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Free Boundary Value Problems for the Stationary Navier-Stokes Equations in Domains with Noncompact Boundaries

R. S. GELLRICH

A free boundary problem for an incompressible viscous fluid in a domain with noncompact boundaries is considered; the upper boundary is to be determined by equilibrium conditions involving the fluid stress tensor and its surface tension. It is proved that if the data of the problem are regular, then the free boundary, the velocity vector and the pressure are regular. Furthermore the exponential decay of the solution is shown.

Key words: *Navier-Stokes Equations, free surfaces, noncompact boundaries*
AMS subject classification: 76 D05

1. Introduction

We consider stationary flows of an incompressible viscous fluid that occupies a three-dimensional semi-infinite domain Ω between a fixed bottom Γ_- and a free upper surface Γ_+, which is governed by surface tension. Both surfaces approach horizontal planes at infinity. The flow is driven by an outer pressure gradient $-\nabla p_a = a$ (a = const), the gravity g and an outer force f_s. It can be described by the following system:

$$\left.\begin{array}{ll}\textbf{(N1)} & -\nu\Delta u + (u\cdot\nabla)u + \nabla p = f_s \\ \textbf{(N2)} & \nabla\cdot u = 0\end{array}\right\} \quad \text{in } \Omega,$$

$$\textbf{(B1)} \quad u = 0 \qquad \text{on } \Gamma_-,$$

$$\left.\begin{array}{ll}\textbf{(B2)} & un = 0 \\ \textbf{(B3)} & t^{(i)}T(u,p)n = 0 \quad (i = 1,2) \\ \textbf{(B4)} & nT(u,p)n = -p_a + gh + 2\kappa H\end{array}\right\} \quad \text{on } \Gamma_+,$$

$$\textbf{(B5)} \quad u(x,y) \longrightarrow q(y) \qquad \text{for } |x| \longrightarrow \infty.$$

Here $\Omega = \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}\colon\ -b(x) < y < h(x)\}$ is the domain occupied by the fluid, ν is the viscosity and κ is the capillary constant. The stress tensor T is defined by $T_{i,j}(u,p) = -p\delta_{ij} + \nu(\partial_i u_j + \partial_j u_i)$. The vectors $t^{(i)}$ and n are the tangents and the normal to Γ_+ and H is the mean curvature of

R. S. Gellrich: Universität des Saarlandes, FB Mathematik, Bau 27, Saarbrücken, Germany

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Γ_+. The system (N) is the Navier-Stokes system of equations for the velocity u and the pressure p; the boundary conditions (B2) and (B3) are of mixed type, and equation (B4) is an additional equation determining the free boundary Γ_+. The limit velocity q is the equilibrium velocity on a strip of height b_0 ($b(x) \longrightarrow b_0$ for $|x| \longrightarrow \infty$).

The corresponding instationary problem was considered by Beale [9]. He used Lagrangian coordinates, therefore his method cannot be used in our context. The stationary problem on bounded domains was studied by Bemelmans [10]. Amick and Fraenkel [4,6,7] considered stationary flows in unbounded channels. Gerhardt [13,14] derived decay estimates for an exterior capillary problem.

The main result of the present paper is the following

Theorem: *Let* $r_0 \in \mathbb{R}^+$, $f_s \in W_2^k(\mathbb{R}^3) \cap C^{k,\alpha}(\mathbb{R}^3)$ $(k \in \mathbb{N},\ 0 < \alpha < 1)$ *and*

$$|D^\mu f_s(x,y)|,\ [D^\mu f_s]_\alpha(x,y) \le c_\mu \exp(-c_2|x|) \quad \text{for } |x| \ge r_0\,, \tag{1.1}$$

$$|D^\beta(b(x)-b_0)|,\ [D^\beta(b(x)-b_0)]_\alpha \le c_\beta \exp(-c_2|x|) \quad \text{for } |x| \ge r_0 \tag{1.2}$$

$(|\mu| \le k,\ |\beta| \le k+3,\ c_\mu,\ c_\beta,\ c_2 > 0)$. *If the $C^{0,\alpha}$-norm of f_s is sufficiently small, then there is a $\nu_0 > 0$, such that the problem* (N), (B) *has exactly one solution* $(u,p,h) \in C^{k+2,\alpha} \times C^{k+1,\alpha} \times C^{k+3,\alpha}$ *for all* $\nu > \nu_0$. *Furthermore we have*

$$|D^\gamma(u(x,y) - q(y))| \le c_\gamma \exp(-c_2|x|) \quad \text{for } |\gamma| \le k+2, \tag{1.3}$$

$$|D^\sigma(\nabla p(x,y) - \nabla p_a)| \le c_\sigma \exp(-c_2|x|) \quad \text{for } |\sigma| \le k, \tag{1.4}$$

$$|D^\tau h(x)| \le c_\tau \exp(-c_2|x|) \quad \text{for } |\tau| \le k+3 \tag{1.5}$$

in $\{(x,y) \in \Omega \cup \Gamma :\ |x| \ge r_0\}$, *with* $c_\gamma,\ c_\sigma,\ c_\tau > 0$.

The proof is divided into three parts: First we consider the Navier-Stokes equations (N) with the boundary conditions (B1) - (B3) and (B5) in a fixed domain $\hat\Omega$ (Section 3). We prove the existence of a weak solution $u = v + g$ in this domain $\hat\Omega$. The velocity field g is a solenoidal function, which satisfies the boundary conditions (B1), (B2) and (B5), see Definition 3.1 below. With this function g and the assumptions on the surfaces h, b and on the force density f_s we get an a priori estimate for the Dirichlet norm of v. Then we can conclude that a weak solution of the Navier-Stokes equations (N) exists. With the regularity results of Agmon, Douglis and Nirenberg [3] and Solonnikov and Ščadilov [22] we show higher regularity of this solution. Finally we get the exponential decay in (1.3) and (1.4) with the help of Green's function for the linearized problem. When we show the exponential decay, the nonlinearity of (N1) plays an important role.

Next (Section 4) we consider the problem for the free surface

$$H(x,h(x)) + ch(x) + f(x) = 0 \qquad \text{on } \mathbb{R}^2$$
$$h(x) \longrightarrow 0 \qquad \text{for } |x| \longrightarrow \infty,$$

where c is a positive constant and f and its derivatives up to the order $k+1$ decay exponentially for $|x| \longrightarrow \infty$. With the help of an appropriate variational problem we show the existence of a solution. The exponential decay is shown as follows: we get the decay of the function h with a maximum principle and that of the first derivative with a method that Trudinger [23] used to show the boundedness of the derivative. The decay of the higher derivatives is shown with the help of Schauder's interior estimates.

In the last part (Section 5) we consider the full problem (N), (B). The existence proof is based on the following successive approximation: the Navier-Stokes equations (N) with the boundary conditions (B1) - (B3) and (B5) are solved in a domain Ω^0, for example a strip of height b_0. The solution (u^1,p^1) inserted into equation (B4) leads to the surface Γ_+^1, and so we get a new domain Ω^1. In this domain we solve the Navier-Stokes equations once more and put this new solution again into the surface equation (B4) and so on. The convergence of this sequence (u^m,p^m,h^m) is shown with a fixed point argument. The exponential decay of the limit function follows directly from the uniform estimates for the approximating sequence.

2. Preliminaries

2.1. Notations. In what follows, the derivatives of a function

$$f : \Omega \to \mathbb{R}^n,\ \Omega = \{ z = (x,y) \in \mathbb{R}^2\times \mathbb{R} : -b(x) < y < h(x) \}$$

are denoted by

$$D_i f = \frac{\partial}{\partial x_i} f = \partial_i f \qquad (i = 1,2) \qquad \text{and} \qquad D_3 f = \frac{\partial}{\partial y} f = \partial_3 f .$$

In the strip $S = \{ \zeta = (\xi,\eta) \in \mathbb{R}^2\times \mathbb{R}: -b_0< \eta < 0\}$ we use coordinates ξ and η and denote the corresponding partial derivatives by

$$\nabla_i g = \frac{\partial}{\partial \xi_i} g \qquad (i = 1,2) \qquad \text{and} \qquad \nabla_3 g = \frac{\partial}{\partial \eta} g .$$

By "div", "curl" and "DIV", "CURL" we mean the divergence and rotation on Ω

and S, respectively. As usual we use D^α and ∇^α for the partial derivatives of order $|\alpha|$, $\alpha \in \mathbb{N}^3$. In $\mathbb{R}^n$ all integrals are taken with respect to the n-dimensional Lebesgue measure. In particular

$$\int_U f(z)\,dz = \int_U f(x,y)\,dx\,dy = \int_U f(z) \qquad \text{for } U \subset \Omega$$

$$\int_V g(\zeta)\,d\zeta = \int_V g(\xi,\eta)\,d\xi\,d\eta = \int_V g(\zeta) \qquad \text{for } V \subset S\,.$$

Let U be an open set in $\mathbb{R}^n$; by the Sobolev space $W^m_p(U)$ ($m \in \mathbb{N}_0$, $p \geq 1$) we mean the real Banach space of functions f such that f and its generalized derivatives $D^\alpha f$ ($|\alpha| \leq m$) belong to $L_p(U)$. The norm is defined by $\| f \|_{m,p,U} = \left(\sum_{|\alpha|\leq m} \int_U |D^\alpha f|^p\,dz \right)^{1/p}$. The space $J(\Omega)$ denotes the set of infinitely differentiable vector fields $v = (v_1, v_2, v_3)$, that are solenoidal and the horizontal coordinates (x_1, x_2) have compact support in Ω and which satisfy on $\Gamma_\pm$ the boundary conditions $v_{|\Gamma_-} = 0$ and $vn_{|\Gamma_+} = 0$. The real Hilbert space $H(\Omega)$ is the completion of $J(\Omega)$ in the Dirichlet norm $\| u \|_\Omega = \left(\int_\Omega |Du|^2\,dz \right)^{1/2}$. The inner product is definded by $\langle u,v\rangle_\Omega = \int_\Omega \partial_i u_j\, \partial_i v_j\,dz$.

Let U be a function space; then U_{sol} is the intersection of U with all solenoidal functions. We define $V(\Omega)$ to be the closure of

$$I(\Omega) = C^\infty_{sol}(\Omega) \cap \{ \phi : \text{supp } \phi \text{ is compact in } \Omega \text{ in } (x_1,x_2)\text{-direction} \}$$
$$\cap \{ \phi : \phi = \text{curl}\phi = 0 \text{ on } \Gamma_- ;\ \phi t^{(1)} = 0,\ \text{curl}\phi\ n = 0 \text{ on } \Gamma_+ \}$$

in the norm $\|\phi\|^2_{V(\Omega)} = \int_\Omega |\Delta\phi|^2 dz$. Furthermore we define for functions $u,v,w \in W^1_2(\Omega)$ the bilinear form

$$[u,v]_\Omega = \frac{1}{2}\int_\Omega D(u):D(v)\,dz = \frac{1}{2}\int_\Omega (\partial_i v_j + \partial_j v_i)(\partial_i u_j + \partial_j u_i)\,dz$$
$$= \int_\Omega (\partial_i v_j \partial_i u_j + \partial_i v_j \partial_j u_i)\,dz$$

and the triple product $\{u,v,w\}_\Omega = \int_\Omega u(v\cdot\nabla)w\,dz$, whenever the vector fields u,v and w are such that the integrals are defined. If v is in $H(\Omega)$, we get $\{u,v,w\}_\Omega = \int_\Omega u_i v_j \partial_j w_i \partial z = -\int_\Omega v_j \partial_j u_i w_i dz = -\{w,v,u\}_\Omega$, and therefore $\{u,v,u\}_\Omega = 0$. With Hölder's inequality we get

$$|\,\{u,v,w\}_\Omega\,| \leq \|u\|_{L_4(\Omega)} \|v\|_{L_4(\Omega)} \|w\|_\Omega\,.$$

The corresponding expressions on S are defined analogously. In Section 3 we need the space $\tilde{H}$. This function space is the closure of

$J = C^{\infty}(\Omega) \cap \{V : \text{supp } V \text{ is compact in } \Omega \text{ in } (x_1,x_2)\text{-direction}\}$

$\cap \{V : V = 0 \text{ on } \Sigma_-;\ VN = 0 \text{ on } \Sigma_+\}$

in the Dirichlet norm, where N is the outer normal to Σ_+.

