

About Integral Equivalence between Linear and Nonlinear Operator Impulsive Differential Equations in a Banach Space

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After an introduction into the problems of impulsive operator differential equations sufficient conditions for the integral and the asymptotic equivalence between linear and nonlinear equations of this kind are presented. These conditions guarantee that for bounded solutions of the linear equation there are also bounded solutions of the corresponding nonlinear equation.

Key words: Abstract impulsive differential equations, integral equivalence, asymptotic equivalence, exponential dichotomy, fixed point theorem of Schauder.

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1. Introduction

Many processes considered in natural science and technology are characterized by shortly acting impulses in their in general evolutionary development. The impulsive differential equations are an adequate mathematical apparatus to describe such processes. Thus the state y of dynamical systems can often be described by differential equations of the form

$$\frac{dy}{dt} = f(t, y(t)), \quad t \geq 0. \quad (1)$$

If there impulsive alterations ξ_j of the state caused by external effects occur for fixed moments $t_j > 0$, then (1) changes into the impulsive differential equation

$$\begin{aligned} \frac{dy}{dt} &= f(t, y(t)) \Big|_{t \neq t_j}, \quad t \geq 0 \\ \Delta y(t_j) &= y(t_j + 0) - y(t_j - 0) = \xi_j. \end{aligned}$$

Using the vector notation we also integrate systems of differential equations.

As an example we consider a mechanical system consisting of a hull S with mass M and a rigid body K of mass m which is connected with S by a buffer [5:pp.11-13]. Let the motion of the hull S be rectilinear and caused by an uncontrollable exterior force $\sigma = \sigma(t)$ depending on the time t . Let the force g effecting the body K depend only on the displacement y_1 of the body with respect to the hull and on the relative velocity $y_1' = \dot{y}_1/dt$ (see the figure below).

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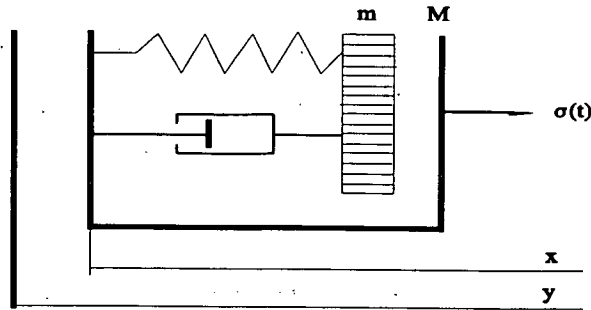


Figure: Mechanical system with a buffer

If z denotes the displacement of S with respect to an inertial system, then the motion of the system will be described by the differential equation

$$Mz'' + m(y_1'' + z'') = \sigma, \quad m(y_1'' + z'') = g(y_1, y_1').$$

Eliminating z'' and using the abbreviations $u = -(M + m)g/(Mm)$, $F = -\sigma/M$ the equation

$$y_1'' + u(y_1, y_1') = F \quad (2)$$

arises which expresses the motion of K with respect to S . (A similar equation will be obtained if we know the acceleration z'' of S instead of the force σ .) If now the unaccelerated system ($F \equiv 0$) suffers shock effects for fixed moments t_j , then instantaneous increments

$$\Delta y_1'(t_j) = y_1'(t_j+0) - y_1'(t_j-0) = R_j(y_1(t_j-0), y_1'(t_j-0)) \quad (3)$$

of the velocity y_1' occur. Here R_j describes the dependency of the velocity jump on the starting conditions. The equations (2) and (3) lead to the system

$$y_1' = y_2|_{t=t_j}, \quad y_2' = -u(y_1, y_2)|_{t=t_j},$$

$$\Delta y_1(t_j) = y_1(t_j+0) - y_1(t_j-0) = 0,$$

$$\Delta y_2(t_j) = y_2(t_j+0) - y_2(t_j-0) = R_j(y_1(t_j-0), y_2(t_j-0))$$

of impulsive differential equations which is easily solvable for linear forces g (and functions u , respectively). Under certain assumptions then also the existence of a solution in the nonlinear case with a similar behaviour for great t as in the linear case follows. We will present such assumptions in this paper.

The mathematical investigation of impulsive differential equations starts with the papers [10] and [12]. In the publications [1] - [3] and [14] they are treated for the first time in abstract spaces. Relations between the solutions of linear and nonlinear equations are studied in [11], [7], [13], [8] and [3].

Here we compare the linear impulsive differential equation

$$\frac{dx}{dt} = A(t)x(t)|_{t=t_j}, \quad (4a)$$

$$(\Delta x)(t_j) = x(t_j+0) - x(t_j-0) = I_j x(t_j-0) \quad (4b)$$

with the nonlinear impulsive differential equation

$$\frac{dy}{dt} = A(t)y(t) + f(t, Ty(t))|_{t=t_j}, \quad (5a)$$

$$(\Delta y)(t_j) = y(t_j+0) - y(t_j-0) = (I_j + H_j)y(t_j-0). \quad (5b)$$

In the following $R_+ = [0, \infty)$ denotes the positive real half-axis, X a Banach space and $L(X)$ the space of all linear and bounded operators from X into itself. Further we assume:

$$x, y: R_+ \rightarrow X, \quad A: R_+ \rightarrow L(X), \quad I_j, H_j \in L(X) \quad (j = 1, 2, 3, \dots)$$

$$f: R_+ \times X \rightarrow X, \quad T: X \rightarrow X, \quad 0 < t_1 < t_2 < t_3 < \dots, \quad t_j \rightarrow \infty \quad (j \rightarrow \infty).$$

If the equations (4) and (5) are *integral equivalent*, then the existence of a bounded solution $x=x(t)$ of (4) also will imply the existence of a bounded solution of the more complicated equation (5). If (4) and (5) are additionally *asymptotic equivalent*, then both solutions will not much differ for great t . If we can solve (4) explicitly, then the solution of (5) can be approximated for great t by a known function. Hence it is important to look for sufficient conditions which guarantee the integral equivalence and the asymptotic equivalence of (4) and (5).

It will be shown that a solution of (5) satisfies under certain assumptions a fixed point equation which contains an integral equivalent solution of (4). The existence of a fixed point is proven by using the well-known theorem of Schauder.

2. Definitions and auxiliary theorems

This section contains some notions and relations for impulsive differential equations and a criterion about the compactness of related function sets.

Definition 1: The function $u = u(t)$ is said to be a *solution* of the impulsive differential equation (4) or (5) if it fulfils for $t \notin \{t_j\}$ the equation (4a) or (5a) and for $t \in \{t_j\}$ the jump condition (4b) or (5b).

