w_{k\bar{z}} - Q^1_{kk} w_{kz} - Q^2_{kk}\bar{w}_{k\bar{z}} &= A^1_{kk} w_k + A^2_{kk}\bar{w}_k + A^3_{kk} s_k + A_k \\ A_k :&= A^4_{k1} + \sum_{\ell\neq k}[Q^1_{k\ell} w_{\ell z} + Q^2_{k\ell}\bar{w}_{\ell\bar{z}} + A^1_{k\ell} w_\ell + A^2_{k\ell}\bar{w}_\ell + A^3_{k\ell} s_\ell] \end{aligned} \tag{2.10}$$

$$\begin{aligned} s_{ky} &= B^1_{kk} w_k + B^2_{kk}\bar{w}_k + B^3_{kk}\bar{s}_k + B_k \\ B_k :&= B_{k4} + \sum_{\ell\neq k}[B^1_{k\ell} w_\ell + B^2_{k\ell}\bar{w}_\ell + B^3_{k\ell} s_\ell] \end{aligned} \tag{2.11}$$

and satisfies the initial and boundary condition

$$\begin{aligned} \operatorname{Re}[\overline{\lambda_{kk}(z)}\, w_k(z)] &= R_k(z) + H_k(z) \\ R_k(z) &:= P_k(z) - \sum_{\ell\neq k} \operatorname{Re}[\overline{\lambda_{k\ell}(z)}\, w_\ell(z)] \end{aligned} \quad (z \in \Gamma) \tag{2.12}$$

$$\operatorname{Im}[\overline{\lambda_{kk}(a_j)}\, w_k(a_j)] = B^j_k := b^j_k - \sum_{\ell\neq k} \operatorname{Im}[\overline{\lambda_{k\ell}(a_j)}\, w_\ell(a_j)],\ j \in \{j\} \tag{2.13}$$

$$a_{kk}(z)s_k(z) = S_k(z) := Q_k(z) - \sum_{\ell\neq k} a_{k\ell}(z)s_\ell(z),\ z \in \gamma. \tag{2.14}$$

We first discuss $[w_1, s_1]$. From Condition C, it can be seen that A_1, R_1, B_1, B^j_1 in (2.10) – (2.14) satisfy

$$
\begin{aligned}
L_{p_0}[A_1,\bar{D}] &\le L_{p_0}(A_1^4,\bar{D}) + \sum_{\ell=2}^{n} \{q_{1\ell}L_{p_0}(w_{\ell z},\bar{D}) \\
&\quad + [L_{p_0}(A_{1\ell}^1,\bar{D}) + L_{p_0}(A_{1\ell}^2,\bar{D})]C(w_\ell,\bar{D}) + L_{p_0}(A_{1\ell}^3,\bar{D})C(s_\ell,\bar{D})\} \\
&\le k_0 + 2\varepsilon[L_{p_0}(w_z,\bar{D}) + C(w,\bar{D}) + C(s,\bar{D})] \le k_0 + 2\varepsilon(X+Y) \\
C_\beta[R_1,\Gamma] &\le C_\beta(P_1,\Gamma) + \sum_{\ell=2}^{n} C_\beta(\lambda_{1\ell},\Gamma)C(w_\ell,\Gamma) \le k_1 + \varepsilon C(w,\bar{D}) \le k_1 + \varepsilon X \\
|B_1^j| &\le |b_1^j| + \sum_{\ell=2}^{n} C(\lambda_{1\ell},\Gamma)C(w_\ell,\Gamma) \le k_2 + \varepsilon X \qquad (2.15) \\
C_\beta[B_1,\bar{D}] &\le C_\beta(B_1^4,\bar{D}) + \sum_{\ell=2}^{n}\{B_{1\ell}^1 w_\ell + B_{1\ell}^2\bar{w}_\ell + B_{1\ell}^3 s_\ell\} \le k_0 + 2\varepsilon(X+Y) \\
C_\beta[S_1,\gamma] &\le C_\beta(Q_1,\gamma) + \varepsilon C(s,\gamma) \le k_3 + \varepsilon Y.
\end{aligned}
$$