2.2. Construction of a map F. Sometimes it is useful to consider the strip $S = \{ \zeta = (\xi,\eta) \in \mathbb{R}^2 \times \mathbb{R} : -b_0 < \eta < 0\}$ instead of the domain $\Omega = \{ z = (x,y) \in \mathbb{R}^2 \times \mathbb{R} : -b(x) < y < h(x)\}$. Therefore we construct a map $F : S \longrightarrow \Omega$ with

$$F : (\xi_1,\xi_2,\eta) \longmapsto \left(\xi_1,\xi_2, \frac{b(\xi) + h(\xi)}{b_0}\eta + h(\xi)\right).$$

Then F maps the upper boundary $\Sigma_+ = \{(\xi,\eta) \in \mathbb{R}^2 \times \mathbb{R}: \eta = 0\}$ of S on Γ_+ and the lower boundary $\Sigma_- = \{(\xi,\eta) \in \mathbb{R}^2 \times \mathbb{R} : \eta = -b_0\}$ on Γ_-.

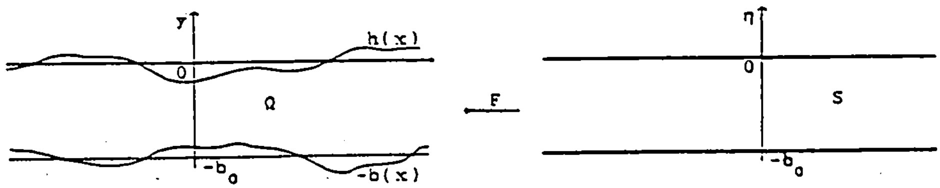

Fig. 2.1. The domains Ω and S

Now we list some properties of the map F:

$$\nabla F = I + E_1 \quad \text{and} \quad DF^{-1} = I + E_2 ,$$

where E_i $(i = 1,2)$ are 3x3-matrices with coefficients of order $O(Dh, Db, h, b-b_0)$ as $|x| \longrightarrow \infty$. The derivatives of a function $f : S \longrightarrow \mathbb{R}^n$ transform as follows:

$$D^{\alpha}f = \nabla^{\alpha}f + \sum_{|\beta|\le|\alpha|-1} E_{\beta}\, \nabla^{\beta}\nabla_3 f ,$$

with E_{β} of order $O(D^{\gamma}h, D^{\gamma}(b-b_0))$ as $|x| \longrightarrow \infty$ $(|\gamma| \le |\alpha|)$.

The functions that appear in the Navier-Stokes equations are transformed in the following way:

$$\left.\begin{aligned} U(\zeta) &= DF^{-1}(F(\zeta))\, u(F(\zeta)) \det DF(\zeta) \\ P(\zeta) &= p(F(\zeta)) \\ F_s(\zeta) &= DF^{-1}(F(\zeta))\, f_s(F(\zeta)) \end{aligned}\right\} \quad \text{for } \zeta \in S .$$

By this U is defined such that div $u = 0$ in Ω is transformed into DIV $U = 0$ in S. Furthermore we have $U = 0$ on Σ_-; $UN = 0$ on Σ_+ and $u(F(\zeta)) = U(\zeta) +$

$EU(\zeta)$, where N is the outer normal to Σ_+ and E is a 3x3-matrix with coefficients of order $O(\nabla^\alpha h, \nabla^\alpha(b-b_0))$ as $|\xi| \longrightarrow \infty$ ($|\alpha| \le 1$).

The following lemma involves the form of Ω more closely.

Lemma 2.1. *Let Ω be a domain as defined above. Then for every $v \in H(\Omega)$ and $V = DF^{-1}v \cdot \det \nabla F$ we have:*

a) $\|v\|_{L_2(\Omega)} \le C\|v\|_\Omega$

b) $\|\nabla V\|_{L_2(S)} \le C\|v\|_\Omega$ and $\|\nabla v\|_{L_2(S)} \le C\|v\|_\Omega$

c) $\displaystyle\int_\Omega \frac{|v|^2}{\sqrt{-\eta}} \cdot |\det(DF^{-1})| = \int_S \frac{|v|^2}{\sqrt{-\eta}} \le 4C_1\|\nabla v\|^2_{L_2(S)} \le C_2\|v\|^2_\Omega$

d) $\displaystyle\int_{x_i \in \mathbb{R}} \int_{-b(x)}^{h(x)} |v(x_1,x_2,y)|^2 \longrightarrow 0$ as $|x_j| \longrightarrow \infty$ $(i,j = 1,2;\quad i \ne j)$.

Proof. a) The first estimate is Poincaré's inequality, which is true because Ω is contained in a strip of finite width and v is equal to zero at the lower boundary.

b) This inequality follows directly from the transformation of the derivatives and part a).

c) Let $v \in J(\Omega)$, then we get with partial integration

$$\int_{-b_0}^{0} \frac{|v|^2}{\sqrt{-\eta}}\,d\eta = -2|v|^2\sqrt{-\eta}\,\Big|_{-b_0}^{0} + 4\int_{-b_0}^{0} |v|\,|\nabla_\eta v|\sqrt{-\eta}\,d\eta$$

$$\le 4C(b_0)\int_{-b_0}^{0} \frac{|v|}{(-\eta)^{1/4}}\,|\nabla_\eta v|\,d\eta$$

$$\le 4C(b_0)\left(\int_{-b_0}^{0} \frac{|v|^2}{(-\eta)^{1/2}}\,d\eta\right)^{1/2}\left(\int_{-b_0}^{0} |\nabla_\eta v|^2 d\eta\right)^{1/2}$$

and so

$$\int_{-b_0}^{0} \frac{|v|^2}{\sqrt{-\eta}}\,d\eta \le C(b_0)\int_{-b_0}^{0} |\nabla_\eta v|^2 d\eta\,.$$

Integrating with respect to ξ and extending the result to $H(\Omega)$ by continuity, we obtain the result on S. Transformation of the integrals to Ω and part b) show the result on Ω.

d) This expression refers to the L_2-trace on a plane $x_j \equiv$ const. Again we take $v \in J(\Omega)$, then we get for a fixed $\tilde{x}_1$

$$|v(\tilde{x}_1, x_2, y)|^2 \le 2 \int_{\tilde{x}_1}^{\infty} v \partial_1 v \, dx_1 \le \int_{\tilde{x}_1}^{\infty} \left(|v|^2 + |\partial_1 v|^2\right) dx_1 .$$

Let $\Omega_0 = \{(x,y) \in \Omega\colon |x_1| > \tilde{x}_1\}$, then we get with b)

$$\int_{x_2 \in \mathbb{R}} \int_{-b(\tilde{x}_1, x_2)}^{h(\tilde{x}_1, x_2)} |v(\tilde{x}_1, x_2, y)|^2 \le \int_{\Omega_0} \left(|v|^2 + |\partial_1 v|^2\right) \le C \, \|v\|^2_{\Omega_0} .$$

$\|v\|_{\Omega_0}$ tends to zero as $\tilde{x}_1 \longrightarrow \infty$, therefore this is true for the integral on the left-hand side. We get the corresponding result for $x_2 \equiv$ const by changing the roles of x_1 and x_2. The result for $v \in H(\Omega)$ follows by continuity ∎

Later we need a mollifier with support near the boundary.

Lemma 2.2. *For every* $\varepsilon > 0$ *there is a mollifier* $\mu(\cdot;\varepsilon) \in C^\infty(\mathbb{R};[0,1])$ *with* supp $\partial_t \mu \subset (0,\varepsilon]$; $\mu(0;\varepsilon) = 1$, $\mu(\varepsilon;\varepsilon) = 0$, $\mu(t;\varepsilon) \le \varepsilon/t^{1/4}$ *and* $|\partial_t \mu(t;\varepsilon)| \le \varepsilon/t^{1/4}$ *for* $t > 0$.

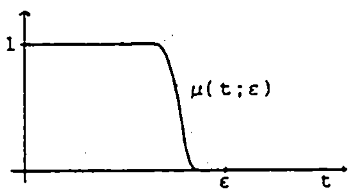

Fig. 2.2. The mollifier $\mu(t;\varepsilon)$

Proof. For every $\alpha > 0$ and $\delta \in (0, 1/4)$ let $\tau(t) = \tau(t;\alpha,\delta)$ be a C^∞-mollifier corresponding to Fig. 2.3. The function τ should have the following properties: $0 \le \tau(t) \le t^{-1/4}$ everywhere; $\tau(t) = t^{-1/4}$ in $[2\alpha\delta, (1-2\delta)\alpha]$ and $\tau(t) = 0$ for $t \le \alpha\delta$ and $t \ge (1-\delta)\alpha$.

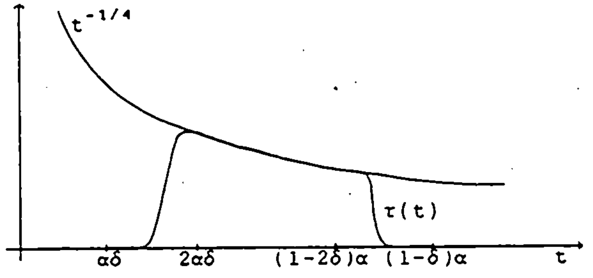

Fig. 2.3. The mollifier $\tau(t)$

Let $T = \int_0^\alpha \tau(s)\, ds$, then we define $\mu(t;\alpha,\delta) = 1 - \frac{1}{T} \int_0^t \tau(s)\, ds$. Because of

$$T > \int_{2\alpha\delta}^{\alpha(1-2\delta)} s^{-1/4} ds = 4/3 s^{3/4} \Big|_{2\alpha\delta}^{\alpha(1-2\delta)} = 4/3 \left[(\alpha(1-2\delta))^{3/4} - (2\alpha\delta)^{3/4} \right]$$

we define δ by $4/3\big((\alpha(1-2\delta))^{3/4} - (2\alpha\delta)^{3/4}\big) = 1/\varepsilon$ such that $1/T < \varepsilon$ and $|\partial_t \mu| = \tau(t)/T \le \varepsilon t^{-1/4}$. If we take $\alpha(\varepsilon) = \varepsilon/(1-\delta)$ and $\mu(t;\varepsilon) = \mu(t;\alpha(\varepsilon), \delta(\varepsilon))$, we get $\operatorname{supp} \partial_t \mu \subset (0,\varepsilon]$, and for $t \in \operatorname{supp} \partial_t \mu$ we get $\varepsilon t^{-1/4} \ge 1 \ge \mu(t;\varepsilon)$ ∎

3. THE FLOW IN A FIXED DOMAIN

Now we consider the Navier-Stokes equations

$$\text{(N)} \qquad \left.\begin{aligned} -\nu\Delta u + (u\cdot\nabla)u + \nabla p &= f_s \\ \operatorname{div} u &= 0 \end{aligned}\right\} \quad \text{in } \Omega$$

with boundary conditions

$$\begin{array}{llll} (\tilde{B}1) & u = 0 & & \text{on } \Gamma_- \\ (\tilde{B}2) & un = 0 & & \text{on } \Gamma \\ (\tilde{B}3) & t^{(i)}T(u,p)n = 0 & (i = 1,2) & \text{on } \Gamma_+ \\ (\tilde{B}5) & u \longrightarrow q & & \text{as } |x| \longrightarrow \infty \end{array}$$

with $\Omega = \{(x,y) \in \mathbb{R}^2 \times \mathbb{R} \ :\ -b(x) < y < h(x)\}$, $h,b \in C^{3+m,\alpha}(\mathbb{R}^2)$ $(m \in \mathbb{N},\ 0 < \alpha < 1)$ where the functions b and h are known. Furthermore the derivatives of h and b-b_0 up to the order m+3+α are bounded by $\exp(-c_2|x|)$ as $|x| \longrightarrow \infty$ (c_2 = const). The force density f_s is of class $C^{m,\alpha}$ with all derivatives bounded by $\exp(-c_2|x|)$, too.

3.1. Existence and uniqueness of solutions. We will show the existence of a weak solution to this problem with the help of an a priori bound.

Definition 3.1. Let Ω be a domain as described before. A vector field $g \in C^\infty(\Omega\cup\Gamma)$ will be called *flux carrier*, if it satifies

$$\text{(G)} \qquad \begin{array}{llll} \operatorname{div} g = 0 & \text{in } \Omega, & gn = 0 & \text{on } \Gamma_+, \\ g = 0 & \text{on } \Gamma_-, & g \longrightarrow q & \text{as } |x| \longrightarrow \infty . \end{array}$$

Definition 3.2. A velocity field $u = g + v$ is a *weak solution* of (N), $(\tilde{B})$, if g is a flux carrier, $v \in H(\Omega)$ and

$$\frac{\nu}{2} \int_\Omega D(u):D(w) - \langle u,u,w\rangle_\Omega = \int_\Omega f_s w , \qquad \forall\, w \in J(\Omega) \tag{3.1}$$

or equivalently

$$\frac{\nu}{2} \int_\Omega D(v):D(w) - \langle v,g+v,w\rangle_\Omega - \langle g,v,w\rangle_\Omega$$
$$= - \frac{\nu}{2} \int_\Omega D(g):D(w) + \langle g,g,w\rangle_\Omega + \int_\Omega f_s w , \qquad \forall\, w \in J(\Omega). \tag{3.1'}$$

Equations (N1) correspond to (3.1): if u is a classical solution we obtain (3.1) upon multiplying (N1) by any $w \in J(\Omega)$ and integrating by parts. The converse, that (3.1) implies (N1), will be shown in Subsection 3.2.