In the following we assume the solutions u of the impulsive equation to be continuous from the left. Hence we have $u(t_j - 0) = u(t_j)$. Instead of $u(t_j)$ we shall shortly write u_j . Let $U_k(t, \tau)$ be the *evolution operator* of the differential equation (4a) in the interval $(t_{k-1}, t_k]$ with $t_0 = 0$ and let $I \in L(X)$ be the identical operator. Besides we introduce the operators $Q_j = I_j + I$ ($j = 1, 2, 3, \dots$).

Lemma 1: *The evolution operator $W(t, \tau)$ of the impulsive equation (4) has for $t \geq \tau > 0$ the form*

$$W(t, \tau) = \begin{cases} U_n(t, \tau) & \text{for } t_n < \tau \leq t \leq t_{n+1}, \\ U_{n+1}(t, t_n) Q_n U_n(t_n, \tau) & \text{for } t_{n-1} < \tau \leq t_n < t \leq t_{n+1}, \\ U_{n+1}(t, t_n) \left(\prod_{j=n}^{k+1} Q_j U_j(t_j, t_{j-1}) \right) Q_k U_k(t_k, \tau) & \text{for } t_{k-1} < \tau \leq t_k < t_n < t \leq t_{n+1}. \end{cases}$$

The proof of the Lemma is simple. We want to pass it over here. It is also easy to see by Lemma 1 that $W(t, \tau)$ satisfies the equations

$$W'(t, \tau) = A(t)W(t, \tau), \quad t \notin \{t_j\}, \quad (6a)$$

$$W(t_j + 0, \tau) = Q_j W(t_j, \tau). \quad (6b)$$

We shall $W(t, 0)$ abbreviate by $W(t)$. For a projector $P_1 \in L(X)$ we define

$$P_2 = I - P_1, \quad W_i(t, s) = W(t) P_i W^{-1}(s), \quad t \geq 0, \quad s \geq 0, \quad i \in \{1, 2\}, \quad (7)$$

$$J_1(t) = \{j > 0: t_j < t\}, \quad J_2(t) = \{j > 0: t \leq t_j\}$$

and

$$G(P_i, t, d) = \int_0^t \|W_1(t, s)\|^d ds + \int_t^\infty \|W_2(t, s)\|^d ds + \sum_{j \in J_1(t)} \|W_1(t, t_j)\|^d + \sum_{j \in J_2(t)} \|W_2(t, t_j)\|^d \quad (8)$$

where d is any positive constant. Finally we use for $p \geq 1$ and functions $z: R_+ \rightarrow X$ the notation

$$L_p(R_+, K) = \left\{ z: \left(\int_0^\infty |z(t)|^p dt \right)^{1/p} \leq K \right\}. \quad (9)$$

Definition 2: The impulsive equations (4) and (5) are called *p-integral equivalent* if there exists a bounded solution $y = y(t)$ of (5) to each bounded solution $x = x(t)$ of (4) and reversely a bounded solution $x = x(t)$ of (4) to each bounded solution $y = y(t)$ of (5) so that $\|x(t) - y(t)\| \in L_p(\mathbb{R}_+)$ holds in both cases.

Definition 3: The impulsive equations (4) and (5) are called *p-asymptotic equivalent* if they are *p-integral equivalent* and if additionally the corresponding solutions $x(t)$, $y(t)$ fulfil the relation $\lim_{t \rightarrow \infty} \|x(t) - y(t)\| = 0$.

We consider the set $S(\mathbb{R}_+, X)$ of all functions $u: \mathbb{R}_+ \rightarrow X$ which are continuous for $t \notin \{t_j\}$ and continuous from the left for $t \in \{t_j\}$, where discontinuities of the first kind can occur in t_j . $S(\mathbb{R}_+, X)$ is a linear space which can be metrized by

$$\rho(x, y) = \sup_{0 < \lambda < \infty} (1 + \lambda)^{-1} \frac{\max_{0 \leq t \leq \lambda} \|x(t) - y(t)\|}{1 + \max_{0 \leq t \leq \lambda} \|x(t) - y(t)\|}$$

This metric induces the so-called compact-open topology Ω that we always want to use in $S(\mathbb{R}_+, X)$. The corresponding convergence is the uniform convergence on each compact interval of \mathbb{R}_+ .

Lemma 2: A set $\mathcal{F} \subseteq S(\mathbb{R}_+, X)$ is relatively compact if and only if the following conditions are satisfied:

1. The functions in \mathcal{F} are equicontinuous, i.e., for all $\varepsilon > 0$ there exists a number $\delta > 0$ in such a way that for all $k \geq 1$, $u \in \mathcal{F}$ and t', t'' in $(t_{k-1}, t_k]$ with $|t' - t''| < \delta$ the inequality $\|u(t') - u(t'')\| < \varepsilon$ holds.

2. For all $t \in \mathbb{R}_+$ the subset $R_t = \{u(t) : u \in \mathcal{F}\}$ is relatively compact in X .

The proof of the Lemma is a direct consequence of the Theorem of Ascoli (see, e.g., [9, p.47]).

3. Integral equivalent equations

At first we formulate conditions, which ensure the integral equivalence of the equations (4) and (5). Although these conditions seem to be relatively complicated and difficult to verify; they are adequate under the intended generality of the problem. An example at the end of this section shows that the quoted conditions can be verified under rather natural assumptions. The notations and concepts are given in section 2.

Theorem 1: *Let the following assumptions be fulfilled for the impulsive equations (4) and (5):*

I. It holds:

- (V1) *A is bounded on each compact interval of \mathbb{R}_+ , f satisfies locally the conditions of Carathéodory, T is continuous.*
- (V2) *The bounded operators Q_j^{-1} ($j = 1, 2, 3, \dots$) and therefore also the operators $W^{-1}(t) = W^{-1}(t, 0)$ exist.*

II. Further, it holds for every closed and bounded central ball $B \subset X$:

- (V3) *For all $\hat{u} \in X$, $\|T\hat{u}\| \leq k(\hat{u})\|\hat{u}\|$ with a functional k bounded on B .*
- (V4) *The sets $f(\mathbb{R}_+ \times B)$ and $\bigcup_{j=1}^{\infty} Q_j^{-1}H_j(B)$ are relatively compact in X .*

III. Finally, it holds for a projector $P_1 \in L(X)$, for a function $\omega(t, s): \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which is monotonically non-descending in s for fixed t and for suitable numbers $a \in (1, 2)$, $b \geq 1$, $m_1 > 0$, $m_2 > 0$ and $M > 0$ by observing the notations (7) - (9) and $\bar{a} = a/(a-1)$:

- (V5) $\sup_{t \in \mathbb{R}_+} [G(P_1, t, a)]^{1/a} \leq m_1$, $G(P_1, \cdot, b) \in L_1(\mathbb{R}_+)$,
- (V6) $\int_0^{\infty} |P_2 W^{-1}(s)| ds + \sum_{j=1}^{\infty} |P_2 W^{-1}(t_j)| < \infty$,
- (V7) $|f(t, T\hat{u})| \leq \omega(t, |T\hat{u}|)$ for almost all $t \in \mathbb{R}_+$ and all $\hat{u} \in X$,
- (V8) $\omega(t, c) \in L_1(\mathbb{R}_+, m_2) \cap L_1(\mathbb{R}_+, m_2)$ for all $c \geq 0$,
- (V9) $m_3(c) := \sup_{t \in \mathbb{R}_+} \omega(t, c) < \infty$ for all $c \geq 0$,
- (V10) $(\sum_{j=1}^{\infty} |Q_j^{-1}H_j|^1)^{1/a} \leq M < 1/(4m_1)$.

Then (4) and (5) are p-integral equivalent with $p = b/(2 - a)$.

Proof: 1) Let x be a bounded solution of (4). Then by (V10) there is a number

$$\bar{\rho} \geq 2m_1m_2(1-4m_1M)^{-1} > 0, \tag{10}$$

so that $x(t)$ lies for all t in the closed central ball $B_{\bar{\rho}}$ with the radius $\bar{\rho}$. Therefore x belongs to the Ω -closed and convex set $D_{\bar{\rho}} = \{u \in S(\mathbb{R}_+, X) : u(t) \in B_{\bar{\rho}} \text{ for all } t\}$. By assumption (V3) the number $\chi_{\rho} = \sup \{k(\hat{u}) : \|\hat{u}\| \leq \rho\}$, $\rho = 2\bar{\rho}$ exists. By using the abbreviations

$$\bar{c} = \rho\chi_{\rho}, u_j = u(t_j), h_j = Q_j^{-1}H_j \in L(X), l_j = \|h_j\|, l = \sup_j l_j < \infty \tag{11}$$

the assumptions (V3), (V7) - (V10) supply for $u \in D_{\bar{\rho}}$ the inequalities

$$\begin{aligned} \|Tu(t)\| &\leq k(u(t))\|u(t)\| \leq \bar{c} \quad \text{for all } t, \\ \|f(t, Tu(t))\| &\leq \omega(t, \|Tu(t)\|) \leq \omega(t, \bar{c}) \leq m_2(\bar{c}) \quad \text{for almost all } t, \end{aligned} \tag{12}$$

$$\|h_j(u_j)\| \leq l_j\|u_j\| \leq \rho l_j \leq \rho l \quad \text{for all } j > 0$$

and the relation

$$\omega(t, \bar{c}) \in L_{\bar{c}}(\mathbb{R}_+, m_2). \tag{13}$$

Now we define on $D_{\bar{\rho}}$ the operator

$$\begin{aligned} (Qu)(t) &= x(t) + \int_0^t W_1(t,s)f(s, Tu(s))ds - \int_t^{\infty} W_2(t,s)f(s, Tu(s))ds \\ &\quad + \sum_{j \in I_1(t)} W_1(t, t_j)h_j(u_j) - \sum_{j \in I_2(t)} W_2(t, t_j)h_j(u_j). \end{aligned} \tag{14}$$

The existence of the second integral in (14) follows from the estimates

$$\begin{aligned} \int_t^{\infty} \|W_2(t,s)f(s, Tu(s))\| ds &\leq \int_t^{\infty} \|W_2(t,s)\| \|f(s, Tu(s))\| ds \\ &\leq \int_t^{\infty} \|W_2(t,s)\| \omega(s, \bar{c}) ds \leq \left(\int_t^{\infty} \|W_2(t,s)\|^a ds \right)^{1/a} \left(\int_t^{\infty} \omega^b(s, \bar{c}) ds \right)^{1/a} \leq m_1m_2 \end{aligned} \tag{15}$$

by using (V5), (12), (13) and the inequality of Hölder for integrals. The second sum in (14) exists because of the estimates

$$\begin{aligned} \sum_{j \in I_2(t)} \|W_2(t, t_j)h_j(u_j)\| &\leq \sum_{j \in I_2(t)} \|W_2(t, t_j)\| \|h_j(u_j)\| \\ &\leq \rho \sum_{j \in I_2(t)} \|W_2(t, t_j)\| l_j \leq \rho \left(\sum_{j \in I_2(t)} \|W_2(t, t_j)\|^a \right)^{1/a} \left(\sum_{j \in I_2(t)} l_j^a \right)^{1/a} \leq \rho m_1M \end{aligned} \tag{16}$$

in consideration of (V5), (V10), (12) and of the inequality of Hölder for sums. Analogously

to (15) and (16) the inequalities

$$\int_0^t \|W_1(t,s)f(s, Tu(s))\| ds \leq m_1 m_2, \quad (15')$$

$$\sum_{j \in I_1(t)} \|W_1(t, t_j) h_j(u_j)\| \leq \rho m_1 M \quad (16')$$

are obtained.

2) Now we show by the Theorem of Schauder-Tichonov (see, e.g., [9, p.627]) that the operator Q from (14) possesses a fixed point in D_ρ . For this purpose we prove the following properties of Q :

- Q maps D_ρ in itself.
- Q is continuous on D_ρ (with respect to Ω).
- QD_ρ is relatively compact (with respect to Ω).

To a): Let u be an arbitrary, fixed element of D_ρ . Then we obtain from (14) together with the relations $\|x(t)\| \leq \bar{\rho}$, $\rho = 2\bar{\rho}$, (10), (15), (15'), (16) and (16') the estimates

$$\begin{aligned} \|(Qu)(t)\| &\leq \|x(t)\| + \int_0^t \|W_1(t,s)f(s, Tu(s))\| ds + \int_1^{\infty} \|W_2(t,s)f(s, Tu(s))\| ds \\ &\quad + \sum_{j \in I_1(t)} \|W_1(t, t_j) h_j(u_j)\| + \sum_{j \in I_2(t)} \|W_2(t, t_j) h_j(u_j)\| \\ &\leq \bar{\rho} + 2m_1 m_2 + 2\rho m_1 M = \bar{\rho}(1 + 4m_1 M) + 2m_1 m_2 \leq 2\bar{\rho} = \rho. \end{aligned}$$

It is easy to see that Qu belongs also to $S(\mathbb{R}_+, X)$. Hence Qu is again in D_ρ .

