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Schur Algorithm for the Integral Representations of Lacunary Hankel Forms

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A Schur type algorithm for the lacunary Nehari problem making use of the extensions of certain isometries is shown. A parametrization of the solution set is also obtained. A constructive method that provides the solutions by a sequence of Schur type parameters is developed. In the case of the classical Nehari problem, this algorithm gives the classical Schur parameters for the Carathéodory-Fejér interpolation problem. Here we propose another way to solve this problem, namely as an application of the Nehari problem via the problem of the extension of isometries associated to it. This point of view will lead in a forthcoming paper to the generalization of the results to the matricial case.

Key words: *Schur algorithm, Hankel form, generalized Toeplitz form, unitary extension of isometries, Nehari problem*

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Introduction

The *Carathéodory-Fejér interpolation problem*, also called the *Schur problem*, consists in characterizing the complex sequences $\{a_0, \ldots, a_N\}$ for which there exists a function φ analytic in the unit disc $\mathbf{D}$ such that $\|\varphi\|_\infty \leq 1$, and whose first non-negative Fourier coefficients are given by that sequence, i.e., $\widehat{\varphi}(n) = a_n$, for $0 \leq n \leq N$. The Schur algorithm solves this problem and gives necessary and sufficient conditions for the existence of solutions by means of a family of parameters (called *Schur parameters*). These parameters give a complete description of the Taylor coefficients of each solution, and also provide a parametrization of all the solutions.

The Carathéodory-Fejér problem has derived a wide investigation (see, e.g., [3]) and matricial generalizations. We can mention the works of Dym [9], Dubovoj, Fritzsche and Kirstein [8].

The *N-reduced Nehari problem*, which is equivalent to the Carathéodory-Fejér problem, consists in characterizing the complex sequences $\{s_{-N}, \ldots, s_{-1}\}$ for which there exists a function $f \in L^\infty(\mathbf{T})$ such that $\|f\|_\infty \leq 1$ and $\widehat{f}(n) = s_n$, for $-N \leq n < 0, \widehat{f}(n) = 0$, for $n < -N$.

The next statement, equivalent to Paley lacunary inequality, provides another interpolation problem, namely the problem to find the set of all functions in $L^\infty(\mathbf{T})$ whose non-negative Fourier coefficients are given by a lacunary sequence:

If $\{n_k\}_{k=0}^\infty$ is a strictly increasing sequence of non-negative integers with the property $n_{k+1} > \lambda n_k$ $(\lambda > 1)$ for all k, then for each square summable sequence $v = \{v_k\}_{k=0}^\infty$ there exists a bounded function g such that $\|g\|_\infty \leq C(\lambda)\|v\|_2$, and $\widehat{g}(n_k) = v_k$, for all k, while $\widehat{g}(n) = 0$, for all other $n \geq 0$.

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Nehari [12] discussed an explicit procedure in order to obtain the function g, given $\{n_k\}$ and $\{v_k\}$, via the Schur algorithm. This same algorithm is used by Fournier [10], who obtains bounds for $C(\lambda)$ in some cases.

Here we are going to state the next Schur type problem for lacunary sequences related with the previous statement.

The problem. *Let $\{a_n\}_{n\geq 0} \in \ell^2$ be a sequence such that $a_n = 0$, if $n \neq n_k$; we define the set $\Sigma(a) = \{\Phi \in L^\infty(\mathbf{T}) : \|\Phi\|_\infty \leq 1, \widehat{\Phi}(n) = a_n, \forall n \geq 0\}$. The goal is to*

(i) find necessary and sufficient conditions for $\Sigma(a) \neq \emptyset$;

(ii) furnish a description of all functions $\Phi \in \Sigma(a)$, when $\Sigma(a) \neq \emptyset$.

In order to get a parametrization of all solutions, there are formulas as the ones obtained in [1] and [2], but here we are going to use a Schur algorithm that allows us to solve the reduced problem (with only a finite number of coefficients non-zero) and give the general solution, by a limit process. Here we make a wide use of the theory of generalized resolvents and the theory of generalized spectral functions of isometric operators. With this purpose, in Section 1, a description of the generalized resolvents of an isometric operator and its expansion in Taylor series is given. In Section 2 a constructive parametrization formula for the generalized resolvent of the class of associated isometries is obtained. This formula is applied in order to parametrize the solution set of the generalized Bochner theorem and as a particular case, the Nehari theorem. At the end of this paper, we will develop an algorithm for constructing all the solutions. The results of this paper can be generalized to the matricial and two-parametric cases; we will study these questions in a forthcoming work.

Basic notations used throughout the text follow: $\mathbf{Z}_1 = \{n \in \mathbf{Z} : n \geq 0\}, \mathbf{Z}_2 = \mathbf{Z} \setminus \mathbf{Z}_1$; $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}, \mathbf{T} = \partial\mathbb{D}$, dt is the normalized Lebesgue measure on $\mathbf{T}$; $\widehat{f}(n) = \int_0^{2\pi} e^{int} f(t)dt$ (respectively $\widehat{\mu}(n) = \int_0^{2\pi} e^{int} d\mu$) denotes the Fourier transform of the function f (resp. of the measure μ). For $1 \leq p \leq \infty$, $H^p(\mathbf{T}) = \{f \in L^p(\mathbf{T}) : \widehat{f}(n) = 0$, for $n < 0\}$. For $\mathcal{M}, \mathcal{N}$ two Hilbert spaces, $\mathcal{M} \vee \mathcal{N}$ is the minimal closed space spanned by $\mathcal{M}$ and $\mathcal{N}$, $L(\mathcal{M}, \mathcal{N})$ stands for the space of all bounded linear operators from $\mathcal{M}$ to $\mathcal{N}$.

1. Description of the Generalized Resolvents of an Isometric Operator

Let $\mathcal{H}$ be a Hilbert space and $U : \mathcal{H} \to \mathcal{H}$ a closed isometric operator with domain $\mathcal{D}$ and range Δ. The orthogonal complements $\mathcal{M} = \mathcal{H} \ominus \mathcal{D}$ and $\mathcal{N} = \mathcal{H} \ominus \Delta$ are called the *defect subspaces* of U, and the numbers $m = \dim \mathcal{M}, n = \dim \mathcal{N}$ are called the *defect indices* of U.

Definition 1.1: A unitary operator $\widetilde{U} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ is a *unitary extension* of U if $\mathcal{H}$ is a closed subspace of $\widetilde{\mathcal{H}}$ and $\widetilde{U}|_{\mathcal{D}} = U$. Moreover, if $\widetilde{\mathcal{H}} = \bigvee_{n \in \mathbf{Z}} \widetilde{U}^n(\mathcal{H}), \widetilde{U}$ is called a *minimal unitary extension* of U.

We identify two unitary extensions $\widetilde{U}_1 : \widetilde{\mathcal{H}}_1 \to \widetilde{\mathcal{H}}_1$ and $\widetilde{U}_2 : \widetilde{\mathcal{H}}_2 \to \widetilde{\mathcal{H}}_2$ if there exists a unitary isomorphism $\varphi : \widetilde{\mathcal{H}}_1 \to \widetilde{\mathcal{H}}_2$ which leaves invariant the elements of $\mathcal{H}$ and $\varphi\widetilde{U}_1 = \widetilde{U}_2\varphi$.

Definition 1.2: If $\{\widetilde{E}_t : 0 \leq t \leq 2\pi\}$ is a spectral function of $\widetilde{U}$, a *generalized* *spectral function* of U is the family of operators $\{E_t : 0 \leq t \leq 2\pi\}$ in $\mathcal{H}$ defined by $E_t h = P_{\mathcal{H}} \widetilde{E}_t h$, for all $h \in \mathcal{H}$, where $P_{\mathcal{H}}$ is the orthogonal projection from $\widetilde{\mathcal{H}}$ onto $\mathcal{H}$. The *generalized resolvent* of U is the family of operators $\{R_z : |z| \neq 1\}$ in $\mathcal{H}$ defined by $R_z h = P_{\mathcal{H}}(I - z\widetilde{U})^{-1} h$, for all $h \in \mathcal{H}$.

REMARKS: (a) The set of all generalized resolvents $\{R_z\}$ of U can be described by the formula $R_z = \int_0^{2\pi} \frac{dE_t}{1-ze^{it}}$, where $\{E_t\}$ is the generalized spectral function. (b) If one of the defect indices of U is not zero, then U has infinitely many spectral functions and corresponding generalized resolvents.

If we use the notation

$$U_{11} = P_{\mathcal{N}} \widetilde{U}|_{\widetilde{\mathcal{H}} \ominus \mathcal{H}} \qquad U_{12} = P_{\mathcal{N}} \widetilde{U}|_{\mathcal{M}}$$
$$U_{21} = P_{\widetilde{\mathcal{H}} \ominus \mathcal{H}} \widetilde{U}|_{\widetilde{\mathcal{H}} \ominus \mathcal{H}} \quad U_{22} = P_{\widetilde{\mathcal{H}} \ominus \mathcal{H}} \widetilde{U}|_{\mathcal{M}},$$

it is easy to prove (see [9]) that $\vartheta : \mathbb{D} \to L(\mathcal{M}, \mathcal{N})$ defined by $\vartheta(z) = U_{12} + zU_{11}(I - zU_{21})^{-1}U_{22}$ is an analytic function and, for each $z \in \mathbb{D}$, $\vartheta(z)$ is a contractive operator.