Remark. The velocity $v \in H(\Omega)$ carries no flux, i.e. $\int_A vn\, d\sigma = 0$ for every cross-section A of Ω.

Lemma 3.1. *Let Ω be a domain as described before and define*

$$\nu_0 := \sup_{u\in H(S)\setminus\{0\}} \frac{\langle q,u,u\rangle_S}{[u,u]_S} .$$

For $\nu > \nu_0$ there exists a flux carrier g such that

$$\frac{\nu}{2}\int_\Omega D(v):D(v) - \langle g,v,v\rangle_\Omega = -\frac{\nu}{2}\int_\Omega D(g):D(v) + \langle g,g,v\rangle_\Omega + \int_\Omega f_s v \tag{3.2}$$

implies $\|v\|_\Omega \le C$ for any $v \in H(\Omega)$, where C depends on Ω, ν and a (see Section 1).

Proof. i) We construct g as follows: On a compact subset of $\Omega \cup \Gamma$ the velocity g consists of two parts, having their supports near the upper and the lower surfaces, respectively. At large distances g is the slightly disorted equilibrium velocity. To this end, we use some mollifiers: for any $\varepsilon > 0$ let $\mu(\cdot;\varepsilon) \in C^\infty([0,\infty);[0,1])$ be a mollifier for extending the

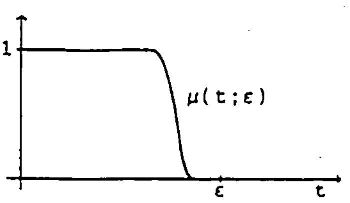

Fig. 3.1. The mollifier $\mu(t;\varepsilon)$

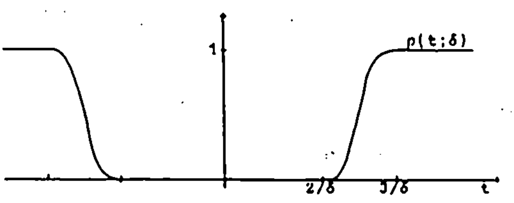

Fig. 3.2. The mollifier $\rho(t;\delta)$

boundary-value functions: $\mu(t;\varepsilon) \equiv 0$ for $t \geq \varepsilon$; $\mu(0;\varepsilon) = 1$, $\partial_t\mu(0;\varepsilon) = 0$ and $\mu(t;\varepsilon)$, $|\partial_t\mu(t;\varepsilon)| \leq \varepsilon t^{-1/4}$. The existence of μ was shown in Lemma 2.2. Furthermore we use $\rho(\cdot;\delta) \in C^\infty(\mathbb{R};[0,1])$ with $\rho(t;\delta) = 0$ for $|t| \leq 2/\delta$; $\rho(t;\delta) = 1$ for $|t| \geq 3/\delta$ and $|\partial_t\rho(t;\delta)| \leq c\delta$, $|\partial_t\partial_t\rho(t;\delta)| \leq c\delta^2$ for $2/\delta \leq |t| \leq 3/\delta$.

We set $g = \nabla F\cdot G\cdot\det DF^{-1}$, where $G = \mathrm{ROT}\, Q$ and $Q = Q_1 + Q_2$ is a vector potential in S. We define

$$Q_1 = \left\{ \begin{pmatrix} 0 \\ \frac{a}{6\nu} b_0^3\mu(-\eta;\varepsilon) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{a}{6\nu} b_0^3\ \mu(b_0+\eta;\varepsilon) \\ 0 \end{pmatrix} \right\} \left[1 - \rho(|\xi|;\delta)\right],$$

$$Q_2 = Q_a\rho(|\xi|;\delta),\ \text{with}\ Q_a = \left(0,\ -\frac{a}{2\nu}\left(\frac{1}{3}\eta^3 - b_0^2\eta\right) + \frac{a}{6\nu} b_0^3\ ,\ 0 \right)^T$$

and get $G_1 = \mathrm{ROT}\, Q_1$, $G_2 = \mathrm{ROT}\, Q_2$. Then $G = G_1 + G_2$ satisfies: $G = 0$ on Σ_-; $GN = 0$ on Σ_+; $\mathrm{DIV}\, G = 0$ in S and $G(\xi,\eta) \longrightarrow q$ as $|\xi| \longrightarrow \infty$. With the transformation of the velocity we see that g satisfies (G).

First we estimate the term $\{g,v,v\}_\Omega$ of equation (3.2). We get

$$|g_1|^2 \leq \varepsilon^2 C_3|\det DF^{-1}|\left(\frac{1}{(-\eta)^{1/4}} + \frac{1}{(b_0+\eta)^{1/4}}\right)^2,$$

for $|\nabla_1\rho| \leq 1$. This implies

$$|\{g_1,v,v\}_\Omega| \leq \varepsilon C_4\left[\int_\Omega|\det DF^{-1}|\left(\frac{|v|^2}{(-\eta)^{1/2}} + \frac{|v|^2}{(b_0+\eta)^{1/2}}\right)\right]^{1/2}\|v\|_\Omega$$

and with Lemma 2.1 and Korn's inequality [9] we have

$$|\{g_1,v,v\}_\Omega| \leq \varepsilon C_5\|v\|_\Omega^2 \leq \varepsilon C_6[v,v]_\Omega\ . \tag{3.3}$$

For the corresponding term with g_2 we get

$$\{g_2,v,v\}_\Omega = \int_\Omega \rho q_1\det DF^{-1}\left(v_1\partial_1v_1 + \nabla_1F_3v_1\partial_1v_3\right)$$

$$+ \int_\Omega \gamma(\eta)\nabla_3F_3\nabla_1\rho v_1\partial_1v_3\det DF^{-1}$$

with $\gamma(\eta) = \frac{a}{2\nu}\left(\frac{1}{3}[\ b_0^3 - \eta^3\ ] + b_0^3\eta\right)$. Because of $|\nabla_1\rho| \leq c\delta$ the last integrand is small on the support of $\rho(\cdot;\delta)$. It follows that

$$\{g_2,v,v\}_\Omega = \{\rho q_\Omega,v,v\}_\Omega + \int_\Omega R_1\ , \tag{3.4}$$

where the term R_1 consists of all terms, which are small on the support of $\rho(\cdot;\delta)$. The function q_Ω is the equilibrium velocity q of S transformed to

Ω. With Lemma 2,1 and Korn's inequality we get

$$| \int_\Omega R_1 | \le \beta(\delta)\|v\|^2_{1,2,\Omega} \le C_7\beta(\delta)[v,v]_\Omega ,$$

with $\beta(\delta) \longrightarrow 0$ as $\delta \longrightarrow 0$. Let

$$\bar{\nu}(\delta) := \sup_{u\in H(\Omega)\setminus\{0\}} \frac{\{\rho q_\Omega, u, u\}_\Omega}{[u,u]_\Omega} . \tag{3.5}$$

Like Amick [4] we can show that $\bar{\nu}(\delta) \longrightarrow \nu_0$ as $\delta \longrightarrow 0$. Collecting (3.3) - (3.5) we get $|\{g,v,v\}_\Omega| \le [\,\varepsilon C_6 + \bar{\nu}(\delta) + C_7\beta(\delta)\,][v,v]_\Omega$. If we choose $\varepsilon = \varepsilon_0$ and $\delta = \delta_0$ sufficiently small and $\nu > \nu_0$, we get

$$|\{g,v,v\}_\Omega| \le 1/2(\nu + \nu_0)[v,v]_\Omega . \tag{3.6}$$

ii) With $v = V + EV$ and $g = G + EG$ for the velocities, the corresponding formulas for the transformation of the derivatives and partial integration we get for the terms on the right-hand side of (3.2)

$$|\int_\Omega f_s v| \le \|f_s\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)} \le C\|v\|_\Omega ,$$

$$\left|\int_\Omega D(g):D(v)\right| \le \left|\int_{\Sigma_+} VT(G)N - \int_S \Delta G\; V\right| + \left|\int_S E_{\alpha\beta}\nabla^\alpha G\nabla^\beta V\right| ,$$

$$|\{g,g,v\}_\Omega| \le |-\{V,G,G\}_S| + \left|\int_S E_{ijk,\alpha} G_i G_j \nabla^\alpha V_k\right| ,$$

where

$$E_{\alpha\beta},\ E_{ijk,\alpha} = O(\nabla^\gamma h, \nabla^\gamma(b-b_0)) \quad (|\alpha|,|\beta| \le 1,\ |\gamma| \le 2;\ i,j,k \in \{1,2,3\}).$$

Now we divide the domain S into two parts, the bounded part $S' = \{(\xi,\eta) \in S: |\xi| < 3/\delta_0\}$ and the unbounded part $S'' = S\setminus S'$. The function g and its derivatives have bounded integrals on S'. Poincaré's inequality and the bounded embedding $W^1_2(S') \subset L_2(\partial S' \cap \Sigma_+)$ lead to

$$\left|\int_{S'} \Delta G\; V\right| + \left|\{V,G,G\}_{S'}\right| + \left|\int_{\partial S' \cap \Sigma_+} VT(G)N\right| \le C\|\nabla V\|_{L_2(S')} .$$

In S'' we have $G = q$, therefore

$$\int_{\partial S'' \cap \Sigma_+} VT(G)N = \int_{\partial S'' \cap \Sigma_+} VT(q)N = 0$$

and

$$\int_{S''} \Delta G\, V = C(G)\int_{S''} V_1 = \int_{\xi_1=-\infty}^{\infty} \int_{X(\xi_1)} VN = 0 ,$$

because V carries no flux. Here $X(\xi_1)$ is a part of the (ξ_1,ξ_2,η)-plane for every $\xi_1 \in \mathbb{R}$, which lies in S and has $N = (1,0,0)$ as normal vector. The triple product $\langle V,G,G\rangle_S$ is zero because $G = q$ in S''.

By h, $b-b_0 \in W_2^3(\mathbb{R}^2)$ we get for the remaining terms

$$\left|\int_S E_{\alpha\beta}\nabla^\alpha G\nabla^\beta V\right| + \left|\int_S E_{ijk,\alpha}G_i G_j \nabla^\alpha V_k\right| \le C\|\nabla V\|_{L_2(S)},$$

where $C = C\big(|G|_{C^1(S\cup\Sigma)}, \|h\|_{3,2,\mathbb{R}^2}, \|b-b_0\|_{3,2,\mathbb{R}^2}\big)$. So we get for the terms at the right-hand side of (3.2)

$$\left|-\frac{\nu}{2}\int_\Omega D(g):D(v) + \langle g,g,v\rangle_\Omega + \int_\Omega f_s v\right| \le C\|\nabla V\|_{L_2(S)} \le C\|v\|_\Omega . \tag{3.7}$$

Collecting the estimates (3.6), (3.7) and the definition of $[v,v]_\Omega$ it follows that $[v,v]_\Omega \le 2C/(\nu-\nu_0)\|v\|_\Omega$. With Korn's inequality we get the assertion $\|v\|_\Omega \le C(\nu,\nu_0,h,b,b_0,a)$ ∎

Theorem 3.2. *Problem* (N), ($\tilde{B}$) *has a weak solution* u *for every viscosity* $\nu > \nu_0$.

Proof. Let $\{S_m\}_{m\in\mathbb{N}}$ be an expanding sequence of simply connected bounded subdomains of S such that $S_m \longrightarrow S$ as $m \longrightarrow \infty$ (see Fig. 3.3) and ∂S_m is of class C^3. The sequence $\{\Omega_m\}_{m\in\mathbb{N}} = \{F(S_m)\}_{m\in\mathbb{N}}$ of bounded C^3 subdomains of Ω converges to Ω as $m \longrightarrow \infty$.