To b): We consider a sequence (u_n) of elements u_n from D_ρ and an element u from D_ρ with the property $\sup_{t \in M} \|u_n(t) - u(t)\| \rightarrow 0$ ($n \rightarrow \infty$) for each compact interval $M \subset \mathbb{R}_+$. At first we win by means of the inequality of Hölder and of (V5)

$$\begin{aligned} &\int_0^t \|W_1(t,s)(f(s, Tu_n(s)) - f(s, Tu(s)))\| ds \\ &\leq \int_0^t \|W_1(t,s)\| \|f(s, Tu_n(s)) - f(s, Tu(s))\| ds \\ &\leq \left(\int_0^t \|W_1(t,s)\|^a ds \right)^{1/a} \left(\int_0^t \|f(s, Tu_n(s)) - f(s, Tu(s))\|^a ds \right)^{1/a} \\ &\leq m_1 \left(\int_0^{\infty} \|f(s, Tu_n(s)) - f(s, Tu(s))\|^a ds \right)^{1/a} \end{aligned}$$

and

$$\int_1^{\infty} \|W_2(t,s)(f(s, Tu_n(s)) - f(s, Tu(s)))\| ds \leq m_1 \left(\int_0^{\infty} \|f(s, Tu_n(s)) - f(s, Tu(s))\|^a ds \right)^{1/a},$$

respectively, as well as

$$\sum_{j \in J_1(t)} \|W_1(t, t_j)(h_j(u_{nj}) - h_j(u_j))\| \leq \sum_{j \in J_1(t)} \|W_1(t, t_j)\| \|h_j(u_{nj} - u_j)\|$$

$$\leq \left(\sum_{j \in J_1(t)} \|W_1(t, t_j)\|^{\bar{a}} \right)^{1/\bar{a}} \left(\sum_{j \in J_1(t)} \|h_j(u_{nj} - u_j)\|^{\bar{a}} \right)^{1/\bar{a}} \leq m_1 \left(\sum_{j=1}^{\bar{n}} \|h_j(u_{nj} - u_j)\|^{\bar{a}} \right)^{1/\bar{a}}$$

and

$$\sum_{j \in J_2(t)} \|W_2(t, t_j)(h_j(u_{nj}) - h_j(u_j))\| \leq m_j \left(\sum_{j=1}^{\bar{n}} \|h_j(u_{nj} - u_j)\|^{\bar{a}} \right)^{1/\bar{a}},$$

respectively, where $u_{nj} = u_n(t_j)$. Therefore we get with the aid of definition (14)

$$\|(Qu_n)(t) - (Qu)(t)\| \leq 2m_1 \left(\int_0^{\bar{n}} \|f(s, Tu_n(s)) - f(s, Tu(s))\|^{\bar{a}} ds \right)^{1/\bar{a}} + 2m_1 \left(\sum_{j=1}^{\bar{n}} \|h_j(u_{nj} - u_j)\|^{\bar{a}} \right)^{1/\bar{a}}$$

Let $\varepsilon > 0$ be arbitrarily given. Now we choose $t' > 0$ so great that

$$\int_{t'}^{\bar{n}} \omega^{\bar{a}}(s, \bar{c}) ds \leq \left(\frac{\varepsilon}{16m_1} \right)^{\bar{a}}, \quad \sum_{j \in J_2(t')} 1_j^{\bar{a}} \leq \left(\frac{\varepsilon}{16m_1 \rho} \right)^{\bar{a}}$$

holds. This leads to the estimate

$$\begin{aligned} & \|(Qu_n)(t) - (Qu)(t)\| \\ & \leq 2m_1 \left(\int_0^{t'} \|f(s, Tu_n(s)) - f(s, Tu(s))\|^{\bar{a}} ds + \int_{t'}^{\bar{n}} \|f(s, Tu_n(s)) - f(s, Tu(s))\|^{\bar{a}} ds \right)^{1/\bar{a}} \\ & + 2m_1 \left(\sum_{j \in J_1(t')} \|h_j(u_{nj} - u_j)\|^{\bar{a}} + \sum_{j \in J_2(t')} \|h_j(u_{nj} - u_j)\|^{\bar{a}} \right)^{1/\bar{a}} \\ & \leq 2m_1 \left(\int_0^{t'} \|f(s, Tu_n(s)) - f(s, Tu(s))\|^{\bar{a}} ds + \int_{t'}^{\bar{n}} (\|f(s, Tu_n(s))\| + \|f(s, Tu(s))\|)^{\bar{a}} ds \right)^{1/\bar{a}} \\ & + 2m_1 \left(\sum_{j \in J_1(t')} \|h_j(u_{nj} - u_j)\|^{\bar{a}} + \sum_{j \in J_2(t')} (\|h_j(u_{nj})\| + \|h_j(u_j)\|)^{\bar{a}} \right)^{1/\bar{a}}. \end{aligned}$$

Now there is a natural number $n_0 = n_0(t', \varepsilon)$ so that for all $n \geq n_0$ and $s \in [0, t']$ the inequalities

$$\|f(s, Tu_n(s)) - f(s, Tu(s))\| \leq \frac{\varepsilon}{8m_1(t')^{1/\bar{a}}}, \quad \|h_j(u_{nj} - u_j)\| \leq \frac{\varepsilon}{8m_1(i(t'))^{1/\bar{a}}}$$

are satisfied, where $i(t)$ denotes the numbers of points t_j in the interval $(0, t]$. Hence we have

for $n \geq n_0$ by use of (12)

$$\begin{aligned} \|(Qu_n)(t) - (Qu)(t)\| &\leq 2m_1 \left(\int_0^{t'} \left(\frac{\varepsilon}{8m_1(t')^{1/\bar{\alpha}}} \right)^{\bar{\alpha}} ds + \int_{t'}^{\infty} 2^{\bar{\alpha}} \omega^{\bar{\alpha}}(s, \bar{c}) ds \right)^{1/\bar{\alpha}} \\ &\quad + 2m_1 \left(\sum_{j \in J_1(t')} \left(\frac{\varepsilon}{8m_1(i(t'))^{1/\bar{\alpha}}} \right)^{\bar{\alpha}} + \sum_{j \in J_2(t')} (2\rho)^{\bar{\alpha}} 1_j^{\bar{\alpha}} \right)^{1/\bar{\alpha}} \\ &\leq 2 \cdot 2m_1 \left(\left(\frac{\varepsilon}{8m_1} \right)^{\bar{\alpha}} + \left(\frac{\varepsilon}{8m_1} \right)^{\bar{\alpha}} \right)^{1/\bar{\alpha}} = 2^{1/\bar{\alpha}-1} \varepsilon \leq \varepsilon. \end{aligned}$$

