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Exact Difference Schemes and Difference Schemes of Arbitrary Given Degree of Accuracy for Generalized One-Dimensional Third Boundary Value Problems

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A variational problem is formulated which is a generalization of the third boundary value problem for one-dimensional equations. Conditions of existence and uniqueness of solutions are considered. The numerical approximation of arbitrary given degree of accuracy (truncated difference schemes) regarded in this paper is based on the three-point difference relations for the exact solution (exact difference scheme).

Key words: *Exact and truncated three-point difference schemes, generalized Cauchy problems, stability of coefficients*

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0. Introduction

In [1,2] exact three-point difference schemes and truncated difference schemes of arbitrary prescribed degree of accuracy for the Dirichlet problem for one-dimensional equations in generalized formulation have been regarded. The book [1] contains a bibliography concerning the history of development and application of the exact and truncated difference schemes since their appearance at the end of the fifties (for linear one-dimensional equations with piecewise smooth coefficients). It is sufficient to mention only their great advantages for the construction of the difference schemes of great rate of accuracy for various practical problems and their significance for the theory of finite difference methods for one-dimensional equations and partial differential equations with generalized solutions to understand their role in modern numerical analysis. We study the exact and truncated difference schemes for a generalization of the third boundary value problem and extend some assumptions and results of [1,2] in a natural way. Throughout this paper we denote by c various constants independing of the meshsize h.

1. Formulation of the problem. Existence and uniqueness of solution

Let us consider the bilinear forms

$$\begin{aligned} a(u,v) &= a^{o}_{[0,1]}(u,v) + Q(1)u(1)v(1) - Q(0)u(0)v(0) + \varkappa_1 u(1)v(1) + \varkappa_0 u(0)v(0) \\ a^{o}_{[\alpha,\beta]}(u,v) &= \int_\alpha^\beta \big[k(x)u'(x)v'(x) - Q(x)(u(x)v(x))'\big]\,dx \end{aligned} \qquad (1.1)$$

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and the linear functionals

$$l(v) = l^{o}_{[0,1]}(v) - \mu_o v(0) - \mu_1 v(1) \quad \text{and} \quad l^{o}_{[\alpha,\beta]}(v) = \int_{\alpha}^{\beta}[f_0(x)v(x) - f_1(x)v'(x)]\,dx \tag{1.2}$$

defined on the space $W_2^1(0,1)$. Here $\varkappa_1, \varkappa_2 \geq 0$ and μ_1, μ_2 are given real numbers and f_0, f_1, k, Q are given functions satisfying the following conditions, where $K := \{v \in W_2^1(0,1)\colon v(x) \geq 0\}$:

(C$_1$) *k is measurable and* $0 < k_0 \leq k(x) \leq k_1 < +\infty$ for some constants k_0, k_1

(C$_2$) $Q \in W_p^{\lambda}(0,1)$ with $p \geq 2$ and $1/2 < \lambda \leq 1$

(C$_3$) $f_0 \in L_q(0,1)$ with $q \geq 2$ and $f_1 \in W_r^{\vartheta}(0,1)$ with $r \geq 2$, $0 < \vartheta \leq 1$

(C$_4$) $-\int_0^1 Q(x)v'(x)\,dx + Q(1)v(1) - Q(0)v(0) \geq q_0 \int_0^1 v(x)\,dx$ for all $v \in K$, $q_0 > 0$ some constant.

The problem now reads as follows:

(P$_1$) Find $u \in W_2^1(0,1)$ such that $a(u,v) = l(v)$ for all $v \in W_2^1(0,1)$.

This problem is equivalent to the following minimization problem of functionals:

(P$_2$) Find $u \in W_2^1(0,1)$ such that $J(u) = \min\{J(v)\colon v \in W_2^1(0,1)\}$, $J(v) := a(v,v) - 2\,l(v)$.

Now the following statement holds.

Theorem 1.1: *Suppose the conditions* (C$_1$) - (C$_4$) *are satisfied and* $q_0, \varkappa_0, \varkappa_1$ *are not equal to zero simultaneously. Then the problems* (P$_1$) and (P$_2$) *have a unique solution.*

Proof: The bilinear form $a = a(u,v)$ is symmetric and $W_2^1(0,1)$-elliptic. This statement for $q_0 > 0$ follows immediately from (C$_1$) and (C$_6$):

$$a(v,v) \geq k_0 |v|^2_{W_2^1} + q_0 \|v\|^2_{L_2} \geq \min(k_0, q_0)\|v\|^2_{W_2^1}.$$

If $q_0 = 0$ and for example $\varkappa_0 \neq 0$, then using the inequalities

$$v^2(x) = \left(\int_0^x v'(s)\,ds + v(0)\right)^2 \leq 2\left(|v|^2_{W_2^1} + v^2(0)\right) \quad \text{and} \quad \|v\|^2_{L_2} \leq 2\left(|v|^2_{W_2^1} + v^2(0)\right)$$

for arbitrary $\varepsilon \in (0, k_0)$ we can obtain

$$\begin{aligned} a(v,v) &\geq k_0 |v|^2_{W_2^1} + \varkappa_0 v^2(0) \\ &\geq (k_0 - \varepsilon)|v|^2_{W_2^1} + \min(\varepsilon, \varkappa_0)\left(|v|^2_{W_2^1} + v^2(0)\right) \\ &\geq \min(k_0 - \varepsilon, 0.5\min(\varepsilon, \varkappa_0))\|v\|^2_{W_2^1}. \end{aligned}$$

The Sobolev imbedding theorem and Cauchy-Schwarz-Bunyakowski equality lead to the following estimates as well:

$$|a(u,v)| \le k_1|u|_{W_2^1}|v|_{W_2^1} + \|Q\|_{L_2}\Big(\int_0^1 (u'(x)v(x)+u(x)v'(x))^2dx\Big)^{1/2}$$
$$+(2\|Q\|_C + \varkappa_0 + \varkappa_1)\|u\|_C\|v\|_C$$
$$\le k_1|u|_{W_2^1}|v|_{W_2^1} + \sqrt{2}\|Q\|_{L_2}\big(\|v\|_C^2|u|_{W_2^1}^2 + \|u\|_C^2|v|_{W_2^1}^2\big)^{1/2}$$
$$+(2\|Q\|_C + \varkappa_0 + \varkappa_1)\|u\|_C\|v\|_C$$
$$\le c\,\|u\|_{W_2^1}\|v\|_{W_2^1},$$
$$|l(v)| = \|f_0\|_{L_2}\|v\|_{L_2} + \|f_1\|_{L_2}|v|_{W_2^1} + |\mu_0|\,\|v\|_C + |\mu_1|\,\|v\|_C \le c\|v\|_{W_2^1},$$

where the constants c are independent of u, v. These inequalities signify the continuity of the bilinear form $a = a(u,v)$ and the linear form $l = l(v)$. The desired result thus follows as a consequence of Lax-Milgram's lemma [1] ∎

