Zeitschrift für Analysis
und ihre Anwendungen
Vol. 12 (1993), 457 - 469

Schur Functions, Schur Parameters and Orthogonal Polynomials on the Unit Circle

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Properties of Schur functions on the unit circle and asymptotic behaviour of corresponding Schur parameters are investigated. Connection between the Schur parameters and the reflection coefficients of a certain system of orthogonal polynomials on the unit circle is used.

Key words: *Schur functions, orthogonal polynomials, reflection coefficients, modulus of continuity*
AMS subject classification: 30D50, 42C05

1. Introduction

In his celebrated paper [13] I. Schur investigated a class of functions which are now generally known as Schur functions. A *Schur function* (*S-function*) $f = f(z)$ is an analytic function in the open unit disk $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ with modulus not exceeding unity. Schur used a "continued fraction-like" algorithm of consecutive linear fractional transformations of the kind

$$f_{n+1}(z) = \frac{f_n(z) - f_n(0)}{z(1 - \overline{f_n(0)}f_n(z))} \quad (n \in \mathbb{N}_0 := \{0,1,2,...\};\ f_0 := f).$$

This algorithm leads to the infinite sequence of S-functions $\{f_n\}_{n=0}^{\infty}$ unless f_0 is a Blaschke product. The latter will be excluded throughout the present paper. The *Schur parameters* (*S-parameters*) $\gamma_n = f_n(0)$ satisfy $|\gamma_n| < 1$ and arise in various problems of complex analysis and its applications (see [3, 4]).

It is a remarkable fact that for any sequence $\{\gamma_n\}$ with $|\gamma_n| < 1$ there exists a unique Schur function f with S-parameters γ_n. Thereby the problem of describing the relations between the Schur parameters and corresponding Schur functions arises naturally. Investigation of this problem is the main goal of our paper. We show that a certain asymptotic behaviour of the Schur parameters provides specific smoothness properties of the corresponding Schur function in $\mathbb{D}$ and vice versa.

The first substantial contribution to the problem was made by Ya. L. Geronimus [5, 7]. He discovered that S-parameters were exactly those occuring in the recurrence relations for the orthogonal polynomials on the unit circle. This fact allowed him to obtain some results on the

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ISSN 0232-2064 / $ 2.50

problem in question. In Section 2 of this paper we give a slighly modified proof of the main Geronimus' theorem. We would also like to mention the paper [2] where the extreme points of the set of all Schur functions were characterized in terms of their Schur parameters. In Section 3 examples of some simple transforms of S-parameters and corresponding S- and C-functions as well as some particular examples are given. Recently S.V. Khrushchev proved that if $f(e^{i\vartheta}) \in \mathrm{Lip}\,\alpha$, then $\gamma_n = O(n^{-\alpha}\ln n)$ as $n \to \infty$. In Section 4 this result will be improved and generalized (see Theorems 2 and 3 below). At last in Section 5 we prove converse theorems for Schur parameters.

We shall use the following generally accepted notations:

S - the set of Schur functions f which are analytic in $\mathbb{D}$ and $\|f\| = \sup_{z\in D}|f(z)| \le 1$.

C - the set of Carathéodori (C-) functions F which are analytic in $\mathbb{D}$ and $\mathrm{Re}\,F(z) \ge 0$.

$C_{2\pi}$ - the set of 2π-periodic and continuous functions p on $\mathbb{R}$ with $\|p\|_{2\pi} = \sup_{|\vartheta|\le\pi}|p(\vartheta)|$.

$C_{2\pi}^{(m)}$ - the set of functions $p \in C_{2\pi}$ which are differentiable m times and $p^{(m)} \in C_{2\pi}$.

$E_n^*(h)$ - the best approximation to a function $h \in C_{2\pi}$ by the class T_n of trigonometric polynomials of degree at most n: $E_n^*(h) = \inf_{t_n\in T_n}\|h(\vartheta) - t_n(\vartheta)\|_{2\pi}$.

$\omega(t,f)$ - the modulus of continuity of the function $f \in C_{2\pi}$: $\omega(t,f) = \sup_{0<\tau\le t}\|\Delta_\tau f\|_{2\pi}$, $\Delta_\tau f(\vartheta) = f(\vartheta+\tau) - f(\vartheta)$; $f \in \mathrm{Lip}\,\alpha$ $(0 < \alpha \le 1)$ if $\omega(t,f) = O(t^\alpha)$ as $t \to 0$.

W - the set of absolutely convergent Fourier series.

$C(f)$ - positive constants depending on a function f.

$\tilde{f}$ - the conjugate function (cf. [17: Chapter 7, §1]).

2. Orthogonal polynomials on the unit circle $\mathbb{D}$

Our reasoning is based on the theory of orthogonal polynomials on the unit circle $\mathbb{D}$ (see [16: Chapters X - XI] and [6: Chapter 8]).

Let $d\sigma$ be a finite positive Borel measure on the interval $[0,2\pi)$ with an infinite set as its support, such that $\sigma([0,2\pi)) = 2\pi$. Let $\{\varphi_n\}$, $\varphi_n(z) = x_n z^n + \ldots$ with $x_n > 0$, be the unique system of orthogonal polynomials on $\mathbb{D}$, associated with this measure, i.e. (δ_{nm} - the Kronecker symbol)

$$\frac{1}{2\pi}\int_0^{2\pi}\varphi_n(e^{i\vartheta})\overline{\varphi_m(e^{i\vartheta})}\,d\sigma(\vartheta) = \delta_{nm}.$$

The monic orthogonal polynomials Φ_n and the reverse polynomials Φ_n^* are defined by

$$\Phi_n(z) = x_n^{-1}\varphi_n(z) = z^n + \ldots \quad \text{and} \quad \Phi_n^*(z) = z^n\overline{\Phi_n}(z^{-1}).$$

In the theory of orthogonal polynomials on the unit circle an essential role is played by the dual pairs of recurrence formulas

$$\Phi_{n+1}(z) = z\,\Phi_n(z) - \overline{a}_n\Phi_n^*(z) \quad (n \in \mathbb{N}) \tag{2.1}$$

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - z a_n\Phi_n(z) \quad (n \in \mathbb{N}) \tag{2.2}$$

(cf. [6: Chapter 8, formulas (8.1)]). Here $a_n = -\overline{\Phi_{n+1}(0)}$ are the parameters of the orthogonal polynomials Φ_n (*OP-parameters*; the values $-\overline{a}_n$ are generally called the *reflection coefficients*) and $|a_n| < 1$. The most remarkable fact is that for any given sequence $\{a_n\} \subset \mathbb{C}$ under the only restriction $|a_n| < 1$ there exists a unique measure $(2\pi)^{-1}d\sigma$ of unit total mass such that for the associated system of orthogonal polynomials $\{\Phi_n\}$ the equality $a_n = -\overline{\Phi_{n+1}(0)}$ holds. This result is usually referred to as *J. Favard's theorem for the unit circle*.