Thus we obtain for each compact interval $M \subset \mathbb{R}_+$ the relation

$$\sup_{t \in M} \|(Qu_n)(t) - (Qu)(t)\| \rightarrow 0 \quad (n \rightarrow \infty).$$

To c): We can verify immediately that the function $v = Qu$ fulfils the equation

$$v'(t) = A(t)v(t) + f(t, Tu(t)) \quad (17)$$

for $t \notin \{t_j\}$. Namely, under consideration of (4a), (6a), (14) and $W_1(t, t) + W_2(t, t) = I$ we find

$$\begin{aligned} v'(t) &= x'(t) + W_1(t, t)f(t, Tu(t)) + W_2(t, t)f(t, Tu(t)) \\ &\quad + A(t) \int_0^t W_1(t, s)f(s, Tu(s)) ds - A(t) \int_t^{\infty} W_2(t, s)f(s, Tu(s)) ds \\ &\quad + A(t) \sum_{j \in J_1(0)} W_1(t, t_j)h_j(u_j) - A(t) \sum_{j \in J_2(0)} W_2(t, t_j)h_j(u_j) \\ &= A(t)x(t) + f(t, Tu(t)) \\ &\quad + A(t) \int_0^t W_1(t, s)f(s, Tu(s)) ds - A(t) \int_t^{\infty} W_2(t, s)f(s, Tu(s)) ds \\ &\quad + A(t) \sum_{j \in J_1(0)} W_1(t, t_j)h_j(u_j) - A(t) \sum_{j \in J_2(0)} W_2(t, t_j)h_j(u_j) \\ &= A(t)(Qu)(t) + f(t, Tu(t)) = A(t)v(t) + f(t, Tu(t)). \end{aligned}$$

Therefore we have also

$$|v'(t)| \leq |A(t)||v(t)| + |f(t, Tu(t))|$$

for $t \notin \{t_j\}$. It follows from (V1) and (12) that the derivatives of the functions v in QD_ρ are uniformly bounded on each compact interval $M \subset \mathbb{R}_+$ so that QD_ρ contains equicontinuous

functions. Now we show that

$$R_t = \{v(t) : v \in QD_\rho\} = \{(Qu)(t) : u \in D_\rho\}$$

is a relatively compact subset of X for each fixed $t \in R_+$. At first we prove the relative compactness of the set

$$R^{(2)}(t) = \left\{ \int_t^\infty P_2 W^{-1}(s) f(s, Tu(s)) ds : u \in D_\rho \right\}.$$

(V6) and (12) imply the boundedness of $R^{(2)}(t)$. Observing the estimate $\|Tu(s)\| \leq \bar{c}$ in (12) the set $\{Tu(s) : u \in D_\rho, s \in R_+\}$ is contained in the central ball $B_{\bar{c}} \subset X$. Because of (V1) and (V4) there exists a sequence (f_m) of continuous, finite-dimensional functions f_m which converges on $R_+ \times B_{\bar{c}}$ uniformly to f (see, e.g., [9, p.626]). The sets

$$R_m^{(2)}(t) = \left\{ \int_t^\infty P_2 W^{-1}(s) f_m(s, Tu(s)) ds : u \in D_\rho \right\} \subset X$$

are finite-dimensional. For sufficiently great m they are also bounded and therefore relatively compact. The Theorem of Hausdorff (see, e.g., [9, p.19]) implies that $R^{(2)}(t)$ is then relatively compact, too. On the other hand this means that the set

$$W(t)R^{(2)}(t) = \left\{ \int_t^\infty W_2(t,s) f(s, Tu(s)) ds : u \in D_\rho \right\}$$

is relatively compact. Analogously we get the relative compactness of

$$W(t)R^{(1)}(t) = \left\{ \int_0^t W_1(t,s) f(s, Tu(s)) ds : u \in D_\rho \right\}.$$

Taking (V6) and (12) into account we find

$$\sum_{j \in I_2(t)} \|P_2 W^{-1}(t_j) h_j(u_j)\| \leq \rho \sum_{j \in I_2(t)} \|P_2 W^{-1}(t_j)\| < \infty.$$

Using (V4) we see that also the set

$$W(t) \left\{ \sum_{j \in I_2(t)} P_2 W^{-1}(t_j) h_j(u_j) : u \in D_\rho \right\} = \left\{ \sum_{j \in I_2(t)} W_2(t, t_j) h_j(u_j) : u \in D_\rho \right\}$$

is relatively compact. Analogous considerations show that $\{\sum_{j \in I_1(t)} W_1(t, t_j) h_j(u_j) : u \in D_\rho\}$ is a relatively compact set. By (14) then R_t is relatively compact, too. Finally Lemma 2 supplies the relative compactness of QD_ρ . Since the conditions a), b) and c) are verified, Q possesses a fixed point $y=y(t)$ in D_ρ .

3) Now we show that

$$y(t) = (Qy)(t) = x(t) + \int_0^t W_1(t,s) f(s, Ty(s)) ds - \int_t^\infty W_2(t,s) f(s, Ty(s)) ds + \sum_{j \in I_1(t)} W_1(t, t_j) h_j(y_j) - \sum_{j \in I_2(t)} W_2(t, t_j) h_j(y_j) \tag{18}$$

is a solution of the impulsive differential equation (5). Let be $t \notin \{t_j : j > 0\}$. For $u = y$ we have $v = Qu = Qy = y$. Thus y satisfies the differential equation (5a) because of (17).