2. Exact difference scheme

Let us untroduce the grids $\overline{\omega}_h = \{x_i = ih\colon\ i = 0(1)N,\ h = 1/N\}$ and $\omega_h = \overline{\omega}_h\setminus\{0,1\}$. Let $G_i = G_i(x,\xi)$ from $\overset{\circ}{W}{}_2^1(e_i)$, $e_i = (x_{i-1}, x_{i+1})$ be the solution of the problem [1]

$$\int_{x_{i-1}}^{x_{i+1}}\Big[k(\xi)\frac{dG_i(x,\xi)}{d\xi}\eta'(\xi) - Q(\xi)\frac{d(G_i(x,\xi)\eta(\xi))}{d\xi}\Big]d\xi = \eta(x) \quad \text{for all } \eta \in \overset{\circ}{W}{}_2^1(e_i),\ x\in e_i. \tag{2.1}$$

We choose in (1.7) $v(\xi) = G_i(x,\xi)$ for $\xi \in e_i$ and $v(\xi) = 0$ for $\xi \in [0,1]\setminus e_i$. Then we obtain the following three-point relation connecting the exact solution $u = u(x)$ in three neighbouring nodes:.

$$u(x_i) = a_i u(x_{i+1}) + b_i u(x_{i-1}) + f_i,\quad i = 1(1)(N-1), \tag{2.2}$$

where

$$a_i = 0{,}5 - \frac{1}{2h}\int_{x_{i-1}}^{x_{i+1}}\Big(k(\xi)G_i'(x_i,\xi) - Q(\xi)((\xi - x_{i-1})G_i(x_i,\xi))'\Big)d\xi$$

$$b_i = 0{,}5 + \frac{1}{2h}\int_{x_{i-1}}^{x_{i+1}}\Big(k(\xi)G_i'(x_i,\xi) + Q(\xi)((x_{i+1} - \xi)G_i(x_i,\xi))'\Big)d\xi$$

$$f_i = \int_{x_{i-1}}^{x_{i+1}}\Big(G_i(x_i,\xi)f_0(\xi) - d/d\xi\, G_i(x_i,\xi)f_1(\xi)\Big)d\xi.$$

Let us introduce the functions v_1^i and v_2^i, $i = 0(1)N$ which are the solutions of the following generalized Cauchy problems ($j = 1,2$; $x_{-1} = x_0 = 0$, $x_{N+1} = x_N = 1$):

(P_3) Find $v_j^i \in W_2^1(e_i)$ such that

$$a^o_{e_i}(v_1^i,\eta) + \eta(x_{i-1}) = 0,\ \ v_1^i(x_{i-1}) = 0 \ \text{ for all } \eta \in W_2^1(e_i) \text{ with } \eta(x_{i+1}) = 0$$

$$a^o_{e_i}(v_2^i,\eta) + \eta(x_{i+1}) = 0,\ \ v_2^i(x_{i+1}) = 0 \ \text{ for all } \eta \in W_2^1(e_i) \text{ with } \eta(x_{i-1}) = 0.$$

Similarly to [1: pp. 66 - 73] one can prove that every problem (P_3) has a unique solution. Moreover the functions v_j^i have the following properties:

1. The function v_1^i is monotonely increasing on the interval $(x_{i-1}, x_{i+1}]$, $v_1^i(x) > 0$, and the function v_2^i is monotonely decreasing on the interval $[x_{i-1}, x_{i+1})$, $v_2^i(x) > 0$.

2. The relations

$$v_1^i(x_{i+1}) = v_2^i(x_{i-1}) \text{ for } i = 0(1)N \qquad \text{and} \qquad v_2^i(x_i) = v_1^{i+1}(x_{i+1}) \text{ for } 1(1)(N-2)$$

$$v_1^i(x_{i+1}) = v_1^i(x) + v_2^i(x) - v_2^i(x)\int_{x_{i-1}}^{x}(v_1^i(\xi))'Q(\xi)\,d\xi - v_1^i(x)\int_{x}^{x_{i+1}}(v_2^i(\xi))'Q(\xi)\,d\xi \text{ for } i = 0(1)N$$

hold where x is an arbitrary point of the interval e_i. Set in (P_1)

$$v(\xi) = \begin{cases} 0 & \text{for } \xi \in (h,1) \\ v_2^o(\xi) & \text{for } \xi \in e_o \end{cases} \qquad \text{and} \qquad \bar{u}(\xi) = u(\xi) - h^{-1}(h-\xi)u(0).$$

Then $\bar{u}(0) = 0$ and

$$a^o_{e_o}(\bar{u}, v_2^o) + \lambda_o u(0) + \chi_o = 0, \tag{2.3}$$

where

$$\begin{aligned} \lambda_o &= h^{-1}a^o_{e_o}(h-\xi, v_2^o) + (x_o - Q(0))v_2^o(0) \\ &= h^{-1}\int_0^h\left(-k(\xi)\frac{dv_2^o}{d\xi} - Q(\xi)\frac{d((h-\xi)v_2^o)}{d\xi}\right)d\xi + (x_o - Q(0))v_2^o(0) \end{aligned} \tag{2.4}$$

$$\chi_o = -l^o_{e_o}(v_2^o) + \mu_o v_2^o(0) = \int_0^h \frac{dv_2^o(\xi)}{d\xi}f_1(\xi)\,d\xi - \int_0^h v_2^o(\xi)f_o(\xi)\,d\xi + \mu_o v_2^o(0).$$