The polynomials $\psi_n(z) = z^n + \ldots$ and their reverse $\psi_n^*(z)$ are defined as solutions of the recurrence relations (2.1) - (2.1) with a_n, replaced by $-a_n$ and are called *polynomials of the second kind*. The polynomials ψ_n are orthogonal with respect to a certain measure $d\tilde{\sigma}$ and connected with Φ_n by the equation

$$\Phi_n^*(z)\psi_n(z) + \psi_n^*(z)\Phi_n(z) = 2h_n z^n \quad \text{where } h_n = \|\Phi_n\|_\sigma^2 = \frac{1}{2\pi}\int_0^{2\pi}|\Phi_n(e^{i\vartheta})|^2 d\sigma(\vartheta) = \varkappa_n^{-2}.$$

Hence, in particular, we have $\mathrm{Re}(\psi_n^*(e^{i\vartheta})/\Phi_n^*(e^{i\vartheta})) = h_n/|\Phi_n(e^{i\vartheta})|^2 > 0$ so that $F_n = \psi_n^*/\Phi_n^* \in C$. It is well-known (cf. [6: Chapter 8, formula (8.10)]) that

$$F(z) := \lim_{n\to\infty}\frac{\Psi_n^*(z)}{\Phi_n^*(z)} = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\vartheta}+z}{e^{i\vartheta}-z}d\sigma(\vartheta) \tag{2.3}$$

uniformly on the compact subsets of $\mathbb{D}$. Thus the measure $d\sigma$ can be recovered from the polynomials ψ_n^*, Φ_n^* (and hence from the OP-parameters a_n) by the inversion formula

$$\frac{\sigma(t+0)+\sigma(t-0)}{2} = \text{const} + \lim_{r\to 1-0}\int_0^t \mathrm{Re}\,F(re^{i\vartheta})\,d\vartheta.$$

Ya. L. Geronimus was the first who discovered a tight connection between the Schur functions and orthogonal polynomials on the unit circle. Let f be an S-function with S-parameters γ_n and $F(z) = (1+zf(z))(1-zf(z))^{-1}$. It is obvious that

$$F(0) = 1 \qquad \text{and} \qquad \mathrm{Re}F(z) = (1-|zf(z)|^2)|1-zf(z)|^{-2} > 0 \quad (|z|<1),$$

that is $F \in C$. According to Riesz-Herglots theorem [17: Chapter 4, Theorem 6.26],

$$F(z) = \frac{1+zf(z)}{1-zf(z)} = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\vartheta}+z}{e^{i\vartheta}-z}d\sigma(\vartheta) \tag{2.4}$$

where the support $d\sigma$ is infinite unless f is a finite Blaschke product. Let Φ_n be the orthogonal polynomials with respect to $d\sigma$ and $a_n = -\overline{\Phi_{n+1}(0)}$. The following Ya. L. Geronimus' theorem plays a crucial role in the whole subject.

Theorem (Geronimus [5: Theorem IX, 2°] and [7: Theorem 18.2]): *The equality $a_n = \gamma_n$ is true for all $n \in \mathbb{N}_0$.*

Proof: We start out from the formula for the polynomials of the second kind (cf. [6: Chapter 1, formula (1.13)]):

$$\psi_n(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\vartheta}+z}{e^{i\vartheta}-z}(\Phi_n(e^{i\vartheta}) - \Phi_n(z))d\sigma(\vartheta). \quad (n \in \mathbb{N})$$

or, in other words,

$$Q_n(z) := F(z)\Phi_n(z) + \psi_n(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\vartheta}+z}{e^{i\vartheta}-z}\,\Phi_n(e^{i\vartheta})d\sigma(\vartheta).$$

Applying to the both sides of this equality the "$*$-transform" we get

$$Q_n^*(z) = z^n\overline{Q}_n(z^{-1}) = F(z)\Phi_n^*(z) - \psi_n^*(z) = \frac{z^n}{2\pi}\int_0^{2\pi}\frac{e^{i\vartheta}+z}{e^{i\vartheta}-z}\,\overline{\Phi_n(e^{i\vartheta})}d\sigma(\vartheta).$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi}\Phi_n(e^{i\vartheta})e^{-ik\vartheta}d\sigma(\vartheta) = \begin{cases} 0 & \text{for } k = 0,1,\dots,n-1 \\ h_n & \text{for } k = n \end{cases}$$

we can find the Taylor coefficients of the function Q_n:

$$Q_n(z) = \frac{1}{2\pi}\int_0^{2\pi}\Phi_n(e^{i\vartheta})\left\{1 + 2\sum_{k=1}^{\infty}\left(\frac{z}{e^{i\vartheta}}\right)^k\right\}d\sigma(\vartheta) = 2h_n z^n + O(z^{n+1}), \quad |z| < 1. \tag{2.5}$$

Next

$$Q_n^*(z) = \frac{z^n}{2\pi}\int_0^{2\pi}\overline{\Phi_n(e^{i\vartheta})}\left\{1 + \frac{2z}{e^{i\vartheta}} + O(z^2)\right\}d\sigma(\vartheta) = \frac{2z^{n+1}}{2\pi}\int_0^{2\pi}\overline{\Phi_n(e^{i\vartheta})}e^{-i\vartheta}d\sigma(\vartheta) + O(z^{n+2}).$$

It directly follows from the formula (2.1) for $z = e^{i\vartheta}$ that $\frac{1}{2\pi}\int_0^{2\pi}e^{i\vartheta}\Phi_n(e^{i\vartheta})d\sigma(\vartheta) = \bar{a}_n h_n$ and therefore

$$Q_n^*(z) = 2a_n h_n z^{n+1} + O(z^{n+2}). \tag{2.6}$$

Consider the functions $\chi_n(z) = Q_n^*(z)/zQ_n(z)$ ($n \in \mathbb{N}_0$). It is obvious that

$$\chi_0(z) = \frac{1}{z}\,\frac{F(z)-1}{F(z)+1} = f(z).$$

From the recurrence formulas for Φ_n and ψ_n we further have

$$\chi_{n+1}(z) = \frac{Q_{n+1}^*(z)}{zQ_{n+1}(z)} = \frac{Q_n^*(z) - a_n zQ_n(z)}{z(zQ_n(z) - \overline{a}_n Q_n^*(z))} = \frac{1}{z}\,\frac{\chi_n(z) - \chi_n(0)}{1 - \overline{\chi_n(0)}\chi_n(z)} = f_{n+1}(z).$$

Taking into account the relations (2.5) and (2.6) we obtain $a_{n+1} = \gamma_{n+1}$. Hence the assertion of the theorem is verified ∎

We deduce the following result due to D. W. Boyd from the theorem just proved.