REMARK: The function ϑ is called the *characteristic function* associated with $\widetilde{U}$. Brodskii and Shvartsman [4] proved that there exists a bijection between the set of all (essentially different) minimal unitary extensions and the set of all the contractive analytic functions $\vartheta : \mathbb{D} \to L(\mathcal{M}, \mathcal{N})$.

LEMMA 1.3 (see [11]): *If $\widetilde{U} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ is a minimal unitary extension of U and ϑ is as above, then*

(a) $\vartheta(z) = P_{\mathcal{N}} \widetilde{U}(I - zP_{\widetilde{\mathcal{H}} \ominus \mathcal{H}} \widetilde{U})^{-1}|_{\mathcal{M}}$.

(b) If $\widehat{\vartheta}(n) = P_{\mathcal{N}} \widetilde{U}(P_{\widetilde{\mathcal{H}} \ominus \mathcal{H}} \widetilde{U})^n|_{\mathcal{M}}$, *then* $\vartheta(z) = \sum_{n \geq 0} z^n \widehat{\vartheta}(n), |z| < 1$.

(c) $(UP_{\mathcal{D}} + \vartheta(z)P_{\mathcal{M}})|_{\mathcal{H}} = \left[P_{\mathcal{H}} \widetilde{U}(I - zP_{\widetilde{\mathcal{H}} \ominus \mathcal{H}} \widetilde{U})^{-1}\right]|_{\mathcal{H}}$.

(d) $P_{\mathcal{H}} \widetilde{U}(I - z\widetilde{U})^{-1}|_{\mathcal{H}} = \left\{(UP_{\mathcal{D}} + \vartheta(z)P_{\mathcal{M}})\left[I - z(UP_{\mathcal{D}} + \vartheta(z)P_{\mathcal{M}})\right]^{-1}\right\}|_{\mathcal{H}}$.

We will use the previous properties in order to get a parametrization of all the generalized resolvents of U. Furthermore, due to the equality $R_{1/\bar{z}} = I - R_z^*$, whenever $|z| \neq 1, z \neq 0$, it suffices to establish the formula for all the values $z \in \mathbb{D}$.

PROPOSITION 1.4: *The generalized resolvent of U can be written as*

$$R_z = P_{\mathcal{H}}(I - z\widetilde{U})^{-1}|_{\mathcal{H}} = P_{\mathcal{H}}\left[I - z(UP_{\mathcal{D}} + \vartheta(z)P_{\mathcal{M}})\right]^{-1}|_{\mathcal{H}} \text{ for } |z| < 1. \tag{1.1}$$

Proof: Since $\widetilde{U} = P_{\widetilde{\mathcal{H}} \ominus \mathcal{H}} \widetilde{U} + P_{\mathcal{H}} \widetilde{U}$, we can write

$$\begin{aligned}(I - z\widetilde{U})^{-1} &= \left[I - zP_{\widetilde{\mathcal{H}} \ominus \mathcal{H}}(\widetilde{U} - zP_{\mathcal{H}} \widetilde{U})\right]^{-1} \\ &= \left\{[I - zP_{\mathcal{H}} \widetilde{U}(I - zP_{\widetilde{\mathcal{H}} \ominus \mathcal{H}} \widetilde{U})^{-1}](I - zP_{\widetilde{\mathcal{H}} \ominus \mathcal{H}} \widetilde{U})\right\}^{-1} \\ &= (I - zP_{\widetilde{\mathcal{H}} \ominus \mathcal{H}} \widetilde{U})^{-1}\left[I - zP_{\mathcal{H}} \widetilde{U}(I - zP_{\widetilde{\mathcal{H}} \ominus \mathcal{H}} \widetilde{U})^{-1}\right]^{-1}.\end{aligned}$$

So

$$\begin{aligned} R_z &= P_{\mathcal{H}}(I - z\widetilde{U})^{-1}|_{\mathcal{H}} \\ &= P_{\mathcal{H}}(I - zP_{\widetilde{\mathcal{H}}\ominus\mathcal{H}}\widetilde{U})^{-1}\left[I - zP_{\mathcal{H}}\widetilde{U}(I - zP_{\widetilde{\mathcal{H}}\ominus\mathcal{H}}\widetilde{U})^{-1}\right]^{-1}|_{\mathcal{H}} \\ &= P_{\mathcal{H}}\left[I - zP_{\mathcal{H}}\widetilde{U}(I - zP_{\widetilde{\mathcal{H}}\ominus\mathcal{H}}\widetilde{U})^{-1}\right]^{-1}|_{\mathcal{H}} \\ &= P_{\mathcal{H}}\left[I - z(UP_{\mathcal{D}} + \vartheta(z)P_{\mathcal{M}})\right]^{-1}|_{\mathcal{H}} \ \blacksquare \end{aligned}$$

Remark: A similar result, proved with a different technique, was obtained by Chumakin [5]: Every generalized resolvent R_z of U is representable in the form $R_z = [I - z(U \oplus \Phi_z)]^{-1}$, for $|z| < 1$, where Φ_z is some operator-valued function of parameter z, analytic in $\mathbf{D}$, whose values for any z are contractive operators from $\mathcal{M}$ into $\mathcal{N}$.

Now, using formula (1.1), a representation formula for the generalized resolvent of U by means of its Fourier series is obtained.

Proposition 1.5: *If $\widehat{R}(n) = P_{\mathcal{H}}\widetilde{U}^n|_{\mathcal{H}}$ are the Fourier coefficients of the generalized resolvent of U for $|z| < 1$ and $\widehat{\Phi}(n)$ are the coefficients of the associated characteristic function ϑ, then R_z has the following expansion in Fourier series:*

$$R_z = I + \sum_{n\geq 1} z^n\left(\widehat{R}(n-1)[UP_{\mathcal{D}} + \widehat{\vartheta}(0)P_{\mathcal{M}}] + \sum_{k=0}^{n-2}\widehat{R}(k)\widehat{\vartheta}(n-k-1)P_{\mathcal{M}}\right). \quad (1.2)$$

Proof: By its own definition,

$$R_z = P_{\mathcal{H}}(I - z\widetilde{U})^{-1}|_{\mathcal{H}} = \sum\nolimits_{n\geq 0} z^n P_{\mathcal{H}}\widetilde{U}^n|_{\mathcal{H}} = \sum\nolimits_{n\geq 0} z^n \widehat{R}(n).$$

Furthermore, if we call

$$A(z) = z\,(UP_{\mathcal{D}} + \vartheta(z)P_{\mathcal{M}}) = zUP_{\mathcal{D}} + \sum\nolimits_{n\geq 0} z^{n+1}\widehat{\vartheta}(n)P_{\mathcal{M}} = \sum\nolimits_{n\geq 0} z^n\widehat{A}(n),$$

it results that

$$\widehat{A}(n) = \begin{cases} 0 & \text{if } n = 0 \\ UP_{\mathcal{D}} + \widehat{\vartheta}(0)P_{\mathcal{M}} & \text{if } n = 1 \\ \widehat{\vartheta}(n-1)P_{\mathcal{M}} & \text{if } n > 1. \end{cases}$$

If we denote $G(z) = (I - A(z))^{-1}$, we can obtain

$$[G(I-A)]\widehat{\ }(n) = \sum\nolimits_{k=0}^{n}\widehat{G}(k)(\widehat{I-A})(n-k) = \begin{cases} I & \text{if } n = 0 \\ 0 & \text{if } n > 0. \end{cases}$$

For $n = 0$, $\widehat{G}(0)(\widehat{I-A})(0) = I$; so $\widehat{G}(0) = I$.
For $n = 1$, $\widehat{G}(0)(\widehat{I-A})(1) + \widehat{G}(1)(\widehat{I-A})(0) = 0$, which leads to

$$-UP_{\mathcal{D}} - \widehat{\vartheta}(0)P_{\mathcal{M}} + \widehat{G}(1) = 0, \text{ and } \widehat{G}(1) = UP_{\mathcal{D}} + \widehat{\vartheta}(0)P_{\mathcal{M}}.$$

For $n > 1, \sum_{k=0}^{n} \widehat{G}(k)\widehat{(I-A)}(n-k) = 0$, which leads to

$$\begin{aligned}\widehat{G}(n) &= -\sum_{k=0}^{n-1} \widehat{G}(k)\widehat{(I-A)}(n-k)\\ &= \widehat{G}(n-1)(UP_{\mathcal{D}} + \widehat{\vartheta}(0)P_{\mathcal{M}}) + \sum_{k=0}^{n-2} \widehat{G}(k)\widehat{\vartheta}(n-k-1)P_{\mathcal{M}}.\end{aligned}$$