By virtue of the definition of the function v_2^o relation (2.3) yields $-\bar{u}(h) + \lambda_o u(0) + \chi_o = 0$ or

$$\bar{\Lambda}_o u := -u(h) + \lambda_o u(0) + \chi_o = 0. \tag{2.5}$$

We take into account the equality $\int_0^h k(\xi)(dv_2^o/d\xi)\,d\xi = \int_0^h Q(\xi)(d(\xi v_2^o)/d\xi)\,d\xi - h$ which is a a consequence of the second equation (P_3) with the substitution ξ for $\eta(\xi)$. Then one can transform λ_o in the following way:

$$\lambda_o = 1 - \int_0^h Q(\xi)\frac{dv_2^o(\xi)}{d\xi}\,d\xi + (x_o - Q(0))v_2^o(0),$$

and (2.5) can be rewritten in the form

$$\Lambda_0 u = h^{-1}\bar{\Lambda}_0 u = -u_x(0) + \varkappa_0^h u(0) + \mu_0^h = 0 \tag{2.6}$$

$$\begin{aligned} \varkappa_0^h &= -h^{-1}\int_0^h Q(\xi)\frac{dv_2^0(\xi)}{d\xi}d\xi + (\varkappa_0 - Q(0))h^{-1}v_2^0(0) \\ \mu_0^h &= h^{-1}\int_0^h\left(f_1(\xi)\frac{dv_2^0(\xi)}{d\xi}d\xi - f_0(\xi)v_2^0(\xi)\right)d\xi + \mu_0 h^{-1}v_2^0(0). \end{aligned} \tag{2.7}$$

Analogously, the substitution $v(\xi) = 0$ for $\xi \in [0,1-h]$, $v(\xi) = v_1^N(\xi)$ for $\xi \in e_N$ in (P_3) leads to the equation

$$\Lambda_1 u = u_{\bar{x}}(1) + \varkappa_1^h u(1) + \mu_1^h = 0 \tag{2.8}$$

where

$$\begin{aligned} \varkappa_1^h &= -h^{-1}\int_{1-h}^1 Q(\xi)\frac{dv_1^N(\xi)}{d\xi}d\xi + (\varkappa_1 + Q(1))h^{-1}v_1^N(1) \\ \mu_1^h &= h^{-1}\int_{1-h}^1\left(\frac{dv_1^N(\xi)}{d\xi}f_1(\xi) - v_1^N(\xi)f_0(\xi)\right)d\xi + \mu_1 h^{-1}v_1^N(1). \end{aligned} \tag{2.9}$$

Taking into account the properties of v_j^i and the representation of Green's function [1: p. 72]

$$G_i(x,\xi) = \frac{1}{v_1^i(x_{i+1})}\begin{cases} v_1^i(x)v_2^i(\xi) & \text{for } x_{i-1} \le x \le \xi \\ v_1^i(\xi)v_2^i(x) & \text{for } \xi \le x \le x_{i+1} \end{cases}$$

the relations (2.2) can be rewritten in the divergence form (see [1: pp. 76 - 80])

$$\Lambda u = (a u_{\bar{x}})_x - du = -\varphi(x), \quad x \in \omega_h \tag{2.10}$$

where, for $i = 1(1)(N-1)$,

$$\begin{aligned} a_i &= a(x_i) = h/v_1^i(x_i), \quad d_i = d(x_i) = T_1^{x_i}(Q), \quad \varphi_i = \varphi(x_i) = T^{x_i}(f) \equiv T_1^{x_i}(f_1) + T_0^{x_i}(f_0) \\ T_1^{x_i}(w) &= -\frac{1}{hv_1^i(x_i)}\int_{x_{i-1}}^{x_i}(v_1^i(\xi))'w(\xi)d\xi - \frac{1}{hv_2^i(x_i)}\int_{x_i}^{x_{i+1}}(v_2^i(\xi))'w(\xi)d\xi \\ T_0^{x_i}(w) &= -\frac{1}{hv_1^i(x_i)}\int_{x_{i-1}}^{x_i}v_1^i(\xi)w(\xi)\,d\xi + \frac{1}{hv_2^i(x_i)}\int_{x_i}^{x_{i+1}}v_2^i(\xi)w(\xi)d\xi. \end{aligned} \tag{2.11}$$

Thus the exact solution of (P_2) and (P_3) satisfies the three-point difference equations (2.6), (2.8) and (2.11) representing exact difference schemes.

3. Truncated difference schemes

In order to calculate the coefficients of the exact difference schemes one needs to solve problem (P_3). But this problem has in principle just the same complexity as the original one. For this reason our aim in this section is the construction of an algorithm for approximate computation of the coefficients of a scheme of the form (2.6), (2.8), (2.10).

We first set in (P_3)

$$\eta(\xi) = \begin{cases} \int_\xi^x k^{-1}(t)\,dt & \text{for } \xi \in [x_{i-1}, x] \\ 0 & \text{for } \xi \in [x, x_{i+1}] \end{cases} \quad \text{and} \quad \eta(\xi) = \begin{cases} 0 & \text{for } \xi \in [x_{i-1}, x] \\ \int_x^\xi k^{-1}(t)\,dt & \text{for } \xi \in [x, x_{i+1}]. \end{cases}$$

After simple transformations for the functions v_j^i we obtain the integral equations

$$\begin{aligned} &-v_1^i(x) - \int_{x_{i-1}}^{x} k^{-1}(t) \int_{x_{i-1}}^{t} (Q(\xi) - Q(t))(v_1^i(\xi))' d\xi\, dt + \int_{x_{i-1}}^{x} k^{-1}(\xi)\,d\xi = 0 \\ &-v_2^i(x) - \int_{x}^{x_{i+1}} k^{-1}(t) \int_{t}^{x_{i+1}} (Q(\xi) - Q(t))(v_2^i(\xi))' d\xi\, dt + \int_{x}^{x_{i+1}} k^{-1}(\xi)\,d\xi = 0. \end{aligned} \tag{3.1}$$