Theorem (Boyd [2: Lemma/p. 146]): *Let f be an S-function with S-parameters γ_n. Then*

$$\prod_{k=0}^{\infty}(1 - |\gamma_k|^2) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\ln(1 - |f(e^{i\vartheta})|^2)\,d\vartheta.\right. \tag{2.7}$$

Proof: Let F be the C-function given by (2.4) and let a_n be the OP-parameters associated with the measure $d\sigma$. We start from the formula (cf. [6: Chapter 8, formula (8.14)])

$$\prod_{k=0}^{\infty}(1-|a_k|^2) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\ln\sigma'(\vartheta)\,d\vartheta\right\} \tag{2.8}$$

(if $\ln\sigma' \notin L^1$, then both sides in this formula are zero). Since $\sigma'(\vartheta) = \operatorname{Re}F(e^{i\vartheta}) = (1-|f(e^{i\vartheta})|^2) \times |1-e^{i\vartheta}f(e^{i\vartheta})|^{-2}$ holds a.e., we have

$$\int_0^{2\pi}\ln\sigma'(\vartheta)\,d\vartheta = \int_0^{2\pi}\ln\frac{1-|f(e^{i\vartheta})|^2}{|1-e^{i\vartheta}f(e^{i\vartheta})|^2}\,d\vartheta.$$

But the function $h(z) = 1 - zf(z)$ is outer, so that $\int_0^{2\pi}\ln|1-e^{i\vartheta}f(e^{i\vartheta})|^2 d\vartheta = 0$ (cf. [11: Chapter 5, first theorem]). The relation (2.7) now immediately follows from (2.8) and Geronimus' theorem ■

G. Szegö developed an important theory for orthogonal polynomials on the unit circle in the case when $d\sigma$ belongs to the Szegö class, i.e. $\ln\sigma' \in L^1[0,2\pi)$. Here $p = \sigma'$ is well-defined a.e. and integrable in $[0,2\pi)$. Ya. L. Geronimus (cf. [6: Chapter 8, Theorem 8.2]) proved that the inclusion $\ln p \in L^1[0,2\pi)$ is equivalent to the condition $\sum_{k=0}^{\infty}|a_k|^2 < \infty$ and $\lim_{n\to\infty}\varkappa_n^{-2} = \varkappa^{-2} = \prod_{k=0}^{\infty}(1-|a_k|^2) > 0$ holds. Under the condition $\ln p \in L^1[0,2\pi)$ the principal tool is the Szegö function $D(d\sigma, z) = D(z)$ which is defined by

$$D(z) = \exp\left\{\frac{1}{4\pi}\int_0^{2\pi}\frac{e^{i\vartheta}+z}{e^{i\vartheta}-z}\ln p(\vartheta)\,d\vartheta\right\}.$$

It is well-known (cf. [6: Chapter 2, formula (2.4)]) that

1. $D \in H^2$; the non-tangential boundary value of D exists a.e. on $\mathbb{D}$ and $|D(e^{i\vartheta})|^2 = p(\vartheta)$ a.e.

2. If $p(\vartheta) \ge \mu > 0$ a.e., then $|D(z)| \ge \mu^{1/2}$ in $\mathbb{D}$.

It is more convenient for us to deal with the function $\pi = D^{-1}$. Then we have, uniformly on compact subsets of $\mathbb{D}$, $\pi(z) = \lim_{n\to\infty}\varphi_n^*(z) = \lim_{n\to\infty}\varkappa_n\Phi_n^*(z)$. Under certain additional assumptions on the measure this convergence takes place on the unit circle.

3. Examples

We consider here some simple transforms of S-parameters and corresponding S- and C-functions. We also bring a few particular examples (with regard to examples 1 - 6 see also [13: §§ 14, 15]).

Let f be an S-function with S-parameters γ_n, F the C-function defined by (2.4) and let Φ_n (ψ_n) be the orthogonal polynomials of first (second) kind associated with $d\sigma$.

Example 1: Let $\hat{\gamma}_n = \varepsilon\gamma_n$ ($n \in \mathbb{N}_0$), where $\varepsilon = e^{i\omega}$, $\omega = \overline{\omega}$. It is easy to check that the polynomials

$$\hat{\Phi}_n(\cdot,\varepsilon) = \tfrac{1}{2}(1+\bar{\varepsilon})\Phi_n + \tfrac{1}{2}(1-\bar{\varepsilon})\psi_n \qquad \left(\hat{\Psi}_n(\cdot,\varepsilon) = \tfrac{1}{2}(1-\bar{\varepsilon})\Phi_n + \tfrac{1}{2}(1+\bar{\varepsilon})\psi_n\right)$$

satisfy the recurrence formulas (2.1) with $a_n = \gamma_n$ replaced by $\hat{\gamma}_n$ ($-\hat{\gamma}_n$). Therefore the orthogonal polynomials $\hat{\Phi}_n(\cdot,\varepsilon)$ are associated with the transformed measure $d\hat{\sigma}$. According to (2.3)

$$\hat{F}(z,\varepsilon) = \lim_{n\to\infty}\frac{\hat{\Psi}_n^*(z,\varepsilon)}{\hat{\Phi}_n^*(z,\varepsilon)} = \lim_{n\to\infty}\frac{(1-\varepsilon)\Psi_n^*(z)+(1+\varepsilon)\psi_n^*(z)}{(1+\varepsilon)\Phi_n^*(z)+(1-\varepsilon)\psi_n^*(z)} = \frac{F(z)-\mathrm{i}\,\mathrm{tg}\,\omega/2}{1-\mathrm{i}F(z)\mathrm{tg}\,\omega/2}$$

$$\hat{f}(z,\varepsilon) = \frac{1}{z}\frac{\hat{F}(z,\varepsilon)-1}{\hat{F}(z,\varepsilon)+1} = \varepsilon f(z).$$