Let be $t = t_n$ ($n > 0$ fixed). Then by (18) there follows in consideration of (6b) and $W_i(t, s) = W(t)P_iW^{-1}(s)$, $i \in \{1, 2\}$ on the one hand

$$\begin{aligned} y(t_n+0) &= x(t_n+0) + Q_n \int_0^{t_n} W_1(t_n, s) f(s, Ty(s)) ds \\ &\quad - Q_n \int_{t_n}^{\infty} W_2(t_n, s) f(s, Ty(s)) ds + Q_n \sum_{j \in J_1(t_n)} W_1(t_n, t_j) h_j(y_j) \\ &\quad - Q_n \sum_{j \in J_2'(t_n)} W_2(t_n, t_j) h_j(y_j) + Q_n W_1(t_n, t_n) h_n(y_n) \end{aligned}$$

with the definition $J_2'(t) = J_2(t) \setminus \{t\} = \{j > 0 : t < t_j\}$ and on the other hand

$$\begin{aligned} y(t_n) &= x(t_n) + \int_0^{t_n} W_1(t_n, s) f(s, Ty(s)) ds - \int_{t_n}^{\infty} W_2(t_n, s) f(s, Ty(s)) ds \\ &\quad + \sum_{j \in J_1(t_n)} W_1(t_n, t_j) h_j(y_j) - \sum_{j \in J_2'(t_n)} W_2(t_n, t_j) h_j(y_j) . \end{aligned}$$

This implies in view of $Q_n = I_n + I$ and (4b) the expression

$$\begin{aligned} y(t_n+0) - y(t_n) &= I_n(x_n) \\ &\quad + I_n \int_0^{t_n} W_1(t_n, s) f(s, Ty(s)) ds - I_n \int_{t_n}^{\infty} W_2(t_n, s) f(s, Ty(s)) ds \\ &\quad + I_n \sum_{j \in J_1(t_n)} W_1(t_n, t_j) h_j(y_j) - I_n \sum_{j \in J_2'(t_n)} W_2(t_n, t_j) h_j(y_j) \\ &\quad + Q_n W_1(t_n, t_n) h_n(y_n) + Q_n W_2(t_n, t_n) h_n(y_n) . \end{aligned}$$

Finally with the aid of

$$W_1(t_n, t_n) h_n(y_n) + W_2(t_n, t_n) h_n(y_n) = h_n(y_n) = (Q_n^{-1} H_n) y_n$$

we get the result

$$y(t_n+0) - y(t_n) = I_n y_n + Q_n (Q_n^{-1} H_n) y_n = (I_n + H_n) y_n .$$

Hence $y=y(t)$ fulfils also the jump condition (5b).

4) Let $y \in D_p$ be a bounded solution of (5). If x is separated in (18), then equation

$$\begin{aligned}
 x(t) = y(t) &- \int_0^t W_1(t,s)f(s,Ty(s)) ds + \int_t^{\infty} W_2(t,s)f(s,Ty(s)) ds \\
 &- \sum_{j \in I_1(t)} W_1(t,t_j)h_j(y_j) + \sum_{j \in I_2(t)} W_2(t,t_j)h_j(y_j)
 \end{aligned}
 \tag{18'}$$

arises. We can show similarly as in part 3 that the function x defined in this way is a bounded solution of (4).

5) Now we prove that the relation $\|x(t) - y(t)\| \in L_\rho(\mathbb{R}_+)$ follows by (18) and (18'), respectively. At first in view of (15), (15'), (16) and (16') the estimate

$$\begin{aligned}
 \|x(t) - y(t)\| &\leq \int_0^t \|W_1(t,s)f(s,Ty(s))\| ds + \int_t^{\infty} \|W_2(t,s)f(s,Ty(s))\| ds \\
 &\quad + \sum_{j \in I_1(t)} \|W_1(t,t_j)h_j(y_j)\| + \sum_{j \in I_2(t)} \|W_2(t,t_j)h_j(y_j)\| \\
 &\leq K_1 + K_2 + S_1 + S_2
 \end{aligned}
 \tag{19}$$

holds with

$$\begin{aligned}
 K_1 = K_1(t) &= \int_0^t \|W_1(t,s)\| \omega(s, \bar{c}) ds, \quad K_2 = K_2(t) = \int_t^{\infty} \|W_2(t,s)\| \omega(s, \bar{c}) ds, \\
 S_1 = S_1(t) &= \rho \sum_{j \in I_1(t)} \|W_1(t,t_j)\|_j, \quad S_2 = S_2(t) = \rho \sum_{j \in I_2(t)} \|W_2(t,t_j)\|_j,
 \end{aligned}
 \tag{20}$$

where $\bar{c} = \rho \sup \{k(t): \|u\| \leq \rho\}$ as in (11). Next we define numbers $\alpha, \beta, \gamma > 0$ according to

$$\alpha = p, \quad \frac{1}{\beta} = \frac{1}{a} - \frac{b}{ap}, \quad \frac{1}{\gamma} = \frac{1}{a} - \frac{1}{p}$$

and choose $p = b/(2-a) > 1$. Then we have $1/\alpha + 1/\beta + 1/\gamma = 1$. Further we put $\beta' = ap/\beta, \gamma' = ap/\gamma$. Now the (generalized) inequality of Hölder supplies in consideration of (V5), (V8) and (V9) with $m_3 = m_3(\bar{c})$ the estimate

$$\begin{aligned}
 K_1^p &= \left(\int_0^t \|W_1(t,s)\|^{bp} \omega^{ap}(s, \bar{c}) \|W_1(t,s)\|^{a/\beta} \omega^{a/\gamma}(s, \bar{c}) ds \right)^p \\
 &\leq \left(\int_0^t \|W_1(t,s)\|^{ba/p} \omega^{aa/p}(s, \bar{c}) ds \right)^{p/\alpha} \left(\int_0^t \|W_1(t,s)\|^a ds \right)^{p/\beta} \left(\int_0^t \omega^a(s, \bar{c}) ds \right)^{p/\gamma} \\
 &\leq m_1^{\beta'} m_2^{\gamma'} \int_0^t \|W_1(t,s)\|^b \omega^a(s, \bar{c}) ds \leq m_1^{\beta'} m_2^{\gamma'} m_3^a \int_0^t \|W_1(t,s)\|^b ds.
 \end{aligned}
 \tag{21}$$

By (V5) the function K_1^p belongs to $L_1(\mathbb{R}_+)$. Therefore we have also $K_1 \in L_p(\mathbb{R}_+)$. Analogously we can show

$$K_2^p \leq m_1^{\beta'} m_2^{\gamma'} m_3^{\alpha} \int_0^{\infty} \|W_2(t,s)\|^p ds \quad (22)$$

and $K_2 \in L_p(\mathbb{R}_+)$. Using (V5), (V10) and (11) we obtain beyond it

$$\begin{aligned} S_1^p &= \rho^p \left(\sum_{j \in J_1(t)} \|W_1(t,t_j)\|^{b\beta} l_j^{a\beta} \|W_1(t,t_j)\|^{a\beta} l_j^{a\gamma} \right)^p \\ &\leq \rho^p \left(\sum_{j \in J_1(t)} \|W_1(t,t_j)\|^{b\alpha/p} l_j^{a\alpha/p} \right)^{p/\alpha} \left(\sum_{j \in J_1(t)} \|W_1(t,t_j)\|^a \right)^{p/\beta} \left(\sum_{j \in J_1(t)} l_j^a \right)^{p/\gamma} \\ &\leq \rho^p m_1^{\beta'} M^{\gamma'} \sum_{j \in J_1(t)} \|W_1(t,t_j)\|^b l_j^a \leq \rho^p m_1^{\beta'} M^{\gamma'} l^a \sum_{j \in J_1(t)} \|W_1(t,t_j)\|^b. \end{aligned} \quad (23)$$