According to *Lemma* 2.1, it can be obtained

$$
\begin{aligned}
L_{p_0}[|w_{1\bar{z}}| + |w_{1z}|,\bar{D}] + C_\beta[w_1,\bar{D}] &\le 2M_1\left[\sum_{j=0}^{3} k_j + 3\varepsilon(X+Y)\right] \\
&\le N_1[1+\varepsilon(X+Y)], \qquad (2.16) \\
C_\beta[S_1,\bar{D}] + C[S_{1y},\bar{D}] &\le 2M_2\left[\sum_{j=0}^{3} k_j + 3\varepsilon(X+Y)\right] \\
&\le N_1'[1+\varepsilon(X+Y)],
\end{aligned}
$$

where $N_1 = 2M_1\left(\sum_{j=0}^3 k_j + 3\right)$, $N_1' = 2M_2\left(\sum_{j=0}^3 k_j + 3\right)$.

Next, we consider $[w_2,\ s_2]$. From (2.10) - (2.14) ($k=2$) and Condition C, we have

$$
\begin{aligned}
L_{p_0}[A_2,D] &\le L_{p_0}(A_2^4,\bar{D}) + \sum_{\ell\ne 2}\Big\{q_{2\ell}L_{p_0}(w_{\ell z},\bar{D}) \\
&\quad + [L_{p_0}(A_{z\ell}^1,\bar{D}) + L_{p_0}(A_{z\ell}^2,\bar{D})]C(w_\ell,\bar{D})L_{p_0}(A_{2\ell}^3,\bar{D})C(s_\ell,\bar{D})\Big\} \\
&\le k_0 + L_{p_0}(w_{1z},\bar{D}) + 2k_0[C(w_1,\bar{D}) + C(s_1,\bar{D})] + 2\varepsilon(X+Y) \\
C_\beta[R_2,\Gamma] &\le C_\beta(P_2,\Gamma) + \sum_{\ell\ne 2} C_\beta(\lambda_{2\ell},\Gamma)C(w_\ell,\Gamma) \le k_1 + k_1 C(w_1,\bar{D}) + \varepsilon X \\
|B_2^j| &\le |b_2^j| + \sum_{\ell\ne 2} C(\lambda_{2\ell},\Gamma)C(w_\ell,\Gamma) \le k_2 + k_1 C(w_1,\bar{D}) + \varepsilon X \qquad (2.17) \\
C_\beta[B_2,\bar{D}] &\le C_\beta(B_2^4,\bar{D}) + \sum_{\ell ne 2}\{B_{1\ell}^1 w_\ell + B_{2\ell}^2\bar{w}_\ell + B_{2\ell}^3 s_\ell\} \\
&\le k_0 + 2k_0[C(w_1,\bar{D}) + C(s_1,\bar{D})] + 2\varepsilon(XY) \\
C_\beta[S_2,\gamma] &\le C_\beta(Q_2,\gamma) + \sum_{\ell\ne 2} C(a_{2\ell},\Gamma)C(s_\ell,\Gamma) \le k_3 + k_1 C(s_1,\bar{D}) + \varepsilon Y.
\end{aligned}
$$

Similarly to (2.16), it can be derived

$$
\begin{aligned}
&L_{p_0}[|w_{2\bar z}| + |w_{2z}|, \bar D] + C_\beta[w_2, \bar D] \\
&\le 2M_1 \left[\sum_{j=0}^{3} k_j + 3\varepsilon(X+Y) + (4k_0 + 2k_1 + 1)[L_{p_0}(|w_{1z}|, \bar D) + C(w_1, \bar D) + C(s_1, \bar D)]\right] \\
&\le 2M_1 \left[\sum_{j=0}^{3} k_j + 3\varepsilon(X+Y) + (4k_0 + 2k_1 + 1)N_1[1 + \varepsilon(X+Y)]\right] \le N_2[1 + \varepsilon(X+Y)], \\
&C_\beta[s_2, \bar D] + C[s_{2y}, \bar D] \\
&\le 2M_2 \left[\sum_{j=0}^{3} k_j + 3\varepsilon(X+Y) + (4k_0 + 2k_1 + 1)[L_{p_0}(|w_{1z}|, \bar D) + C(w_1, \bar D) + C(s_1, \bar D)]\right] \\
&\le N_2'[1 + \varepsilon(X+Y)],
\end{aligned}
\tag{2.18}
$$

