Zeitschrift für Analysis
und ihre Anwendungen
Vol. 12 (1993), 535 - 548

Fractal Interpolation Functions from R^n into R^m and their Projections

D. P. HARDIN and P. R. MASSOPUST

We construct fractal interpolation functions from $\mathbb{R}^n \to \mathbb{R}^m$, and consider the projections of their graphs onto $\mathbb{R}^n \times \mathbb{R}$. Since these projections still depend continuously on all the variables we refer to them as hidden variable fractal interpolation surfaces. The hidden variable fractal interpolation surfaces carry additional free parameters and are thus more general than, for instance, the fractal surfaces defined earlier by the authors. These free parameters may prove useful in approximation-theoretic considerations. A formula for the box dimension of a hidden variable fractal interpolation surface is presented. This dimension parameter could be used to distinguish different textures on natural surfaces.

Key words: *Iterated function systems, fractal functions and surfaces, attractors, box dimensions*
AMS classification: *26A15, 26A18, 26B05, 41A46, 41A63, 58F12*

1 Introduction

In this paper we use the theory of iterated function systems to construct fractal interpolation functions from $\mathbb{R}^n \to \mathbb{R}^m$ and consider the projections of their graphs onto $\mathbb{R}^n \times \mathbb{R}$. These projections depend continuously on all the variables, and following [3], are called *hidden variable fractal interpolation surfaces.*

The fractal surfaces constructed and investigated in [4] and [5] can be used for interpolation and approximation purposes, in particular, for image compression and pattern recognition. The existence of a free parameter in the construction of such fractal surfaces allows for some freedom in the description of the underlying model. The new hidden variable fractal interpolation surfaces carry $m^2 - 1$ additional free parameters which could be used to better model or approximate natural surfaces, exhibiting a highly complex fine structure.

The dimension of a function is a measure for its smoothness and related to the approximation order in many applications. For instance, it is a well-known fact that a function $f \in \mathrm{Lip}^\alpha$, $0 < \alpha < 1$, has box dimension $\leq 2 - \alpha$. We derive a formula for the box dimension of the graphs of hidden variable fractal interpolation surfaces. This dimension parameter can also serve as a means to distinguish or describe different textures on natural surfaces.

The organization of this paper is as follows: In Section 2 we briefly introduce and review the theory of iterated function systems, fractal functions and fractal surfaces. Section 3 is devoted to the construction of fractal functions from simplices in $\mathbb{R}^n$ into $\mathbb{R}^m$. In Section 4 we prove a result about the oscillation of such a fractal function. Section 5 introduces the hidden variable fractal interpolation surfaces, and there we prove the main theorem giving the box dimension of the graph of a hidden variable fractal interpolation surface. Examples of them close out this section. In the Appendix we give the proof of a rather technical lemma needed to prove Theorem 6 in Section 5.

D. P. Hardin: Vanderbilt Univ., Dep. Math., Nashville, Tenn. 37240, USA
P. R. Massopust: Vanderbilt Univ., Dep. Math., Nashville, Tenn. 37240, USA

ISSN 0232-2064 / $ 2.50

2 Preliminaries

In this section we briefly review some of the relevant results from the theory of iterated function systems, fractal functions, and fractal surfaces.

2.1 Iterated function systems and fractal interpolation functions

Let $X = (X, d)$ be a compact metric space or a closed subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, with metric d. Denote by $H(X)$ the set of all non-empty compact subsets of X. It is easy to show that $(H(X), h)$ becomes a complete metric space when endowed with the metric $h : H(X) \times H(X) \to \mathbb{R}_0^+$

$$h(A, B) = \max\{\sup\{d(x, B) : x \in A\},\ \sup\{d(A, y) : y \in B\}\}.$$

This metric is called the *Hausdorff metric on* $H(X)$. Let $\mathrm{w} = \{w_i : X \to X : i = 1, \ldots, N\}$ be a collection of continuous functions on X. The pair (X, w) is called an *iterated function system* on X (see [2]). If the maps w_i are contractive, then (X, w) is called *hyperbolic*. If we define a set-valued map $W : H(X) \to H(X)$ by $W(A) = \cup_i w_i(A)$, for all $A \in H(X)$, then W is a contraction on $H(X)$, thus possessing a unique fixed point A^*, called the *attractor* of the iterated function system (X, w). This fixed point satisfies

$$A^* = W(A^*) = \bigcup_{i=1}^{N} w_i(A^*).$$

For a more detailed and elaborate presentation of iterated function systems we refer the reader to [1, 2].

One can associate a code space with an iterated function system in the following way: Let A^* be the attractor of an iterated function system (X, w). Define $\Sigma^* = \Sigma_N^* = \{1, \ldots, N\}^{\mathbb{N}}$. Note that Σ^* endowed with the Fréchet metric $|\cdot, \cdot| : \Sigma^* \times \Sigma^* \to \mathbb{R}_0^+$,

$$|\mathrm{i}, \mathrm{j}| = \sum_{m=1}^{\infty} \frac{|i_m - j_m|}{(N+1)^m},$$

with $\mathrm{i} = (i_m)_{m \in \mathbb{N}}$, $\mathrm{j} = (j_m)_{m \in \mathbb{N}}$, is a compact metric space, homeomorphic to the classical Cantor set on N symbols. Σ^* is called the *code space* for the iterated function system (X, w) and its elements are called *codes*. We will denote the codes by small bold-faced latin characters such as i, j, etc. If $\mathrm{i} = (i_1, i_2, \cdots i_m, \cdots) \in \Sigma^*$, then $\mathrm{i}(m) = (i_1 i_2 \cdots i_m)$ is called a *finite code of length* m. The set of all finite codes is denoted by Σ. If $\mathrm{i}(m)$ is a finite code, we will write f_{i} for $f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_m}$ (function composition) and $a_{\mathrm{i}} = a_{i_1} a_{i_2} \ldots a_{i_m}$ for products of real numbers indexed by the components of i.

To each $\mathrm{i} \in \Sigma^*$ we can associate a point $S(\mathrm{i}) \in A^*$ via $S(\mathrm{i}) = w_{\mathrm{i}}(A)$, and the map $S : \Sigma^* \to A^*$ is surjective (however, S is not injective, in general). Thus, we will identify points on A^* with points in Σ^*. Note that, if $\mathrm{i}(m)$ is the initial segment of $\mathrm{i} \in \Sigma$, then $\lim_{m \to \infty} w_{\mathrm{i}(m)}(A) = A^*$.

Now let $X = [0, 1] \times \mathbb{R}$ and let $\mathbf{I} := \{1, \ldots, N\}$. Consider a set of $N + 1$ interpolation points $\mathcal{T} = \{(x_j, y_j) : 0 = x_0 < \cdots < x_N = 1,\ y_j \in \mathbb{R},\ j \in \mathbf{J} = \{0\} \cup \mathbf{I}\}$. Define maps $w_i : X \to X$ by

$$w_i(x, y) = \begin{pmatrix} a_i & 0 \\ c_i & s_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d_{i-1} \\ e_{i-1} \end{pmatrix}$$

where $0 < |s_i| < 1$ is given, and a_i, c_i, d_i, e_i are determined by the conditions $w_i(0, y_0) = (x_{i-1}, y_{i-1})$ and $w_i(1, y_N) = (x_i, y_i)$ yielding

$$\begin{aligned} a_i &= x_i - x_{i-1}, & c_i &= y_i - y_{i-1}, \\ d_i &= x_{i-1}, & e_{i-1} &= y_{i-1} - s_i y_0. \end{aligned}$$

Let $\mathcal{D}$ be the metric on $\mathbb{R}^2$ defined by $\mathcal{D}((x,y),(\tilde{x},\tilde{y})) := |x-\tilde{x}| + \beta|y-\tilde{y}|$, where β is chosen so that $0 < \beta < \min_i\{(1-a_i)/(1+|c_i|)\}$. Then it is easy to verify that w_i is a contraction in the metric $\mathcal{D}$. Hence $(X, \mathbf{w})$ is a hyperbolic iterated function system possessing a unique fixed point G.