Example 2: Let $\hat{\gamma}_n = \varepsilon^{n+1}\gamma_n$, $|\varepsilon| = 1$ $(n \in \mathbb{N}_0)$. As in the Example 1 we have

$$\hat{\Phi}_n(z,\varepsilon) = \varepsilon^{-n}\Phi_n(\varepsilon z),\ \hat{\Phi}_n^*(z,\varepsilon) = \Phi_n^*(\varepsilon z) \quad \text{and} \quad \hat{\Psi}_n(z,\varepsilon) = \varepsilon^{-n}\psi_n(\varepsilon z),\ \hat{\Psi}_n^*(z,\varepsilon) = \psi_n^*(\varepsilon z)$$

$$\hat{F}(z,\varepsilon) = F(\varepsilon z),\quad \hat{f}(z,\varepsilon) = \varepsilon f(\varepsilon z).$$

Example 3: Consider a composition of the transforms from Examples 1 and 2: $\hat{\gamma}_n = \varepsilon^{n+1}\gamma_n$ and $\tilde{\gamma}_n = \varepsilon^{-1}\hat{\gamma}_n = \varepsilon^n\gamma_n$ $(n \in \mathbb{N}_0)$. We have $\hat{f}(z,\varepsilon) = \varepsilon f(\varepsilon z)$ and $\tilde{f}(z,\varepsilon) = \varepsilon^{-1}\hat{f}(z,\varepsilon) = f(\varepsilon z)$.

Example 4: Let $k \ge 2$ be a positive integer and $\hat{\gamma}_n = 0$ if $n \not\equiv -1 (\mathrm{mod}\, k)$, $\hat{\gamma}_n = \gamma_{m-1}$ if $n = km-1$, $m \in \mathbb{N}$ $(n \in \mathbb{N}_0)$. It easily follows from (2.1) that

$$\hat{\Phi}_{km+k-1}(z) = z\hat{\Phi}_{km+k-2}(z) = \ldots = z^{k-1}\hat{\Phi}_{km}(z)$$

$$\hat{\Phi}^*_{km+k-1}(z) = \hat{\Phi}^*_{km+k-2}(z) = \ldots = \hat{\Phi}^*_{km}(z)$$

The same relations are valid for $\hat{\psi}_n$. Applying induction on m we obtain

$$\hat{\Phi}_{km}(z) = \Phi_m(z^k) \quad \text{and} \quad \hat{\psi}_{km}(z) = \psi_m(z^k).$$

Hence

$$\hat{F}(z) = \lim_{n\to\infty}\left(\hat{\Psi}_n^*(z)/\hat{\Phi}_n^*(z)\right) = F(z^k) \quad \text{and} \quad \hat{f}(z) = z^{k-1}f(z^k).$$

Example 5 (shift transform): Let $\hat{\gamma}_n = \gamma_{n+1}$ $(n \in \mathbb{N}_0)$. From the structure of the Schur algorithm we deduce that

$$\hat{f}(z) = f_1(z) = \frac{1}{z}\frac{f(z)-f(0)}{1-\overline{f(0)}f(z)} = \frac{1}{z}\frac{f(z)-\gamma_0}{1-\overline{\gamma}_0 f(z)}.$$

In a similar way we have for $\hat{\gamma}_n = \gamma_{n-1}$

$$\hat{f}(z) = \frac{zf(z)+\gamma_{-1}}{1+\overline{\gamma}_{-1}zf(z)} \tag{3.1}$$

for given complex number γ_{-1}, $|\gamma_{-1}| < 1$.

Example 6 (cf. [5: Theorem III]): Let $\gamma_k = \gamma_{k+1} = \ldots = 0$. Then $\Phi_k^* = \Phi_{k+1}^* = \ldots$ and $\psi_k^* = \psi_{k+1}^* = \ldots$ so that $F = \psi_k^*/\Phi_k^*$. Note that the measure $d\sigma$ is now absolutely continuous and $\sigma'(\vartheta) = |\Phi_k(e^{i\vartheta})|^{-2}\prod_{n=0}^{k-1}(1-|\gamma_n|^2)$. In this case the S-function f may be calculated explicitly; for $k = 2$ we have $f(z) = (\gamma_0+\gamma_1 z)/(1+\overline{\gamma}_0\gamma_1 z)$.

Example 7 (cf. [7: § 24]): Let $\gamma_n = (n+\alpha)^{-1}$ ($\alpha > 1$, $n \in \mathbb{N}_0$). As is known (cf. [6: Chapter 8, formula (8.3')]) the difference equation of second order

$$\gamma_n y_{n+2} - (\gamma_n + \gamma_{n+1} z) y_{n+1} + \gamma_{n+1} z(1 - |\gamma_n|^2) y_n = 0 \tag{3.2}$$

has two linearly independent polynomial solutions $y_n = \Phi_n^*$ and $y_n = \psi_n^*$ with initial data $y_0 = 1$, $y_1 = 1 - \gamma_0 z$ and $y_0 = 1, y_1 = 1 + \gamma_0 z$, respectively. The equation (3.2) can be solved explicitly now. In fact, since $\gamma_n \neq 0$ then we have

$$y_{n+2} = (1 + \gamma_n^{-1}\gamma_{n+1} z) y_{n+1} - \gamma_n^{-1}\gamma_{n+1} z(1 - |\gamma_n|^2) y_n.$$

Denote $b_n = \gamma_{n-1}^{-1}\gamma_n z$ (here one has to take $\gamma_{-1} = (\alpha - 1)^{-1}$). It can be readily checked that $b_{n+1}(1 - |\gamma_n|^2) = b_n$ so that $y_{n+2} - y_{n+1} = b_{n+1} y_{n+1} - b_n y_n$. Hence $y_{n+1} - b_n y_n = y_1 - b_0 y_0$, i.e.

$$\Phi_{n+1}^*(z) - b_n \Phi_n^*(z) = 1 - z \quad \text{and} \quad \psi_{n+1}^*(z) - b_n \psi_n^*(z) = 1 + \alpha^{-1} z(2 - \alpha).$$