where

$$
N_2 = 2M_1 \left[\sum_{j=0}^{3} k_j + 2(4k_0 + 2k_1 + 1)\, N_1 + 3\right]
$$

$$
N_2' = 2M_2 \left[\sum_{j=0}^{3} k_j + 2(4k_0 + 2k_1 + 1)\, N' + 3\right].
$$

Moreover, we can obtain

$$
\left.
\begin{aligned}
L_{p_0}\,[|w_{k\bar z}| + |w_{kz}|, \bar D] + C_\beta(w_k, \bar D) \le N_k\,[1 + \varepsilon(X+Y)] \\
C_\beta\,[s_k, \bar D] + C\,[s_{ky}, \bar D] \le N_k'\,[1 + \varepsilon(X+Y)]
\end{aligned}
\right\} \quad (\ell < k \le n)
\tag{2.19}
$$

in which N_k, N_k' are constants only depending on M_1, M_2, k_0, k_1, k_3. Combining (2.16), (2.18), and (2.19), it follows that

$$
X + Y \le \sum_{k=1}^{n} (N_k + N_k')\,[1 + \varepsilon(X+Y)]. \tag{2.20}
$$

Choosing the positive number ε small enough such that $1 - \varepsilon\, \sum_{k=1}^{n}(N_k + N_k') \ge \frac{1}{2}$, we conclude

$$
X + Y \le \frac{\sum_{k=1}^{n}(N_k + N_k')}{1 - \varepsilon\, \sum_{k=1}^{n}(N_k + N_k')} \le 2 \sum_{k=1}^{n} (N_k + N_k'). \tag{2.21}
$$

Thus, (2.8) and (2.9) are established.

§3. The solvability of the initial and boundary value problem

We first discuss a special case:

$$
\begin{aligned}
&Q_{k\ell}^j = Q_{k\ell}^j(z, w_z)\ (j = 1, 2) \\
&A_{k\ell}^j = A_{k\ell}^j(z),\ B_{k\ell}^j = B_{k\ell}^j(z)\ (j = 1, \ldots, 4;\ 1 \le k,\ \ell \le n)
\end{aligned}
$$

of the system (1.1), (1.2), and denote such system by

$$w_{\bar z} = \tilde F(z, w, w_z, s),\ s_y = \tilde G(z, w, s). \tag{3.1}$$

Theorem 3.1: *Suppose that the system (3.1) satisfies Condition C and the positive constant ε is sufficiently small. Then Problem A for (3.1) is solvable.*

Proof: To use the imbedding method, we introduce the following initial–boundary value problem (*Problem B*):

$$w_{\bar z} = t\tilde F(z, w, w_z, s) + A(z),\ A(z) := (A_1(z), \ldots, A_n(z))',\ z \in D \tag{3.2}$$

$$s_y = t\tilde G(z, w, s) + B(z),\ B(z) := (B_1(z), \ldots, B_n(z))',\ z \in D \tag{3.3}$$

$$\operatorname{Re}[\overline{\lambda(z)}\, w(z)] = P(z) + H(z),\ z \in \Gamma \tag{3.4}$$

$$\operatorname{Im}[\overline{\lambda(a_j)}\, w(a_j)] = b^j,\ j \in \{j\},\ \kappa_k \ge 0,\ 1 \le k \le n \tag{3.5}$$

$$a(z)\, s(z) = Q(z),\ z \in \gamma \tag{3.6}$$

where $0 \le t \le 1$, $A_k \in L_{p_0}(\bar D)$ $(2 < p_0 \le p)$, $B_k \in C_\beta(\bar D)$ $(1 \le k \le n)$, for convenience, sometimes we denote them by $A \in L_{p_0}(\bar D)$, $B \in C_\beta(\bar D)$, respectively. When $t = 0$, using Lemma 3.1 in Ref. [2], we see that the boundary value problem (3.2), (3.4), (3.5) has a unique solution w, and similarly we can prove that the initial value problem (3.3), (3.6) has a solution s. Hence in this case, $[w, s]$ is a unique solution of Problem B.