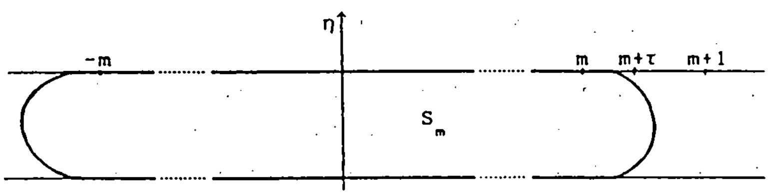

Fig. 3.3. A domain S_m

We consider the problem of finding a weak solution (u_m,p_m) of (N) in Ω_m with boundary coditions

$$u_m = 0 \qquad \text{on } \Gamma_-^m = \partial\Omega_m \cap \Gamma_-,$$

$$u_m n = 0, \qquad t^{(1)}T(u_m,p_m)n = 0 \qquad \text{on } \Gamma_+^m = \partial\Omega_m \cap \Gamma_+,$$

$$\alpha(|x|)t^{(1)}T(u_m,p_m)n + (1-\alpha(|x|))u_m = (1-\alpha(|x|))g \qquad \text{on } \partial\Omega_m\setminus\Gamma^m,$$

with a $C^\infty(\mathbb{R}^+;[0,1])$-function $\alpha = \alpha(t)$ such that, for $1 \gg \tau > 0$, $\alpha(t) \equiv 1$ for $m \le t \le m+\tau$ and $\alpha(t) \equiv 0$ for $m + 2\tau \le t$. So we are looking for a function $u_m = v_m + g$, $v_m \in H(\Omega_m)$ and g as before, which satisfies

$$\frac{\nu}{2}\int_{\Omega_m} D(u_m):D(\varphi) = \{u_m,u_m,\varphi\}_{\Omega_m} + \int_{\Omega_m} f_s\varphi \ , \qquad \forall\ \varphi \in J(\Omega_m).$$

Standard methods show the existence of a weak solution u_m in Ω_m. Because Ω_m is bounded for fixed m, we have $u_m \in H(\Omega_m)$. The domain Ω_m lies in a strip of finite width, so with Poincaré's inequality it follows $\|\omega\|_{L_2(\Omega_m)} \le C\|\omega\|_{\Omega_m}$ for all $\omega \in H(\Omega_m)$ and we conclude that $H(\Omega_m)$ is continously embedded in $L_2(\Omega_m)$. This allows the analogue of (3.1) to be extended to all test functions in $H(\Omega_m)$. Choosing $w = v_m$ we get

$$\frac{\nu}{2}\int_{\Omega} D(v_m):D(v_m) - \{g,v_m,v_m\}_{\Omega} = -\frac{\nu}{2}\int_{\Omega} D(g):D(v_m) + \{g,g,v_m\}_{\Omega} + \int_{\Omega} f_s v_m,$$

where $v_m \in H(\Omega)$, if we set $v_m = 0$ in $\Omega\backslash\Omega_m$. Thus each v_m satisfies (3.2) in Lemma 3.1, so that $\|v_m\|_{\Omega}$ is bounded independently of m. Hence there exists a subsequence $\{v_{m_n}\}$ and an element $v \in H(\Omega)$, such that $v_{m_n} \longrightarrow v$ weakly in $H(\Omega)$ as $m_n \longrightarrow \infty$. Now we have to show that v is a solution of (3.1'). For simplicity we now write $\{v_m\}$ instead of $\{v_{m_n}\}$. The function v_m is a solution of

$$\frac{\nu}{2}\int_{\Omega_m} D(v_m):D(\varphi) - \{v_m,g+v_m,\varphi\}_{\Omega_m} - \{g,v_m,\varphi\}_{\Omega_m}$$
$$= -\frac{\nu}{2}\int_{\Omega_m} D(g):D(\varphi) + \{g,g,\varphi\}_{\Omega_m} + \int_{\Omega_m} f_s\varphi \ , \qquad \forall\ \varphi \in J(\Omega_m).$$

For any given $w \in J(\Omega)$ we have supp $w \subset \Omega_k$ for some k, so that v_m satisfies (3.1') for that w, if $m \ge k$. With this fixed w the linear terms on the left-hand side of (3.1') define a bounded linear functional, say $f_{w,g}: H(\Omega) \longrightarrow \mathbb{R}$. Then $f_{w,g}(v_m) \longrightarrow f_{w,g}(v)$ as $m \longrightarrow \infty$ by the definition of weak convergence. For the nonlinear part we have $\tau_m = \{v,v,w\}_{\Omega} - \{v_m,v_m,w\}_{\Omega}$ and therefore

$$|\tau_m| \le \Big(\|v-v_m\|_{L_4(\Omega_k)}\|v\|_{L_4(\Omega_k)} + \|v_m\|_{L_4(\Omega_k)}\|v-v_m\|_{L_4(\Omega_k)}\Big)\|w\|_{\Omega_k}$$

For the bounded domain Ω_k the embedding $W_2^1(\Omega_k) \subset L_4(\Omega_k)$ is compact. So v_m converges strongly to v in $L_4(\Omega_k)$ and with Poincaré's inequality and Lemma 3.1 we get

$$|\tau_m| \le C_1\|w\|_{\Omega_k}\Big(\|v\|_{1,2,\Omega_k} + \|v_m\|_{1,2,\Omega_k}\Big)\|v-v_m\|_{L_4(\Omega_k)} \le C_2\|v-v_m\|_{L_4(\Omega_k)},$$

where C_2 is independent of m. With $v_m \longrightarrow v$ as $m \longrightarrow \infty$ in $L_4(\Omega_k)$ it follows $\{v,v,w\}_{\Omega} - \{v_m,v_m,w\}_{\Omega} \longrightarrow 0$ as $m \longrightarrow \infty$. Therefore v is a solution of (3.1'); and $u = g + v$ is a weak solution of the Navier-Stokes equations ∎

With standard methods one can show

Theorem 3.3. *The weak solution of* (N), ($\tilde{B}$) *is unique for large values of* ν *and small values of* $\|f_s\|_{L_2(\Omega)}$ *and a.*

To exclude that a classical solution is in the generally, greater space

$$H^\bullet(\Omega) = \{ u \in W_2^1(\Omega): un = 0 \text{ on } \Gamma_+,\ u = 0 \text{ on } \Gamma_-,\ \operatorname{div} u = 0 \}$$

we show that for our type of domain the spaces $H^\bullet(\Omega)$ and $H(\Omega)$ are identical. For that we need some notations:

$$I(\Omega) = \{ u \in C^\infty(\Omega): un = 0 \text{ on } \Gamma_+,\ u = 0 \text{ on } \Gamma_-,$$
$$\text{supp } u \text{ is compact in } (x_1,x_2)\text{-direction in } \Omega \},$$

$$I^\sigma(\Omega) = \{ u \in I(\Omega): \operatorname{div} u = 0 \},$$

$$H(\Omega) = \overline{I^\sigma(\Omega)}^D, \quad H^\bullet(\Omega) = \{ u \in \overline{I(\Omega)}^D: \operatorname{div} u = 0 \}.$$

Furthermore we need the domains

$$A^R = \{z \in \mathbb{R}^3:\ R < |z| < 2R\}, \quad \tilde{\Omega}^R = \Omega \cap A^R \quad \text{and} \quad \Omega^R = \tilde{\Omega}^R \setminus \Omega_\varepsilon,$$

where Ω_ε is a small set such that Ω^R is of $C^{1,1}$ type. Boundary pieces of Ω^R are denoted by Γ_-^R for the lower boundary, Γ_+^R for the upper boundary, $S^R = \{z \in \bar{\Omega}:\ |z| = R\}$ and S^{2R}, respectively, for the lateral boundaries. $\partial\tilde{\Omega}^R = \Gamma_-^R \cup \Gamma_+^R \cup S^R \cup S^{2R}$ is the boundary of $\tilde{\Omega}^R$, and $\partial\Omega^R$ is the boundary of the smoothed domain (cf. Fig. 3.4).

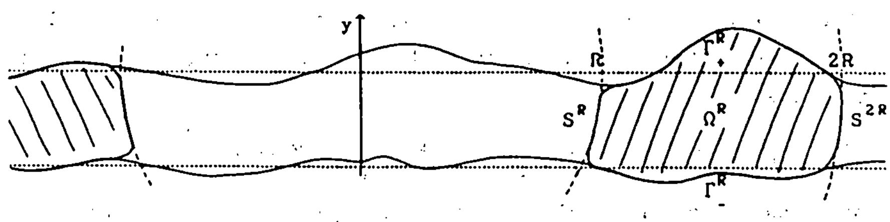

Fig. 3.4. The domains Ω_R and $\tilde{\Omega}^R$

Theorem 3.4. *Assume the following:*

1) For every $R \geq R_0 > 0$, Ω^R *is a connected domain and for every* $\varphi \in \hat{L}_2(\Omega^R) = \{\psi \in L_2(\Omega^R):\ (\psi,1) = 0\}$ *there is a* $u \in \mathring{W}_2^1(\Omega^R)$ *such that*

(i) $\operatorname{div} u = \varphi$ *in* Ω^R *and (ii)* $\|Du\|_{L_2(\Omega^R)} \leq C_1 R \|\varphi\|_{L_2(\Omega^R)}$.

2) There is a $C_2 > 0$ *such that* $\|w\|_{L_2(\Omega^R)} \le C_2\|Dw\|_{L_2(\Omega^R)}$, *for* $w \in H^*(\Omega^R)$.

3) For every domain $\Omega_{2R} = \{ z \in \Omega : |z| < R \}$, $H(\Omega_{2R}) = H^*(\Omega_{2R})$.

Then $H(\Omega) = H^*(\Omega)$.

Proof. Because $H(\Omega) \subset H^*(\Omega)$ it remains to show $H^*(\Omega) \subset H(\Omega)$. Let $v \in H^*(\Omega)$ and $\zeta \in C^\infty(\mathbb{R})$ be a cut-off function with $\zeta(t) = 1$ for $0 \le t \le 1$ and $\zeta(t) = 0$ for $t \ge 2$.

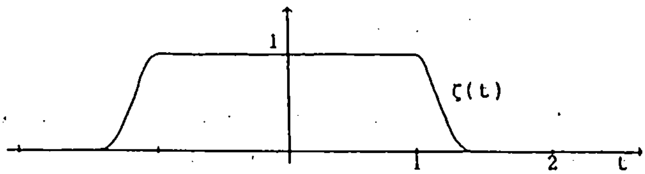

Fig. 3.5. The function $\zeta(t)$

The function ζ is monotonically decreasing in (1,2) with $|\zeta'| \le C_3$. We construct for the vector $v \in H^*(\Omega)$ a function v^R by $v^R(z) = v(z)\zeta^R(|z|) + u^R(z)$, where $\zeta^R(r) = \zeta(r/R)$ and u^R is an element of $\mathring{W}^1_2(\Omega^R)$ with $\mathrm{div}(u^R) = -\nabla\zeta^R v$ in Ω^R and $\|Du^R\|_{L_2(\Omega^R)} \le C_1 R\|\nabla\zeta^R v\|_{L_2(\Omega^R)}$. The existence of such a function u^R follows by assertion 1. The function v^R satisfies the boundary conditions (B1), (B2) and is solenoidal because $\mathrm{div}\, v^R = \mathrm{div}(v)\zeta^R + v\nabla\zeta^R + \mathrm{div}\, u^R = 0$. So we get $v^R \in H^*(\Omega)$. Furthermore v^R vanishes for $|z| \ge 2R$. Thus its restriction to Ω_{2R} belongs to $H^*(\Omega_{2R})$. But, according to assertion 3, $H(\Omega_{2R}) = H^*(\Omega_{2R})$, so we can find a sequence $\{w^R_n\} \subset I^\sigma(\Omega_{2R})$ with $w^R_n \xrightarrow[D]{} v^R$ in Ω_{2R}. Extending the vector w^R_n to $\Omega\backslash\Omega_{2R}$ by zero, we get $w^R_n \in I^\sigma(\Omega)$ and $w^R_n \xrightarrow[D]{} v^R$ in Ω. So we have

$$\|v - w^R_n\|_\Omega \le \|v - v^R\|_\Omega + \|w^R_n - v^R\|_\Omega . \tag{3.8}$$

With assertion 1/(ii) and 2 we obtain

$$\|v - v^R\|_\Omega = \|Dv - Dv\zeta^R - v\nabla\zeta^R - Du^R\|_{L_2(\Omega)}$$
$$\le \left(1 + C_3C_2/R + C_1C_3C_2\right)\|Dv\|_{L_2(\Omega\backslash\Omega_R)} \le C_4\|v\|_{\Omega\backslash\Omega_R} .$$

From (3.8) it follows that $\|v - w^R_n\|_\Omega \le C_4\|v\|_{\Omega\backslash\Omega_R} + \|w^R_n - v^R\|_\Omega$. Selecting first a sufficiently large R and then a sufficiently large n we can make the right-hand side of this inequality arbitrarily small; thus, any vector $v \in H^*(\Omega)$ can be approximated by a sequence $\{w_n\}$ in $I^\sigma(\Omega)$ ■

Now we want to apply this theorem to our problem. We have to prove the following assertions:

1) In three dimensions the set Ω^R, $R \ge R_0 \ge 0$, is a connected and bounded $C^{1,1}$-domain. With the results of Giaquinta and Modica [15] one gets for all $\varphi \in \hat{L}_2(\Omega^R)$ a $u \in \mathring{W}_2^1(\Omega^R)$, which satisfies (i) and (ii). The constant C depends on Ω^R by $C(\Omega^R) = C_1 R$.