Let us set in (3.1), for $i = 0(1)N$, $x = x_i + sh$,

$$v_1^i(x_i + sh) = h\,\alpha^{(i)}(s,h), \quad v_2^i(x_i + sh) = h\,\beta^{(i)}(s,h) \qquad \text{and} \quad s \in \begin{cases} [-1,1] & \text{for } i = 1(1)(N-1) \\ [0,1] & \text{for } i = 0 \\ [-1,0] & \text{for } i = N \end{cases}$$

$$\tilde{k}(s) = k(x_i + sh), \qquad \tilde{Q}(s) = Q(x_i + sh)$$

Then we obtain the following equations for the functions $\alpha^{(i)}, \beta^{(i)}$:

$$\begin{aligned} \alpha^{(i)}(s,h) &= h \int_{-1+\delta_{0,i}}^{s} \tilde{k}^{-1}(\xi) \int_{-1+\delta_{0,i}}^{\xi} (\tilde{Q}(\xi) - \tilde{Q}(\eta)) \frac{d\alpha^{(i)}(\eta,h)}{d\eta}\, d\eta\, d\xi + \int_{-1+\delta_{0,i}}^{s} \tilde{k}^{-1}(\xi)\,d\xi \\ \beta^{(i)}(s,h) &= h \int_{s}^{1-\delta_{N,i}} \tilde{k}^{-1}(\xi) \int_{\xi}^{1-\delta_{N,i}} (\tilde{Q}(\xi) - \tilde{Q}(\eta)) \frac{d\beta^{(i)}(\eta,h)}{d\eta}\, d\eta\, d\xi + \int_{s}^{1-\delta_{N,i}} \tilde{k}^{-1}(\xi)\,d\xi \end{aligned} \tag{3.2}$$

where $\delta_{i,j}$ is the Kronecker symbol. Differentiation of (3.2) by s gives a Volterra integral equation for the derivatives (the equality sign means equality of elements in L_2; $i = 1(1)N$):

$$\begin{aligned} \frac{d\alpha^{(i)}(s,h)}{ds} &= h\,\tilde{k}^{-1}(s) \int_{-1+\delta_{0,i}}^{s} (\tilde{Q}(s) - \tilde{Q}(\eta)) \frac{d\alpha^{(i)}(\eta,h)}{d\eta}\, d\eta + \tilde{k}^{-1}(s) \\ \frac{d\beta^{(i)}(s,h)}{ds} &= -h\,\tilde{k}^{-1}(s) \int_{s}^{1-\delta_{N,i}} (\tilde{Q}(s) - \tilde{Q}(\eta)) \frac{d\beta^{(i)}(\eta,h)}{d\eta}\, d\eta - \tilde{k}^{-1}(s). \end{aligned} \tag{3.3}$$

Each of the equations (3.3) has a unique solution in L_2, hence each of the equations (3.2) has a unique solution in W_2^1 (see [1: p. 81] where the similar equations have been regarded, except when $i = 0$ and $i = N$). Formal substitution of the series

$$\alpha^{(i)}(s,h) = \overset{m}{\alpha}{}^{(i)}(s,h) + \sum_{k=m+1}^{\infty} h^{2k}\alpha_k^{(i)}(s,h) \quad \text{with} \quad \overset{m}{\alpha}{}^{(i)}(s,h) = \sum_{k=0}^{m} h^{2k}\alpha_k^{(i)}(s,h)$$
$$\beta^{(i)}(s,h) = \overset{m}{\beta}{}^{(i)}(s,h) + \sum_{k=m+1}^{\infty} h^{2k}\beta_k^{(i)}(s,h) \quad \text{with} \quad \overset{m}{\beta}{}^{(i)}(s,h) = \sum_{k=0}^{m} h^{2k}\beta_k^{(i)}(s,h) \tag{3.4}$$

into (3.3) and comparison of the coefficients results in the recurrence relations ($s \in E_i$, $k \geq 0$)

$$\frac{d\alpha_0^{(i)}(s,h)}{ds} = \frac{1}{\tilde{k}(s)}, \quad \frac{d\alpha_{k+1}^{(i)}(s,h)}{ds} = \frac{1}{h\tilde{k}(s)} \int_{-1+\delta_{0,i}}^{s} (\tilde{Q}(s) - \tilde{Q}(\eta)) \frac{d\alpha_k^{(i)}(\eta,h)}{d\eta}\, d\eta, \; \alpha_k^{(i)}(-1+\delta_{0,i}, h) = 0$$
$$\frac{d\beta_0^{(i)}(s,h)}{ds} = -\frac{1}{\tilde{k}(s)}, \quad \frac{d\beta_{k+1}^{(i)}(s,h)}{ds} = -\frac{1}{h\tilde{k}(s)} \int_{s}^{1-\delta_{N,i}} (\tilde{Q}(s) - \tilde{Q}(\eta)) \frac{d\beta_k^{(i)}(\eta,h)}{d\eta}\, d\eta, \; \beta_k^{(i)}(1,h) = 0 \tag{3.5}$$

where $E_i = [-1+\delta_{0,i}, 1-\delta_{N,i}]$. Using mathematical induction over k one can prove that $\alpha_k^{(i)}$ is a monotonely increasing function on E_i, $\alpha_k^{(i)}(s,h) \geq 0$, and $\beta_k^{(i)}$ is a monotonely decreasing function on E_i, $\beta_k^{(i)}(s,h) \geq 0$. Analogously to [1: p. 82] one can also prove that the series (3.4) with the coefficients satisfying (3.5) uniformly converge to the functions $\alpha^{(i)}$ and $\beta^{(i)}$.

Substituting the functions

$$\overset{m}{v}{}_1^i(x) = \overset{m}{v}{}_1^i(x_i + sh) = h\overset{m}{\alpha}{}^{(i)}(s,h) \qquad \text{and} \quad \overset{m}{v}{}_2^i(x) = \overset{m}{v}{}_2^i(x_i + sh) = h\overset{m}{\beta}{}^{(i)}(s,h)$$

instead of v_j^i in (2.7), (2.9), (2.11) we obtain instead of $\varkappa_j^h$, μ_j^h, a_i, d_i, φ_i the coefficients $\varkappa_j^{(m)}$, $\mu_j^{(m)}$, $a_i^{(m)}$, $d_i^{(m)}$, $\varphi_i^{(m)}$ and the corresponding difference scheme

$$\Lambda^{(m)}y^{(m)} = (a^{(m)}y_{\bar{x}}^{(m)})_x - d^{(m)}y^{(m)} = -\varphi^{(m)}(x), \; x \in \omega_h$$
$$\Lambda_0^{(m)}y^{(m)} = -y_x^{(m)}(0) + \varkappa_0^{(m)}y^{(m)}(0) + \mu_0^{(m)} = 0 \tag{3.6}$$
$$\Lambda_1^{(m)}y^{(m)} = -y_{\bar{x}}^{(m)}(1) + \varkappa_0^{(m)}y^{(m)}(1) + \mu_1^{(m)} = 0,$$