Assuming that Problem B for $t = t_0$ $(0 \le t_0 < 1)$ is solvable, we can verify that there exists a positive number δ, such that Problem B has a unique solution $[w, s]$ on $E = \{|t - t_0| \le \delta,\ 0 \le t \le 1\}$ for any $A_k \in L_{p_0}(\bar D)$, $s_k \in C_\beta(\bar D)$. Here $w \in C_\beta(\bar D) \cap W^1_{p_0}(D)$, $s \in C^*_\beta(\bar D)$. In fact, we may rewrite the system (3.2) and (3.3) as

$$w_{\bar z} - t_0\, \tilde F(z, w, w_z, s) = (t - t_0)\, \tilde F(z, w, w_z, s) + A(z) \tag{3.7}$$

$$s_y - t_0\, \tilde G(z, w, s) = (t - t_0)\, \tilde G(z, w, s) + B(z). \tag{3.8}$$

Choosing any function vectors $w^0 \in C_\beta(\bar D) \cap W^1_{p_0}(D)$, $s^0 \in C^*_\beta(\bar D)$ and substituting w^0, s^0 into the positions of w, s of the right–hand sides in (3.7), (3.8), it is clear that

$$(t - t_0)\, \tilde F(z, w^0, w^0_z, s^0) + A(z) \in L_{p_0}(\bar D) \quad \text{and} \quad (t - t_0)\, \tilde G(z, w^0, s^0) + B(z) \in C_\beta(\bar D).$$

Thus Problem B for (3.7), (3.8) has a unique solution $[w^1, s^1] \in C_\beta(\bar D) \cap W^1_{p_0}(D) \times C^*_\beta(\bar D)$. Iterating successively, we obtain a sequence of solutions $[w^n,\ s^n]$ $(n = 1, 2, \ldots)$ satisfying

$$w^{n+1}_{\bar z} - t_0\, \tilde F(z, w^{n+1}, w^{n+1}_z, s^{n+1}) = (t - t_0)\, \tilde F(z, w^n, w^n_z, s^n) + A(z) \tag{3.9}$$

$$s^{n+1}_y - t_0\, \tilde G(z, w^{n+1}, s^{n+1}) = (t - t_0)\, \tilde G(z, w^n, s^n) + B(z) \tag{3.10}$$

and the initial–boundary condition (3.4), (3.6). It is not difficult to see that $W^{n+1} = w^{n+1} - w^n$, $S^{n+1} = s^{n+1} - s^n$ is a solution of the following initial–boundary value problem:

$$W_{\bar z}^{n+1} - t_0\,[\tilde F(z, w^{n+1}, w_z^{n+1}, s^{n+1}) - \tilde F(z, w^n, w_z^n, s^n)] \qquad (3.11)$$
$$= (t - t_0)\,[\tilde F(z, w^n, w_z^n, s^n) - \tilde F(z, w^{n-1}, w_z^{n-1}, s^{n-1})]$$
$$S_y^{n+1} - t_0\,[\tilde G(z, w^{n+1}, s^{n+1}) - \tilde G(z, w^n, s^n)] \qquad (3.12)$$
$$= (t - t_0)[\tilde G(z, w^n, s^n) - \tilde G(z, w^{n-1}, s^{n-1})]$$

$$\operatorname{Re}[\overline{\lambda(z)}\, W^{n+1}(z)] = H(z),\ z \in \Gamma \qquad (3.13)$$
$$\operatorname{Im}[\overline{\lambda(a_j)}\, W^{n+1}(a_j)] = 0,\ j \in \{j\},\ \kappa_k \geq 0,\ 1 \leq k \leq n \qquad (3.14)$$
$$a(z)\, s(z) = 0,\ z \in \gamma. \qquad (3.15)$$