2) Because of the boundary conditions for $w \in H^*(\Omega)$ the Poincaré inequality is true for w with C_2 independent of R.

3) The identity of the spaces $H(\Omega^{2R})$ and $H^*(\Omega^{2R})$ can be shown like Bemelmans has done in [10; Theorem 4].

3.2. Regularity of the weak solution. In this subsection we examine the regularity of the weak solution. We show the inclusion $u \in C^2(\Omega) \cup C(\Omega \cup \Gamma)$ and that there exists a pressure $p \in C^1(\Omega) \cup C(\Omega \cup \Gamma)$, such that (u,p) satisfies (N),($\tilde{B}$) pointwise. Such results are standard for weak solutions of the steady Navier-Stokes equations and so we will only list the results; for proofs we refer to [12]. Let V be a bounded domain such that $V \subset\subset \Omega$ and $\varphi \in C_0^\infty(V)$. Then we take $\Phi = \operatorname{rot} \varphi \in J(V)$ as test function in (3.1) and partial integration leads to

$$-(\Delta\varphi, \operatorname{rot} w)_V + 1/\nu\{\operatorname{rot} \varphi, w, w\}_V = 1/\nu(\operatorname{rot} \varphi, f_s)_V \qquad \forall\ \varphi \in C_0^\infty(V).$$

This equation can be examined by L_p-estimates (cf. Agmon [2] for the proof of the inner regularity) and by the results of Solonnikov [21] and Solonnikov, Ščadilov [22] for the regularity up to the boundary. Repeated use of embedding theorems and the L_p-theory of Agmon, Douglis and Nirenberg [3] then show the regularity up to the boundary and the following decay result.

Theorem 3.5. *If* $f_s \in C^{m,\alpha}(\Omega \cup \Gamma) \cap W_2^m(\Omega)$ *($m \in \mathbb{N}$, $0 < \alpha < 1$) and u is a weak solution of* (N),($\tilde{B}$), *then*

i) $u \in C^{m+2,\alpha}(\Omega \cup \Gamma)$ *and* $v \in W_s^{m+2}(\Omega)$ $(s \ge 2)$,

ii) there exists a pressure $p \in C^{m+1,\alpha}(\Omega \cup \Gamma)$ *such that* (u,p) *satisfies* (N),($\tilde{B}$) *pointwise,*

iii) for every $\beta, \gamma \in \mathbb{N}^3$ *with* $|\beta| \le m+2$ *and* $|\gamma| \le m$

$$|D^\beta(u(x,y) - q(y))| \longrightarrow 0, \quad \textit{uniformly as } |x| \longrightarrow \infty \textit{ in } \Omega \cup \Gamma,$$

$$|D^\gamma(\nabla p(x,y) - \nabla p_a)| \longrightarrow 0, \quad \textit{uniformly as } |x| \longrightarrow \infty \textit{ in } \Omega \cup \Gamma.$$

The equivalent is true for the corresponding Hölder derivatives.

3.3. Decay rates. We now examine the asymptotic behaviour of the solution of (N), $(\tilde{B})$ as $|x| \longrightarrow \infty$ in Ω, in case $D^{\alpha}(b-b_0)$, $D^{\alpha}h$ and $D^{\beta}f_s$ ($|\alpha| \le k+3$, $|\beta| \le k$) decay like $\exp(-c_2|x|)$ as $|x| \longrightarrow \infty$ ($c_2 > 0$). We show that $D^{\alpha}v$ ($|\alpha| \le k+2$) is bounded by $\exp(-c_2|x|)$ in $\Omega_{\tilde{a}}$, where $\Omega_{\tilde{a}} = F(S_{\tilde{a}})$ with $S_{\tilde{a}} = \{(\xi,\eta) \in S\colon \ \|\xi\|_1 = |\xi_1| + |\xi_2| > \tilde{a}\}$ and $\tilde{a}$ has been chosen so large that $G = q$ in $S_{\tilde{a}}$.

We transform the Navier-Stokes equations to S and define the linear operator L in $S_{\tilde{a}}$, which contains the linear nondecaying part of the first three Navier-Stokes equations $LV = -\nu\Delta V + (q\cdot\nabla)V + (V\cdot\nabla)q$ in $S_{\tilde{a}}$. The Navier-Stokes equations and $\nu\Delta q = -a = \nabla P_a$ and $(q\cdot\nabla)q = 0$ give

$$\begin{aligned}
&LV = -(V\cdot\nabla)V - \nabla(P-P_a) + E_1 && \text{in } S_{\tilde{a}}\,, \\
&V = 0 && \text{on } \Sigma^-_{\tilde{a}} = \Sigma_- \cap \bar{S}_{\tilde{a}} \\
&VN = 0 && \\
&T^{(1)}T(V,P)N = E_0 && \text{on } \Sigma^+_{\tilde{a}} = \Sigma_+ \cap \bar{S}_{\tilde{a}} \\
&|V| \longrightarrow 0 && \text{as } |\xi| \longrightarrow \infty\,,
\end{aligned} \tag{3.9}$$

where E_0 and E_1 contain the terms, which originate from the transformation of the derivatives: $E_0 = O\big(\nabla^{\beta}h, \nabla^{\beta}(b-b_0)\big)\cdot\lambda\big(\nabla^{\gamma}V\big)$, $E_1 = O\big(\nabla^{\alpha}h, \nabla^{\alpha}(b-b_0)\big)$; $\lambda\big(\nabla^{\beta}V, \nabla^{\beta}q, V\nabla^{\gamma}V, q\nabla^{\gamma}q, V\nabla^{\gamma}q, q\nabla^{\gamma}V, \nabla^{\gamma}P\big)$ as $|\xi| \longrightarrow \infty$ ($|\alpha| \le 3$, $|\beta| \le 2$ and $|\gamma| \le 1$). The components of the vector-valued function λ are linear combinations of its arguments. For the investigations of the decay it is more favourable to change to the rotational form of L and to work with the vector potential $\phi \in V(S)$, where $V = \mathrm{ROT}\ \phi$ and $\mathrm{DIV}\ \phi = 0$. We get the operator

$$\begin{aligned}
(\hat{L}\phi)_p := -\Delta^2\phi_p &- \big(\partial_s q_l \partial_{ls}\phi_p - \partial_s q_l \partial_{lp}\phi_s + q_l\partial_{ssl}\phi_p\big) \\
&- \varepsilon_{jsp}\big(\varepsilon_{lkl}\partial_{sk}\phi_l\partial_l q_j + \varepsilon_{lkl}\partial_k\phi_l\partial_{sl}q_j\big) \quad \text{in } S
\end{aligned} \tag{3.10}$$

($p = 1,2,3$) with boundary conditions

$$\begin{aligned}
&\phi = \mathrm{ROT}\ \phi = 0 && \text{on } \Sigma_- \\
&\phi T^{(1)} = \mathrm{ROT}\ \phi\cdot N = 0 && \\
&T^{(1)}\cdot T(\mathrm{ROT}\ \phi)\cdot N = 0 && \text{on } \Sigma_+ \\
&\phi,\ \mathrm{ROT}\ \phi \longrightarrow 0 && \text{as } |x| \longrightarrow \infty.
\end{aligned} \tag{3.11}$$

In order to examine (3.10), (3.11) we consider the adjoint problem. For $\nu > \nu_0$ the corresponding bilinear form is coercive and bounded. Since Dirac's delta distibution is a bounded linear functional in $V(S)$, there exists for each $z_0 \in S$ a Green matrix function $G_{jp}(z_0,\cdot) \in V(S)$ (j, $p = 1$, 2,3) such that

$$\begin{aligned}
&\hat{L}^{*}G(z_0,z) = \delta(z_0-z)I && \text{in } S\\
&G(z_0,z) = \mathrm{ROT}\ G(z_0,z) = 0 && \text{on } \Sigma_-\\
&T^{(1)}\cdot G(z_0,z) = N\cdot\mathrm{ROT}\ G(z_0,z) = 0 && \\
&T^{(1)}\cdot T(\mathrm{ROT}\ G(z_0,z))\cdot N = 0 && \text{on } \Sigma_+\\
&G(z_0,z),\ \mathrm{ROT}\ G(z_0,z) \longrightarrow 0 && \text{as } |x| \longrightarrow \infty.
\end{aligned} \tag{3.12}$$

Then the solution of (3.10), (3.11) is given by $\phi(z_0) = \int_S G(z_0,z)f(z)\,dz$.

For the Green function we have the following decay result.

Theorem 3.6. *Let $S = \mathbb{R}^2 \times (-b_0,0)$ and $\nu > \nu_0$. Then for each $z_0 \in S$:*

a) There exists a Green matrix function $G(z_0,\cdot) \in V(S)$, which solves (3.12).

b) There exist positive constants $C = C(\nu)$ and $c = c(\nu)$ independent of z_0 such that, with $r = |z-z_0|$,

$$|\nabla_0^{\alpha}\nabla^{\beta}G(z_0,z)| \le \begin{cases} C/r^3 & \text{for } \|\xi_0-\xi\|_1 \le 2\\ C\exp(-c\|\xi_0-\xi\|_1) & \text{for } \|\xi_0-\xi\|_1 > 2,\end{cases}$$

where α and β are multi-indices satisfying $|\alpha| + |\beta| = 4$.

c) Analogous bounds hold for derivatives of order 3, 2, 1 and 0 with C/r^3 replaced by C/r^2, C/r, C and C, respectively, for $\|\xi_0-\xi\|_1 \le 2$.

d) For $|\alpha| + |\beta| = k > 4$ there are positive constants C_k such that

$$|\nabla_0^{\alpha}\nabla^{\beta}G(z_0,z)| \le C_k\exp(-c\|\xi_0-\xi\|_1) \qquad \text{for } \|\xi_0-\xi\|_1 > 2.$$

Remark. Amick [5] has shown this result for the Dirichlet problem in two dimensions. All steps of his proof can also be done with our boundary conditions, only slight modifications are necessary.

In what follows let $\nu > \nu_0$ and we consider, for the present, the domain $S_d = \{(\xi,\eta) \in S:\ \|\xi\|_1 > d\}$, where d will be fixed later. We multiply the Navier-Stokes equations (N1) in Ω with a function $h \in \tilde{H}(\Omega)$ and integrate over Ω. By partial integration of the highest order terms and transforming the integrals to S we get

$$-\nu\int_S \Delta H\ V + \int_S H\big((G_s\cdot\nabla)V + (V\cdot\nabla)G_s\big) - \int_{\Sigma_+} VT(H)N$$

$$= \int_S H\big(\nu\Delta G_s - (G_s\cdot\nabla)G_s + f_s\big) + P\ \mathrm{DIV}\ H$$

$$+ \int_S E_1\lambda_1\big(HG_s\nabla^\alpha G_s, HG_s\nabla^\alpha V, HV\nabla^\alpha G_s, HV\nabla^\alpha V, Hf_s, H\nabla^\sigma P\big)$$

$$+ \int_S E_{\alpha\beta}\lambda_2\big(\nabla^\alpha H\nabla^\beta(V+G_s)\big) - \int_S H(V\cdot\nabla)V$$

($|\alpha|$, $|\beta| \le 1$, $|\sigma| = 1$). The terms E_1 and $E_{\alpha\beta}$ are of order $O\big(\nabla^\gamma h, \nabla^\gamma(b-b_0)\big)$ as $|\xi| \longrightarrow \infty$ ($|\gamma| \le 2$). Every term in the following text, which is denoted by E_i ($i \in \mathbb{N}$), is of this form with $|\gamma| \le 3$. To avoid confusion we denote by $G_s(\zeta)$ the vector potential $G(\zeta)$. The variable of integration is z.