Then, it results that

$$\widehat{G}(n) = \widehat{G}(n-1)UP_{\mathcal{D}} + \sum_{k=0}^{n-1} \widehat{G}(k)\widehat{\vartheta}(n-k-1)P_{\mathcal{M}}, \text{ for } n \geq 1,$$

and therefore, for $n \geq 1$,

$$\begin{aligned}\widehat{R}(n) &= \widehat{R}(n-1)UP_{\mathcal{D}} + \sum_{k=0}^{n-1} \widehat{R}(k)\widehat{\vartheta}(n-k-1)P_{\mathcal{M}}\\ &= \widehat{R}(n-1)\left[UP_{\mathcal{D}} + \widehat{\vartheta}(0)P_{\mathcal{M}}\right] + \sum_{k=0}^{n-2} \widehat{R}(k)\widehat{\vartheta}(n-k-1)P_{\mathcal{M}} \ \blacksquare\end{aligned} \tag{1.3}$$

2. Characterization of a Class of Isometries through the Resolvent

It is well known that in certain moment problems there appear isometric operators with some conditions. In this section, we will describe the set of all minimal unitary extensions of these isometries. In the sequel, $U : \mathcal{H} \to \mathcal{H}$ will be an isometric operator for which there exist two fixed elements e_0 and e_{-1} such that $U^n e_0 \in \mathcal{D}$, for all $n \geq 0$, $U^n e_{-1} \in \Delta$, for all $n \leq 0$, and $\mathcal{H}$ is generated by $\{U^n e_0 : n \geq 0\}$ and $\{U^n e_{-1} : n \leq 0\}$. From these hypotheses, we deduce that both defect indices of U are less than or equal to one. If we suppose that there is not a unique solution, then both defect indices are equal to one. In particular, $e_0 \notin \Delta$ and $e_{-1} \notin \mathcal{D}$. In [1] we proved the following

Proposition 2.1: *Every minimal unitary extension $\widetilde{U} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ of $U : \mathcal{H} \to \mathcal{H}$ is uniquely determined (up to unitary equivalences) by $\langle P_{\mathcal{H}}\widetilde{R}_z e_{-1}, e_0\rangle$ with $|z| < 1$, where $\widetilde{R}_z = (I - z\widetilde{U})^{-1}$ is the resolvent of $\widetilde{U}$.*

So, the parametrization problem of U can be reduced to the parametrization problem of $P_{\mathcal{H}}\widetilde{R}_z$; now, we can use formula (1.1) and write

$$R_z e_{-1} = P_{\mathcal{H}}\widetilde{R}_z e_{-1} = (I - zT_z)^{-1} e_{-1} = \sum_{n\geq 0} z^n T_z^n e_{-1} \text{ for } |z| < 1 \tag{2.1}$$

where $T_z = UP_{\mathcal{D}} \oplus \vartheta(z)P_{\mathcal{M}}$ and $\Phi_z = \vartheta(z)P_{\mathcal{M}}$ is a contractive operator from $\mathcal{M}$ onto $\mathcal{N}$.

Let us choose two unitary vectors $u, u_0 \in \mathcal{H}$ which are orthogonal to $\mathcal{D}$ and Δ, respectively. Thus, u and u_0 span the subspaces $\mathcal{M}$ and $\mathcal{N}$, respectively, and we can write $\Phi_z(u) = \varphi(z)u_0$ where $|\varphi(z)| \leq 1$. In particular, if $\varphi(z) \equiv \lambda$ with $|\lambda| = 1$, then each Φ_z is a unitary operator. So, T_z is unitary in $\mathcal{H}$ and R_z is an orthogonal resolvent of U generated by the corresponding unitary extension T_z. Conversely, if T_z is unitary, λ can be obtained in that form and it is possible to define $\Phi_z : \mathcal{M} \to \mathcal{N}$. In conclusion, we have proved the following

PROPOSITION 2.2: *(a) R_z is an orthogonal resolvent of U if and only if $\varphi(z) \equiv \lambda$ with $|\lambda| = 1$, and the number of minimal unitary extensions of U is determined by the different values of λ such that $|\lambda| = 1$. (b) In the general case, if R_z is a generalized resolvent of U, there are as many unitary extensions as analytic functions φ such that $|\varphi(z)| \leq 1$ for $|z| \leq 1$.*

In order to obtain a formula of the resolvent of U, we proceed as follows: Since $\mathcal{H} = \mathcal{M} \oplus \mathcal{D}$, there exist two vectors $v_0, w_0 \in \mathcal{D}$ such that

$$e_{-1} = c_0 u + v_0, \quad u_0 = d_0 u + w_0. \tag{2.2a}$$

By recurrence, we define the numerical sequences $\{c_n\}$ and $\{d_n\}$ and the vectorial ones $\{v_n\}$, $\{w_n\}$, for $n > 0$ as

$$U v_n = c_{n+1} u + v_{n+1}, \; U w_n = d_{n+1} u + w_{n+1} \; (n \geq 0). \tag{2.2b}$$

Also, we construct the polynomial sequence $\{P_n\}_{n \geq 0}$ as

$$P_0(\lambda) = c_0, \; P_n(\lambda) = c_n + \sum_{k=1}^{n} d_{n-k} \lambda P_{k-1}(\lambda) \text{ if } n \geq 1. \tag{2.3}$$

The two next theorems allow us to express the resolvent of U as a function of the sequence $\{P_n\}$ and therefore, to obtain in a constructive form the parametrization of all their unitary extensions.

THEOREM 2.3: *If R_z is the orthogonal resolvent of U with $|z| < 1$, $\{P_n\}$ is the sequence defined in (2.3), and $\{v_n\}, \{w_n\}$ are given by (2.2a) and (2.2b), then*

$$\langle R_z e_{-1}, e_0 \rangle = \sum\nolimits_{n \geq 1} z^n \left(\sum\nolimits_{k=1}^{n} \lambda P_{k-1}(\lambda) \langle w_{n-k}, e_0 \rangle \right) + \sum\nolimits_{n \geq 0} z^n \langle v_n, e_0 \rangle \tag{2.4}$$

where $\lambda \in \mathbf{T}$.

Proof: At first, knowing that $T_z = U P_{\mathcal{D}} \oplus \Phi_z$ and $\Phi_z u = \lambda u_0$ with $|\lambda| = 1$, it is easy to prove by induction that

$$T_z^n e_{-1} = P_n(\lambda) u + \sum\nolimits_{k=1}^{n} \lambda P_{k-1}(\lambda) w_{n-k} + v_n, \text{ for all } n \geq 1.$$

Then, if we apply (2.1), it results

$$\langle R_z e_{-1}, e_0 \rangle = \langle e_{-1}, e_0 \rangle + \sum_{n \geq 1} z^n \left(P_n(\lambda) \langle u, e_0 \rangle + \sum_{k=1}^{n} \lambda P_{k-1}(\lambda) \langle w_{n-k}, e_0 \rangle + \langle v_n, e_0 \rangle \right)$$

which gives (2.4) because $\langle u, e_0 \rangle = 0$, and $\langle e_{-1}, e_0 \rangle = \langle v_0, e_0 \rangle$ ∎

THEOREM 2.4: *Under the conditions and hypotheses of the previous theorem, if R_z is a generalized resolvent of U with $|z| < 1$, then*

$$\langle R_z e_{-1}, e_0 \rangle = \sum_{n \geq 1} z^n \left(\sum_{k=1}^{n} \varphi(z) P_{k-1}(\varphi(z)) \langle w_{n-k}, e_0 \rangle \right) + \sum_{n \geq 0} z^n \langle v_n, e_0 \rangle \tag{2.5}$$

where $\varphi \in H^\infty$ *and* $\|\varphi\|_\infty \leq 1$.

Proof: It is the same as the one of Theorem 2.3 but taking into account Proposition 2.2 ∎

Now, we will obtain another parametrization formula for the generalized resolvent of U by the associated contractive analytic function, using formula (1.3). Lastly, we will see that both formulas are equivalent when we can write the relation between the polynomials $\{P_k\}$ and the Fourier coefficients $\widehat{R}(j)$ of R_z.

PROPOSITION 2.5: *If* R_z *is the generalized resolvent of* U *with* $|z| < 1$, *then*

$$\langle R_z e_{-1}, e_0\rangle = \sum_{n\geq 0} z^n \langle v_n, e_0\rangle + \sum_{n\geq 1} z^n \Big(\sum_{j=1}^{n} \varphi c_{j-1}\langle \widehat{R}(n-j)u_0, e_0\rangle\Big) \tag{2.6}$$

where $\varphi \in H^\infty$ *and* $\|\varphi\|_\infty \leq 1$.