Because of (V5) the function S_1^p lies in $L_1(\mathbb{R}_+)$. Thus $S_1 \in L_p(\mathbb{R}_+)$. Similarly we find

$$S_2^p \leq \rho^p m_1^{\beta'} M^{\gamma'} l^a \sum_{j \in J_2(t)} \|W_2(t,t_j)\|^b \quad (24)$$

and $S_2 \in L_p(\mathbb{R}_+)$. Finally $\|x(t) - y(t)\| \in L_p(\mathbb{R}_+)$ arises in connection with (19) ■

Remark: If X is finite-dimensional or f does not depend on x , the assertion of Theorem 1 remains true without assumption (V4).

Using more special natural conditions we can also show the property of integral equivalence.

Theorem 2: Let the space X be finite-dimensional. Besides let be fulfilled the following assumptions for the impulsive equations (4) and (5):

I. There holds:

(B1) A is bounded on each compact interval of \mathbb{R}_+ and integral-bounded with a constant α , i.e. $\int_0^t \|A(s)\| ds \leq \alpha t$ ($t \geq 0$). Moreover the commutativity relations $A(s)A(t) = A(t)A(s)$, $Q_j A(t) = A(t)Q_j$ are satisfied for all $s, t \geq 0$ and all $j \geq 1$.

(B2) The operators Q_j ($j \geq 1$) are continuously invertible. The sequence (Q_j^{-1}) is norm-bounded.

(B3) T is continuous, f is continuous on $\mathbb{R}_+ \times X$.

II. Further, for each closed bounded central ball $B \subset X$ there holds:

(B4) For all $\hat{u} \in X$, $\|T\hat{u}\| \leq k(\hat{u})\|\hat{u}\|$ with a functional k bounded on B .

(B5) The sets $f(\mathbb{R}_+ \times B)$ and $\bigcup_{j=1}^{\infty} Q_j^{-1} H_j(B)$ are bounded in X .

III. Finally, for a projector $P_1 \in L(X)$, for a monotone non-decreasing and bounded function $\omega^*: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and for suitable numbers $a \in (1, 2)$, $L > 0$, $K > 0$, $\bar{Q} > 0$,

$q > \alpha, l > 0, \lambda > 0, \delta > 0$ and $\gamma \in \mathbb{N}$ there holds with the notations (8), $\bar{a} = a/(a-1)$, $Q = \sup_j \|Q_j^{-1}\|$ and $d(t) = \max \{j : t_j < t\}$:

(B6) Each interval $I \subset \mathbb{R}_+$ of the length l contains at most γ members of the sequence (t_j) .

(B7) $\|W_1(t,s)\| \leq Ke^{-\delta(t-s)} \quad (0 \leq s \leq t)$, $\|W_2(t,s)\| \leq Ke^{-\delta(s-t)} \quad (0 \leq t \leq s)$.

(B8) $\|\prod_{j=d(t)}^1 Q_j^{-1}\| \leq \bar{Q}e^{-qt} \quad (t \geq 0)$.

(B9) $\|f(t, Tu)\| \leq Le^{-\lambda t} \omega^*(\|Tu\|)$ for almost all $t \in \mathbb{R}_+$ and all $u \in X$.

(B10) $\sum_{j=1}^{\infty} \|H_j\|^{\bar{a}} < \left(\frac{1}{4\bar{Q}}\right)^{\bar{a}} \left(2\frac{K}{a\delta} + 2K\frac{\gamma}{1-e^{-\delta al}}\right)^{1/(1-a)}$

Then (4) and (5) are p -integral equivalent with $p \in (1/(2-a), a/(2-a))$.

We omit the proof here. It can be shown that with the exception of the second part of (V5) all assumptions of Theorem 1 are fulfilled. But this part is not necessary, because the relations $K_1, K_2, S_1, S_2 \in L_p(\mathbb{R}_+)$ are in this case essentially a consequence of (B6) and (B7) (see [6]). Impulsive equations of type (4) satisfying the condition (B7) with positive constants K and δ are said to be *exponentially dichotomous*. Examples for such equations are contained in [4].

Example: Let be $X = \mathbb{R}^2$ and $t_j = jh$ ($h \in [0,1]$). We define

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}.$$

where $a_{ij}(t) \in [0, \alpha]$ for $i, j \in \{1,2\}$ and all $t \geq 0$. Further, we assume that the functions a_{ij} fulfil the following properties:

(P1) $a_{11}(t) \geq a_{12}(t), a_{22}(t) \geq a_{21}(t)$ for all $t \geq 0$.

(P2) $a_{12}(t)a_{21}(s) = a_{12}(s)a_{21}(t)$

$$a_{12}(t)a_{11}(s) + a_{12}(s)a_{22}(t) = a_{11}(t)a_{12}(s) + a_{12}(t)a_{22}(s) \quad \text{for all } t, s \geq 0.$$

$$a_{11}(t)a_{12}(s) + a_{21}(t)a_{22}(s) = a_{21}(t)a_{11}(s) + a_{22}(t)a_{21}(s)$$

(P3) There is $t^* > 0$ so that $e^{2\alpha} \max\{a_{11}(t^*), a_{12}(t^*)\} \leq a_{11}(t^*)a_{22}(t^*) - a_{12}(t^*)a_{21}(t^*)$

holds and the spectrum of the matrix

$$Q = \begin{pmatrix} a_{11}(t^*) & a_{12}(t^*) \\ a_{21}(t^*) & a_{22}(t^*) \end{pmatrix}$$

is contained in the left half-plane.