Now we introduce a vector potential $\psi \in \{\ \phi \in C(S\cup\Sigma)$: $\phi = \mathrm{ROT}\ \phi = 0$ on Σ_-; $T^{(1)}\phi = 0$, $\mathrm{ROT}\ \phi\cdot N = 0$ on $E_+\}$. The existence of such a potential is shown in [12]. Let $\mu = \mu(\cdot;d) \in C^\infty(\mathbb{R};[0,1])$ be a mollifier such that $\mu(t;d) = 1$ for $t \ge d$ and $\mu(t;d) = 0$ for $t \le d-3$. We define $S_{d-3,d} = \{(\xi,\eta) \in S$: $d-3 < \|\xi\|_1 < d\}$. Then for $d-3 \ge \tilde{a}$ we have $\mathrm{supp}\ \mu \subset S_{d-3}$ and $\mathrm{supp}\ \mu'$, $\mathrm{supp}\ \mu'' \subset S_{d-3,d}$. So in the above integrals we get $G_s = q$ and therefore $\nu\Delta G_s = \nu\Delta q = -a = \nabla P_a$ and $(G_s\cdot\nabla)G_s = (q\cdot\nabla)q = 0$. With the definition of the operators $\hat{L}^*$ and L^* and $H = \mu\ \mathrm{ROT}\ G$ we get for $z_0 \in S_d$ the representation

$$\psi(z_0) = \int_{S_d} \mathrm{ROT}\ G(z_0,z)\ (V\cdot\nabla)V\ dz + T(z_0), \tag{3.13}$$

where $T = T_1 + T_2$ and

$$T_1(z_0) = -\int_{S_{d-3,d}} \mathrm{ROT}\ L^*(\mu\ \mathrm{ROT}\ G)\ \psi\ dz - \int_{S_{d-3,d}} \mu\ \mathrm{ROT}\ G\ (V\cdot\nabla)V\ dz$$
$$+ \int_{\Sigma^+_{d-3,d}} \lambda_3\big(\nabla^\sigma\mu V\nabla^\alpha G, \nabla^\pi\mu\psi\nabla^\gamma G\big)\ d\sigma(z)$$
$$+ \int_{S_{d-3,d}} E_\sigma\nabla^\sigma\mu\lambda_5\big(\nabla^\delta G\nabla^\alpha V,\ GV\nabla^\beta V,\ \nabla^\delta G,\ \nabla^\beta G\nabla^\alpha P\big)$$
$$+ \lambda_6\big(\nabla^\alpha\mu\nabla^\beta G(P-P_a)\big)\ dz\ ,$$

$$T_2(z_0) = \int_{S_{d-3}} \mu E_2\lambda_4\big(\nabla^\gamma G\nabla^\alpha V,\ \nabla^\alpha GV\nabla^\beta V,\ \nabla^\gamma G,\ \nabla^\alpha G\nabla^\sigma P\big)\ dz$$

($|\alpha|, |\beta|, |\delta| \le 1$, $|\gamma| \le 2$, $|\sigma| = 1$, $1 \le |\pi| \le 2$).

For each fixed $d \ge \tilde{a}$ we define the Banach spaces C and E by

$$C := C_B(S_d\cup\Sigma_d;\mathbb{R}^3),$$

$$E := E(S_d\cup\Sigma_d;\mathbb{R}^3) = \{\ \phi \in C:\ \|\phi\|_E = \sup_{z\in S} |\ \phi(z)\ \exp(c_2(|\xi|-d))| < \infty\ \}.$$

Lemma 3.7. *Define*

$$(A_\alpha\phi)(z_0) := \nabla_0^\alpha \int_{S_d} \mathrm{ROT}\; G(z_0,z)\; (\phi\cdot\nabla)V\; dz \quad \textit{for} \quad z_0 \in S_d \quad (|\alpha| \le 1)$$

for all $\phi \in C$ *and* $\phi \in E$. *Then*

a) $\|A_\alpha\phi\|_C \le \varepsilon(d)\|\phi\|_C$, b) $\|A_\alpha\phi\|_E \le \varepsilon(d)\|\phi\|_E$

and $\varepsilon(d) \longrightarrow 0$ *as* $d \longrightarrow \infty$.

Proof. a) Let $\phi \in C$. Since $G(z_0,z)$ is of class W_2^2 we get

$$|(A_\alpha\phi)(z_0)| \le C_1\|\phi\|_C\|\nabla V\|_{L_2}\|G(z_0,\cdot)\|_{W_2^2} \le C_2\|\nabla V\|_{L_2}\|\phi\|_C ,$$

where the constant C only depends on S. Because $V \in W_2^1(S)$ it follows $\|\nabla V\|_{L_2} = \|\nabla V\|_{0,2,S_d} \longrightarrow 0$ as $d \longrightarrow \infty$. We now show $A_\alpha\phi \in C$. For $z_0 \in S_d$, choose h such that $z_0 + h$ is in S_d, too. Then

$$\begin{aligned} &|(A_\alpha\phi)(z_0+h) - (A_\alpha\phi)(z_0)| \\ &\qquad \le C \int_{S_d} |\nabla_0^\alpha \mathrm{ROT}(G(z_0+h,z) - G(z_0,z))|\,|(\phi\cdot\nabla)V|\; dz \\ &\qquad \le C\|\phi\|_C\|\nabla V\|_{L_p}\|\nabla_0^\alpha(G(z_0+h,\cdot) - G(z_0,\cdot))\|_{W_q^1} , \end{aligned}$$

with $1/p + 1/q = 1$. Since $V \in W_p^1(S)$ $(p \ge 2)$ and $\nabla_0^\tau\nabla^\beta G \in L_q(S)$ $(1 \le q < 3/2$, $|\tau| = 2$, $|\beta| = 1)$ it follows with

$$\|\nabla_0^\alpha(G(z_0+h,\cdot) - G(z_0,\cdot))\|_{W_q^1} \le C|h|\,\|\nabla_0^\tau G(z_0,\cdot)\|_{1,q,S} \longrightarrow 0 \qquad \text{as } h \longrightarrow 0,$$

that $A_\alpha\phi \in C$.

b) Let $\phi \in E$, then $A_\alpha\phi \in C$ by a) and we estimate $\|A_\alpha\phi\|_E$ in two steps.

i) Let $S_1 = \{(\xi,\eta) \in S_d\colon \|\xi-\xi_0\|_1 > 2\}$. Then for $|\alpha| \le 1$

$$\begin{aligned} &\left|\int_{S_1} \nabla_0^\alpha \mathrm{ROT}\; G(z_0,z)\; (\phi\cdot\nabla)V\; dz\right| \\ &\qquad \le C\|\phi\|_E \int_{S_1} \exp\big(-c\|\xi-\xi_0\|_1\big)\; \exp\big(-c_2(|\xi|-d)\big)\,|\nabla V|\; dz \\ &\qquad \le C\|\nabla V\|_C \exp\big(-c_2(|\xi_0|-d)\big)\,\|\phi\|_E , \end{aligned}$$

where $\|\nabla V\|_C = \|\nabla V\|_{C(S_d)} \longrightarrow 0$ as $d \longrightarrow \infty$.

ii) Let $S_2 = \{(\xi,\eta) \in S_d\colon \|\xi-\xi_0\|_1 \le 2\}$. Then for $|\alpha| \le 1$

$$\left|\int_{S_2} \nabla_0^\alpha \mathrm{ROT}\; G(z_0,z)\; (\phi\cdot\nabla)V\; dz\right|$$

$$\le C\|\nabla V\|_{L_2} \|G(z_0,\cdot)\|_{W_2^2} \exp\left(-c_2(-|\xi-\xi_0|+|\xi_0|-d)\right)\|\phi\|_E$$

$$\le C\|\nabla V\|_{L_2} \|G(z_0,\cdot)\|_{W_2^2} \exp\left(-c_2(|\xi_0|-d)\right)\|\phi\|_E ,$$

where $\|G(z_0,\cdot)\|_{W_2^2} \le$ const and $\|\nabla V\|_{L_2} \longrightarrow 0$ as $d \longrightarrow \infty$ ∎

Using Theorem 3.6 a tedious calculation [12; Lemma 3.17] gives

Lemma 3.8. *Let $T(z_0)$ be as in the representation formula (3.13); then $\nabla^\tau T \in E$ ($|\tau| \le 1$).*

Now we are able to proof

Theorem 3.9. *If the distance d is sufficiently large, then the solution V of (3.9) is of class $E(S_d)$.*

Proof. Formula (3.13) and the definition of A_α give

$$\mathrm{ROT}_0\psi(z_0) = V(z_0) = \sum_{|\alpha|=1} a_\alpha\cdot(A_\alpha V)(z_0) + \mathrm{ROT}_0 T(z_0)$$

$$= AV(z_0) + \mathrm{ROT}_0 T(z_0), \quad \text{for } z_0 \in S_d,$$

with $a_\alpha = \pm 1$ or 0. For a Banach space $\mathfrak{B}$ let $\mathcal{L}(\mathfrak{B})$ be the Banach space of bounded linear maps $\mathfrak{B} \longrightarrow \mathfrak{B}$. By Lemma 3.8 we have $\mathrm{ROT}_0 T \in E$. So the equation $V - AV = \mathrm{ROT}\; T$ has a unique solution

a) in C, if d is so large that $\|A\|_{\mathcal{L}(C)} < 1$;

b) in E, if d is so large that $\|A\|_{\mathcal{L}(E)} < 1$.

By Lemma 3.7, the number d can be chosen in that way. Since $E \subset C$, these two solutions are identical. Because $V \in C$ we get the assertion ∎

By continuity of V on the set $\{(\xi,\eta) \in S\colon \tilde{a} \le \|\xi\|_1 \le d\}$ we get $V \in E(S_d)$ for all $d \ge \tilde{a}$. Using this result in (3.9) we obtain $LV - \nabla(P_a - P) = (V\cdot\nabla)V + g$ where $g \in E(S_d)$ for all $d \ge \tilde{a}$. Using [3; Theorem 9.3] in the bounded domains

$$S_{j,k} = \{(\xi,\eta) \in S\colon \quad j-1 < \xi_1 < j+1; \quad k-1 < \xi_2 < k+1\} \qquad (j,k > \tilde{a} + 1)$$

we get for sufficiently large ν

$$|\nabla^\beta V(z)|,\; |\nabla^\gamma (P_a-P)(z)|,\; [\nabla^\beta V]_\alpha(z),\; [\nabla^\gamma (P_a-P)]_\alpha(z) \le C_2\exp\left(-c_2(|\xi|-d)\right),$$

for all $z \in S_d$ ($|\beta| \le 2$, $|\gamma| \le 1$, $0 < \alpha < 1$). By induction and transformation to Ω one gets

Theorem 3.10. *Under the assumptions in the beginning of this section we have for the velocity $v = u - g$ and the pressure p*

$$D^{\beta}v,\ D^{\gamma}(p_a - p),\ [D^{\beta}v]_{\alpha},\ [D^{\gamma}(p_a - p)]_{\alpha} \in E(\Omega_d),$$

for $|\beta| \le k+2$, $|\gamma| \le k+1$, $0 < \alpha < 1$.

4. The equation for the capillary surface

In this section we consider the problem

$$-D_i\left(\frac{D_i h}{\sqrt{1 + |Dh|^2}}\right) + ch + f = 0 \qquad \text{in } \mathbb{R}^n, \tag{4.1}$$

with a constant $c > 0$ and a function $f \in C^1_B(\mathbb{R}^n)$.

4.1. Existence and uniqueness of the solution. First we are looking for a weak solution of (4.1). That means a function $h \in C^{0,1}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$ such that, for any bounded domain $U \subset \mathbb{R}^n$, h belongs to $W^1_1(U)$ and

$$\int_{\mathbb{R}^n} D_i h D_i \varphi \cdot W^{-1}\, dx + c\int_{\mathbb{R}^n} h\varphi\, dx + \int_{\mathbb{R}^n} f\varphi\, dx = 0, \tag{4.2}$$

for every $\varphi \in \overset{\circ}{W}{}^1_1(\mathbb{R}^n)$, $W = (1 + |Dh|^2)^{1/2}$. With the help of the corresponding variational problem

$$J_k(v) = \int_{B_k}\left(1 + |Dv|^2\right)^{1/2} dx + \frac{c}{2}\int_{B_k} v^2 dx + \int_{B_k} fv\, dx \longrightarrow \min \quad \text{in } BV(B_k)$$

in a sequence of bounded domains $B_k \longrightarrow \mathbb{R}^n$ one can show the existence of a solution of (4.2).

Theorem 4.1. *Let $f \in C^1_B(\mathbb{R}^n)$, then equation (4.2) has a solution $h \in C^{0,1}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$.*

Remark: The uniqueness of the solution follows later by Theorem 4.3. There we get $|h| \longrightarrow 0$ as $|x| \longrightarrow \infty$, if $|f(x)|$ is bounded by $c_f \exp(-c_2|x|)$ for $|x| \ge r_0$.