Set $u_n = \gamma_{n-1}^{-1}\Phi_n^*$. For u_n we have the difference equation $u_{n+1} - z u_n = (1 - z)(n + \alpha)$, the general solution of which is

$$u_n = Az^n + n + \alpha - (1 - z)^{-1}; \quad \Phi_n^*(z) = A(n + \alpha - 1)^{-1} z^n + (n + \alpha - 1)^{-1}\left(n + \alpha - \frac{1}{1 - z}\right).$$

Putting $n = 0$ we determine the constant A as $A = z(1 - z)^{-1}$. Finally

$$\Phi_n^*(z) = 1 - \frac{z(z^n - 1)}{(n + \alpha - 1)(z - 1)}.$$

Similar arguments lead to an expression for $\psi_n^*(z)$:

$$\psi_n^*(z) = -\frac{z^{n+1}(\alpha(1 - z) - 2)}{\alpha(n + \alpha - 1)(1 - z)^2} + \frac{(1 - z + 2\alpha^{-1} z)(n + \alpha - (1 - z)^{-1})}{(1 - z)(n + \alpha - 1)}.$$

Thus

$$F(z) = \lim_{n \to \infty}\left(\Psi_n^*(z)/\Phi_n^*(z)\right) = (\alpha(1 - z))^{-1}(\alpha - z(\alpha - 2)) \quad \text{and} \quad f(z) = (\alpha + z - \alpha z)^{-1}. \tag{3.3}$$

We should point out that in this case $\sigma'(\vartheta) = \alpha^{-1}(\alpha - 1)$ and that there is a mass point at $\vartheta = 0$: $\sigma\{0\} = 2\pi\alpha^{-1}$.

The case $\alpha = 2$, $f(z) = (2 - z)^{-1}$ has been examined by Schur [13: p. 144] as well as the example $\gamma_0 = 1/2$, $\gamma_n = 2/(2n+1)$ ($n \in \mathbb{N}$). Using (3.3) with $\alpha = 3/2$ and the shift transform (3.1) with $\gamma_{-1} = 1/2$ we get $f(z) = (1 + z)/2$. In connection with these examples Schur posed the following question: Are there any S-functions f, continuous in the closed unit disk $\overline{\mathbb{D}}$ such that $\|f\| < 1$ and $\sum_{n=0}^{\infty}|\gamma_n| = \infty$? We give an affirmative answer to this question (see Remark 2 after Theorem 1 below).

Example 8 (cf. [10: Chapter 10.10(e)]): Consider the weight function

$$p(\vartheta) = (1 + \rho^2)^{-1}|1 - \rho e^{i\vartheta}|^2 = 1 - \frac{\rho}{1 + \rho^2} e^{i\vartheta} - \frac{\rho}{1 + \rho^2} e^{-i\vartheta} = \mathrm{Re} F(e^{i\vartheta})$$

where $0 < \rho \le 1$ and $F(z) = 1 - 2\rho(1 + \rho^2)^{-1} z$. It is easy to calculate the moment sequence $\{c_k\}$,

$c_k = (2\pi)^{-1}\int_0^{2\pi} e^{-ik\vartheta} d\sigma(\vartheta)$: $c_0 = 1$, $c_1 = -\rho(1+\rho^2)^{-1}$, $c_3 = c_4 = \dots = 0$, and the determinants

$$|c_{i-j+1}|_0^n = (-1)^{n+1}(\rho/(1+\rho^2))^{n+1} \quad \text{and} \quad |c_{i-j}|_0^n = (\rho/(1+\rho^2))^{n+1} U_{n+1}(\lambda)$$

where U_n is the Chebyshev polynomial of the second kind and $\lambda = (2\rho)^{-1}(1+\rho^2)$. Hence

$$\gamma_n = (-1)^n |c_{i-j+1}|_0^n / |c_{i-j}|_0^n = -\left(U_{n+1}(\lambda)\right)^{-1} \quad \text{and} \quad f(z) = \left(z - \frac{1+\rho^2}{\rho}\right)^{-1}.$$

Let us point out that for $\rho < 1$ the parameters γ_n decrease exponentially (cf. Theorem 5 below).

4. Direct theorems for Schur parameters

We adopt the terms "direct theorems" here (and "converse theorems" later in Section 5) from the approximation theory.

Let f be an S-function with boundary values $f(e^{i\vartheta})$. In what follows we deal with the "regular" case

(R) $f \in C_{2\pi}$, $\|f\| = \|f(e^{i\vartheta})\|_{2\pi} < 1$.

Theorem 1: *Let the S-function f satisfy* (R). *If*

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \omega\left(\frac{1}{n}, f\right) < \infty, \tag{4.1}$$

then for the S-parameters γ_n

(G) $\sum_{n=0}^{\infty} |\gamma_n| < \infty$

holds. Conversely, *the hypothesis* (G) *yields* (R).

Proof: Under the hypothesis (R) the C-function F in (2.4) is continuous in the closed unit disk $\overline{\mathbb{D}}$ and

$$0 < \mu \le \rho(\vartheta) = \sigma'(\vartheta) = \operatorname{Re} F(e^{i\vartheta}) = \frac{1 - |f(e^{i\vartheta})|^2}{|1 - e^{i\vartheta} f(e^{i\vartheta})|^2} \in C_{2\pi}.$$

For F we have

$$F(e^{i(\vartheta+h)}) - F(e^{i\vartheta}) = 2\frac{e^{i\vartheta}\left(f(e^{i(\vartheta+h)}) - f(e^{i\vartheta})\right) + f(e^{i(\vartheta+h)})\left(e^{i(\vartheta+h)} - e^{i\vartheta}\right)}{\left(1 - e^{i\vartheta} f(e^{i\vartheta})\right)\left(1 - e^{i(\vartheta+h)} f(e^{i(\vartheta+h)})\right)} \tag{4.2}$$

whence it follows that

$$\omega(t,F) \le 2(1 - \|f\|)^{-2}\left(\omega(t,f) + \|f\| \sin\tfrac{t}{2}\right) \le C(f)\omega(t,f).$$

Since $\omega(t,p) \le \omega(t,F)$, then $\omega(t,p) \le C(f)\omega(t,f)$, and therefore the weight function p satisfies (4.1). The well-known Bernstein theorem asserts that the condition (4.1) implies the inclusion $p \in W$, i.e. the Fourier series of p converges absolutely (cf. [17: Chapter 6, Theorem 3.1 and the

Remark following it]). The conclusion (G) now follows immediately from G. Baxter's Theorem [1: Corollary 1.1] (see also [9: Corollary 2.1]) and Geronimus' theorem, proved in Section 2.