We next show that G is the graph of a continuous function $f^* : [0,1] \to \mathbb{R}$ that interpolates $\mathcal{T}$. For this purpose, let $\hat{C}([0,1])$ denote the Banach space of all continuous functions satisfying $f(x_j) = y_j$, $j \in \mathbf{J}$. Let $u_i : [0,1] \to \mathbb{R}$ and $v_i : [0,1] \times \mathbb{R} \to \mathbb{R}$ be given by

$$u_i(x) = a_i x + d_i \qquad \text{and} \qquad v_i(x,y) = c_i x + s_i y + e_i,$$

for $i \in \mathbf{I}$. For $f \in \hat{C}([0,1])$ we define

$$\Phi(f)(x) = v_i(u_i^{-1}(x), f(u_i^{-1}(x))), \qquad \text{for} \quad x \in [x_{i-1}, x_i].$$

Let $s := \max\{|s_i| : i \in \mathbf{I}\}$. It is straight-forward to show that Φ maps $\hat{C}([0,1])$ into itself, and is contractive in the sup-norm with contractivity s. By the Banach Fixed-Point Theorem, Φ has a unique fixed point $f^* \in \hat{C}([0,1])$. It is easy to verify that $W(\text{graph } f^*) = \text{graph } (f^*)$ and hence $G = \text{graph } (f^*)$. Following [1] we call f^* a *fractal interpolation function.* In the next subsection we show how a modification of the above construction can be used to obtain *fractal surfaces* defined on triangular regions of $\mathbb{R}^2$.

2.2 Fractal interpolation surfaces

Let D be a closed non-degenerate triangular region in $\mathbb{R}^2$ and let $S = \{q_1, \ldots, q_m\}$ be m ($m > 3$) distinct points in D such that $\{q_1, q_2, q_3\}$ are the vertices of D. Given real numbers $z_1, \ldots, z_m$ we wish to construct a function f^* such that $f^*(q_i) = z_i$, $i = 1, \ldots, m$, and whose graph is self-affine. We decompose D into N non-degenerate subtriangles $\sigma_1, \ldots, \sigma_N$ whose interiors are non-intersecting. We assume that the set of vertices of $\{\sigma_i\}_{i=1}^N$ equals S. Let $k : \{1, \cdots, N\} \times \{1,2,3\} \to \{1, \cdots, m\}$ be such that $\{q_{k(i,j)}\}_{j=1}^3$ gives the vertices of σ_i.

Let $i \in \{1, \ldots, N\}$. Since D and σ_i are non-degenerate there exists a unique invertible affine map $u_i : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying

$$u_i(q_j) = q_{k(i,j)}, \qquad j = 1,2,3. \tag{1}$$

Let s_i be given such that $|s_i| < 1$ and $v_i : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$v_i(x,y,z) = a_i x + b_i y + s_i z + c_i, \tag{2}$$

where a_i, b_i and c_i are uniquely determined by

$$v_i(q_j, z_j) = z_{k(i,j)}, \qquad j = 1,2,3. \tag{3}$$

Let $C(D)$ be the Banach space of all real-valued continuous functions defined on D endowed with the sup-norm. Define a mapping $\Phi : C(D) \to \mathbb{R}^D$ by

$$\Phi(f)(x) = v_i(u_i^{-1}(x), f(u_i^{-1}(x))), \qquad \text{for } x \in \sigma_i. \tag{4}$$

We next find conditions so that Φ takes $C(D)$ into an appropriate subspace of itself. With the above definitions for u_i and v_i we have $\Phi(f)|_{\sigma_i} \in C(\sigma_i)$ and $\Phi(f)(q_{k(i,j)}) = z_{k(i,j)}$, whenever $f \in C(D)$ and $f(q_j) = z_j$, $j = 1,2,3$. If σ_i and $\sigma_{i'}$ are adjacent triangles with common edge $\overline{q_j q_{j'}}$, it remains to be determined whether $\Phi(f)$ is well-defined along $\overline{q_j q_{j'}}$, that is, whether $\Phi(f)$ satisfies the "join-up" condition:

$$v_i(u_i^{-1}(x),\ f(u_i^{-1}(x))) = v_{i'}(u_{i'}^{-1}(x),\ f(u_{i'}^{-1}(x))), \qquad \text{for all } x \in \overline{q_j q_{j'}}. \tag{5}$$

Equation (5) is satisfied if, for instance, the boundary data $\{(q_j, z_j) : q_j \in \partial D\}$ is assumed to be coplanar (see [8]) or if the graph associated with the triangulation $\{\sigma_i\}_{i=1}^N$ has *chromatic*

number 3 (recall that the chromatic number of a graph is the least number of symbols required to label the vertices of the graph in such a way that any two adjacent vertices, i.e., vertices that are joined by an edge, have distinct labels, see [4]).

Now let P be a non-vertical plane in $\mathbb{R}^3$, and let $C_P(D)$ denote the collection of all continuous functions $f : D \to \mathbb{R}$ such that $(x, f(x)) \in P$, for all $x \in D$. Let $C_B(D)$ denote the set of all continuous functions $f : D \to \mathbb{R}$ such that $f(q_j) = z_j$, $q_j \in D$, $j = 1, \ldots, m$.

Theorem 1. *Suppose the points $\{(q_j, z_j) : q_j \in \partial D\}$ are contained in a plane $P \subseteq \mathbb{R}^3$. Let Φ be defined as in (4) where u_i and v_i are defined as in (1), (2) and (3). Then $\Phi : C_P(D) \to C_P(D)$ is well-defined and contractive in the sup-norm. The unique fixed point of Φ is the graph of a continuous function $f^* : D \to \mathbb{R}$ satisfying $f^*(q_j) = z_j$, $j = 1, \cdots, m$.*

Let D be a polygonal region, $\{\sigma_i\}_{i=1}^N$ a triangulation of D consisting of 2-simplices σ_i with non-intersecting interiors whose union is D, and $Q = \{q_1, \ldots, q_m\}$ be a set of vertices of $\{\sigma_i\}_{i=1}^N$. Let $\{\tau_k\}_{k=1}^M$ be M simplices each of which is a union of some subset of $\{\sigma_i\}_{i=1}^N$. Order Q such that $\{q_1, \ldots, q_n\} \subseteq Q$ are all the vertices of $\{\tau_k\}_{k=1}^M$. We call $\ell : \{1, \cdots, m\} \to \{1, \cdots, n\}$ a *σ-labelling associated with $\{\tau_k\}_{k=1}^M$ and $\{\sigma_i\}_{i=1}^N$* if $\{q_{\ell(j)}, q_{\ell(j')}, q_{\ell(j'')}\}$ are the vertices of some τ_k whenever $\{q_j, q_{j'}, q_{j''}\}$ are the vertices of some σ_i. Let u_i and v_i be the unique affine maps satisfying

$$u_i(q_{\ell(j)}) = q_j \qquad \text{and} \qquad v_i(q_{\ell(j)}, z_{\ell(j)}) = z_j. \tag{6}$$

Thus $u_i(\tau_k) = \sigma_i$, for some τ_k.

Theorem 2. *Let ℓ be a σ-labelling associated with the triangulations $\{\sigma_i\}_{i=1}^N$ and $\{\tau_k\}_{k=1}^M$. Let u_i and v_i be the unique affine maps defined by (6) with $s_i = s$, $0 < |s| < 1$. Let Φ be defined as in (4). Then $\Phi : C_B(D) \to C_B(D)$ is well-defined and contractive in the sup-norm with contractivity s. The unique fixed point of Φ is the graph of a continuous function $f^* : D \to \mathbb{R}$, satisfying $f^*(q_j) = z_j, j = 1, \cdots, m$.*

The interested reader is referred to [4, 8] for the proof of Theorems 1 and 2.

The function f^* is called a *fractal interpolation function* and its graph a *fractal interpolation surface*. For a more detailed discussion of these fractal surfaces and their properties we again refer to [4] or [8].

2.3 Dimension

We recall that the *box dimension* (sometimes also called the *fractal dimension* or *capacity*) of a bounded set $S \subseteq \mathbb{R}^n$ is defined as

$$d = \lim_{\varepsilon \to 0+} \frac{\log \mathcal{N}_\varepsilon(S)}{-\log \varepsilon}, \quad \text{provided this limit exists,} \tag{7}$$

where $\mathcal{N}_\varepsilon(S) = \min\{\text{card } C_\varepsilon : C_\varepsilon \text{ is a cover of } S \text{ consisting of } n\text{-dimensional } \varepsilon\text{-balls}\}$.