The higher regularity can be shown with the help of the gradient estimates of Bombieri, De Giorgi and Miranda (see, e.g. [10]) and the theory of quasilinear elliptic equations (cf. [19]). One gets (cf. [12])

Theorem 4.2. *If $f \in C^1(\mathbb{R}^n)$, then the solution h of equation (4.2) is of class $C^{k,\alpha}(\mathbb{R}^n)$ $(k \le 2)$ and $\|h\|_{C^{k,\alpha}(\mathbb{R}^n)} \le C(\|f\|_{C^1(\mathbb{R}^n)})$. If $f \in C^{k-2,\alpha}(\mathbb{R}^n)$ $(k \ge 3)$, then h is of class $C^{k,\alpha}(\mathbb{R}^n)$ and $\|h\|_{C^{k,\alpha}(\mathbb{R}^n)} \le C(\|f\|_{C^{k-2,\alpha}(\mathbb{R}^n)})$.*

4.2. Decay estimates. To show a decay estimate for h we construct appropriate barrier functions. To prove an upper estimate we set

$$\delta_\varepsilon = c_1\exp(-c_2|x|) + \varepsilon|x| \qquad (\varepsilon > 0),$$

where the constants c_1, c_2 and ε should be determined such that $A\delta_\varepsilon + c\delta_\varepsilon + f \ge 0$ for $|x| > r_0$ and $\delta_\varepsilon \ge h(x)$ for $|x| = r_0 > 0$. A is the negativ minimal surface operator $Ah = -D_i\left(D_ih/\sqrt{1 + |Dh|^2}\right)$. Suppose for a moment that we have chosen the constants appropriately, then with a maximum principle [20; Theorem 31] we get

$$\delta_\varepsilon(x) \ge h, \qquad \text{for } |x| \ge r_0 . \tag{4.3}$$

Thus h is bounded from above by δ_ε outside of B_{r_0}. If we are able to bound the constants c_1 and c_2 uniformly with respect to ε then, going to zero with ε, we obtain the desired result for h. For brevity we set $\delta_0 := c_1\exp(-c_2|x|)$. A calculation shows

$$A\delta_\varepsilon = -\frac{1}{W}\,\frac{n-1}{|x|}\,(-c_2\delta_0 + \varepsilon) - \frac{c_2^2\delta_0}{W^3} .$$

With $1 \le W^2 \le 1 + 2(c_1c_2)^2$ for $\varepsilon \le c_1c_2$ we conclude

$$A\delta_\varepsilon + c\delta_\varepsilon + f \ge$$
$$\left(c - c_2^2 + \frac{n-1}{|x|}\,\frac{c_2}{\sqrt{1 + 2(c_1c_2)^2}}\right)\delta_0 + \varepsilon|x|\left(c - \frac{n-1}{|x|^2}\right) + f. \tag{4.4}$$

If $-f \le (c - c_2^2)\delta_0$, the right-hand side of (4.4) is non-negative for $|x|^2 \ge (n-1)/c_2^2 =: r_0^2$. To find a lower bound for h we set $\tilde\delta_\varepsilon := -\delta_\varepsilon$ and get the requirement $f \le (c - c_2^2)\delta_0$. The constant c_1 can be determined as follows: With an estimate of Concus and Finn (see, e.g. [12; Theorem 4.5]) it follows that $|h| \le (n + c_f)/c + 1$. Thus with (4.3) we get $c_1 = (n + c_f)/c + 1$ and we have shown

Theorem 4.3. *If $f \in C_B^1(\mathbb{R}^n)$ with $|f(x)| \le c_f\exp(-c_2|x|)$ for $|x| \ge r_0$, and $c > c_2^2$, $c_f = (c - c_2^2)c_1$, then the solution h of (4.1) satisfies $|h(x)| \le c_1\exp(-c_2|x|)$ for $|x| \ge r_0$, where $c_1 = (n+c_f)/c + 1$.*

For the gradient we get

Theorem 4.4. *Let* $c > 0$ *and* $f \in C^1_B(\mathbb{R}^n)$ *with* $|D^\gamma f(x)| \le c_f^\gamma \exp(-c_2|x|)$ *for* $|x| \ge r_0$ $(|\gamma| \le 1)$, *then we get for the solution* h *of (4.1)* $|D^\beta h(x)| \le c_\beta \exp(-c_2|x|)$ *for* $|x| \ge r_0$, *where* $c_\beta = c_\beta(\|f\|_{C^1(\mathbb{R}^n)})$ $(|\beta| = 1)$.

Proof. Let us introduce some definitions: We denote by $\mathcal{S} := \{(x,h(x)) \in \mathbb{R}^{n+1};\ x \in \mathbb{R}^n\}$ the graph of h over $\mathbb{R}^n$. The outward normal vector ν at a point $(x,h(x))$ is then defined by

$$\nu := W^{-1}(-D_1h, \dots, -D_nh, 1)\ , \qquad W = \sqrt{1 + |Dh|^2}\ .$$

Furthermore, we define the differential operators $\delta_i = D_i - \nu_i\nu_kD_k$ $(i = 1,2,\dots,n+1)$ and $\mathcal{D} = \delta_i\delta_i$. Let $y_0 = (x_0,h(x_0)) \in \mathcal{S}$ be arbitrary and $\mathcal{S}_R$ the intersection of $\mathcal{S}$ with the $(n+1)$-dimensional ball in $\mathbb{R}^{n+1}$ of radius $R = R_0$ = const and center y_0. Furthermore, let $\mathcal{S}^*_R$ be the projection of $\mathcal{S}_R$ on $\mathbb{R}^n$, then $\mathcal{S}^*_R$ is contained in the ball $B_R = B_R(x_0)$ (cf. Fig. 4.1). $\mathcal{H}_n$ is the n-dimensional Hausdorff measure.

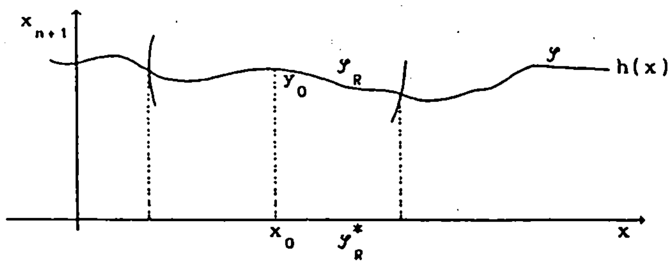

Fig. 4.1

Let w be defined by $w := -\log \nu_{n+1}$. From [23; Corollary 4] we deduce for $R > 1/8$

$$w(y_0) \le C_3 \int_{\mathcal{S}_R} w \, d\mathcal{H}_n + C_4 G\mathcal{H}_n(\mathcal{S}_R)\ , \tag{4.5}$$

where $G = R^2\sup\{(-\mathcal{D}w,0);\ x \in \mathcal{S}_R\}$ and the constants C_3 and C_4 depend on n, $c\|h\|_{C(\mathcal{S}_R)}$ and $\|f\|_{C(\mathcal{S}_R)}$. With [23; Lemma 3] we get $\mathcal{D}w \ge |\delta w|^2 - n\nu_iD_iH \ge \partial_ih\partial_if\ /\ W$ on $\mathcal{S}$. Hence we obtain

$$G = R^2\sup\{(-\mathcal{D}w,0);\ x \in \mathcal{S}_R\} \le R^2\sup\{|\partial_if|;\ x \in \mathcal{S}_R\}\ \sup\{|\partial_ih|;\ x \in \mathcal{S}_R\}.$$

Rewriting the integral $\int_{\mathcal{S}_R} w\, d\mathcal{H}_n$ as $\int_{\mathcal{S}^*_R} wW\, dx$ and observing that $w \le |Dh|^2$

and that W is uniformly bounded, we deduce for the first term of the right-hand side of (4.5)

$$C_3 \int_{\mathcal{S}_R} w \, d\mathcal{H}_n \le C_5 \int_{B_R} |Dh|^2 \, dx \, .$$

For the second term we get with $\mathcal{H}_n(\mathcal{S}_R) = \int_{\mathcal{S}^*_R} W \, dx \le C_6 \text{vol } B_R$ the estimate

$$C_4 G\mathcal{H}_n(\mathcal{S}_R) \le C_7 R^2 \|f\|_{C^1(B_R)} \|Dh\|_{C(B_R)} \text{vol } B_R \, .$$

Because of Theorem 4.2, the constants C_5, C_6 and C_7 depend on n, c and $\|f\|_{C^1(B_R)}$. Then we obtain from (4.5)

$$w(y_0) \le C_5 \int_{B_R} |Dh|^2 \, dx + C_8 R^{n+2} \|f\|_{C^1(B_R)} \|Dh\|_{C(B_R)} \, . \tag{4.6}$$

So it remains to estimate the integral $\int_{B_R} |Dh|^2 \, dx$. Let η, $0 \le \eta \le 1$, be a cut-off function being equal to 1 in B_R and zero outside B_{2R} such that $|D\eta| \le 1/R$. Multiplying equation (4.1) by ηh and integrating partially we deduce

$$\int_{B_{2R}} \left\{ (D_i h)^2 \eta W^{-1} + D_i h D_i \eta h W^{-1} + (ch^2 + fh)\eta \right\} dx = 0 \, . \tag{4.7}$$

We have

$$\int_{B_R} |Dh|^2 \, dx \le K_1 \int_{B_{2R}} \frac{|Dh|^2}{\sqrt{1 + |Dh|^2}} \eta \, dx \, , \tag{4.8}$$

where K_1 depends on $\|f\|_{C^1(\mathbb{R}^n)}$. Taking the boundedness of h into account and inserting (4.7) into (4.8), we obtain

$$\int_{B_R} |Dh|^2 \, dx \le K_2 \int_{B_{2R}} \Big(|D\eta|\,|h|\,|Dh| + |f|\,|h|\,|\eta| \Big) \, dx$$

and therefore

$$\int_{B_R} |Dh|^2 \, dx \le K_3 \text{vol } B_{2R} \times \left(\frac{1}{R} \|h\|_{C(B_{2R})} \|Dh\|_{C(B_{2R})} + \|f\|_{C(B_{2R})} \|h\|_{C(B_{2R})} \right), \tag{4.9}$$

where the constant K_3 depends on $\|f\|_{C^1(B_{2R})}$ and n, but not on R.

On the other hand, we deduce from Theorem 4.3

$$\|h\|_{C(B_{2R})} \le c_1 \exp(-c_2(|x_0| - 2R)) \le \tilde{c}_1(R) \exp(-c_2 |x_0|) . \tag{4.10}$$

Furthermore, we have

$$\|f\|_{C^1(B_{2R})} \le \tilde{c}_f(R, \|f\|_{C^1(\mathbb{R}^n)}) \exp(-c_2|x_0|) \ . \tag{4.11}$$

Using (4.9), (4.10) and (4.11) in (4.6) we get

$$\begin{aligned} w(y_0) \le & \Big(C_5K_3\big(\tilde{c}_1/R\|Dh\|_{C(B_{2R})} + \tilde{c}_f\tilde{c}_1\exp(-c_2|x_0|)\big) \\ & \qquad + C_8\tilde{c}_fR^2\|Dh\|_{C(B_{2R})}\Big) R^n\exp(-c_2|x_0|) \\ & \le C_9(R)\exp(-c_2|x_0|)\Big(\|Dh\|_{C(B_{2R})} + \exp(-c_2|x_0|)\Big) \ . \end{aligned}$$

Moreover, since $t/2 \le \log(1+t)$ if $0 \le t \le 1$, we derive

$$|Dh(x_0)|^2 \le 4C_9(R)\exp(-c_2|x_0|)\Big(\|Dh\|_{C(B_{2R})} + \exp(-c_2|x_0|)\Big)$$

and finally, for fixed R, we get $|Dh(x_0)| \le C_{10}\exp(-c_2|x_0|)$ for $|x_0| \ge r_0$ ∎

For estimating the higher derivatives we transform equation (4.1) in such a way that the linear terms of the second order derivatives are separated from the other terms. We get the equation $\Delta h = (ch + f)W^3 + D_ihD_jhD_{ij}h - D_{ii}hD_jhD_jh$. Schauder's interior estimates and induction then show (cf. [12; Theorem 4.9])

Theorem 4.5. *Let h be the solution of (4.1).*

a) If $f \in C^1(\mathbb{R}^n)$ and $|D^\gamma f(x)| \le c_f^\gamma\exp(-c_2|x|)$ for $|x| \ge r_0$ ($|\gamma| \le 1$), then

$$|D^\beta h(x)|,\ [D^\beta h]_\alpha(x) \le c_\beta\exp(-c_2|x|) \qquad \text{for } |x| \ge r_0 \quad (|\beta| \le 2),$$

where $c_\beta = c_\beta(\|f\|_{C^1(\mathbb{R}^n)})$ and $0 < \alpha < 1$.

b) If $|D^\gamma f(x)|$, $[D^\gamma f]_\alpha(x) \le c_f^\gamma\exp(-c_2|x|)$ for $|x| \ge r_0$ ($3 \le k \in \mathbb{N}$, $|\gamma| \le k-2$, $0 < \alpha < 1$), then

$$|D^\beta h(x)|,\ [D^\beta h]_\alpha(x) \le c_\beta\exp(-c_2|x|) \qquad \text{for } |x| \ge r_0 \quad (|\beta| \le k),$$

where $c_\beta = c_\beta(\|f\|_{C^{k-2,\alpha}(\mathbb{R}^n)})$.