which is called the *truncated difference scheme of rank m* for problem (P_1). If $\sigma^{(i)} = \sum_{k=0}^{\infty} h^{2k} \times \sigma_k^{(i)}$ denotes any of the functions $\alpha^{(i)}$ or $\beta^{(i)}$, then the following estimates hold:

$$0 \leq \sigma_k^{(i)}(s,h) \leq k_0^{-(k+1)} h^{k(\lambda-1/p-1)} |Q|_{\lambda,p,e_i}^{k} (1/k!)^{(p-2)/(2p)}$$
$$0 \leq \sigma^{(i)}(s,h) - \overset{m}{\sigma}{}^{(i)}(s,h) = \sum_{k=m+1}^{\infty} h^{2k}\sigma_k^{(i)}(s,h) \leq c_{m+1} h^{(m+1)(1+\lambda-1/p)} |Q|_{\lambda,p,e_i}^{m+1} \tag{3.7}$$
$$\left\| \frac{d\sigma^{(i)}(s,h)}{ds} - \frac{d\overset{m}{\sigma}{}^{(i)}(s,h)}{ds} \right\|_{0,2,E_\sigma} \leq c_{m+1} h^{(m+1)(1+\lambda-1/p)} |Q|_{\lambda,p,e_i}^{m+1}, \; \left\| \frac{d\sigma^{(i)}(s,h)}{ds} \right\|_{0,2,E_\sigma} \leq c_0$$

where $E_\alpha = [-1,0]$, $E_\beta = [0,1]$,

$$c_m = k_o^{-1} \sum_{k=m}^{\infty} \left(k_o^{-1}|Q|_{\lambda,p,\Omega}\, 2^{1/p-\lambda-1}\right)^{k-m} (1/k!)^{(p-2)/p}$$

$\Omega = (0,1)$, $e_i = (x_{i-1}, x_i)$, $|\cdot|_{s,t,D}$ and $\|\cdot\|_{s,t,D}$ denote semi-norm and norm, respectively, in the Sobolev space $W_t^s(D)$, $W_2^0(D) = L_2(D)$. These estimates are proved in [1: pp. 83 - 88] for $i = 1(1)(N-1)$ and in the same way one can prove them also for $i = 0, N$. The estimates (3.7) show that the functions $\overset{m}{\sigma}{}^{(i)}$approximate the functions $\sigma^{(i)}$and for this reason one can expect the nearness of the coefficients of exact and truncated difference schenes.

4. Coefficients stability of difference scheme with boundary condition of the third kind

Let y be the solution of the problem

$$\Lambda y = (a y_{\bar{x}})_x - dy = -\varphi(x), \quad x \in \omega_h$$
$$-y_x(0) + \varkappa_o^h y(0) + \mu_o^h = 0, \quad y_{\bar{x}}(1) + \varkappa_1^h y(1) + \mu_1^h = 0 \tag{4.1}$$

and y' the solution of a problem of the same kind but with the perturbed coefficients $\tilde{a}$, $\tilde{d}$, φ, $\tilde{\varkappa}_o^h, \tilde{\mu}_o^h$ $(j = 0,1)$. Suppose that the following conditions hold:

$$0 < k_o \le a(x), \tilde{a}(x) \le k_1, \quad 0 < q_o \le d(x), \tilde{d}(x) \le q_1, \quad 0 \le \tilde{\varkappa}_j^h, \varkappa_j^h \le \bar{\varkappa} \quad (j = 0,1). \tag{4.2}$$

Let us rewrite the problem (4.1) in the index form

$$\Lambda y_i = a_i y_{i-1} - (d_i + a_i + a_{i+1}) y_i + a_{i+1} y_{i+1} = -h^2 \varphi_i, \quad i = 1(1)(N-1) \tag{4.3}$$

$$y_1 = (1 + h\varkappa_o^h) y_o + h\mu_o^h, \quad y_{N-1} = (1 + h\varkappa_1^h) y_N + h\mu_1^h. \tag{4.4}$$

Eliminating the unknowns y_o, y_N from (4.3) with $i = 1$ and $i = N$ and using (4.4) we obtain the following problem with Dirichlet boundary conditions:

$$\Lambda' y' \equiv (a' y'_{\bar{x}})_x - d' y' = -\varphi'(x), \quad x \in \omega_h, \quad y_o' = y_N'$$

where

$$a_i' = a_i \ (i = 2(1)(N-2)), \quad a_1' = d_1 + \frac{a_1 \varkappa_o^h h}{1 + h\varkappa_o^h}, \quad a_{N-1}' = a_{N-1}, \quad a_N' = d_{N-1} + \frac{a_N \varkappa_1^h h}{1 + h\varkappa_1^h}$$

$$\varphi_i' = \varphi_i \ (i = 2(1)(N-2)), \quad \varphi_1' = \varphi_1 - \frac{a_1 \mu_o^h}{h(1 + h\varkappa_o^h)}, \quad \varphi_{N-1}' = \varphi_{N-1} - \frac{a_N \mu_1^h}{h(1 + h\varkappa_1^h)} \tag{4.6}$$

$$y_i' = y_i \ \ (i = 1(1)(N-1)); \quad d_i' = d_i \ (i = 2(1)(N-2)), \quad d_1' = 0, \ d_{N-1}' = 0.$$

In the same way one can obtain a similar problem for $y_i'' = \tilde{y}_i = y_i$, $i = 1(1)(N-1)$. It is easy to see that in view of (4.2) the following conditions are fulfilled:

$$0 < k_o' := \min\{k_o, q_o\} \le a'(x), a''(x) \le k_1' := \max\{k_1, q_1 + k_1\bar{\varkappa}) \ \text{and} \ 0 \le d'(x), d''(x). \tag{4.7}$$