Recall that a *similitude* $f : \mathbf{R}^n \to \mathbf{R}^n$ is a mapping satisfying

$$\|f(x) - f(x')\| = s\,\|x - x'\|,$$

for all $x, x' \in \mathbb{R}^n$ and some *scaling factor* $s \in \mathbb{R}$. It was shown in [6] that such a mapping S is of the form

$$S = s\,O + t_v,$$

where O is an orthonormal transformation on $\mathbb{R}^n$ and $t_v : \mathbb{R} \to \mathbb{R}^n$, $x \mapsto x + v$, the translation by $v \in \mathbb{R}^n$.

Theorem 3. *Let u_i and v_i be defined as in (1) and (2), respectively. Suppose u_i is a similitude on $\mathbb{R}^2$ with scaling factor a_i, $i = 1,\dots,N$. If $\{(q_j, z_j) : j = 1,\dots,m\}$ is not coplanar and $\sum_{i=1}^{N} |s_i| a_i > 1$, then the box dimension d of graph (f^*) is the unique positive solution of*

$$\sum_{i=1}^{N} |s_i| a_i^{d-1} = 1; \tag{8}$$

otherwise $d = 2$.

Proof. See [4] or [8] (this theorem is a special case of Theorem 5.1 in [4] and generalizes Theorem 5 in [8] to a certain extent)■

3 The construction of fractal interpolation surfaces in $\mathbb{R}^{n+m}$

In this section we construct fractal functions which map a polyhedron $D \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$. This construction is given in a general setting.

Let D be a polyhedron made up of finitely many n-simplices $\sigma_i \subseteq \mathbb{R}^n$, $i = 1,\dots,N$. Denote by Q the set of all vertices $q_j \in D$, $j = 1,\dots,M$. Let $\{(q_j, z_j) \in D \times \mathbb{R}^m : j = 1,\dots,M\}$ be a given set of interpolation or data points. Let τ_k be an n-simplicial complex in D that is a union of some of the σ_i's, $k = 1,\dots,K$. After relabelling — if necessary — we write $q_1,\dots,q_L$ for the vertices of τ_k. A function $\ell : \{1,\dots,M\} \to \{1,\cdots,L\}$ is called a *labelling map* if whenever $q_{j_1},\dots,q_{j_{n+1}}$ are the vertices of some σ_i, then $q_{\ell(j_1)},\dots,q_{\ell(j_{n+1})}$ are the vertices of some τ_k.

Now we are ready to set up the maps that will define the fractal function. Let $u_i : \mathbb{R}^n \to \mathbb{R}^n$ be the unique affine map such that

$$u_i(q_{\ell(j)}) = q_j, \qquad \text{for all } q_j \in \sigma_i, \tag{9}$$

$i = 1,\cdots,N$. The maps u_i can be represented as

$$u_i(x) = A_i x + D_i, \tag{10}$$

where $A_i \in M_{nn}$, the algebra of all $n \times n$ matrices over $\mathbb{R}$, and $D_i \in \mathbb{R}^n$. Let $B \in M_{mm}$ and suppose that the spectral radius s of B is less than one. Note that there exists a norm $\|\cdot\|_B$ on $\mathbb{R}^m$ such that the induced matrix norm of B equals s. Let $v_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be the unique affine map of the form

$$v_i(x, y) = C_i x + B y + E_i, \tag{11}$$

where $C_i \in M_{mn}$, $E_i \in \mathbb{R}^m$ are such that

$$v_i(q_{\ell(j)}, z_{\ell(j)}) = z_j, \tag{12}$$

for all j such that $q_j \in \sigma_i$, and for all $i = 1,\dots,N$. Let $\mathcal{C} = \{f \in C(D, \mathbb{R}^m) : f(q_j) = z_j,\ j = 1,\dots,M\}$. Define a norm $\|\cdot\|_\infty$ on $\mathcal{C}$ by $\|f\|_\infty = \sup\{\|f(x)\|_B : x \in D\}$ and let $\Phi : \mathcal{C} \to C(D, \mathbb{R}^m)$ be defined as

$$(\Phi f)(x) = v_i(u_i^{-1}(x), f(u_i^{-1}(x))), \qquad \text{for all } x \in \sigma_i, \tag{13}$$

$i = 1,\cdots,N$.

Theorem 4. *The mapping Φ in (13) is well-defined, maps $\mathcal{C}$ into itself, and is contractive in $\|\cdot\|_\infty$.*

Proof. Clearly $\Phi(f)$ is continuous on each σ_i. Let $\phi_i = \Phi(f)|_{\sigma_i}$, $i = 1, \ldots, N$. Suppose σ_i and $\sigma_{i'}$ intersect along a face, i.e., $\sigma_i \cap \sigma_{i'} = F$, where F is a p-simplex with $p < n$. To prove that Φ is well-defined it suffices to show that $\phi_i(F) = \phi_{i'}(F)$. Note that $\phi_i(q_j) = z_j = \phi_{i'}(q_j)$ for each vertex $q_j \in F$. But (9) and the fact that each $x \in F$ is a linear combination of the vertices of F imply that $\phi_i(x) = \phi_{i'}(x)$, for all $x \in F$. Note that $\Phi(f)(q_j) = z_j$, $j = 1, \ldots, M$, by (9) and (12). Therefore, Φ maps C into itself. Now let $f, g \in C$. Then

$$\begin{aligned}\|\Phi(f) - \Phi(g)\|_\infty &= \sup_{1\le i\le N}\{\|v_i(u_i^{-1}(x), f(u_i^{-1}(x))) - v_i(u_i^{-1}(x), g(u_i^{-1}(x)))\|_B : x \in D\} \\ &= \sup_{1\le i\le N}\{\|B(f(u_i^{-1}(x)) - g(u_i^{-1}(x)))\|_B : x \in D\} \le s\|f - g\|_\infty \blacksquare\end{aligned}$$

By the Banach Fixed-Point Theorem Φ has a unique fixed point $f^* \in C$, and if $f \in C$, then $\Phi^n(f)$ converges uniformly to f^*. The graph G^* of f^* is the attractor of the iterated function system $(D \times \mathbb{R}^m; w_1, \ldots, w_N)$ where

$$w_i\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u_i(x) \\ v_i(x, y) \end{pmatrix} \qquad (i = 1, \ldots, N). \tag{14}$$

The associated set-valued map $W : H(\mathbb{R}^n \times \mathbb{R}^m) \to H(\mathbb{R}^n \times \mathbb{R}^m)$ satisfies $W(\text{graph } f) = \text{graph } \Phi(f)$, for any $f \in C$, and, in particular,

$$W(\text{graph } f^*) = \text{graph } f^*. \tag{15}$$

As above we call f^* a *fractal interpolation function.* We refer to its graph as a *fractal interpolation surface.*

Throughout this paper we use the notation $(\Phi_{\mathbf{i}} f)(x) = v_{\mathbf{i}}(u_{\mathbf{i}}^{-1}(x), f(u_{\mathbf{i}}^{-1}(x)))$ where $v_{\mathbf{i}}(x, y)$ is such that $w_{\mathbf{i}}(x, y) = (u_{\mathbf{i}}(x), v_{\mathbf{i}}(x, y))$, for all finite $\mathbf{i} \in \Sigma$.

4 The oscillation of f^*

In this section and throughout the remainder of this paper we will assume that $D = \sigma$, where σ is an n-dimensional simplex with the property

$$\sigma = \bigcup_{i=1}^{N} \sigma_i, \tag{16}$$

for some $1 < N \in \mathbb{N}$, and where each σ_i is similar to σ. (That such a simplex exists follows from the theory of Coxeter groups. At this point we refer the interested reader to [5] for a more detailed description of foldable figures and their relation to Coxeter groups.) This then implies that if u_i is of the form (9), A_i has to be a similitude with scaling factor $a_i < 1$, i.e., $A_i = a_i\Theta_i$, where Θ_i is an orthogonal transformation on $\mathbb{R}^n$. Let us also assume that B is a similitude, i.e. $B = s\Theta$, Θ an orthonormal transformation on $\mathbb{R}^m$.

Definition 1. An ε-cover $\mathcal{C}_\varepsilon$ of a bounded set $S \subseteq \mathbb{R}^n$ is called *admissible* if it is of the form

$$\mathcal{C}_\varepsilon = \{B_\varepsilon(r_\alpha) : r_\alpha, r_{\alpha'} \in S \text{ and } |r_\alpha - r_{\alpha'}| \ge \varepsilon/(2\sqrt{n}) \text{ for all } r_\alpha \ne r_{\alpha'}\}, \tag{17}$$

where $B_\varepsilon(r_\alpha)$ denotes the n-dimensional ball of radius ε centered at $\alpha \in S$.