Proof (Sketch): At first, we can prove by induction that, for $1 \leq m \leq n$,

$$\widehat{R}(n)e_{-1} = \widehat{R}(n-m)Uv_{m-1} + \sum_{j=1}^{m} c_{j-1} \sum_{k=0}^{n-j} \widehat{R}(k)\widehat{\vartheta}(n-k-j)u.$$

Afterwards, for $m = n$,

$$\begin{aligned}\langle \widehat{R}(n)e_{-1}, e_0\rangle &= \langle Uv_{n-1}, e_0\rangle + \sum_{j=1}^{n} c_{j-1} \sum_{k=0}^{n-j} \langle \widehat{R}(k)\widehat{\vartheta}(n-k-j)u, e_0\rangle \\ &= \langle v_n, e_0\rangle + \sum_{j=1}^{n} c_{j-1}\langle \widehat{R}(n-j)\varphi u_0, e_0\rangle.\end{aligned}$$

So

$$\begin{aligned}\langle R_z e_{-1}, e_0\rangle &= \langle e_{-1}, e_0\rangle + \sum_{n\geq 1} z^n \Big(\langle v_n, e_0\rangle + \sum_{j=1}^{n} \varphi c_{j-1}\langle \widehat{R}(n-j)u_0, e_0\rangle\Big) \\ &= \langle v_0, e_0\rangle + \sum_{n\geq 1} z^n \Big(\langle v_n, e_0\rangle + \sum_{j=1}^{n} \varphi c_{j-1}\langle \widehat{R}(n-j)u_0, e_0\rangle\Big)\end{aligned}$$

which leads to the desired result ∎

PROPOSITION 2.6: *The polynomial family* $\{P_k\}$ *and the Fourier coefficients* $\widehat{R}(j)$ *of the generalized resolvent of* U *are related by the formula*

$$\sum_{k=1}^{n} P_{k-1}(\varphi)\langle w_{n-k}, e_0\rangle = \sum_{j=1}^{n} c_{j-1}\langle \widehat{R}(n-j)u_0, e_0\rangle, \text{ for } n \geq 1. \tag{2.7}$$

Proof: If we define the polynomial sequence

$$Q_0(\varphi) = d_0, \; Q_n(\varphi) = d_n + \sum_{k=0}^{n-1} d_k \varphi Q_{n-k-1}(\varphi) \text{ if } n \geq 1,$$

then we can easily obtain the following relation between $\{P_n\}$ and $\{Q_n\}$:

$$P_n(\varphi) = \sum_{k=0}^{n-1} c_k \varphi Q_{n-k-1}(\varphi) + c_n, \text{ for all } n \geq 0.$$

So, the expression on the left-hand side of (2.7) can be written as

$$\sum_{k=1}^{n} P_{k-1}(\varphi)\langle w_{n-k}, e_0\rangle = \sum_{k=1}^{n} c_{k-1}\langle w_{n-k}, e_0\rangle + \sum_{k=2}^{n}\sum_{j=0}^{k-2} c_j\varphi Q_{k-j-2}(\varphi)\langle w_{n-k}, e_0\rangle.$$

On the other hand, it can be proved by induction that, for $1 \leq m \leq k$,

$$\widehat{R}(k)u_0 = (UP_D + \widehat{\vartheta}(0)P_{\mathcal{M}})^{k-m}\left(Q_m(\varphi)u + w_m + \sum_{j=0}^{m-1} \varphi Q_j(\varphi)w_{m-j-1}\right).$$

Thus, if $m = k$, then

$$\widehat{R}(k)u_0 = Q_k(\varphi)u + w_k + \sum_{j=0}^{k-1} \varphi Q_j(\varphi)w_{k-j-1}.$$

So

$$\langle \widehat{R}(k)u_0, e_0\rangle = \langle w_k, e_0\rangle + \sum_{j=0}^{k-1} \varphi Q_j(\varphi)\langle w_{k-j-1}, e_0\rangle, k \geq 1.$$

Thus, the expression on the right-hand side of (2.7) can be written as

$$\sum_{j=1}^{n} c_{j-1}\langle \widehat{R}(n-j)u_0, e_0\rangle = \sum_{j=1}^{n} c_{j-1}\langle w_{n-j}, e_0\rangle + \sum_{j=1}^{n-1}\sum_{k=0}^{n-j-1} c_{j-1}\varphi Q_k(\varphi)\langle w_{n-k-j-1}, e_0\rangle.$$

Interchanging the order of the last sum we arrive at the result ∎

CONCLUSION. From (2.7) we can deduce that the parametrizations (2.5) and (2.6) are the same.

3. Liftings of a Weakly Positive Measure Matrix

Formula (2.5) leads to a parametrization of all the positive liftings of a weakly positive measure matrix. For this purpose, we establish a close connection between the unitary extensions of an isometric operator and the positive liftings of a measure matrix. At first, we bring out some preliminary definitions.

DEFINITION 3.1: (a) A 2×2 Hermitian matrix $M = (\mu_{\alpha\beta})_{\alpha,\beta=1,2}$, whose elements are finite complex measures on $\mathbb{T}$, is said to be *positive*, $(\mu_{\alpha\beta}) \geq 0$, if the numerical matrix $(\mu_{\alpha\beta}(\Delta))$ is positive definite for every Borel set Δ of $\mathbb{T}$. This is equivalent to $M(f_1, f_2) \equiv \sum_{\alpha,\beta=1,2} \int_0^{2\pi} f_\alpha \bar{f}_\beta d\mu_{\alpha\beta} \geq 0$, for all $(f_1, f_2) \in \mathcal{P} \times \mathcal{P}$, where $\mathcal{P} = \{f : \mathbb{T} \to \mathbb{C} : f(t) = \sum_{-N}^{N} \widehat{f}(n)e_n(t), e_n(t) = e^{int}\}$ is the space of the trigonometric polynomials in $\mathbb{T}$.

(b) We say that the matrix $M = (\mu_{\alpha\beta})$ is *weakly positive*, and write $(\mu_{\alpha\beta}) \succ 0$, if $M(f_1, f_2) \geq 0$, for all $(f_1, f_2) \in \mathcal{P}_1 \times \mathcal{P}_2$, where $\mathcal{P}_1 = \{f \in \mathcal{P} : \widehat{f}(n) = 0 \text{ for } n < 0\}$ and $\mathcal{P}_2 = \{f \in \mathcal{P} : \widehat{f}(n) = 0 \text{ for } n \geq 0\}$ are the subspaces of $\mathcal{P}$ of the analytic and the conjugate analytic polynomials, respectively.

Definition 3.2: A sesquilinear form $B : \mathcal{P} \times \mathcal{P} \to \mathbb{C}$ is said to be a *Toeplitz form* if $B(\tau f, \tau g) = B_0(f, g)$, for all $(f, g) \in \mathcal{P} \times \mathcal{P}$, where $\tau f(t) = e^{it} f(t)$. If B is a Toeplitz form and $B_0 = B|_{\mathcal{P}_1 \times \mathcal{P}_2}$, then B_0 is called a *Hankel form* and one has $B_0(\tau f, g) = B_0(f, \tau^{-1} g)$, for all $(f, g) \in \mathcal{P}_1 \times \mathcal{P}_2$.

If B_1, B_2 are Toeplitz forms and B_0 is a Hankel form, we say that B_0 is *weakly bounded* by (B_1, B_2) and write $B_0 \prec (B_1, B_2)$ if

$$B_1, B_2 \geq 0, |B_0(f,g)|^2 \leq B_1(f,f) B_2(g,g), \text{ for all } (f,g) \in \mathcal{P}_1 \times \mathcal{P}_2.$$

If $B_0 \prec (B_1, B_2)$, we define the matrix $(B_{\alpha\beta})_{\alpha,\beta=1,2}$ where $B_{\alpha\alpha} = B_\alpha$ ($\alpha = 1, 2$), $B_{12} = B_0, B_{21} = B_0^*$ and say that a form $B : \mathcal{P} \times \mathcal{P} \to \mathbb{C}$ given by $B(f, g) = B_{\alpha\beta}(f, g)$, for $(f, g) \in \mathcal{P}_\alpha \times \mathcal{P}_\beta$, is a *generalized Toeplitz form.*

The next theorem has been stated by Cotlar and Sadosky in different ways (see [6]) and has provided several extensions of classical results.

Theorem 3.3 (Generalized Bochner Theorem): *If B is a generalized Toeplitz form, then there exists $(\mu_{\alpha\beta}) \geq 0$. such that*

$$B(f,g) = \int_0^{2\pi} f(t)\overline{g(t)} d\mu_{\alpha\beta}(t), \text{ for } (f,g) \in \mathcal{P}_\alpha \times \mathcal{P}_\beta \, (\alpha, \beta = 1, 2). \tag{3.1}$$

When (3.1) is satisfied, we say that B is *the associated form* to $M = (\mu_{\alpha\beta})$. Another form of expressing this theorem is in terms of a lifting property:

Theorem 3.4 (Lifting of weakly positive measure matrix): *Given the matrix $M = (\mu_{\alpha\beta}) \succ 0$, there exists $M' = (\mu'_{\alpha\beta}) \geq 0$ such that*

$$M(f_1, f_2) = M'(f_1, f_2), \text{for all } (f_1, f_2) \in \mathcal{P}_1 \times \mathcal{P}_2. \tag{3.2}$$

From (3.2) and a theorem of F. and M. Riesz, we can deduce that there exists $h \in H^1(\mathbb{T})$ such that

$$\mu'_{11} = \mu_{11}, \; d\mu'_{12} = d\mu_{12} + \bar{h} dt, \; d\mu'_{21} = d\mu_{21} + h dt, \; \mu'_{22} = \mu_{22}. \tag{3.3}$$

Now, the problem of parametrizing all the positive liftings of M can be related to the problem of parametrizing the unitary extension of a certain isometric operator, as follows:

Assume that B is the form associated to $M \succ 0$. It defines in $\mathcal{P}$ an inner product by $\langle e_n, e_k \rangle = B(e_n, e_k)$, for each $(n, k) \in \mathbb{Z} \times \mathbb{Z}$. Thus, we obtain a Hilbert space $\mathcal{H}$ such that $\mathcal{P}$ is a dense subspace. Let $\mathcal{H}_{-1}$ and $\mathcal{H}_0$ denote the closed subspaces of $\mathcal{H}$ spanned by $\{e_k : k \neq -1\}$ and $\{e_k : k \neq 0\}$, respectively, and define the right shift operator U in $\mathcal{H}$ by $U e_k = e_{k+1}$, whose domain and range are $\mathcal{D} = \mathcal{H}_{-1}$ and $\Delta = \mathcal{H}_0$. It is immediate that U satisfies the conditions of Section 2 because $U^n e_0 = e_n$ if $n \geq 0$ and $U^n e_{-1} = e_{n-1}$ if $n \leq 0$.