Now we use the estimates (see [2])

$$\|\rho^{-1/p_0}(x)z(x)\|_{\infty,\omega} \le \frac{k_1'}{k_0'}\Big(\|\varphi''-\varphi'\|_{1,\omega} + \|y'\|_{\infty,\omega}\|d''-d'\|_{1,\omega} + \Big(\frac{\|d''\|_{1,\omega}}{k_0'}+1\Big)\|y_{\bar{x}}'\|_{p_1,\omega^+}\|a''-a'\|_{p_2,\omega^+}\Big)$$

$$0 \le G(x,\xi) \le \frac{k_1'}{k_0'}\begin{cases} x(1-\xi) & \text{for } x \le \xi \\ \xi(1-x) & \text{for } x \ge \xi \end{cases}, \qquad \max_\xi G(x,\xi) \le \frac{k_1'}{k_0'}\rho(x) \tag{4.8}$$

$$|G_{\bar{x}}(x,\xi)| \le 2(k_0')^{-1}, \quad |G_{\bar{\xi}}(x,\xi)| \le \frac{k_1'}{k_0'}\Big(\frac{\|\bar{d}\|_{1,\omega}}{k_0'}\rho(x) + \begin{cases} x, & \text{for } 0 \le \xi \le x \\ 1-x, & \text{for } \xi \le x \le 1 \end{cases}\Big)$$

where $G = G(x,\xi)$ is the Green function of the operator Λ'' or Λ', $\bar{d}$ denotes d'' or d', respectively, $z = y''-y'$, $\rho(x) = x(1-x)$,

$$\|\bar{d}\|_{p,\omega} = \Big(\sum_{\xi\in\omega} h(d''(\xi))^p\Big)^{1/p} \quad \text{and} \quad \|\bar{d}\|_{\infty,\omega} = \max_{\xi\in\omega}|d(\xi)|,$$

$1/p_0 + 1/p_1 + 1/p_2 = 1$, $p_i \ge 1$ $(i = 1,2,3)$, $\omega^+ = \omega \cup \{1\}$. Using the representations

$$y'(x) = \sum_{\xi\in\omega} G'(x,\xi)\varphi'(\xi)h \quad \text{and} \quad y_{\bar{x}}'(x) = \sum_{\xi\in\omega} G_{\bar{x}}'(x,\xi)\varphi'(\xi)h$$

and the estimates (4.8) one can obtain

$$\|y'(x)\|_{\infty,\omega} \le \frac{k_1'}{k_0'}\|\rho\varphi'\|_{1,\omega}, \ \|y_{\bar{x}}'\|_{\infty,\omega^+} \le \frac{2}{k_0'}\|\varphi'\|_{1,\omega}, \ \|y_{\bar{x}}'(x)\|_{p_1,\omega^+} \le \|y_{\bar{x}}'(x)\|_{\infty,\omega^+}. \tag{4.10}$$

For $p_1 \in [1,2]$ the last estimate can be improved in the following way. Let us multiply the equation $\Lambda' y' = (a'y_{\bar{x}}')_x - d'y' = -\varphi'$ by y' and sum by parts over ω. Then

$$\begin{aligned} k_0'\|y_{\bar{x}}'\|_{2,\omega^+}^2 &\le (\Lambda'y',y') \equiv \sum_{x\in\omega} h\Lambda'y'(x)y'(x) \le |(\varphi',y')| \\ &= \Big|\Big(\varphi'(x), \sum_{\xi\in\omega} hG'(x,\xi)\varphi'(\xi)\Big)\Big| \le \sum_{x\in\omega} h|\varphi'(x)| \sum_{\xi\in\omega} h(G'(x,\xi))^{s_1}(G'(x,\xi))^{s_2}|\varphi'(\xi)| \\ &\le \sum_{x\in\omega} h|\varphi'(x)| \max_\xi (G'(x,\xi))^{s_1} \sum_{\xi\in\omega} h \max_x (G'(x,\xi))^{s_2}|\varphi'(\xi)| \\ &\le \Big(\frac{k_1'}{k_0'}\Big)^2 \|\varphi'\rho^{s_1}\|_{1,\omega}\|\varphi'\rho^{s_2}\|_{1,\omega}, \end{aligned} \tag{4.11}$$

where $s_1 + s_2 = 1$. Hence we have

$$\|y_{\bar{x}}'\|_{p_1,\omega^+}^2 \le c\,\|\varphi'\rho^{s_1}\|_{1,\omega}\|\varphi'\rho^{s_2}\|_{1,\omega} \quad (p_1 \in [1,2],\ s_1 + s_2 = 1),$$

in particular

$$\|y_{\bar{x}}'\|_{p_1,\omega^+} \le c\,\|\varphi'\rho^{1/2}\|_{1,\omega} \quad (p_1 \in [1,2]). \tag{4.12}$$

Returning to the old notations in (4.8) and taking (4.9) - (4.12) into account we obtain the following inequalities of coefficients stability:

$$\begin{aligned}
&\|\rho^{-1/p_0}(y-\tilde{y})\|_{\infty,\omega} \\
&\le c\Big(\|\varphi-\tilde{\varphi}\|_{1,\omega} + \big(\|\varphi\|_{1,\omega} + h(|\mu_0^h|+|\mu_1^h|)\big)\|d-\tilde{d}\|_{1,\omega} \\
&\quad + \big(\|\varphi\|_{1,\omega} + h^{s(p_1)}(|\mu_0^h|+|\mu_1^h|)\big)\Big(\|a-\tilde{a}\|_{p_2,\omega^+} \\
&\quad + h^{1/p_2}\sum_{i=0}^{1}\big(|d_{i(N-2)+1}-\tilde{d}_{i(N-2)+1}| + h|\varkappa_i^h-\tilde{\varkappa}_i^h| + h|a_{i(N-1)+1}-\tilde{a}_{i(N-1)+1}|\big)\Big) \\
&\quad + \sum_{i=0}^{1}\big(|\mu_i^h|\,|a_{i(N-1)+1}-\tilde{a}_{i(N-1)+1}| + h|\mu_i^h|\,|\varkappa_i^h-\tilde{\varkappa}_i^h| + |\mu_i^h-\tilde{\mu}_i^h|\big)\Big)
\end{aligned}$$

$$p_j \ge 1 \ (j=0,1,2), \quad s(p_1) = \begin{cases} 0 & \text{for } p_1 > 2 \\ 0{,}5 & \text{for } 1 \le p_1 \le 2 \end{cases}$$

$$|y_0-\tilde{y}_0| = |(1+h\varkappa_0^h)^{-1}((y_1-\tilde{y}_1) - h(\mu_0^h-\tilde{\mu}_0^h))| \le |y_1-\tilde{y}_1| + h|\mu_0^h-\tilde{\mu}_0^h|$$

$$|y_N-\tilde{y}_N| = |(1+h\tilde{\varkappa}_1^h)^{-1}((y_{N-1}-\tilde{y}_{N-1}) - h(\mu_1^h-\tilde{\mu}_1^h))| \le |y_{N-1}-\tilde{y}_{N-1}| + h|\mu_1^h-\tilde{\mu}_1^h|.$$