Remark. These admissible covers will also be used in the next section to calculate the box dimension of the projections of graph f^* onto $\mathbb{R}^n \times \mathbb{R}$.

Definition 2. Let $S \subseteq \mathbb{R}^n$ be the domain of a function $f : \mathbb{R}^n \to \mathbb{R}^m$. The *oscillation of f over $B \subseteq S$* is defined as

$$\omega(f; B) = \sup_{x,x' \in B} \|f(x) - f(x')\|, \tag{18}$$

and the *ε-oscillation of f over S* as

$$\Omega_\varepsilon(f; S) = \inf \sum_{B \in C_\varepsilon} \omega(f; B) \tag{19}$$

where the infimum is taken over all admissible ε-covers C_ε of S.

Now let $S = D$, the domain of $f^* : D \to \mathbb{R}^m$. Recall that M is the number of interpolation points (q_j, z_j). We need the following lemma.

Lemma 1. *Suppose that*

(a) *the set of interpolation points $\{(q_j, z_j) : j = 1, \ldots, M\}$ is not contained in any n-dimensional hyperplane of $\mathbb{R}^{n+m}$, and*
(b) $\sum_{i=1}^N s a_i^{n-1} > 1$.

Then

$$\lim_{\varepsilon \to 0+} \varepsilon^{n-1} \Omega_\varepsilon(f^*; D) = +\infty. \tag{20}$$

Proof. Condition (a) implies that there exists an $\tilde{x} \in \mathring{D}$ (where $\mathring{D}$ denotes the interior of D) such that $V = \|f^*(\tilde{x}) - \pi(\tilde{x})\| > 0$, where $\pi : \mathbb{R}^n \to \mathbb{R}^m$ is the unique affine map such that $\pi(q_j) = z_j$, for each of the $n+1$ vertices $q_1, \ldots, q_{n+1}$ of D. Let D' be a closed and connected subset of $\mathring{D}$ such that whenever $x \in D$, there is an $x' \in D'$ with $\|f^*(x) - f^*(x')\| \le V/2$. Let $\eta > 0$ be the distance between D' and ∂D. Let $0 < \varepsilon < \eta \underline{a}/2$ ($\underline{a} = \min\{a_i : i = 1, \ldots, N\}$) and let Σ_ε be the collection of all finite codes $\mathbf{i} \in \Sigma$ such that

$$2\varepsilon \le \eta\, a_{\mathbf{i}} \le 2\varepsilon/\underline{a} \tag{21}$$

holds for $\mathbf{i}$ but no curtailment of it. Since $\pi(\tilde{x})$ is in the convex hull of $\{f^*(q_j) : j = 1, \ldots, n+1\}$ we have $\pi(\tilde{x}) = \sum_{j=1}^{n+1} \alpha_j f^*(q_j)$ where $\alpha_j \ge 0$ and $\sum_j \alpha_j = 1$. Let $x_j \in D'$ be such that $\|f^*(q_j) - f^*(x_j)\| \le V/2$, for $j = 1, \ldots, n+1$. Then

$$\omega(f^*; D') \ge \sum_j \alpha_j \|f^*(\tilde{x}) - f^*(x_j)\| \ge \eta \|f^*(\tilde{x}) - \pi(\tilde{x})\| - \|\textstyle\sum_j \alpha_j (f^*(q_j) - f^*(x_j))\| \ge V/2.$$

Using equation (13) we have

$$\begin{aligned}
\omega(f^*; u_{\mathbf{i}}(D')) &\ge \|f^*(u_{\mathbf{i}}(\tilde{x})) - \textstyle\sum_j \alpha_j f^*(u_{\mathbf{i}}(x_j))\| \\
&\ge \|(\Phi_{\mathbf{i}} f^*)(u_{\mathbf{i}}(\tilde{x})) - (\Phi_{\mathbf{i}} \pi)(u_{\mathbf{i}}(\tilde{x}))\| - \|(\Phi_{\mathbf{i}} \pi)(u_{\mathbf{i}}(\tilde{x})) - \textstyle\sum_j \alpha_j (\Phi_{\mathbf{i}} f^*)(u_{\mathbf{i}}(x_j))\| \\
&= \|B_{\mathbf{i}}\| (\|f^*(\tilde{x}) - \pi(\tilde{x})\| - \|\pi(\tilde{x}) - \textstyle\sum_j \alpha_j f^*(x_j)\|) \ge s_{\mathbf{i}} (V/2).
\end{aligned}$$

Note that $\Omega_\varepsilon(f^*; u_{\mathbf{i}}(D')) \ge \omega(f^*; u_{\mathbf{i}}(D'))$ since f^* is continuous and $u_{\mathbf{i}}(D')$ connected. Therefore, using (21)

$$\Omega_\varepsilon(f^*; D) \ge \Omega_\varepsilon(f^*; \bigcup_{\mathbf{i}} u_{\mathbf{i}}(D')) = \sum_{\mathbf{i}} \Omega_\varepsilon(f^*; u_{\mathbf{i}}(D')) \ge \left(\sum_{\mathbf{i}} s_{\mathbf{i}} \right) (V/2). \tag{22}$$

From (21) it also follows that $(a_{\mathbf{i}}/\varepsilon)^{n-1} \le (2/\eta \underline{a})^{n-1} =: \gamma^{-1}$. Thus

$$\frac{1}{2} \sum_{\mathbf{i}} s_{\mathbf{i}} V \ge \frac{\gamma}{2} \sum_{\mathbf{i}} \left(a_{\mathbf{i}}^{n-1} s_{\mathbf{i}} \right) V \varepsilon^{-n+1} = \frac{\gamma}{2} \sum_{\mathbf{i}} \left(a_{\mathbf{i}}^{d-1} s_{\mathbf{i}} \right) \left(a_{\mathbf{i}}^{n-d} V \right) \varepsilon^{-n+1}, \tag{23}$$

where d is the unique positive solution of $\Sigma_i s\, a_i^{d-1} = 1$. Note that $d > n$ by condition (b).

We define a probability measure μ on Σ by $\mu(\mathbf{i}) = s_{\mathbf{i}} a_{\mathbf{i}}^{d-1}$, for any cylinder set $\mathbf{i}$. Since Σ_ε partitions Σ, we have $1 = \Sigma_{\mathbf{i}\in\Sigma_\varepsilon}\mu(\mathbf{i}) = \Sigma_{\mathbf{i}\in\Sigma_\varepsilon} s_{\mathbf{i}} a_{\mathbf{i}}^{d-1}$. Therefore, $\Omega_\varepsilon(f^*; D) \geq (\gamma/2)(\overline{a}^{k(n-d)}V)\varepsilon^{-n+1}$, where $\overline{a} = \max_i\{a_i\}$ and $k = \min\{|\mathbf{i}| : \mathbf{i} \in \Sigma_\varepsilon\}$. Since $k \to \infty$ as $\varepsilon \to 0+$, the result follows ■

Theorem 5. *Assume that the hypotheses of Lemma 1 are satisfied. Then there exist positive constants ε_0, k_1, and k_2 such that*

$$k_1\varepsilon^{-\delta} \leq \Omega_\varepsilon(f^*; D) \leq k_2\varepsilon^{-\delta}, \tag{24}$$

for all $0 < \varepsilon < \varepsilon_0$, where δ is the unique positive solution of $\sum_{i=1}^N s a_i^\delta = 1$.

Proof. Let $i \in \{1, \ldots, N\}$, let $0 < \varepsilon < 1$, and suppose $B_{\varepsilon/a_i}(r) \subseteq D$. By (13) we have

$$\begin{aligned}\omega(f^*; u_i(B_{\varepsilon/a_i}(r))) &= \sup_{x,x'\in B_\varepsilon(u_i(r))} \|v_i(x, f^*(x)) - v_i(x', f^*(x))\| \\ &\leq \|B\|\omega(f^*; B_{\varepsilon/a_i}(r)) + \frac{2\varepsilon}{a_i}\|C_i\|. \end{aligned}\tag{25}$$

Let $D_i = \{x \in u_i(D) : \operatorname{dist}(x, \partial u_i(D)) \geq 2\varepsilon\}$. Note that $D_i \neq \emptyset$ for ε small enough. Furthermore, if C_ε is an admissible ε-cover of a set S and x is a point not covered by C_ε, then $C_\varepsilon \cup B_\varepsilon(x)$ is an admissible ε-cover of $S \cup \{x\}$. Thus, any admissible cover of a set S may be extended to an admissible cover of a superset of S.