Proposition 3.5: *There is a bijection between the set of all positive liftings of M and the set of all minimal unitary extensions $\widetilde{U}$ of the isometry U.*

Proof: Since $\langle e_n, e_k\rangle = B(e_n, e_k) = \mu_{\alpha\beta}(e_{n-k})$ if $(n,k) \in \mathbf{Z}_\alpha \times \mathbf{Z}_\beta$ $(\alpha,\beta = 1,2)$, we can deduce that μ_{11} and μ_{22} are uniquely determined by U:

$$\mu_{11}(e_k) = \begin{cases} \langle e_k, e_0\rangle = \langle U^k e_0, e_0\rangle & \text{if } k \geq 0 \\ \langle e_0, e_{-k}\rangle = \langle e_0, U^{-k} e_0\rangle & \text{if } k < 0 \end{cases}$$

$$\mu_{22}(e_k) = \begin{cases} \langle e_{-1}, e_{-k-1}\rangle = \langle e_{-1}, U^{-k} e_{-1}\rangle & \text{if } k \geq 0 \\ \langle e_{k-1}, e_{-1}\rangle = \langle U^k e_{-1}, e_{-1}\rangle & \text{if } k < 0. \end{cases}$$

However, μ_{12} is defined only in $\mathcal{P}_1$ and μ_{21} in $\mathcal{P}_2$:

$$\mu_{12}(e_k) = \langle e_0, e_{-k}\rangle = \langle e_0, U^{-k+1} e_{-1}\rangle = \langle U^{k-1} e_0, e_{-1}\rangle \text{ if } k > 0,$$

$$\mu_{21}(e_k) = \langle e_k, e_0\rangle = \langle U^{k+1} e_{-1}, e_0\rangle \text{ if } k < 0.$$

In order to complete the lifting, it is enough to determine $\mu_{21}'(e_k)$ for $k \geq 0$.

If we associate to each extension $\widetilde{U}$ defined in $\widetilde{H}(H \subset \widetilde{H})$ its spectral measure $\{\widetilde{E}_t : t \in [0, 2\pi]\}$ by $\widetilde{U}^k = \int_0^{2\pi} e^{ikt} d\widetilde{E}_t$, then the next numerical measure matrix can be defined:

$$\begin{pmatrix} \langle \widetilde{E}(\Delta) e_0, e_0\rangle & \langle \widetilde{E}(\Delta) e_0, e_{-1}\rangle \\ \langle \widetilde{E}(\Delta) e_{-1}, e_0\rangle & \langle \widetilde{E}(\Delta) e_{-1}, e_{-1}\rangle \end{pmatrix}.$$

This matrix is positive, that is $|\langle \widetilde{E}(\Delta) e_{-1}, e_0\rangle|^2 \leq \langle \widetilde{E}(\Delta) e_0, e_0\rangle \langle \widetilde{E}(\Delta) e_{-1}, e_{-1}\rangle$, because $\widetilde{E}(\Delta)$ are orthogonal projections. Taking into account that

$$\mu_{11}(e_k) = \int_0^{2\pi} e^{ikt} d_t \mu_{11}, \text{ and } \mu_{11}(e_k) = \langle \widetilde{U}^k e_0, e_0\rangle = \int_0^{2\pi} e^{ikt} d_t \langle \widetilde{E}_t e_0, e_0\rangle,$$

we can assert that $\langle \widetilde{E}(\Delta) e_0, e_0\rangle = \mu_{11}(\Delta)$. Analogously, $\langle \widetilde{E}(\Delta) e_{-1}, e_{-1}\rangle = \mu_{22}(\Delta)$ and $\langle \widetilde{E}(\Delta) e_{-1}, e_0\rangle$ extends to $\mu_{21}(\Delta)$. So, we can say that $\mu_{21}'(\Delta) = \langle \widetilde{E}(\Delta) e_{-1}, e_0\rangle$. Then, parametrizing $\mu_{21}'(e_k)$ for $k \geq 0$ is equivalent to parametrizing $\langle \widetilde{U}^{k+1} e_{-1}, e_0\rangle$ for $k \geq 0$ ∎

Applying the resolvent formula,

$$\langle \widetilde{R}_z e_{-1}, e_0\rangle = \int_0^{2\pi} \frac{d\langle \widetilde{E}_t e_{-1}, e_0\rangle}{1 - ze^{it}} = \int_0^{2\pi} \frac{d\mu_{21}'(t)}{1 - ze^{it}},$$

we can see that the Stieltjes transform of μ_{21}', defined by the expression on the right-hand side of the previous formula, leads to the parametrization of $\widetilde{U}$. Therefore, the parametrization of μ_{21}' is given by (2.4) if R_z is an orthogonal resolvent, and by (2.5) if R_z is a generalized resolvent.

As we have seen in (3.3), $d\mu_{21}' = d\mu_{21} + h(t)dt$ where $h \in H^1(\mathbf{T})$. So, the transform of μ_{21}' will be equal to the transform of μ_{21} plus the transform of h. According to

$$\int_0^{2\pi} \frac{h(t)dt}{1 - ze^{it}} = \int_{\mathbf{T}} \frac{h(u)du}{-iu(1 - z/u)} = \frac{-1}{i} \int_{\mathbf{T}} \frac{h(u)du}{u - z} = -h(z),$$

the Stieltjes transform of h is h itself by the Cauchy integral formula. Letting in (2.5) $\varphi \equiv 0$, we obtain a particular positive lifting of μ_{21}, called ν, whose transform will be $\langle R_z^{(\nu)} e_{-1}, e_0\rangle = \sum_{n\geq 0} z^n \langle v_n, e_0\rangle$ for $|z| < 1$. Moreover, from (3.3) there is an absolutely continuous function h_0 such that $d\nu = d\mu_{21} + h_0(t)dt$. As $d\mu'_{21} - d\nu = (h - h_0)(t)dt$, their transforms are

$$h - h_0 = \sum_{n\geq 1} z^n \Big(\sum_{k=1}^{n} \varphi P_{k-1}(\varphi)\langle w_{n-k}, e_0\rangle\Big), \|\varphi\| \leq 1. \qquad (3.4)$$

In brief, we can state the following result.

Theorem 3.6: *Let $M = (\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ be a weakly positive measure matrix on $\mathbf{T}$ with more than one positive lifting. The parametrization of all the positive liftings of M comes from the sequences $\{c_n\}, \{d_n\}$ and the polynomial family $\{P_n\}$ defined in (2.2) and (2.3) by the matrix $M' = (\mu'_{\alpha\beta})$ which has the form expressed in (3.3) where h is indicated in (3.4).*

4. Schur Algorithm for the Nehari Problem

Here we are going to obtain an alternative algorithm for the Nehari problem, as an application of the procedure developed in previous sections, which allows a clear geometrical interpretation. We start remembering some previous definitions and facts.

Definition 4.1: A complex function φ defined in the unit circle $\mathbf{D}$ belongs to the *Schur class* S if φ is analytic and $\|\varphi\|_\infty \leq 1$. We also say that a finite or infinite sequence $\{s_0, s_1, ...\}$ is a *Schur sequence* if there exists a function φ in the Schur class such that $\widehat{\varphi}(n) = s_n$, for $n = 0, 1, ...$

As it is well known, the classical Carathéodory-Fejér problem consists in finding necessary and sufficient conditions for a prescribed sequence $\{s_0, s_1, ...\}$ of complex numbers to be a Schur sequence. The Schur algorithm solves this problem; the main features of this algorithm are the next ones [15]:

Every solution can be uniquely parametrized by a complex sequence $\{\sigma_n\}_{n\geq 0}$ with $|\sigma_n| \leq 1$. More precisely, this sequence is either finite with $|\sigma_n| < 1$, for $0 \leq n < N$, and $|\sigma_N| = 1$ or infinite with $|\sigma_n| < 1$, for $n \in \mathbf{N}$. Furthermore, Schur constructed an algorithm for computing these parameters. Taking into account that an infinite sequence $\{s_0, ..., s_N, ...\}$ is a Schur sequence if and only if $\{s_0, ..., s_N\}$ also is a Schur sequence for all N, we can associate to each problem the so-called *N-reduced Schur problem*, which consists in finding φ in the Schur class such that $\widehat{\varphi}(n) = s_n$, for $0 \leq n \leq N$. So, the solution of the non-reduced problem can be obtained by a limit procces. Although the reduced problem has no unique solution in general, the non-reduced problem has always a unique solution. In particular, we can point that if there exists N such that $|\sigma_N| = 1$, the solution is unique and rational and has a degree less than or equal to N.