5. Estimate of accuracy of truncated difference schemes

We first prove that the coefficients of the exact three-point and truncated difference schemes satisfy conditions under which the estimates (4.13) are realized. Taking (C_1) and the non-negativity of the functions α_k^i and β_k^i into account, we have

$$v_1^i(x_i) = h\alpha^{(i)}(0,h) \ge h\overset{m}{\alpha}{}^{(i)}(0,h) \ge h\overset{0}{\alpha}{}^{(i)}(0,h) = h\int_{-1}\tilde{k}^{-1}(s)\,ds = h\int_{-1}k^{-1}(x_i+sh)\,ds \ge h\,k_1^{-1}.$$

Together with (3.7) this leads to the relations

$$0 < \frac{1}{c_0} \le \frac{1}{\alpha^{(i)}(0,h)} = \frac{h}{v_1^i(x_i)} = a(x_i) \le a^m(x_i) = \frac{1}{\overset{m}{\alpha}{}^{(i)}(0,h)} \le \frac{1}{\overset{0}{\alpha}{}^{(i)}(0,h)} \le k_1. \tag{5.1}$$

Analogously

$$0 < \frac{1}{c_0} \le \frac{1}{\beta^{(i)}(0,h)} \le \frac{1}{\overset{m}{\beta}{}^{(i)}(0,h)} \le \frac{1}{\overset{0}{\beta}{}^{(i)}(0,h)} \le k_1. \tag{5.2}$$

The formula for the coefficients $d(x)$ can be transformed in the following way:

$$d(x_i) = T_1^{x_i}(Q) = -\frac{1}{h}\int_0^h \frac{d\bar{v}(\xi)}{d\xi}Q(\xi)\,d\xi\,,\ \text{where}\ \bar{v}(\xi) = \begin{cases} v_1^i(\xi)/v_1^i(x_i)\ \text{for}\ \xi \in (x_{i-1}, x_i) \\ v_2^i(\xi)/v_2^i(x_i)\ \text{for}\ \xi \in (x_i, x_{i+1}) \\ \qquad 0\ \text{for}\ \xi \in [0,1]\setminus(x_{i-1}, x_{i+1}). \end{cases}$$

Due to the condition (C_4) (we assume further that $q_0 > 0$) and the boundary conditions $\bar{v}(0) = \bar{v}(1) = 0$ we obtain

$$d(x_i) \ \ge\ \frac{q_0}{h}\int_0^1 \bar{v}(\xi)\,d\xi = \frac{q_0}{h v_1^i(x_i)}\int_{x_{i-1}}^{x_i} v_1^i(\xi)\,d\xi + \frac{q_0}{h v_2^i(x_i)}\int_{x_i}^{x_{i+1}} v_1^i(\xi)\,d\xi$$

$$= \frac{q_0}{\alpha^{(i)}(0,h)}\int_{-1}^{0}\alpha^{(i)}(s,h)\,ds + \frac{q_0}{\beta^{(i)}(0,h)}\int_0^1 \beta^{(i)}(s,h)\,ds.$$

Taking into account

$$\alpha^{(i)}(s,h) \ \ge\ \alpha_0^{(i)}(s,h) = \int_{-1}^{s}\tilde{k}^{-1}(\eta)\,d\eta \ \ge\ (1+s)k_1^{-1},$$

we have $\int^{0}\alpha^{(i)}(s,h)ds \ge 0{,}5\ k_1^{-1}$. In complete analogy with this result we may prove that

$$\int_0^1 \beta^{(i)}(s,h)\,ds \ \ge 0{,}5\,k_1^{-1}, \quad \int_0^1 \overset{m}{\beta}{}^{(i)}(s,h)\,ds \ \ge 0{,}5\,k_1^{-1}, \quad \int_{-1}^{0} \overset{m}{\alpha}{}^{(i)}(s,h)\,ds \ \ge 0{,}5\,k_1^{-1}$$

and together with (5.1), (5.2) we have

$$d(x_i) \ \ge\ q_0(c_0 k_1)^{-1}. \tag{5.3}$$

Let us consider the coefficients $\varkappa_0^h$ and $\varkappa_0^{(m)}$. Conditions (C_4) yields

$$\begin{aligned} \varkappa_0^h \ &=\ -h^{-1}\int_0^h Q(\xi)\frac{dv_2^0(\xi)}{d\xi}\,d\xi \ -\ h^{-1}Q(0)v_2^0(0) + h^{-1}\varkappa_0 v_2^0(0) \\ &\ge\ h^{-1}q_0\int_0^h v_2^0(\xi)\,d\xi + h^{-1}\varkappa_0 v_2^0(0) = q_0 h\int_0^h \beta^{(0)}(s,h)\,ds \ + h^{-1}\varkappa_0 v_2^0(0) \ \ge\ \frac{q_0 h}{2k_1} + \frac{\varkappa_0}{k_1}. \end{aligned} \tag{5.4}$$

On the other hand,

$$|\varkappa_0^h| \ \le\ \left|h^{-1}\int_0^h Q(\xi)\frac{dv_2^0(\xi)}{d\xi}\,d\xi\right| + \left(\|Q\|_{C[0,h]} + \varkappa_0\right)c_0.$$

The first term can be represented in the form

$$\varkappa_{01}^h \ \equiv\ h^{-1}\int_0^h Q(\xi)\frac{dv_2^0(\xi)}{d\xi}\,d\xi = \varkappa_{01} + \varkappa_{02},$$

where

$$\varkappa_{01} = h^{-1}\int_0^h\Big[Q(\xi) - h^{-1}\int_0^h Q(\eta)\,d\eta\Big]\frac{dv_2^o(\xi)}{d\xi}\,d\xi = h^{-2}\int_0^h\frac{dv_2^o(\xi)}{d\xi}\,d\xi\int_0^h\Big[Q(\xi) - Q(\eta)\Big]d\eta\,d\xi$$

$$\varkappa_{02} = h^{-2}\int_0^h Q(\eta)\,d\eta\int_0^h\frac{dv_2^o(\xi)}{d\xi}\,d\xi = -\beta^{(o)}(0,h)h^{-1}\int_0^h Q(\eta)\,d\eta.$$