The converse statement in Theorem 1 is due to I. Schur (cf. [13: p. 143]), who has proved that under the assumption (G)

$$\|f\|^2 \le \frac{A-1}{A} \qquad \text{where } A = \prod_{k=0}^{\infty} \frac{1+|\gamma_k|}{1-|\gamma_k|}. \tag{4.3}$$

and so Theorem 1 is completely proved ∎

Remark 1: Ya. L. Geronimus [7: §27, Theorem 27.1 and § 28] has obtained the sharp estimate for $\|f\|$, f satisfying condition (G): $\|f\| \le (A-1)(A+1)^{-1}$. Equality here holds for the function $f(z) = (A-1)(A+1)^{-1}z$.

Remark 2: The hypothesis (4.1) cannot be rejected. Indeed, it is well-known (cf. [17: Chapter 5, §4]) that the function g, $g(\vartheta) = \operatorname{Re}\sum_{n=1}^{\infty} n^{-1}\exp\{in \ln n + in\vartheta\}$, is an element of $\mathrm{Lip}\,\frac{1}{2}$ and not of W. For the function p we have $p(\vartheta) = C + g(\vartheta) \ge \mu > 0$ for an appropriate constant C and it has the same properties, as g. According to Privalov's theorem (cf. [17: Chapter 3, Theorem 13.29]) for the conjugate function we have $\tilde{p} \in \mathrm{Lip}\,\frac{1}{2}$. Therefore the C-function F,

$$F(z) = \frac{1}{2\pi M}\int_0^{2\pi} \frac{e^{i\vartheta}+z}{e^{i\vartheta}-z}\, p(\vartheta)\,d\vartheta, \quad M = \frac{1}{2\pi}\int_0^{2\pi} p(\vartheta)\,d\vartheta$$

is continuous in $\overline{\mathbb{D}}$, $F(0) = 1$ and $\operatorname{Re} F(z) \ge \mu_1 > 0$. It means that the S-function f,

$$f(z) = \frac{1}{z}\,\frac{F(z)-1}{F(z)+1} \tag{4.4}$$

satisfies (R). According to the above-mentioned theorems of G. Baxter and Ya. L. Geronimus (G) is false now. So an affirmative answer on Schur's question (see Example 7 in Section 3) is obtained.

Theorem 2: *Let the S-function f satisfy* (R) *and assume that* $\omega(t,f)t^{-1} \in L^1(0,1)$. *Then*

$$|\gamma_n| \le C(f)\left(\int_0^{1/n} \frac{\omega(t,f)}{t}\,dt + \frac{1}{n}\int_{1/n}^{1} \frac{\omega(t,f)}{t^2}\,dt\right).$$

Proof: As in the proof of Theorem 1 we have $\omega(t,p) \le C(f)\omega(t,f)$. Set $q = \ln p$, so that (see the end of Section 2) $\pi(e^{i\vartheta}) = D^{-1}(e^{i\vartheta}) = \exp\{-\frac{1}{2}(q(\vartheta) + i\tilde{q}(\vartheta))\}$ holds a.e. From the elementary inequality $|\ln\chi_1 - \ln\chi_2| \le \mu^{-1}|\chi_1 - \chi_2|$ $(\chi_1, \chi_2 \ge \mu > 0)$ we deduce $|q(\vartheta+h) - q(\vartheta)| \le \mu^{-1}|p(\vartheta+h) - p(\vartheta)|$ and hence $\omega(t,q) \le \mu^{-1}\omega(t,p)$ so that $\omega(t,q)t^{-1} \in L^1$. By means of the well-known Zygmund inequality [17: Chapter 3, Theorem 13.30] we obtain

$$\omega(t,\tilde{q}) \le C(f)B(t), \quad B(t) = \int_0^t \frac{\omega(x,p)}{x}\,dx + t\int_t^1 \frac{\omega(x,p)}{x^2}\,dx.$$

Applying the inequality $|z_1 - z_2| \le |\ln z_1 - \ln z_2|\max\{|z_1|,|z_2|\}$ $(|z_1|,|z_2| \ne 0)$ and taking into account that $|\pi(e^{i\vartheta})| \le \mu^{-1/2}$, we get

$$|\pi(e^{i(\vartheta+h)}) - \pi(e^{i\vartheta})| \le \frac{1}{2\mu^{1/2}}\left\{|q(\vartheta+h) - q(\vartheta)| + |\tilde{q}(\vartheta+h) - \tilde{q}(\vartheta)|\right\}$$

and $\omega(t,\pi) \le C(f)(\omega(t,p) + B(t))$. But

$$\int_0^t \frac{\omega(x,p)}{x}dx \ge \int_{t/2}^t \frac{\omega(x,p)}{x}dx \ge \ln 2\cdot\omega(t/2,p) \ge \frac{\ln 2}{2}\omega(t,p)$$

so that

$$\omega(t,\pi) \le C(f)B(t). \tag{4.5}$$

The quantity $\delta_n = \varkappa^2 - \varkappa_n^2$ plays an important role in the theory of orthogonal polynomials on the unit circle (and also in the linear prediction theory). As is known (cf. [6: Chapter 2, formulas (2.8), (2.18) - (2.20)]),

$$\delta_n = \varkappa^2 \inf_{G_n\in T_n} \|\pi - G_n\|_\sigma = \varkappa^2 \inf_{G_n\in T_n}\left(\int_0^{2\pi} |\pi(e^{i\vartheta}) - G_n(e^{i\vartheta})|^2 d\sigma(\vartheta)\right)^{1/2}$$

and the following two-sided estimate $\varkappa^2\sum_{k=n}^\infty |a_k|^2 \le \delta_n^2 \le \sum_{k=n}^\infty |a_k|^2$ $(a_k = \gamma_k!)$ holds. Therefore by Jackson's theorem (cf. [17: Chapter 3, Theorem 13.6])

$$\delta_n \le \varkappa^2 M E_n^*(\pi) \le C(f)\omega(1/n,\pi), \quad M = \|p\|_{2\pi}. \tag{4.6}$$

Using (4.5), (4.6) we finally obtain

$$|\gamma_n| = |a_n| \le \varkappa\delta_n \le C(f)\omega(1/n,\pi) \le C(f)B(1/n). \tag{4.7}$$

Hence the assertion of Theorem 2 is verified ∎

Corollary: *If under the assumptions of Theorem 2* $f(e^{i\vartheta}) \in \mathrm{Lip}\,\alpha$, $0 < \alpha \le 1$, *then* $\gamma_n = O(n^{-\alpha})$ *for* $0 < \alpha < 1$ and $\gamma_n = O(n^{-1}\ln n)$ for $\alpha = 1$.