By Condition C, we have

$$L_{p_0}\,[\tilde F(z, w^n, w_z^n, s^n) - \tilde F(z, w^{n-1}, w_z^{n-1}), \bar D)] \qquad (3.16)$$
$$\leq L_{p_0}[W^n(z), \bar D] + 2k_0\,[C_\beta(W^n(z), \bar D) + C_\beta[(S^n(z), \bar D)]$$
$$C_\beta\,[\tilde G(z, w^n, s^n) - G(z, w^{n-1}, s^{n-1}), \bar D] \qquad (3.17)$$
$$\leq 2k_0\,[C_\beta(W^n(z), \bar D) + C_\beta(S^n(z), \bar D)].$$

With the method used in the proof of Theorem 2.2, we can obtain

$$L(W^{n+1}, S^{n+1}) = L_{p_0}\,[|W_{\bar z}^{n+1}| + |W_z^{n+1}|, \bar D] + C_\beta\,[W^{n+1}, \bar D] + C_\beta^*(S^{n+1}, \bar D) \qquad (3.18)$$
$$\leq |t - t_0|\, M_5 L(W^n, S^n),$$

where $M_5 = M_5(q_0, p_0, k', \beta, \kappa)$, q_0, k', κ are as stated in Theorem 2.2. Choosing $\delta = 1/[2(M_5 + 1)]$, then for any $t \in E$, we have

$$L(W^{n+1}, S^{n+1}) \leq \frac{1}{2} L(W^n, S^n) \leq \frac{1}{2^n}\, L(W^1, S^1)$$
$$\leq \frac{1}{2^N}\, L(W^1, S^1),\ \text{ if } n > N + 1, \qquad (3.19)$$

and

$$L(w^n - w^m, s^n - s^m) \leq \frac{1}{2^{N-1}}\, L(w^1 - w^0, s^1 - s^0),\ \text{if } n \geq m > N + 1. \qquad (3.20)$$

This shows that $L(w^n - w^m, s^n - s^m) \to 0$ as $n, m \to \infty$. Hence there exist $w^* \in C_\beta(\bar D) \cap W_{p_0}^1(D)$, $s^* \in C_\beta^*(\bar D)$ such that $L(w^n - w^*, s^n - s^*) \to 0$ as $n \to \infty$, and $[w^*, s^*]$ is just a solution of Problem B for (3.2), (3.3) on E. Thus we see that when $t = 0, 1, \ldots, [\frac{1}{\delta}]\ \delta, 1$, Problem B for (3.2), (3.3) is solvable. In particular, when $t = 1$, $A(z) = 0$, $B(z) = 0$, Problem A for (3.1) is solvable.

Next, we prove the solvability of Problem A for (1.1), (1.2).

Theorem 3.2:. *If the composite type system (1.1), (1.2) satisfies Condition C and the positive constant ε is small enough, then Problem A for (1.1), (1.2) has a solution $[w, s]$.*

Proof: We introduce a bounded and closed convex set E in the Banach space $C(\bar{D})\times C(\bar{D})$, the elements of which are vectors $\omega[w, s]$ satisfying the condition

$$C[w(z), \bar{D}] \leq M_3,\ C[s(z), \bar{D}] \leq M_4, \tag{3.21}$$

where M_3, M_4 are the constants as stated in (2.8), (2.9). We select any vector $\Omega = [W, S]$, and substitute W, S into the proper positions of the system (3.1), (3.2). On the basis of Theorem 3.1, we see that there exists a solution $\omega := [w, s]$ of the following initial–boundary value problem:

$$\begin{aligned} &w_{\bar{z}} = f(z, w, W, w_z, s, S)\\ &\quad f = Q^1(z, W, w_z, S)\, w_z + Q^2(z, W, w_z, S)\, \bar{w}_{\bar{z}} + A^1(z, W, S)\, w\\ &\quad\ + A^2(z, W, S,)\, \bar{w} + A^4(z, W, S)\bar{w} + A^4(z, W, S) \end{aligned} \tag{3.22}$$