Let C_{ε/a_i} be an admissible ε/a_i-cover of D. If we apply u_i to this cover we obtain an admissible ε-cover C_ε^i of $u_i(D)$. Let $\hat{C}_\varepsilon^i = \{B_\varepsilon \in C_\varepsilon^i : B_\varepsilon \cap D_i \neq \emptyset\}$. Note that $\hat{C}_\varepsilon = \bigcup_i \hat{C}_\varepsilon^i$ is an admissible ε-cover of $\bigcup_i D_i$ and may be extended to an admissible ε-cover C_ε of D by adding ε-balls with centers in $D - \bigcup_i D_i$, as described above. Therefore,

$$\Omega_\varepsilon(f^*; D) \leq \sum_{B_\varepsilon\in C_\varepsilon} \omega(f^*, B_\varepsilon) = \sum_{B_\varepsilon\in\hat{C}_\varepsilon} \omega(f^*, B_\varepsilon) + \sum_{B_\varepsilon\in C_\varepsilon-\hat{C}_\varepsilon} \omega(f^*, B_\varepsilon).$$

It follows from (17) and $\operatorname{vol}_n(D - \bigcup_i D_i) \leq 4\varepsilon \operatorname{vol}_{n-1}(\bigcup_i \partial u_i(D))$ that there exists a positive constant c_0 such that $C_\varepsilon - \hat{C}_\varepsilon$ contains at most $c_0\varepsilon^{-n+1}$ ε-balls. Thus

$$\sum_{B_\varepsilon\in C_\varepsilon-\hat{C}_\varepsilon} \omega(f^*, B) \leq 2\|f^*\|c_0\varepsilon^{-n+1}.$$

Furthermore, if $B_\varepsilon \in \hat{C}_\varepsilon^i$, then $B_\varepsilon = u_i(B_{\varepsilon/a_i})$, for some $B_{\varepsilon/a_i} \in C_{\varepsilon/a_i}$. Hence

$$\sum_{B\varepsilon\in\hat{C}_\varepsilon} \omega(f^*; B_\varepsilon) = \sum_i \sum_{B_{\varepsilon/a_i}\in C_{\varepsilon/a_i}} \omega(f^*; B_{\varepsilon/a_i}) \leq \sum_i \sum_{B_{\varepsilon/a_i}\in C_{\varepsilon/a_i}} \left(s\omega(f^*; B_{\varepsilon/a_i}) + \frac{2\varepsilon}{a_i}\|C_i\|\right).$$

Note that by (17) $\sum_{B_{\varepsilon/a_i}\in C_{\varepsilon/a_i}} (2\varepsilon/a_i)\|C_i\| \leq c_1\varepsilon^{-n+1}$, for some $c_1 > 0$. Therefore,

$$\Omega_\varepsilon(f^*; D) \leq \sum_i \sum_{B_{\varepsilon/a_i}\in C_{\varepsilon/a_i}} s\omega(f^*; B_{\varepsilon/a_i}) + c_2\varepsilon^{-n+1},$$

for some $c_2 > 0$.

Since the above inequality holds for any admissible ε/a_i-cover we have

$$\Omega_\varepsilon(f^*; D) \leq \sum_i s\Omega_{\varepsilon/a_i}(f^*; D) + c_2\varepsilon^{-n+1}. \tag{26}$$

On the other hand, if C_ε is an admissible cover of $\sigma_i = u_i(D)$, $i = 1, \cdots, N$, then we have again by (13) (assuming $s \neq 0$)

$$\begin{aligned}\omega(f^*, B_{\varepsilon/a_i}(u_i^{-1}(r_\alpha))) &= \sup_{x,x' \in B_\varepsilon(r_\alpha)} \|(v_i^{-1}(x, f^*(x))) - v_i^{-1}(x', f(x'))\| \\ &\leq \|B^{-1}\|\omega(f^*; B_\varepsilon(r_\alpha)) + 2\|B^{-1}C_iA_i\|\varepsilon,\end{aligned}$$

for all $B_\varepsilon(r_\alpha) \in C_\varepsilon$ where v_i^{-1} is such that $w_i^{-1}(\cdot, *) = (u_i^{-1}(\cdot), v_i^{-1}(\cdot, *))$. Thus

$$\sum_i \Omega_{\varepsilon/a_i}(f^*; D) \leq \sum_i s^{-1}\Omega_\varepsilon(f^*; \sigma_i) + c_0\left(\sum_i \|B^{-1}C_iA_i\|\right)\varepsilon^{-n+1},$$

that is,

$$\Omega_\varepsilon(f^*; D) \geq \sum_i s\Omega_{\varepsilon/a_i}(f^*; D) - c_1\varepsilon^{-n+1}, \tag{27}$$

for $c_1 = c_0(\sum_i \|B^{-1}C_iA_i\|) > 0$. Note that (27) holds trivially for $s = 0$. Hence, combining (26) and (27), we have

$$\sum_i s\Omega_{\varepsilon/a_i}(f^*, D) - c_1\varepsilon^{-n+1} \leq \Omega_\varepsilon(f^*; D) \leq \sum_i s\Omega_{\varepsilon/a_i}(f^*; D) + c_2\varepsilon^{-n+1}. \tag{28}$$

Now let $\gamma = \sum_{i=1}^N sa_i^{n-1}$ and let $\overline{a} = \max\{a_i : i = 1, \cdots, N\}$. By Lemma 1 we can select $\varepsilon_0 > 0$ small enough so that $\Omega_\varepsilon(f^*; D) \geq [2c_1/(\gamma - 1)]\varepsilon^{-n+1}$, for $\varepsilon_0 < \varepsilon \leq \varepsilon_0/\underline{a}$. Choose $K_1 > 0$ small enough so that $K_1\varepsilon_0^{-\delta} \leq [c_1/(\gamma - 1)]\varepsilon^{-n+1}$ and $K_2 > 0$ large enough so that $\Omega_\varepsilon(f^*, D) \leq [c_2/(1 - \gamma)]\varepsilon^{-n+1} + K_2\varepsilon^{-\delta}$ for $\varepsilon_0 \leq \varepsilon \leq \varepsilon_0/\underline{a}$. Define functions $\underline{\varphi}, \overline{\varphi} : (0, \varepsilon_0] \to \mathbb{R}$ by

$$\underline{\varphi}(\varepsilon) = \left(\frac{c_1}{\gamma - 1}\right)\varepsilon^{-n+1} + K_1\varepsilon^{-\delta} \qquad \text{and} \qquad \overline{\varphi}(\varepsilon) = \left(\frac{c_2}{1 - \gamma}\right)\varepsilon^{-n+1} + K_2\varepsilon^{-\delta},$$

respectively. It follows that, for all $\varepsilon_0 \leq \varepsilon \leq \varepsilon_0/\underline{a}$, $\underline{\varphi}(\varepsilon) \leq \Omega_\varepsilon(f^*; D) \leq \overline{\varphi}(\varepsilon)$. Note that

$$\underline{\varphi}(\varepsilon) = \sum_i s\underline{\varphi}(\varepsilon/a_i) - c_1\varepsilon^{-n+1} \qquad \text{and} \qquad \overline{\varphi}(\varepsilon) = \sum_i s\overline{\varphi}(\varepsilon/a_i) + c_2\varepsilon^{-n+1}.$$

If $\overline{a}\varepsilon_0 \leq \varepsilon \leq \varepsilon_0$, then $\varepsilon_0 \leq \varepsilon/a_i \leq \varepsilon/\underline{a}$ and

$$\Omega_\varepsilon(f^*; D) \leq \sum_i s\Omega_{\varepsilon/a_i}(f^*; D) + c_2\varepsilon^{-n+1} \leq \sum_i s\overline{\varphi}(\varepsilon/a_i) + c_2\varepsilon^{-n+1} = \overline{\varphi}(\varepsilon).$$

Similarly, $\underline{\varphi}(\varepsilon) \leq \Omega_\varepsilon(f^*; D)$ for $\overline{a}\varepsilon_0 \leq \varepsilon \leq \varepsilon_0$.