Theorem 3: *Let the S-function f satisfy condition* (R) *and* $f(e^{i\vartheta}) \in C_{2\pi}^{(m)}$, *where* $m \ge 1$ *is an integer. Then* $|\gamma_n| \le C(f,m)(\ln n/n^m)\,\omega(1/n, f^{(m)})$. *If in addition* $f^{(m)}(e^{i\vartheta}) \in \mathrm{Lip}\,\alpha$, $0 < \alpha < 1$, *then* $|\gamma_n| \le C(f,m)n^{-(m+\alpha)}$.

Proof: We begin with the second statement. P.K. Suetin [15: Lemma 1.4] proved that if $p \in C_{2\pi}^{(m)}$ and $p^{(m)} \in \mathrm{Lip}\,\alpha$, $0 < \alpha < 1$, then $\pi^{(m)} \in \mathrm{Lip}\,\alpha$ in the closed unit disk $\bar{\mathbb{D}}$. By (4.7) we have

$$|\gamma_n| \le C(f)\omega(1/n,\pi) \le C(f,m)n^{-m}\omega(1/n,\pi^{(m)}) \le C(f,m)n^{-(m+\alpha)}.$$

Next we turn to the general case. Under the hypothesis of the present theorem the C-function F in (2.4) belongs to $C_{2\pi}^{(m)}$. If we differentiate (4.2) m times with respect to ϑ, we obtain

$$\Delta_h F^{(m)}(e^{i\vartheta}) = \frac{g_{m+1}(e^{i\vartheta},h)(e^{i\vartheta}-1) + \sum_{k=0}^\infty g_j(e^{i\vartheta},h)\Delta_h f^{(j)}(e^{i\vartheta})}{(1 - e^{i\vartheta}f(e^{i\vartheta}))^{2m}(1 - e^{i(\vartheta+h)}f(e^{i(\vartheta+h)}))^{2m}} \tag{4.8}$$

where the functions $g_0(e^{i\vartheta},h), \dots, g_{m+1}(e^{i\vartheta},h)$ are continuous and depend on $f, f', \dots, f^{(m)}$ only.

It readily follows from (R) and (4.8) that $\omega(t, F^{(m)}) \le C(f,m)\omega(t, f^{(m)})$ and hence

$$\omega(t, p^{(m)}) \le \omega(t, F^{(m)}) \le C(f,m)\omega(t, f^{(m)}), \quad p(\vartheta) = \operatorname{Re} F(e^{i\vartheta}). \tag{4.9}$$

We can now apply a theorem due to B. L. Golinsky [9: Theorem 1.2] which asserts that

$$\|\pi(e^{i\vartheta}) - \varphi_n^*(e^{i\vartheta})\|_{2\pi} \le C(f,m)(\ln n/n^m)\,\omega(1/n, p^{(m)}). \tag{4.10}$$

By means of the inequalities (4.9) and (4.10) we obtain (cf. (4.6)) the inequality

$$|\gamma_n| \le C(f)E_n^*(\pi) \le C(f,m)(\ln n/n^m)\,\omega(1/n, f^{(m)}).$$

Thus the proof is completed ∎

5. Converse theorems for Schur parameters

In this section we show that a certain decay of the Schur parameters provides some smoothness properties of the corresponding S-function.

Theorem 4: *Let $f \in S$. If, for some integer $m \in \mathbb{N}_0$, $\gamma_n = O(n^{-(m+\alpha+1)})$ for $n \to \infty$ $(0 < \alpha \le 1)$, then*

$$f(e^{i\vartheta}) \in C_{2\pi}^{(m)} \quad \text{and} \quad \omega(t, f^{(m)}) = \begin{cases} O(t^\alpha) & \text{for } 0 < \alpha < 1 \\ O(t\ln(1/t)) & \text{for } \alpha = 1 \end{cases} \quad (t \to 0)$$

Proof: Let us denote $\tau_n = \sum_{k=n}^{\infty}|\gamma_k|$. We obviously have

$$\tau_n = O(n^{-(m+\alpha)}) \text{ as } n \to \infty. \tag{5.1}$$

In particular, f satisfies (G) (and hence (R) by Theorem 1); the Szegö function D is continuous and does not vanish in the closed unit disc $\overline{\mathbb{D}}$; $0 < \mu \le \rho(\vartheta) = |\pi(e^{i\vartheta})|^{-2} \in C_{2\pi}$, $\pi = D^{-1}$. As is known (cf. [6: Chapter 8; Theorem 8.5]) under the condition (G) $\sup_n \|\varphi_n^*(e^{i\vartheta})\|_{2\pi} \le C(f)$ and $\|\pi(e^{i\vartheta}) - \varphi_n^*(e^{i\vartheta})\|_{2\pi} \le C(f)\tau_n$. We proceed with estimating the value $\omega(t, g^{(m)})$ where $g = \rho^{-1}$. We have

$$\begin{aligned} |g(\vartheta) - |\varphi_n^*(e^{i\vartheta})|^2| &= ||\pi(e^{i\vartheta})|^2 - |\varphi_n^*(e^{i\vartheta})|^2| \\ &\le |\{|\pi(e^{i\vartheta})| - |\varphi_n^*(e^{i\vartheta})|\}\{|\pi(e^{i\vartheta})| + |\varphi_n^*(e^{i\vartheta})|\}| \\ &\le C(f)|\pi(e^{i\vartheta}) - \varphi_n^*(e^{i\vartheta})|. \end{aligned}$$

By using (5.1) we get from the previous relations

$$E_n^*(g) \le C(f)\|\pi(e^{i\vartheta}) - \varphi_n^*(e^{i\vartheta})\|_{2\pi} \le C(f)\tau_n \le C(f,m)n^{-(m+\alpha)} \tag{5.2}$$

The following theorem due to S. B. Stechkin [14: Theorem 11] asserts that if for some number $m \in \mathbb{N}_0$

$$\sum_{k=0}^{\infty} k^{m-1}E_k^*(g) < \infty \tag{5.3}$$

holds, then $g \in C_{2\pi}^{(m)}$ and

$$\omega\left(\frac{1}{n}, g^{(m)}\right) \le C(m)\left\{\frac{1}{n}\sum_{k=0}^{n}(k+1)^m E_k^*(g) + \sum_{k=n+1}^{\infty} k^{m-1}E_k^*(g)\right\}.$$

In our case (5.3) is true by virtue of (5.2) and consequently $\omega(1/n, g^{(m)}) \le C(f, m)n^{-\alpha}$, $0 < \alpha \le 1$. The latter is obviously equivalent to $\omega(t, g^{(m)}) = O(t^\alpha)$ as $t \to 0$. We are within a few steps from estimating the value $\omega(t, f^{(m)})$.