$$\begin{aligned} &s_y = g(z, w, W, s, S)\\ &\quad g = B^1(z, W, S)\, w + B^2(z, W, S)\, \bar{w} + B^3(z, W, S)\, s + B^4(z, W, S) \end{aligned} \tag{3.23}$$

and (1.6), (1.7), (1.10). Following Theorem 2.2, the solution $\omega = [w, s]$ satisfies the estimates (2.8) and (2.9), therefore $\omega \in E$. Moreover, we can see that the mapping $\omega = T(\Omega)$ from $\Omega \in E$ onto $\omega \in E$ is a compact set in E. In the following, we shall show that $\omega = T(\Omega)$ is a continuous mapping in E. Choosing a sequence of vectors $[W^n, S^n]$ $(n = 0, 1, 2, \ldots)$ in E so that $C[W^n - W^0, \bar{D}] \to 0$, $C[S^n - S^0, \bar{D}] \to 0$ as $n \to \infty$, and denoting $\omega^n = (w^n, s^n) = T(W^n, S^n)$ $(n = 0, 1, 2, \ldots)$, it is evident that

$$(w^n - w^0)_{\bar{z}} = f(z, w^n, W^n, w^n_z, s^n, S^n) - f(z, w^0, W^0, w^0_z, s^0, S^0) \tag{3.24}$$

$$(s^n - s^0)_y = g(z, w^n, W^n, s^n, S^n) - g(z, w^0, W^0, s^0, S^0) \tag{3.25}$$

$$\operatorname{Re}[\overline{\lambda(z)}(w^n(z) - w^0(z))] = H(z),\ z \in \Gamma \tag{3.26}$$

$$\operatorname{Im}[\overline{\lambda(a_j)}(w^n(a_j) - w^0(a_j))] = 0,\ j \in \{j\},\ \kappa_k \geq 0,\ 1 \leq k \leq n \tag{3.27}$$

$$a(z)\, s(z) = 0,\ z \in \gamma \tag{3.28}$$

and the complex equation (3.25) can be written as

$$\begin{aligned} &[w^n - w^0]_{\bar{z}} - [f(z, w^n, W^n, w^n_z, s^n, S^n) - f(z, w^0, W^n, w^0_z, s^0, S^n)] = c^n\\ &c^n = f(z, w^0, W^n, w^0_z, s^0, S^n) - f(z, w^0, W^0, w^0_z, s^0, S^0). \end{aligned} \tag{3.29}$$

Applying the method in the proof of Theorem 2.2 of Chapter 4 in Ref. [5], we can verify that $L_{p_0}[c^n, \bar{D}] \to 0$ as $n \to \infty$. Similarly to the proof of Theorem 2.1, it can be derived that

$C[w^n - w^0, \bar{D}] \to 0$, $C[s^n - s^0, \bar{D}] \to 0$, as $n \to \infty$. By means of the Schauder fixed-point theorem, there exists a vector $\omega = [w, s] \in B$ such that $\omega = T(\omega)$, and $\omega = [w, s]$ is just a solution of Problem A for the system (1.1) and (1.2).

Finally, we assume that $F(x, w, w_z, s)$ satisfies

$$F(z, w^1, U, s^1) - F(z, w^2, U, s^2) = A_\star^1(w^1 - w^2) - A_\star^2(s^1 - s^2) \tag{3.30}$$

for any vectors w^j, $s^j \in C_\beta(\bar{D})$ $(j = 1, 2)$ and $U \in L_{p_0}(\bar{D})$ $(2 < p_0 < p)$, where $A_\star^1$, $A_\star^2$ satisfy the condition similar to A^1, A^3 in Condition C. Using the method as stated in the proof of Theorem 2.2, we can prove the following theorem.

Theorem 3.3: *Let the system (1.1), (1.2) satisfy Condition C, (3.30), and the positive constant ε is sufficiently small. Then the solution of Problem A for (1.1), (1.2) is unique.*

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