Now, if $\underline{\varphi}(\varepsilon) \leq \Omega_\varepsilon(f^*; D) \leq \overline{\varphi}(\varepsilon)$ holds for $(\overline{a})^\ell\varepsilon_0 \leq \varepsilon \leq \varepsilon_0$, it must hold for $(\overline{a})^{\ell+1}\varepsilon_0 \leq \varepsilon \leq \varepsilon_0$. Therefore, $\underline{\varphi}(\varepsilon) \leq \Omega_\varepsilon(f^*; D) \leq \overline{\varphi}(\varepsilon)$ for all $0 < \varepsilon \leq \varepsilon_0$. Since $\delta > n - 1$, there exist positive constants k_1 and k_2 such that $k_1\varepsilon^{-\delta} \leq \Omega_\varepsilon(f^*; D) \leq k_2\varepsilon^{-\delta}$■

Remark. The inequalities in (24) imply that $\delta = \lim_{\varepsilon \to 0+} \log \Omega_\varepsilon(f^*; D)/(-\log \varepsilon)$. In the next section we will relate this result to (7).

5 Dimension of Projections

In this section we calculate the box dimension of the projections of graph f^* onto $\mathbb{R}^n \times \mathbb{R}$.

We shall assume that $u_i(x) = A_ix + D_i$, where A_i is a similitude of norm $\|A_i\| = a_i$, and $v_i(x, y) = C_ix + By + E_i$, where $B = s\Theta$ for an isometry Θ, $i = 1, \ldots, n$. We consider the components of $f^* = (f_1^*, \ldots, f_m^*)^T$. (Here ^{T} denotes transpose.) In what follows we fix $j \in \{1, \ldots, m\}$.

The graph of f_j^* is the projection of graph f^* onto $\mathbb{R}^n \times 0 \times \ldots \times \mathbb{R} \times \ldots \times 0$, where the factor $\mathbb{R}$ is in the jth position. Since f_j^* still depends continuously on all the variables, we refer to it as a *hidden variable fractal interpolation function*, and to its graph as a *hidden variable fractal interpolation surface*.

Denote by $\lambda_1, \ldots, \lambda_m$ and by $h_1, \ldots, h_m$ the eigenvalues and orthonormal eigenvectors of Θ, respectively. Let us order the eigenvalues of Θ in such a way that $\lambda_1, \ldots, \lambda_\mu$ are all the distinct eigenvalues of Θ, $1 \leq \mu \leq m$. The canonical basis of $\mathbb{R}^m$ is denoted by $\{e_1, \ldots, e_m\}$. Define

$$c_{kj} = \langle h_k, e_j \rangle \tag{29}$$

($\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^m$). We can write

$$f^*(x) = \sum_{k=1}^{m} b_k(x) h_k, \qquad \text{where} \quad b_k(x) = \langle f^*(x), h_k \rangle. \tag{30}$$

Also, $f_j^*(x) = \langle f^*(x), e_j \rangle$. Let $I(\kappa)$ be the set of all indices from $\{1, \ldots, m\}$ indexing the same eigenvalue of Θ, $\kappa \in \{1, \ldots, \mu\}$, and let

$$d_\kappa(x) = \sum_{k \in I(\kappa)} b_k(x) c_{kj}. \tag{31}$$

Theorem 6. *Suppose that* (i) $d_\nu \not\equiv 0$, *for some* $\nu \in \{1, \ldots, \mu\}$ *and* (ii) $\sum_{i=1}^{n} a_i^{n-1} s > 1$. *Then the box dimension* d *of graph* f_j^* *is the unique positive solution of*

$$\sum_{i=1}^{N} a_i^{d-1} s = 1; \tag{32}$$

otherwise $d = n$.

Proof. To prove the theorem we will make use of a special class $\mathcal{K}_\varepsilon, \varepsilon > 0$, of covers of graph f_j^*. The covers $\mathcal{G}_\varepsilon \in \mathcal{K}_\varepsilon$ are defined as follows:

$$\mathcal{G}_\varepsilon = \{B_\varepsilon(r_\alpha) \times [y_\alpha + (k-1)\varepsilon, y_\alpha + k\varepsilon] : y_\alpha \leq \inf_{B_\varepsilon} f_j^*,\ y_\alpha + n_\alpha \varepsilon \geq \sup_{B_\varepsilon} f_j^*\ ;\ k = 1, \ldots, n_\alpha;\ n_\alpha \in \mathbb{N}\},$$

where $\{B_\varepsilon(r_\alpha)\}$ is an admissible ε- cover of D. Let $\mathcal{N}_\varepsilon(f_j^*; D) = \inf\{|\mathcal{G}_\varepsilon| : \mathcal{G}_\varepsilon \in \mathcal{K}_\varepsilon\}$.

Next we show that it suffices to consider covers from $\mathcal{K}_\varepsilon$ to calculate the box dimension of graph f_j^*. Let $\hat{\mathcal{G}}_\varepsilon$ be an arbitrary minimal cover of graph f_j^* consisting of sets of the form $B_\varepsilon(r) \times [z, z+\varepsilon], z \in \mathbb{R}$. Denote by $\hat{\mathcal{N}}_\varepsilon(f_j^*; D)$ the cardinality of this minimal cover. Then we obviously have

$$\hat{\mathcal{N}}_\varepsilon(f_j^*; D) \leq \mathcal{N}_\varepsilon(f_j^*; D). \tag{33}$$

But since $\{B_\varepsilon(r_\alpha)\}$ is an admissible cover of D, we also have that $B_{\varepsilon/2\sqrt{n}}(r_\alpha) \cap B_{\varepsilon/2\sqrt{n}}(r_\beta) = \emptyset$, $r_\alpha \neq r_\beta$, and that the number of balls $B_{\varepsilon/2\sqrt{n}}(r_\alpha)$ contained in the ball $B_{\varepsilon/2\sqrt{n}}(r_\alpha)$ is less than or equal to $\xi = (4\sqrt{n}+1)^n$. Thus any $B_\varepsilon(r)$ meets at most ξ elements of any admissible ε-cover of D. Hence

$$\mathcal{N}_\varepsilon(f_j^*; D) \leq 3\xi\, \hat{\mathcal{N}}_\varepsilon(f_j^*; D). \tag{34}$$

Note that by (17) there is a constant $c > 0$ such $|\mathcal{C}_\varepsilon| \leq c\varepsilon^{-n}$ for any admissible ε-cover of D. Furthermore, from (18), (19) and the definition of $\mathcal{N}_\varepsilon(f_j^*, D)$ we have

$$\varepsilon^{-1}\Omega_\varepsilon(f_j^*, D) - c_1\varepsilon^{-n} \leq \mathcal{N}_\varepsilon(f_j^*, D) \leq \varepsilon^{-1}\Omega_\varepsilon(f_j^*, D) + c_2\varepsilon^{-n} \tag{35}$$

for some positive constants c_1 and c_2.

Since $\Omega_\varepsilon(f_j^*; D) \leq \Omega_\varepsilon(f^*; D)$, Theorem 5 and (34) imply

$$\mathcal{N}_\varepsilon(f_j^*; D) \leq k_2\varepsilon^{-d} \tag{36}$$

where $k_2 > 0$ and d is given by (32). We next derive a lower bound for $\mathcal{N}_\varepsilon(f_j^*; D)$. First we derive some estimates on Ω_ε.

Definition 3. *For $f: D \to \mathbb{R}$ we define*

$$\overline{\Omega}_\varepsilon(f; D) = \sup\left\{\sum_{B_\varepsilon \in C_\varepsilon} \omega(f; B_\varepsilon)\right\} \tag{37}$$

where the supremum is taken over all admissible ε-covers C_ε of D.

Lemma 2. *Let $f: D \to \mathbb{R}$ and let $\varepsilon > 0$ be arbitrary. Then there exist positive constants K, k_1, and k_2 so that*
(i) $\Omega_\varepsilon(f; D) \le \overline{\Omega}_\varepsilon(f; D) \le K\Omega_\varepsilon(f; D)$ *and*
(ii) $k_1\Omega_\varepsilon(f; D) \le \Omega_{c\varepsilon}(f; D) \le k_2\Omega_\varepsilon(f; D)$, *for any $c > 0$.*
Furthermore, k_1 and k_2 depend only on f and c.