1. If we differentiate m times with respect to ϑ the identity $p(\vartheta + h)\Delta_h g(\vartheta) + \Delta_h p(\vartheta)\cdot g(\vartheta) \equiv 0$ we obtain

$$\sum_{k=0}^{m}\binom{m}{k}\left\{p^{(k)}(\vartheta + h)\Delta_h g^{(m-k)}(\vartheta) + \Delta_h p^{(k)}(\vartheta)\cdot g^{(m-k)}(\vartheta)\right\} = 0$$

whence it can be readily deduced that $\omega(t, p^{(m)}) \le C(f, m)\omega(t, g^{(m)}) \le C(f, m)t^\alpha$.

2. By Zygmund's inequality we have

$$\omega(t, \tilde{p}^{(m)}) = \begin{cases} O(t^\alpha) & \text{if } 0 < \alpha < 1 \\ O(-t\ln t) & \text{if } \alpha = 1 \end{cases} \tag{5.4}$$

and the same relation is valid for $\omega(t, F^{(m)})$, since $F(e^{i\vartheta}) = p(\vartheta) + i\tilde{p}(\vartheta)$.

3. Arguments identical to those used in the proof of Theorem 3 (cf. (4.8)) lead to the following identity for the S-function f given by (4.4):

$$\Delta_h f^{(m)}(e^{i\vartheta}) = \frac{G_{m+1}(e^{i\vartheta}, h)(e^{i\vartheta} - 1) + \sum_{j=0}^{m} G_j(e^{i\vartheta}, h)\Delta_h F^{(j)}(e^{i\vartheta})}{(e^{i\vartheta}F(e^{i\vartheta}) + e^{i\vartheta})^{2m}(e^{i(\vartheta+h)}F(e^{i(\vartheta+h)}) + e^{i(\vartheta+h)})^{2m}}$$

where the functions $G_0(e^{i\vartheta}, h), \dots, G_{m+1}(e^{i\vartheta}, h)$ are continuous. So $\omega(t, f^{(m)}) \le C(f, m)\omega(t, F^{(m)})$. The rest is immediate from the latter inequality and (5.4) ∎

Theorem 5: *The relation*

$$r_1 = \overline{\lim_{n\to\infty}} |\gamma_n|^{1/n} < 1 \tag{5.5}$$

holds if and only if the S-function f is analytic in the closed unit disk $\overline{\mathbb{D}}$ *(i.e. f is analytic in the open disk* $\{z \in \mathbb{C}: |z| < 1 + \varepsilon\}$ *for some* $\varepsilon > 0$) and $\|f\| < 1$.

Proof: *Necessity*. According to [12: Theorem 1] the function $\pi = \pi(z)$ is analytic in the disk $\{z \in \mathbb{C}: |z| < r_1^{-1}\}$ and $\lim_{n\to\infty} \Phi_n^*(z) = 1 - z\sum_{k=0}^{\infty} a_k \Phi_k(z) = \varkappa^{-1}\pi(z)$, where the convergence is absolute and uniform in any disk $\{z \in \mathbb{C}: |z| < R\}$, $R < r_1^{-1}$. The same is true for the function $\omega(z) = \lim_{n\to\infty}\psi_n^*(z)$. Since $\pi(z) \neq 0$ for $|z| \le 1$, then both the C-function $F = \pi^{-1}\omega$ and the S-function f (see (4.4)) are analytic in the disk $\{z \in \mathbb{C}: |z| < 1 + \varepsilon\}$ for some $\varepsilon > 0$. The assertion $\|f\| < 1$ follows from (G) by Theorem 1.

Sufficiency. Under the assumptions of the present theorem there exists a number $q \in (0,1)$ and an S-function f_1 such that $f(z) = qf_1(qz)$. We also have $F(z) = F_1(qz)$ for the corresponding C-functions F and F_1. So F is analytic and $\operatorname{Re}F(z) > 0$ in the disk $\{z \in \mathbb{C}: |z| < q^{-1}\}$. Let $F(z) = 1 + 2\sum_{k=1}^{\infty} c_k z^k$, $c_k = u_k + iv_k$, so that $p(\vartheta) = \operatorname{Re}F(e^{i\vartheta}) = 1 + 2\sum_{k=1}^{\infty}(u_k\cos k\vartheta - v_k\sin k\vartheta)$. It is actually not hard to see that p admits an analytic continuation into the strip $|\operatorname{Im}\lambda| < -\ln q$, $\lambda = \vartheta + i\tau$. Since $2p(\lambda) = F(e^{i\lambda}) + \overline{F(e^{i\bar{\lambda}})}$, then $\operatorname{Re}p(\lambda) > 0$ for $|\operatorname{Im}\lambda| < -\ln q$. Thus the function p^{-1} is analytic in the same domain. By [8: Theorem 2] $\gamma_n = a_n = O(q^n)$ as $n \to \infty$ and so the theorem has been proved ∎

Remark: Let $S(\rho)$ be the set of S-functions f satisfying $\overline{\lim}_{n\to\infty} |\gamma_n|^{1/n} \leq \rho$, $0 \leq \rho < 1$, and $R(\rho) = \inf\{r: \text{every } f \in S(\rho) \text{ is analytic in the disk } |z| < r\}$. Example 6 from Section 3 shows that, for all ρ, $R(\rho) = 1$.

Acknowledgement: The author is grateful to Prof. S.V. Khrushchev for the opportunity to get acquainted with the manuscript of his report at the conference on approximation theory (St. Peterburg/Leningrad 1991).

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Received 22.07.1992