5. The solution of the free boundary value problem

Now we come back to our original problem (N), (B). With the help of successive approximation (cf. Lemma 5.2) we show

Theorem 5.1. *Let $f_s \in C^{k,\lambda}(\mathbb{R}^3)$ ($k \in \mathbb{N}$, $0 < \lambda < 1$) be sufficiently small and bounded by $\exp(-c_2|x|)$ for $|x| \ge r_0$. If $\nu > \nu_0$, then there exists one* *and only one solution* $(u,p,h) \in C^{k+2,\lambda}(\Omega\cup\Gamma) \times C^{k+1,\lambda}(\Omega\cup\Gamma) \times C^{k+3,\lambda}(\mathbb{R}^2) =: C$ *of problem* (N), (B). *This solution satisfies*

$$\left.\begin{aligned} |D^{\alpha}(p-p_a)(x,\cdot)| &\le C_1\exp(-c_2|x|) \\ |D^{\beta}v(x,\cdot)| &\le C_2\exp(-c_2|x|) \\ |D^{\gamma}h(x)| &\le C_3\exp(-c_2|x|) \end{aligned}\right\} \text{ for } |x| \ge r_0$$

$(|\alpha| \le 1,\ |\beta| \le 2,\ |\gamma| \le 3)$, *where* C_1, C_2, C_3 *are some positive constants.*

Proof. We choose $\Omega_0 = \{(x,y) \in \mathbb{R}^3:\ -b(x) < y < 0\}$. For $m \ge 0$ we determine (u_{m+1},p_{m+1}) in $\Omega_m = \{(x,y) \in \mathbb{R}^3:\ -b(x) < y < h_m(x)\}$ as solutions of the systems

$$\text{(P1)}\quad \begin{cases} -\nu\Delta u_{m+1} + \nabla p_{m+1} + (u_{m+1}\cdot\nabla)u_{m+1} = f_s & \\ \operatorname{div} u_{m+1} = 0 & \text{in } \Omega_m \\ u_{m+1} = 0 & \text{on } \Gamma_m^- \\ u_{m+1}n_m = 0\ ;\quad t_m^{(i)}T(u_{m+1},p_{m+1})n_m = 0, \quad (i=1,2) & \text{on } \Gamma_m^+ \\ u_{m+1} \longrightarrow q & \text{as } |x| \longrightarrow \infty . \end{cases}$$

We get the new surface Γ_{m+1}^+ and by this the new domain Ω_{m+1} from (u_{m+1}, p_{m+1}) by solving

$$\text{(P2)}\qquad n_m T(u_{m+1},p_{m+1})n_m = -p_a + gh_{m+1} - \kappa D_i\left(\frac{D_i h_{m+1}}{\sqrt{1+|Dh_{m+1}|^2}}\right) \text{ in } \mathbb{R}^2.$$

The existence and uniqueness of the solution of (P1) and (P2) follow by Section 3 and 4. The existence and uniqueness of the solution of problem (N), (B) then ensue from

Lemma 5.2. *If* $f_s \in C^{0,\alpha}(\Omega_m\cup\Gamma_m) \cap L_2(\Omega_m)$ *for all* m, *then the sequence* $\{u_m,p_m,h_m\}_{m\in\mathbb{N}}$ *defined by* Ω_0, (P1) *and* (P2) *converges in* C.

Proof. We first show that the sequence does not leave the class C: in Ω_0 we get a solution (u_1,p_1) of the class $C^{2,\alpha} \times C^{1,\alpha}$. For $(u_m,p_m) \in C^{2,\alpha} \times C^{1,\alpha}$ the solution h_m of (P2) is in $C^{3,\alpha}$. If we now solve (P1) in Ω_m, we get $(u_{m+1},p_{m+1}) \in C^{2,\alpha} \times C^{1,\alpha}$.

To show the convergence of the sequence we first have to show a result about the difference of two solutions of (P1) and (P2) (cf. Lemmas 5.3 and 5.4). The difference of two solutions (u,p) and (v,q) of the Navier-Stokes equations in the domains Ω_A and Ω_B is defined in the strip S. For that reason we transform the equations from Ω to S and get (U,P) and (U,Q) as solutions of the transformed equations

$$\left.\begin{aligned} &\nu(A^1_{jl}\nabla_j\nabla_l U_i + A^2_{jmi}\nabla_m U_j + A^3_l\nabla_l U_i + A^4_{ij}U_j) \\ &\qquad + A^5 U_j\nabla_j U_i + A^6_{jli}U_j U_l + A^7_{ij}\nabla_j P = F_i \\ &\nabla_i U_i = 0 \end{aligned}\right\} \text{ in } S$$

and

$$\left.\begin{aligned} &\nu(B^1_{jl}\nabla_j\nabla_l V_i + B^2_{jmi}\nabla_m V_j + B^3_l\nabla_l V_i + B^4_{ij}V_j) \\ &\qquad + B^5 V_j\nabla_j V_i + B^6_{jli}V_j V_l + B^7_{ij}\nabla_j Q = G_i \\ &\nabla_i V_i = 0 \end{aligned}\right\} \text{ in } S\ ,$$

where the coefficients A and B depend on the transformations $F_A : S \longrightarrow \Omega_A$ and $F_B : S \longrightarrow \Omega_B$ (cf. Subsection 2.2). With [3; Theorem 9.3] it follows that

$$\|U - V\|_{C^{2,\alpha}(S)} + \|P - Q\|_{C^{1,\alpha}(S)} \le$$
$$C_6(\nu)\Big(\|h_A - h_B\|_{C^{3,\alpha}(\mathbb{R}^2)} + \|F - G\|_{C^{0,\alpha}(S)} + \|U - V\|_{C(S)} + \|P - Q\|_{C(S)}\Big),$$

where the constant C_6 tends to zero like $1/\nu$ as $\nu \longrightarrow \infty$. Because $F = (DF_A)^{-1}f_s$ and $G = (DF_B)^{-1}f_s$ we can estimate the term $\|F - G\|_{C^{0,\alpha}(S)}$ by $\|h_A - h_B\|_{C^{3,\alpha}(\mathbb{R}^2)}$, too. The two last terms of the right-hand side are small for sufficienly large ν, so we get

Lemma 5.3. *Let Ω_A und Ω_B be two domains whose surfaces are defined by the functions h_A and h_B and let (u,p) and (v,q) be the solutions of* (N), ($\tilde{\text{B}}$) *in Ω_A and Ω_B, then*

$$\|u - v\|_{C^{2,\alpha}} + \|p - q\|_{C^{1,\alpha}} := \|U - V\|_{C^{2,\alpha}} + \|P - Q\|_{C^{1,\alpha}}$$
$$\le C_7(\nu)\|h_A - h_B\|_{C^{3,\alpha}}\ ,$$

where C_7 tends to zero for large ν.

For the difference of two solutions of the surface problem we get

Lemma 5.4. *Let g and h be the solutions of the surface problems* (4.1) *for the data A and B, then*

$$\|h - g\|_{C^{3,\alpha}(\mathbb{R}^2)} \le K\Big(\|P_A - P_B\|_{C^{1,\alpha}} + \|U_A - U_B\|_{C^{2,\alpha}}\Big) + C(\nu)\|h_A - h_B\|_{C^{3,\alpha}}\ ,$$

where the constants K and C are independent of g, h, A and B.

Proof. We define $w := h - g \in C^{3,\alpha}(\mathbb{R}^2)$ and get

$$(1+(\partial_2 h)^2)\partial_{11}w - 2\partial_1 h\partial_2 h\partial_{12}w + (1+(\partial_1 h)^2)\partial_{22}w$$
$$+ \Big(\partial_{22}g(\partial_1 g + \partial_1 h) - 2\partial_{12}g\partial_2 g - \mathfrak{B}\cdot(\partial_1 h + \partial_1 g)(f(B) + cg)\Big)\partial_1 w$$
$$+ \Big(\partial_{11}g(\partial_2 g + \partial_2 h) - 2\partial_{12}g\partial_1 h - \mathfrak{B}\cdot(\partial_2 h + \partial_2 g)(f(B) + cg)\Big)\partial_2 w$$
$$- c(W(h))^3 w = (W(h))^3(f(A) - f(B)) ,$$

where $\mathfrak{B}$ depends on Dh and Dg. Therefore w is the solution of a linear strictly elliptic equation. With Schauder's interior estimates it follows that

$$\|w\|_{C^{3,\alpha}(\mathbb{R}^2)} \le K_1\|f(A) - f(B)\|_{C^{1,\alpha}(\mathbb{R}^2)}$$
$$\le K_1\Big\{K_2\Big(\|P_A - P_B\|_{C^{1,\alpha}} + \|U_A - U_B\|_{C^{2,\alpha}}\Big) + C_8(\nu)\|h_A - h_B\|_{C^{2,\alpha}}\Big\},$$

where the constants K_1, K_2 and C_8 do not depend on the data A and B. The constant $C_8(\nu)$ tends to zero like $1/\nu$ as $\nu \longrightarrow \infty$. So we have shown the assertion of Lemma 5.4. ∎

Now we continue the proof of Lemma 5.2. We can show that the map T : $(u_m, p_m, h_m) \longrightarrow (u_{m+1}, p_{m+1}, h_{m+1})$ is a contraction for small data. We fix the data f_s, ρ and g, but reserve us the right to choose the viscosity ν suitably in the end. With the Lemmas 5.3 and 5.4 we get

$$\|U_{m+1} - U_m\|_{C^{2,\alpha}} + \|P_{m+1} - P_m\|_{C^{1,\alpha}} \le C_9(\nu)\|h_m - h_{m-1}\|_{C^{3,\alpha}}$$
$$\|h_m - h_{m-1}\|_{C^{3,\alpha}} \le K\Big(\|U_m - U_{m-1}\|_{C^{2,\alpha}} + \|P_m - P_{m-1}\|_{C^{1,\alpha}}\Big) + C(\nu)\|h_{m-1} - h_{m-2}\|_{C^{3,\alpha}}.$$

From the proofs of the lemmas we see that $C_9(\nu)$ and $C(\nu)$ are small for large ν. Therefore, we can choose ν sufficiently large such that $C_9(\nu)K + C(\nu)$ is smaller than one. Thus Lemma 5.2 is proven. ∎

Now we return to the proof of the decay estimates: for the velocity $v_m = u_m - g_m$ and the pressure $p_m - p_a$ in Ω_{m-1} $(m = 1,2,\ldots)$ we have shown in Subsection 3.3 that

$$|D^\alpha(p_m - p_a)(x,\cdot)|,\ |D^\beta v_m(x,\cdot)| \le c_1^m \exp(-c_2|x|) \qquad \text{for } |x| \ge r_0 ,$$
$$[D^\alpha(p_m - p_a)]_\lambda(x,\cdot),\ [D^\beta v_m]_\lambda(x,\cdot) \le c_1^m \exp(-c_2|x|) \qquad \text{for } |x| \ge r_0 ,$$

$(|\alpha| \le 1,\ |\beta| \le 2$ and $0 < \lambda < 1)$ if the force f_s, the surface function h_{m-1}

and their derivatives up to the third order (and the corresponding Hölder derivatives) decay as $\exp(-c_2|x|)$ for $|x| \ge r_0$. Here c_1^m depends on the $C^{3,\alpha}(\mathbb{R}^2)$-norm of the surface function h_{m-1}. Since the sequence $\{h_m\}$ is uniformly bounded, c_1^m $(m = 1,2,..)$ can be estimated by a constant C_1. The constant c_2 is independent of m.

Because $|D^\alpha(p_m - p_a)(x,\cdot)|$ and $|D^\beta v_m(x,\cdot)|$ $(|\alpha| \le 1,\ |\beta| \le 2)$ and the corresponding Hölder derivatives are bounded by $\exp(-c_2|x|)$ for $|x| \ge r_0$, we got in Section 4 for the surface functions h_m $(m = 1,2,\ldots)$

$$|D^\alpha h_m(x)|,\ [D^\alpha h_m]_\lambda(x) \le c_3^m \exp(-c_2|x|) \qquad \text{for } |x| \ge r_0 ,$$

$(|\alpha| \le 3,\ 0 < \lambda < 1)$ where c_3^m is a function of $\|u_m\|_{C^{2,\alpha}(\Omega_{m-1})}$ and $\|p_m - p_a\|_{C^{1,\alpha}(\Omega_{m-1})}$. Because of the uniform boundedness of these norms we can bound them by a constant C_3.

Now we have shown that (v_m,p_m,h_m) $(m = 1,2,..)$ are uniformly exponentially bounded and therefore this is also true for the limit (v,p,h). So Theorem 5.1 is proven ■

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Note: The assumption h → 0 as |x| → ∞ for the solution of the surface equation [12; Chapter 4] is redundant. It is not used in the proof of the existence of a solution and the decay of h. That means, if we first show the decay result for h, we can use it afterwards for the uniqueness proof.

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