On the other hand, the Nehari moment problem (see [12]) consists in finding a function $f \in L^\infty(\mathbf{T})$ such that $\|f\|_\infty \leq 1$ whose negative Fourier coefficients are given by $\{s_n\}_{n<0}$, i.e., $s_n = \widehat{f}(n) = \int_0^{2\pi} e^{-int} f(t)dt, n < 0$.

Let us now state the conditions for the existence of solution in the Nehari problem.

THEOREM 4.2 (Nehari): *A necessary and sufficient condition for the existence of solution to the Nehari problem is*

$$\left|\sum_{m<0}\sum_{n\geq 0} s_{m-n}a_m\bar{b}_n\right|^2 \leq \sum_{m<0}|a_m|^2\sum_{n\geq 0}|b_n|^2, \qquad (4.1)$$

for all finitely supported sequences $\{a_m\}, \{b_n\}$.

The Nehari theorem is a particular case of the Generalized Bochner Theorem 3.3 where

$$B_{11}(f,g) = B_{22}(f,g) = \int f\bar{g}dt, \text{ and } B_{21}(f,g) = \sum_{m<0}\sum_{n\geq 0} s_{m-n}\widehat{f}(m)\overline{\widehat{g}(n)}.$$

A special case of the Nehari problem (in which only a finite number of coefficients is non-zero) is equivalent to one reduced Schur problem, as we see in the following: Given $\{s_0, s_1, \ldots, s_N\}$, if there is a solution $\varphi \in \mathcal{S}$ of the N-reduced Schur problem, then the function $\psi(t) = e^{-i(N+1)t}\varphi(t)$ is a solution of the Nehari problem where the coefficients are zero for $n < -N-1$. Thus, we can associate to each Schur sequence a generalized Toeplitz form B and the solutions are obtained by the method developed in Section 3. A parametrization formula can be constructed through the Stieltjes transform. Next we are going to build an algorithm in order to solve the Carathéodory-Fejér problem and to determine the Schur parameters in a recurrent form. At first, we state the 1-reduced problem as a Nehari problem.

The Case $N = 1$. *Given the sequence* $\{s_n\}_{n<0}$ *where* $s_n = 0$ *if* $n < -1$, *find a function* $\varphi \in L^\infty(\mathbf{T})$ *such that* $\|\varphi\|_\infty \leq 1$ *and* $s_n = \widehat{\varphi}(n)$, *for all* $n < 0$. This wants to say that $z\varphi(z)$ will be analytic with the first coefficient prescribed. The problem can also be stated as follows:

Given the function $f(z) = s_{-1}z^{-1}$, find $h \in H^1(\mathbf{T})$ such that $\|f + h\|_\infty \leq 1$.

Giving f is equivalent to giving the weakly positive measure matrix $(\mu_{\alpha\beta})_{\alpha,\beta=1,2}$ on $\mathbf{T}$, where $\mu_{11} = \mu_{22} = dt, d\mu_{21}(t) = f(t)dt, \mu_{12} = \overline{\mu}_{21}$, and the problem consists in finding a positive matrix $(\mu'_{\alpha\beta})_{\alpha,\beta=1,2}$ such that $\mu'_{11} = \mu'_{22} = dt, \widehat{\mu}'_{21}(n) = \widehat{\mu}_{21}(n)$, if $n < 0$. Owing to the Lifting Theorem 3.4, there must exist a function $h \in H^1(\mathbf{T})$ such that $d\mu'_{21}(t) = d\mu_{21}(t) + h(t)dt$. Thus, the parametrization problem is a particular case of the general problem where the measures are arbitrary. We can provide the solution through the Stieltjes transform of the measure or, equivalently, through the generalized resolvent of the associated isometric operator. Next, the solution of this problem is obtained.

The form B is now

$$B(e_m, e_n) = \begin{cases} s_{-1} & \text{if } m = -1, n = 0 \\ \bar{s}_{-1} & \text{if } m = 0, n = -1 \\ 1 & \text{if } m = n \\ 0 & \text{otherwise.} \end{cases}$$

If we call $c_0 = \sqrt{1 - |s_{-1}|^2}$, then $u = \frac{1}{c_0}(e_{-1} - s_{-1}e_0)$ and $u_0 = \frac{1}{c_0}(e_0 - \bar{s}_{-1}e_{-1})$ are the unitary elements which span $\mathcal{H} \ominus \mathcal{H}_{-1}$ and $\mathcal{H} \ominus \mathcal{H}_0$, respectively. Moreover,

$\langle u, e_{-1}\rangle = c_0$ and $\langle u_0, e_0\rangle = c_0$. Since $e_{-1} = c_0 u + s_{-1} e_0 = c_0 u + v_0$ and $u_0 = -\bar{s}_{-1} u + c_0 e_0 = -\bar{s}_{-1} u + w_0$, it is easy to see that the sequences $\{c_n\}_{n\geq 0}$ and $\{d_n\}_{n\geq 0}$ defined in (2.2) are $\{c_0, 0, 0, \ldots\}$ and $\{-\bar{s}_{-1}, 0, 0, \ldots\}$, respectively, and the sequences $\{v_n\}_{n\geq 0}, \{w_n\}_{n\geq 0}$ have the next particular form: $v_n = s_{-1} e_n, w_n = c_0 e_n, n \geq 0$. Therefore, $\langle v_k, e_0\rangle = s_{-1}\delta_{k0}$ and $\langle w_k, e_0\rangle = c_0 \delta_{k0}$.

Moreover, the polynomial family (2.3) verifies the recurrence law

$$P_0(\lambda) = c_0, \text{ and } P_n(\lambda) = d_0 \lambda P_{n-1}(\lambda),\ n > 0,$$

and we can write the next explicit form for the sequence: $P_n(\lambda) = (d_0\lambda)^n c_0, n \geq 0$. Inserting this into (2.5), we obtain

$$\begin{aligned}
\langle R_z e_{-1}, e_0\rangle &= \sum_{n\geq 1} z^n (-\bar{s}_{-1})^{n-1} [\phi(z)]^n \left(1 - |s_{-1}|^2\right) + s_{-1} \\
&= s_{-1} + \sum_{n\geq 1} z^n (-\bar{s}_{-1})^{n-1} [\phi(z)]^n - \sum_{n\geq 1} z^n (-\bar{s}_{-1})^{n-1} [\phi(z)]^n |s_{-1}|^2 \\
&= s_{-1}\left(1 + \sum_{n\geq 1} z^n (-1)^n (\bar{s}_{-1})^n [\phi(z)]^n\right) \\
&\qquad + z\phi(z) \sum_{n\geq 1} z^{n-1} (-1)^{n-1} (\bar{s}_{-1})^{n-1} [\phi(z)]^{n-1} \\
&= [s_{-1} + z\phi(z)] \sum_{n\geq 0} [-z\bar{s}_{-1}\phi(z)]^n \\
&= \frac{s_{-1} + z\phi(z)}{1 + \bar{s}_{-1} z \phi(z)}.
\end{aligned}$$

In order to obtain an expression for the function h in (3.4), we can write either

$$\mu'_{21}(e_k) = \langle \widetilde{U}^{k+1} e_{-1}, e_0\rangle = \int_0^{2\pi} e^{i(k+1)t} d\langle \widetilde{E}_t e_{-1}, e_0\rangle$$

or $\mu'_{21}(e_k) = \int_0^{2\pi} e^{ikt} d\mu'_{21}(t)$. Then $d\mu'_{21}(t) = e^{it} d\langle \widetilde{E}_t e_{-1}, e_0\rangle$. If we apply the formula of the resolvent:

$$\langle \widetilde{R}_z e_{-1}, e_0\rangle = \int_0^{2\pi} \frac{d\langle \widetilde{E}_t e_{-1}, e_0\rangle}{1 - ze^{it}} = \int_0^{2\pi} \frac{e^{-it} d\mu'_{21}(t)}{1 - ze^{it}}.$$

As $d\mu'_{21} = d\mu_{21} + h(t)dt = f(t)dt + h(t)dt$,

$$\begin{aligned}
\langle \widetilde{R}_z e_{-1}, e_0\rangle &= \int_0^{2\pi} \frac{e^{-it} f(t)dt}{1 - ze^{it}} + \int_0^{2\pi} \frac{e^{-it} h(t)dt}{1 - ze^{it}} \\
&= \int_{\mathbb{T}} \frac{uf(u)du}{-iu(1 - z/u)} + \int_{\mathbb{T}} \frac{uh(u)du}{-iu(1 - z/u)} \\
&= \frac{-1}{i} \int_{\mathbb{T}} \frac{uf(u)du}{u - z} + \frac{-1}{i} \int_{\mathbb{T}} \frac{uh(u)du}{u - z} \\
&= -z\left(f(z) + h(z)\right).
\end{aligned}$$

Combining the previous formulas, we obtain

$$h(z) = \sum_{n\geq 1} z^{n-1}(-\overline{s}_{-1})^{n-1}\,[\phi(z)]^n\,(1-|s_{-1}|^2) = \frac{\phi(z)\,(1-|s_{-1}|^2)}{1+\overline{s}_{-1}z\phi(z)}. \tag{4.2}$$

Observe that this result is the same as the one obtained in the first step of the classical Schur algorithm.