Finally we choose for $x = (x_1, x_2) \in \mathbb{R}^2$, $t \in \mathbb{R}_+$

$$f(t, x) = e^{-2t} \begin{pmatrix} x_1^2 |\sin(x_1 x_2)| + x_2^2 |\cos(x_1 + x_2)| \\ x_1^2 + x_2^2 \end{pmatrix}, \quad T(x) = \begin{pmatrix} |x| a_1(x) \\ |x| a_2(x) \end{pmatrix},$$

where a_i are continuous functions ($i = 1, 2$), $Q_j = I_j + I = Q$ and $H_j = 0$ ($j = 1, 2, 3, \dots$). Then simple calculations show that the conditions (B1) - (B10) of Theorem 2 are satisfied for all numbers $a \in (1, 2)$. Hence (4) and (5) are p -integral equivalent in this case for every $p \geq 1$.

4. Asymptotic integral equivalence

Supposing additional conditions we can ensure the asymptotic equivalence of the equations (4) and (5).

Theorem 3: *Let the assumptions of Theorem 1 be satisfied with $a=b$. Further we assume the following conditions:*

$$(Z1) \quad \lim_{t \rightarrow \infty} \omega(t, c) = 0 \text{ for each constant } c \geq 0.$$

$$(Z2) \quad \lim_{j \rightarrow \infty} I_j = 0.$$

$$(Z3) \quad \lim_{t \rightarrow \infty} \int_0^t \|W_1(t, s)\|^a ds = \lim_{t \rightarrow \infty} \sum_{j \in J_1(t')} \|W_1(t, t_j)\|^a = 0 \text{ for arbitrarily great } r, r' > 0.$$

$$(Z4) \quad \lim_{t \rightarrow \infty} \int_t^\infty \|W_2(t, s)\|^a ds = \lim_{t \rightarrow \infty} \sum_{j \in J_2(t)} \|W_2(t, t_j)\|^a = 0.$$

Then the impulsive differential equations (4) and (5) are p -asymptotic equivalent with $p = a/(2-a)$.

Proof: Observing Definition 3 and relation (19) it is sufficient to show that the limits

$$\lim_{t \rightarrow \infty} K_1(t) = \lim_{t \rightarrow \infty} K_2(t) = \lim_{t \rightarrow \infty} S_1(t) = \lim_{t \rightarrow \infty} S_2(t) = 0$$

hold for the functions K_1 , K_2 , S_1 and S_2 defined in (20). Let $\varepsilon > 0$ be arbitrarily given. Because of the assumptions (Z1), (Z2), (Z4) there is a number $T > 0$ such that for $t \geq T$, $t_j \geq T$ the inequalities

$$\omega^a(t, \bar{c}) \leq \frac{\epsilon^p}{2 \cdot 4^p m_1^{\beta'} m_2^{\gamma'} m_1^a}, \quad l_j^a \leq \frac{\epsilon^p}{2 \cdot 4^p \rho^p m_1^{\beta'} M^{\gamma'} m_1^a}$$

$$\int_t^{\infty} \|W_2(t, s)\|^a ds \leq \frac{\epsilon^p}{4^p m_1^{\beta'} m_2^{\gamma'} m_3^a}, \quad \sum_{j \in J_2(t)} \|W_2(t, t_j)\|^a \leq \frac{\epsilon^p}{4^p \rho^p m_1^{\beta'} M^{\gamma'} 1^a}$$

hold with the notations used in Theorem 1. Now we choose numbers $r, r' \geq T$, for which the limit relations in (Z3) are fulfilled. Then a number $T' \geq \max(r, r')$ exists so that for $t \geq T'$ the inequalities

$$\int_0^r \|W_1(t, s)\|^a ds \leq \frac{\epsilon^p}{2 \cdot 4^p m_1^{\beta'} m_2^{\gamma'} m_3^a}, \quad \sum_{j \in J_1(r')} \|W_1(t, t_j)\|^a \leq \frac{\epsilon^p}{2 \cdot 4^p \rho^p m_1^{\beta'} M^{\gamma'} 1^a}$$

are realized. Now let be $t \geq T'$. By (21) we get with $a = b$ the estimate

$$\begin{aligned} K_1^p &\leq m_1^{\beta'} m_2^{\gamma'} \left(\int_0^r \|W_1(t, s)\|^a \omega^a(s, \bar{c}) ds + \int_t^t \|W_1(t, s)\|^a \omega^a(s, \bar{c}) ds \right) \\ &\leq m_1^{\beta'} m_2^{\gamma'} \left(\frac{\epsilon^p m_3^a}{2 \cdot 4^p m_1^{\beta'} m_2^{\gamma'} m_3^a} + \frac{\epsilon^p m_1^a}{2 \cdot 4^p m_1^{\beta'} m_2^{\gamma'} m_1^a} \right) = \frac{\epsilon^p}{4^p} \end{aligned}$$

where the assumptions (V5) and (V9) are used. Consequently it holds $K_1 \leq \epsilon/4$. Further, by (22) we win with $a = b$ the estimate

$$K_2^p \leq m_1^{\beta'} m_2^{\gamma'} m_3^a \int_t^{\infty} \|W_2(t, s)\|^a ds \leq m_1^{\beta'} m_2^{\gamma'} m_3^a \frac{\epsilon^p}{4^p m_1^{\beta'} m_2^{\gamma'} m_3^a} = \frac{\epsilon^p}{4^p}$$

In view of (23), $a = b$, (V5) and (V10) the inequalities

$$\begin{aligned} S_1^p &\leq \rho^p m_1^{\beta'} M^{\gamma'} \left(\sum_{j \in J_1(r')} \|W_1(t, t_j)\|^a l_j^a + \sum_{j \in J_1^-(t)} \|W_1(t, t_j)\|^a l_j^a \right) \\ &\leq \rho^p m_1^{\beta'} M^{\gamma'} \left(\frac{\epsilon^p 1^a}{2 \cdot 4^p \rho^p m_1^{\beta'} M^{\gamma'} 1^a} + \frac{\epsilon^p m_1^a}{2 \cdot 4^p \rho^p m_1^{\beta'} M^{\gamma'} m_1^a} \right) = \frac{\epsilon^p}{4^p} \end{aligned}$$

follow, where $J_1^-(t) = J_1(t) - J_1(r') = \{j > 0 : r' \leq t_j < t\}$. Finally we deduce from (24) and $a = b$ the estimate

$$S_2^p \leq \rho^p m_1^{\beta'} M^{\gamma'} 1^a \sum_{j \in J_2(t)} \|W_2(t, t_j)\|^a \leq \rho^p m_1^{\beta'} M^{\gamma'} 1^a \frac{\epsilon^p}{4^p \rho^p m_1^{\beta'} M^{\gamma'} 1^a} = \frac{\epsilon^p}{4^p}$$

But for $t \geq T'$ this means $\|x(t) - y(t)\| \leq \epsilon$ ■

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