Proof. (i) Let C_ε be an arbitrary admissible ε-cover of D and let C'_ε be an admissible ε-cover of D such that $\sum_{B' \in C'_\varepsilon} \omega(f; B') \le 2\Omega_\varepsilon(f; D)$. For each $B \in C_\varepsilon$ let $S(B) = \{B' \in C'_\varepsilon : B \cap B' \ne \phi\}$. Then any B' meets at most $\xi = (4\sqrt{n} + 1)^n$ elements of C_ε. Furthermore, since $S(B) \supset B \cap D$, we have $\omega(f; B) \le \sum_{B' \in S(B)} \omega(f; B')$ for any $B \in C_\varepsilon$. Thus

$$\sum_{B \in C_\varepsilon} \omega(f; B) \le \sum_{B \in C_\varepsilon} \sum_{B' \in S(B)} \omega(f; B') \le \xi \sum_{B' \in C'_\varepsilon} \omega(f; B') \le 2\xi\Omega_\varepsilon(f; D).$$

(ii) As in Section 4, it follows that any $B_{c_1\varepsilon}(r)$ meets at most $[2(c_1/c_2 + 1)\sqrt{n} + 1]^n$ elements of an admissible $c_2\varepsilon$-cover of D for any $c_1, c_2 > 0$. Now the result follows as in (i) ■

We obtain a lower bound for $\Omega_\varepsilon(d_\nu; D)$ using a functional inequality as in the proof of Theorem 5. In order to get the induction started we use the following lemma whose proof is given in the appendix.

Lemma 3. *Suppose that conditions (i) and (ii) of Theorem 6 are satisfied. Then*

$$\lim_{\varepsilon \to 0^+} \varepsilon^n N_\varepsilon(f_j^*; D) = +\infty.$$

Observe that

$$f_j^* = \sum_{\kappa=1}^{\mu} d_\kappa. \tag{38}$$

Using Lemma 2(i) and (38) we obtain

$$\Omega_\varepsilon(f_j^*; D) \le \sum_{\kappa=1}^{\mu} \overline{\Omega}_\varepsilon(d_\kappa; D) \le K \sum_{\kappa=1}^{\mu} \Omega_\varepsilon(d_\kappa; D). \tag{39}$$

Lemma 3, (35), and (39) and possibly reindexing yields

$$\limsup_{\varepsilon \to 0^+} \varepsilon^{n-1}\Omega_\varepsilon(d_\nu; D) = +\infty. \tag{40}$$

By taking the inner product of (30) with h_k and using (31) we obtain a functional equation for d_ν:

$$d_\nu(u_i(x)) = \left\langle c_i x + E_i, \sum_{k \in I(\kappa)} h_k c_{kj} \right\rangle + s\lambda_\nu d_\nu(x) \tag{41}$$

for $x \in D$, $i \in \{1, \ldots, N\}$. Arguments similar to those in the derivation of (27) lead to

$$\Omega_\varepsilon(d_\nu; D) \ge \sum_i s\Omega_{\varepsilon/a_i}(d_\nu; D) - c\varepsilon^{-n+1}, \qquad \text{for some } c > 0. \tag{42}$$

By Lemma 3 and Lemma 4(ii) there exists an $\varepsilon_0 > 0$ such that $\Omega_\varepsilon(d_\nu; D) \geq 2c/(\gamma - 1)$ for $\varepsilon_0 \leq \varepsilon \leq q\varepsilon_0/\underline{a}$ where $\gamma = \sum_{i=1}^{N} s a_i^{n-1}$ and $\underline{a} = \min |a_i|$. Thus we can choose $K_1 > 0$ small enough so that

$$\Omega_\varepsilon(d_\nu; D) \geq \left(\frac{c}{\gamma - 1}\right) \varepsilon^{-n+1} + K_1 \varepsilon^{-d+1}, \qquad \text{for } \varepsilon_0 \leq \varepsilon \leq \varepsilon_0/\underline{a}. \tag{43}$$

Using induction and (42) we see that (43) holds for all $0 < \varepsilon \leq \varepsilon_0$. In particular, we have

$$\omega_\varepsilon(d_\nu; D) \geq K_1 \varepsilon^{-d+1} \qquad \text{for } 0 < \varepsilon \leq \varepsilon_0. \tag{44}$$

Let $\mathbf{i} \in \sum$, $|\mathbf{i}| = \ell$. Note that by (30) we have

$$f^*(u_{\mathbf{i}}(x)) = v_{\mathbf{i}}(x, f^*(x)) = c_{\mathbf{i}} x + B^\ell f^*(x) + E_{\mathbf{i}},$$

for $x \in D$. Therefore, equations (30) and (31) imply

$$f_j^*(u_{\mathbf{i}}(x)) - f_j^*(u_{\mathbf{i}}(y)) = \langle C_{\mathbf{i}}(x - y), e_j \rangle + \sum_{\kappa=1}^{\mu} s^\ell \lambda_\kappa^\ell (d_\kappa(x) - d_\kappa(y)),$$

for $x, y \in D$. Hence

$$\Omega_\varepsilon(f_j^*; u_{\mathbf{i}}(D)) \geq -\frac{2\|C_{\mathbf{i}}\|}{a_{\mathbf{i}}} \varepsilon^{n+1} + s^\ell \Omega_{\varepsilon/a_{\mathbf{i}}} \left(\sum_\kappa \lambda_\kappa^\ell d_\kappa; D \right),$$

and by Lemma 2(ii)

$$\Omega_\varepsilon(f_j^*; u_{\mathbf{i}}(D)) \geq -\frac{2\|C_{\mathbf{i}}\|}{a_{\mathbf{i}}} \varepsilon^{-n+1} + k_1 s^\ell \Omega_\varepsilon \left(\sum_\kappa \lambda_\kappa^\ell d_\kappa; D \right), \tag{45}$$

where $k_1 = k_1(a_{\mathbf{i}})$. Therefore

$$\Omega_\varepsilon(f_j^*; D) \geq -c_\ell \varepsilon^{-n+1} + c'_\ell \Omega_\varepsilon \left(\sum_\kappa \lambda_\kappa^\ell d_\kappa; D \right) \tag{46}$$

where $c_\ell, c'_\ell > 0$. To obtain a lower bound for $\Omega_\varepsilon(f_j^*; D)$ we will relate the lower bound for $\Omega_\varepsilon(d_\nu, D)$ to a lower bound for $\Omega_\varepsilon(\sum_\kappa \lambda_\kappa^\ell d_\kappa; D)$. Since the Vandermonde determinant of $\lambda_1, \ldots, \lambda_\mu$ is non-zero, there exist constants $\alpha_0, \ldots, \alpha_{\mu-1}$ such that

$$d_\nu = \sum_{\ell=0}^{\mu-1} \alpha_\ell \left(\sum_\kappa d_\kappa \right). \tag{47}$$

Therefore, by (44) and Lemma 2(i) we have the inequalities

$$\begin{aligned} 0 < K_1 &\leq \liminf_{\varepsilon \to 0^+} \varepsilon^{-d+1} \Omega_\varepsilon(d_\nu; D) = \liminf_{\varepsilon \to 0^+} \varepsilon^{-d+1} \Omega_\varepsilon \left(\textstyle\sum_\ell \alpha_\ell \left(\sum_\kappa \lambda_\kappa^\ell d_\kappa \right); D \right) \\ &\leq \liminf_{\varepsilon \to 0^+} \textstyle\sum_\ell \varepsilon^{-d+1} |\alpha_\ell| \overline{\Omega}_\varepsilon \left(\sum_\kappa \lambda_\kappa^\ell d_\kappa; D \right) \\ &\leq \liminf_{\varepsilon \to 0^+} \textstyle\sum_\ell \varepsilon^{-d+1} |\alpha_\ell| K \Omega_\varepsilon \left(\sum_\kappa \lambda_\kappa^\ell d_\kappa; D \right) \\ &\leq \liminf_{\varepsilon \to 0^+} c'' \textstyle\sum_\ell \varepsilon^{-d+1} \Omega_\varepsilon \left(\sum_\kappa \lambda_\kappa^\ell d_\kappa; D \right), \end{aligned}$$

where $c'' = \max\{K|\alpha_\ell| : \ell = 0, \ldots, \mu - 1\}$. Thus there exists an $\varepsilon_0 > 0$ such that, for each $0 < \varepsilon \leq \varepsilon_0$, there is some $\ell \in \{0, \ldots, \mu - 1\}$ with

$$\varepsilon^{-d+1} \Omega_\varepsilon \left(\sum_\kappa \lambda_\kappa^\ell d_\kappa; D \right) \geq \frac{K_1}{2\mu c''}. \tag{48}$$

Hence, using (46) and (48), there exist $\chi_1, \chi_2 > 0$ such that $\Omega_\varepsilon(f_j^*; D) \geq \chi_1 \varepsilon^{-n+1} + \chi_2 \varepsilon^{-d+1}$, for $0 < \varepsilon \leq \varepsilon_0$■

Example. The following sequence of pictures displays the two projections of a fractal function $f^* : D \to \mathbb{R}^2$, with $D = \{(x,y) \in \mathbb{R}^2 : x, y \geq 0, x + y \leq 1\}$, $Q = \{(j/3, k/3) : j + k \leq 3; j, k = 0, 1, 2, 3\}$, $z_{jk} = (\sin(j/3), \cos(k/3))$, and $s = 4/5$.