The above construction allows us to give conditions for the existence and unicity of solutions. Thus, if $|s_{-1}| \leq 1$, there exists a solution; moreover, $f(z) = s_{-1}z^{-1}$ will be the unique solution only if $|s_{-1}| = 1$.

The General Case. Now the problem is to find the set of all functions $\varphi \in L^\infty(\mathbb{T})$ such that $\|\varphi\|_\infty \leq 1$ and $\widehat{\varphi}(j) = 0$, $j < -N$, $\widehat{\varphi}(-r) = s_{-r}$, $1 \leq r \leq N$. At this end, only 1-reduced problems will be solved, by building a parameter sequence associated to the problem $\sigma_k = \sigma(s_k, s_{k-1}, \ldots, s_{-N}), k = -N, -N+1, \ldots, -1$.

Step 1. *Find* $\varphi \in L^\infty(\mathbb{T})$ *such that* $\|\varphi\|_\infty \leq 1$, *and* $\widehat{\varphi}(j) = 0, j < -N, \widehat{\varphi}(-N) = s_{-N}$. A slight modification of the formula (4.2) provides the set of all solutions. So

$$\varphi(z) = s_{-N}z^{-N} + \sum\nolimits_{n\geq 1}(1-|s_{-N}|)^2(-\overline{s}_{-N})^{n-1}[f_{-N}(z)]^n z^{n-N} \tag{4.3}$$

where f_{-N} belongs to the unit ball of H^∞. The general solution $\varphi(z)$ depends only on a sole parameter $\sigma_{-N} \equiv s_{-N}$.

Step r $(2 \leq r \leq N)$. *Find* $\varphi \in L^\infty(\mathbb{T})$ *such that* $\|\varphi\|_\infty \leq 1$, *and* $\widehat{\varphi}(j) = 0, j < -N, \widehat{\varphi}(-N) = s_{-N}, \widehat{\varphi}(-N+1) = s_{-N+1}, \ldots, \widehat{\varphi}(-N+r-1) = s_{-N+r-1}$. This problem is equivalent to finding, among the functions φ which are solutions of the step $r-1$, those ones that satisfy $\widehat{\varphi}(-N+r-1) = s_{-N+r-1}$. Now, the value $\widehat{\varphi}(-N+r-1)$ depends only on $\widehat{f}_{-N+r-2}(0)$. If we call $\sigma_{-N+r-1} = \widehat{f}_{-N+r-2}(0)$, the problem can be restated as: *Find all the functions* $f_{-N+r-2} \in H^\infty$ *such that* $\|f_{-N+r-2}\|_\infty \leq 1$ *and* $\widehat{f}_{-N+r-2}(0) = \sigma_{-N+r-1}$. So, the general solution is also like (4.2):

$$\begin{aligned} f_{-N+r-2}(z) &= \sigma_{-N+r-1} \\ &\quad + \sum\nolimits_{n\geq 1}(1-|\sigma_{-N+r-1}|)^2(-\overline{\sigma}_{-N+r-1})^{n-1}[f_{-N+r-1}(z)]^n z^n \end{aligned} \tag{4.4}$$

where f_{-N+r-1} belong to the unit ball of H^∞.

In each step, a necessary and sufficient condition for the existence of solutions is $|\sigma_{-N+r-1}| \leq 1$: moreover, if $|\sigma_{-N+r-1}| = 1$, we have unicity. From an idea contained in the mentioned paper of Nehari [12], if the whole sequence $\{s_n\}_{n<0}$ is given, the general solution for the problem can be obtained by means of a limit process.

5. Schur Algorithm for the Lacunary Nehari Problem

The method developed in Section 4 can be applied with some changes to solve the interpolation problem stated in the introduction where the given sequence is lacunary. We will study conditions for the existence and unicity as well as the parametrization of the solution set, by means of a Schur algorithm.

DEFINITION 5.1: A sequence of positive integers $\{n_k\}_{k\geq 0}$ is said to be *λ-lacunary* if $\frac{n_{k+1}}{n_k} > \lambda > 1$, for all k.

The next boundary theorem due to Paley [13] establishes that every lacunary sequence of coefficients of an $H^1(\mathbf{T})$ function belongs to ℓ^2.

THEOREM 5.2 (Paley lacunary inequality): *Let $\{n_k\}_{k\geq 0}$ be a λ-lacunary sequence. There exists $C = C(\lambda)$ such that if $f(t) = \sum_{n\geq 0} c_n e^{int}$ belongs to $H^1(\mathbf{T})$, then $\left(\sum_{k\geq 0} |c_{n_k}|^2\right)^{1/2} \leq C \int_0^{2\pi} |f(t)|dt$.*

Rudin [14] observed that Paley's theorem has an equivalent dual formulation, as follows.

THEOREM 5.3 (Rudin): *If $\{n_k\}_{k\geq 0}$ is a λ-lacunary sequence and $\{v_k\} \in \ell^2$, then there exists $g \in L^\infty(\mathbf{T})$ such that*

$$\widehat{g}(n) = \begin{cases} v_k & \text{if } n = n_k \\ 0 & \text{otherwise} \end{cases} \quad (n \geq 0) \text{ and } \|g\|_\infty \leq C\|v\|_2.$$

The Paley theorem, proved by Fournier [10] in a constructive way, can also be proved as a consequence of the next theorem and from the Generalized Bochner Theorem (see [6,7] and the references quoted there).

THEOREM 5.4: *Given a λ-lacunary sequence $\{n_k\}_{k\geq 0}$, there exists $C = C(\lambda)$ such that if $f(t) = \sum_{n\geq 0} c_n e^{int}$ belongs to $H^2(\mathbf{T})$ and $c_n = 0$ when $n \neq n_k$, then the matrix*

$$\begin{pmatrix} C\|f(t)\|_2 dt & \bar{f}(t)dt \\ f(t)dt & C\|f(t)\|_2 dt \end{pmatrix}$$

is weakly positive.

Proof (Sketch): At first, we consider $\lambda = 2$; thus, $n_{k+1} > 2n_k$. We must prove that, if $f_1(t) = \sum_{n\geq 0} a_n e^{int}$ and $f_2(t) = \sum_{n>0} b_n e^{-int}$ are analytic and anti-analytic polynomials, respectively, then

$$\left|\int f_1(t)\overline{f}_2(t)dt\right| \leq C\|f\|_2 \left(\int |f_1(t)|^2 dt\right)^{1/2} \left(\int |f_2(t)|^2 dt\right)^{1/2}.$$

The expression on the left-hand side is equal to $\left|\sum_k c_{n_k} \left(\sum_{i=0}^{n_k-1} a_i \overline{b}_{n_k-i}\right)\right|$ and we decompose it in two summands; at this end, we call $m_k = \left[\frac{n_k}{2}\right]$. Applying the Schwarz inequality twice, we can obtain that

$$\left|\sum_k c_{n_k} \left(\sum_{i=0}^{m_k} a_i \overline{b}_{n_k-i}\right)\right| \leq \|f\|_2 \left(\int |f_1(t)|^2 dt\right)^{1/2} \left(\int |f_2(t)|^2 dt\right)^{1/2}.$$

In the same way,

$$\left|\sum_k c_{n_k} \left(\sum_{i=m_k+1}^{n_k-1} a_i \overline{b}_{n_k-i}\right)\right| \leq \|f\|_2 \left(\int |f_1(t)|^2 dt\right)^{1/2} \left(\int |f_2(t)|^2 dt\right)^{1/2}.$$

In the case where $\lambda \neq 2$, some terms repeat themselves a fixed number of times; the last result will be multiplied by a constant C. So

$$\left|\int f_1(t)\overline{f}_2(t)f(t)dt\right| \leq C\|f\|_2 \left(\int |f_1(t)|^2 dt\right)^{1/2} \left(\int |f_2(t)|^2 dt\right)^{1/2} \blacksquare$$

In order to solve the problem stated in the introduction, we consider a λ-lacunary sequence $\{n_k\}$ and a sequence $\{a_n\}_{n\geq 0} \in \ell^2$ such that $a_n = 0$, if $n \neq n_k$.