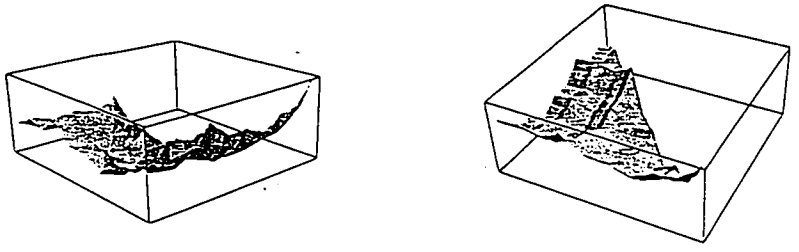

Figure 1: Two views of the projection f_1 of f.

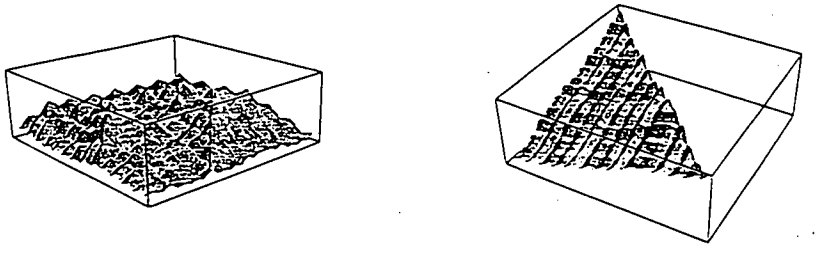

Figure 2: Two views of the projection f_2 of f.

Appendix: Proof of Lemma 2.
Since Θ is an isometry we have $|\lambda_\nu^n| = 1$, for all $n \in \mathbb{N}$. For $\alpha, \beta \in \text{span}\,\{(\lambda_\nu^n)_{n=0}^\infty : \nu = 1, \cdots, \mu\}$ we define

$$(\alpha, \beta) = \lim_{T \to \infty} \frac{1}{T} \sum_{n=0}^{T} \alpha_n \overline{\beta}_n.$$

Now it follows from hypothesis (i) in Theorem 6 and

$$((\lambda_\nu^n), (\lambda_{\nu'}^n)) = \delta_{\nu\nu'x}$$

that there exists an $n_0 \in \mathbb{N}$ with $\sum_\nu d_\nu(x) \lambda_\nu^{n_0} \neq 0$. Assume without loss of generality that $n_0 = 0$. We may also assume that $z_1 = \cdots = z_{n+1} = 0$ in $\mathbb{R}^m$, with one of the vertices being the origin.

Definition 4. *Let $a, b \in \mathbb{R}^{n+1}$ and let π be a plane perpendicular to b containing the point $P \in \mathbb{R}^{n+1}$. Then*

$$Proj_{a;b,P}(x) = x - \frac{\langle x - P, b \rangle}{\langle a, b \rangle} a, \qquad \textit{for all } x \in \mathbb{R}^{n+1}.$$

Since $f_j^* \not\equiv 0$ and continuous, there exists a $b \in \mathbb{R}^n$ such that $\text{Proj}_{(b,0);(b,0),O}$ (graph f_j^*) contains an n-dimensional cube $C_{2\rho}$ of side 2ρ, for some $\rho > 0$ (here O denotes the origin in $\mathbb{R}^{n+1}$). By the Poincaré Recurrence Theorem there exists a subsequence $\{n_k\} \subseteq \mathbb{N}$ such that

$$|\lambda_\nu - \lambda_\nu^{n_k}| < \rho/2c$$

for all $\nu = 1, \cdots, \mu$, and $c = \max_D \sum_\nu |d_\nu(x)|$. Hence

$$|f_j^*(x) - (\Theta^{n_k} f_j^*)(x)| = |\sum_\nu d_\nu(x)(\lambda_\nu - \lambda_\nu^{n_k})| < \sum_\nu d_\nu(x) \cdot \frac{\rho}{2c} \leq \frac{\rho}{2}.$$

Let $\mathrm{i} \in \Sigma$, $|\mathrm{i}| = n_k$, and $a = \omega_{\mathrm{i}}(b,0) - \omega_{\mathrm{i}}(0,0)$. Then

$$\mathrm{Proj}_{a;(b,0),P}(\mathrm{graph}\, f_j^*|_{D_{\mathrm{i}}}) = \mathrm{Proj}_{a;(b,0),P}(\{(u_{\mathrm{i}}(x), f_j^*(u_{\mathrm{i}}(x))) : x \in D\}).$$

After some algebra, this reduces to

$$T^{(\mathrm{i})}\left(x - \frac{\langle x,b\rangle}{\langle a,b\rangle} b, \langle (\Theta^{n_k} f^*)(x), e_j \rangle\right),$$

where $T^{(\mathrm{i})} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ is defined by

$$T^{(\mathrm{i})}(x,y) = (u_{\mathrm{i}}(x), s^{n_k} y + C_{\mathrm{i}} x + E_{\mathrm{i}}).$$

Note that the Jacobian of $T^{(\mathrm{i})}$ is given by

$$\mathrm{Jac}\, T^{(\mathrm{i})} = a_{\mathrm{i}}^{n-1} s^{|\mathrm{i}|}, \quad \mathrm{i} \in \Sigma.$$

Thus

$$\mathrm{vol}\, T^{n_k}(C_{2\rho}) = \mathrm{Jac}\, T^{n_k}\ \mathrm{vol}\, C_{2\rho}.$$

Now, if $\mathcal{N}(f_j^*; D)$ is the cardinality of a minimal ε-cover $\mathcal{C}_\varepsilon \in \mathcal{K}_\varepsilon$ of graph f_j^*, we have that, for $\varepsilon < 2\rho(\underline{a})^{n_k}$,

$$\varepsilon^n \mathcal{N}\varepsilon(f_j^*; D) \geq \sum_{\mathrm{i}} \mathrm{Jac}\, T^{(\mathrm{i})}\, \mathrm{vol}\, C_{2\rho} \geq \left(\sum_{\mathrm{i}=1}^{N} a_{\mathrm{i}}^{n-1} s\right)^{|\mathrm{i}|} \mathrm{vol}\, C_{2\rho} \blacksquare$$

REFERENCES

[1] BARNSLEY, M.F.: *Fractal functions and interpolation.* Constr. Approx. **2** (1986), 303 - 329.

[2] BARNSLEY, M.F. and S. DEMKO: *Iterated function systems and the global construction of fractals.* Proc. Royal Soc. London (Ser. A) **399** (1985), 243 - 275.

[3] BARNSLEY, M.F., ELTON, J., HARDIN, D.P. and P.R. MASSOPUST: *Hidden variable interpolation functions.* SIAM J. Math. Anal. **20** (1989), 1218 - 1242.

[4] GERONIMO, J.S. and D.P. HARDIN: *Fractal interpolation surfaces.* J. Math. Anal. Appl. (to appear).

[5] HOFFMAN, M.E. and W.D. WITHERS: *Generalized Chebyshev polynomials associated with affine Weyl groups.* Trans. Amer. Math. Soc. **308** (1988), 91 - 104.

[6] HUTCHINSON, J.: *Fractals and self-similarity.* Indiana J. Math. **30** (1981), 713 - 747.

[7] JACOBS, K.: *Poincarés Wiederkehrsatz.* In: Selecta Mathematica IV (ed.: K. Jacobs) (Heidelberger Taschenbücher: Vol. 98). Berlin - Heidelberg - New York: Springer - Verlag 1972, pp. 31 - 56.

[8] MASSOPUST, P.R.: *Fractal surfaces.* J. Math. Anal. Appl. **151** (1990), 275 - 290.

Received 07.02.1992; in revised form 04.05.1993