NOTATION: For each $M > 0$, we choose a positive integer p_M such that $p_{M+1} > p_M$ and

$$\left(\sum_{n>p_M} |a_n|^2\right)^{1/2} \leq \frac{1}{CM}. \tag{5.1}$$

For each natural p, we define $f_p(x) = \sum_{0\leq n\leq p} a_n e_n(x)$. Then,

$$\widehat{f}_p(n) = \begin{cases} a_n & \text{if } n \leq p, \\ 0 & \text{if } n > p. \end{cases}$$

Analogously to $\Sigma(a)$, we define the set

$$\Sigma_M(a) = \left\{\Phi \in L^\infty : \|\Phi\|_\infty \leq 1, \widehat{\Phi}(n) = \frac{M}{M+1}\widehat{f}_{p_M}(n), \forall n \geq 0\right\}.$$

LEMMA 5.5: *If $\Phi_M \in \Sigma_M(a)$, for all M, there exists a sub-sequence $\{M_k\}$ such that, if $\Phi = \lim_{k\to\infty} \Phi_{M_k}$, then $\Phi \in \Sigma(a)$.*

Proof. As $\|\Phi_M\|_\infty \leq 1$, the sequence is bounded; so, there is a weakly convergent sub-sequence $\{\Phi_{M_k}\}$. Let $\Phi(x) = \lim_{k\to\infty} \Phi_{M_k}(x)$. For every $n \geq 0$,

$$\begin{aligned}\widehat{\Phi}(n) = \int \Phi(t)e_{-n}(t)dt &= \lim_{k\to\infty} \int \Phi_{M_k}(t)e_{-n}(t)dt \\ &= \lim_{k\to\infty} \widehat{\Phi}_{M_k}(n) = \lim_{k\to\infty} a_n \frac{M_k}{M_k+1} = a_n.\end{aligned}$$

This implies that $\Phi \in \Sigma(a)$ $\blacksquare$

LEMMA 5.6: *If $\Phi \in \Sigma(a)$, then for every $M > 0$, there exists $\Phi_M \in \Sigma_M(a)$ such that $\|\Phi - \Phi_M\| < 2/M$.* In other words, there exists a sub-sequence $\{\Phi_{M_k}\}$ convergent in norm to Φ.

Proof. Let $b_n = \begin{cases} 0 & \text{if } n \leq p_M \\ a_n & \text{if } n > p_M \end{cases}$. Thus, $\{b_n\}$ satisfies also that $b_n = 0$, if $n \neq n_k$ and, by (5.1),

$$\left(\sum_{n\geq 0} |b_n|^2\right)^{1/2} \leq \left(\sum_{n>p_M} |a_n|^2\right)^{1/2} \leq \frac{1}{CM}.$$

By the Rudin theorem, there exists $\Psi \in L^\infty(\mathbf{T})$ such that $\|\Psi\|_\infty \leq C\frac{1}{CM} = \frac{1}{M}$ and $\widehat{\Psi}(n) = \begin{cases} 0 & \text{if } 0 \leq n \leq p_M \\ a_n & \text{if } n > p_M \end{cases}$. If we define $\Phi_M = \frac{M}{M+1}(\Phi - \Psi)$, it results that

$$\widehat{\Phi}_M(n) = \begin{cases} \frac{M}{M+1}a_n & \text{if } 0 \leq n \leq p_M \\ 0 & \text{if } n > p_M \end{cases} \text{ and } \|\Phi_M\|_\infty \leq \left(1 + \frac{1}{M}\right)\frac{M}{M+1} = 1.$$

So, $\Phi_M \in \Sigma_M(a)$ and

$$\|\Phi - \Phi_M\|_\infty = \left\|\frac{1}{M+1}\Phi + \frac{M}{M+1}\Psi\right\| \leq \frac{1}{M+1} + \frac{1}{M+1} = \frac{2}{M+1} < \frac{2}{M} \blacksquare$$

THEOREM 5.7: *$\Sigma(a) \neq \emptyset$ if and only if $\Sigma_M(a) \neq \emptyset$, for all M.*

Proof. (a) If $\Sigma(a) \neq \emptyset$, let $\Phi \in \Sigma(a)$. By Lemma 5.6, for every $M > 0$, there exists $\Phi_M \in \Sigma_M(a)$ and $\Phi = \lim_{k\to\infty} \Phi_{M_k}(x)$, where $\{\Phi_{M_k}\}$ is some sub-sequence of $\{\Phi_M\}$. (b) If there exists $\Phi_M \in \Sigma_M(a)$, for all M, we can choose a convergent sub-sequence $\{\Phi_{M_k}\}$. If $\Phi(x) = \lim_{k\to\infty} \Phi_{M_k}(x)$, by Lemma 5.5, $\Phi \in \Sigma(a)$ $\blacksquare$

CONCLUSION. In order to show that $\Sigma(a) \neq \emptyset$, it is enough to see that $\Sigma_M(a) \neq \emptyset$, for all M, and in order to parametrize $\Sigma(a)$, it is enough to parametrize each $\Sigma_M(a)$, for all M. By definition, the problem of parametrizing $\Sigma_M(a)$ can be converted into a Schur reduced problem; so an algorithm can be constructed and a sequence of parameters which generate the general solution can be obtained. As for each $\Sigma_M(a)$, the given coefficients depend on M, the Schur parameters sequence will have an almost triangular form, like this:

$$\begin{array}{ll} \Sigma_1(a) & \sigma_1^1 \ldots \sigma_{p_1}^1 \\ \Sigma_2(a) & \sigma_1^2\sigma_2^2 \ldots \sigma_{p_2}^2 \\ \vdots & \vdots \\ \Sigma_M(a) & \sigma_1^M\sigma_2^M \ldots \sigma_M^M \ldots \sigma_{p_M}^M \end{array}$$

The differences between these results and those from the Schur algorithm are that for each M, the set of solutions is different, because the value of the coefficients changes and the set $\Sigma_M(a)$ is not included in $\Sigma_{M+1}(a)$. The constructive method developed in Section 4 will give all solutions of the previous problem, generating for each $M > 0$ a set of Schur parameters; the existence and unicity of the solutions will depend on their values. At first, we state the problem like a Nehari problem:

Given the sequence $\{s_0, \ldots, s_{p_M}, 0, \ldots\}$ where $s_k = \frac{M}{M+1}a_k$ $(0 \leq k \leq p_M)$, find a function $\Phi \in L^\infty(\mathbf{T})$ such that $\|\Phi\|_\infty \leq 1$ and $s_n = \widehat{\Phi}(n)$, for all $n \geq 0$.

If we define $\Psi(z) = z^{-1}\Phi\left(\frac{1}{z}\right)$, we have

$$\widehat{\Psi}(n) = \begin{cases} s_{-n-1} & \text{if } -p_M - 1 \leq n \leq -1 \\ 0 & \text{if } n < -p_M - 1 \end{cases}$$

The problem of finding Ψ is now a (p_M+1)-special Nehari problem. A constructive method for getting all solutions can be obtained as an application of the general method developed in Section 3.

The constructive algorithm is:

Step 1. *Find a function* $\Psi \in L^\infty(\mathbb{T})$ *such that* $\|\Psi\|_\infty \leq 1, \widehat{\Psi}(-p_M-1) = s_{p_M}, \widehat{\Psi}(j) = 0, j < -p_M - 1$. The solution, as the one obtained in Section 4, is

$$z^{p_M+1}\Psi(z) = s_{p_M} + \sum\nolimits_{n\geq 1} (1-|s_{p_M}|^2)(-\overline{s}_{p_M})^{n-1} z^n (\Psi_{p_M}(z))^n$$
$$= \frac{s_{p_M} + z\Psi_{p_M}(z)}{1+\overline{s}_{p_M} z\Psi_{p_M}(z)}$$

where $\Psi_{p_M} \in H^\infty$ is an arbitrary function and $\|\Psi_{p_M}\| \leq 1$. We define $\sigma_{p_M} = s_{p_M}$ as the first parameter associated to the problem.

Step k $(2 \leq k \leq p_M + 1)$. *Find a function* $\Psi \in L^\infty(\mathbb{T})$ *such that*

$$\|\Psi\|_\infty \leq 1, \widehat{\Psi}(-j) = \begin{cases} s_{j-1} & \text{if } j = p(M)-k+2,\ldots,p(M)+1 \\ 0 & \text{if } j > p(M)+1. \end{cases}$$

Here, the solution is

$$z^{p_M+1}\Psi(z) = \sigma_{p_M} + \sum_{n\geq 1}(1-|\sigma_{p_M}|^2)(-\overline{\sigma}_{p_M})^{n-1}z^n \times \left(\sum_{j=p_M-k+2}^{p_M} \sigma_{j-1}z^{p_M-j} + \Psi_{p_M-k+1}(z)z^{k-1}\right)^n$$

where $\Psi_{p_M-k+1} \in H^\infty$ is an arbitrary function such that $\|\Psi_{p_M-k+1}\| \leq 1$, and σ_{p_M-k+1} depends on $\{s_{p_M-k},\ldots,s_{p_M+1}\}$.

Definitely, we obtain the next final result.

THEOREM 5.8: *The set* $\Sigma_M(a)$ *can be parametrized by*

$$z^{p_M+1}\Psi(z) = \sigma_{p_M} + \sum_{n\geq 1}(1-|\sigma_{p_M}|^2)(-\overline{\sigma}_{p_M})^{n-1}z^n\left(\sum_{j=1}^{p_M}\sigma_{j-1}z^{p_M-j} + \Psi_0(z)z^{p_M}\right)^n$$

where $\Psi_0 \in H^\infty$ *is an arbitrary function such that* $\|\Psi_0\| \leq 1$, *and* $\{\sigma_k\}_{k=0}^{p_M}$ *is a sequence of parameters obtained in a recurrent form from the sequence* $\{s_k\}_{k=0}^{p_M}$.

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