Applying the Cauchy-Schwarz-Buniakowski and Hölder (with the exponents $p/2$ and $p/(p-2)$ inequalities as well as the inequalities (3.7) we obtain the following estimate:

$$\begin{aligned}|\varkappa_{01}| &\le h^{-1}\Big(\int_0^h\Big[Q(\xi) - h^{-1}\int_0^h Q(\eta)\,d\eta\Big]^2 d\xi\Big)^{1/2}\Big(\int_0^h\Big[\frac{dv_2^o(\xi)}{d\xi}\Big]^2 d\xi\Big)^{1/2}\\ &\le h^{-2}\Big(h\int_0^h\int_0^h\Big[Q(\xi) - Q(\eta)\Big]^2 d\eta\,d\xi\Big)^{1/2}\Big(\int_0^h\Big[\frac{d\beta^{(o)}(s,h)}{ds}\Big]^2 ds\Big)^{1/2} h^{1/2}\\ &\le c_0\,h^{-1}\Big(\int_0^h\int_0^h\Big|Q(\xi) - Q(\eta)\Big|^p d\eta\,d\xi\Big)^{1/2} h^{(p-2)/p}\\ &\le c_0\,h^{-2/p}\Big(\int_0^h\int_0^h\frac{|Q(\xi) - Q(\eta)|^p}{|\xi-\eta|^{1+\lambda p}}|\xi-\eta|^{1+\lambda p}d\xi\,d\eta\Big)^{1/p} \le c_0\,h^{\lambda-1/p}|Q|_{\lambda,p,e_0}.\end{aligned} \tag{5.5}$$

Since the imbedding $W_p^\lambda \subset C$ for $\lambda \in (p^{-1},1]$ and the estimate (5.2) hold it follows that $|\varkappa_{02}| \le c_0\|Q\|_{C[0,h]}$, hence

$$\varkappa_0^h \le c_0\Big(2\|Q\|_{C[0,h]} + \varkappa_0 + h^{\lambda-1/p}|Q|_{\lambda,p,e_0}\Big).$$

It is easy to verify that the same estimate is valid for $\varkappa_0^{(m)}$. Analogously to (5.4), (5,6) one can also prove the estimates

$$\begin{aligned}&\varkappa_0^{(m)}, \varkappa_1^h, \varkappa_1^{(m)} \ge c_* := \frac{q_0 h}{2k_1} + \frac{\varkappa_0}{k_1},\\ &\varkappa_1^h, \varkappa_1^{(m)} \le c_0\big(2\|Q\|_{C[1-h,1]} + \varkappa_0 + h^{\lambda-1/p}|Q|_{\lambda,p,e_N}\big).\end{aligned} \tag{5.7}$$

Now let us consider the values μ_i^h ($i = 0,1$). For example, the value μ_0^h can be represented as $\mu_0^h = \mu_{00}^h + \mu_{01}^h + \mu_{02}^h$, where

$$\mu_{00}^h = \mu_0 h^{-1}v_2^o(0),\quad \mu_{01}^h = -h^{-1}\int_0^h f_0(\xi)v_2^o(\xi)\,d\xi,\quad \mu_{02}^h = h^{-1}\int_0^h f_1(\xi)\frac{dv_2^o(\xi)}{d\xi}\,d\xi.$$

Using the condition (C_3) and the inequality (5.2) one can estimate μ_{00}^h and μ_{01}^h as follows:

$$\mu_0 k_1^{-1} \le \mu_{00}^h \le c_0 \mu_0$$

$$|\mu_{01}^h| = h^{-1}\left|\int_0^h f_0(\xi) v_2^0(\xi)\, d\xi\right| = \left|\int_0^1 f_0(sh) v_2^0(sh)\, ds\right| = h\left|\int_0^1 f_0(sh)\beta^{(0)}(sh)\, ds\right|$$

$$\le c_0 h \int_0^1 |f_0(sh)|\, ds = c_0 \int_0^h |f_0(\xi)|\, d\xi = c_0\left(\int_0^h |f_0(\xi)|^q d\xi\right)^{1/q} h^{1-1/q} = c_0 h^{1-1/q} \|f_0\|_{0,q,e_0}.$$

Let us represent the value μ_{02}^h in the form $\mu_{02}^h = \mu_{021} + \mu_{022}$ where

$$\mu_{021} = h^{-1}\int_0^h \left[f_1(\xi) - h^{-1}\int_0^h f_1(\eta)\, d\eta\right] \frac{dv_2^0(\xi)}{d\xi}\, d\xi$$

$$\mu_{022} = h^{-2}\int_0^h f_1(\eta)\, d\eta \int_0^h \frac{dv_2^0(\xi)}{d\xi}\, d\xi = -\beta^{(0)}(0,h) h^{-1}\int_0^h f_1(\eta)\, d\eta.$$

The proof of the estimate $|\mu_{021}| \le c_0 h^{\vartheta - 1/r} |f_1|_{\vartheta, r, e_0}$ is completely analogous to that one of (5.5). If $\vartheta - 1/r \le 0$, then the imbedding $W_r^\vartheta \subset W_r^0 \equiv L_r$ is valid. Therefore using Hölder's inequality we obtain

$$|\mu_{022}| \le c_0 h^{-1} \int_0^h |f_1(\eta)|\, d\eta < c_0 h^{-1/r} \|f_1\|_{0,r,e_0}.$$

In the case $\vartheta - 1/r > 0$, due to the imbedding $W_r^\vartheta \subset C$, we have $|\mu_{022}| \le c_0 \|f_1\|_{C[0,h]}$. Hence

$$|\mu_0^h| \le c_0\left(\mu_0 + h^{1-1/q}\|f_0\|_{0,q,e_0} + h^{\vartheta-1/r}|f_1|_{\vartheta,r,e_0} + \mu_0(f_1,\vartheta,r,h)\right)$$

$$\mu_0(f_1,\vartheta,r,h) = \begin{cases} h^{-1/r}\|f_1\|_{0,r,e_0} & \text{for } \vartheta - 1/r \le 0 \\ \|f_1\|_{C[0,h]} & \text{for } \vartheta - 1/r > 0 \end{cases} \tag{5.8}$$

ore more roughly

$$|\mu_0^h| \le c_0 h^{-n_\mu^{(0)}} F_\mu^{(0)}$$

$$n_\mu^{(0)} = \begin{cases} 1/r & \text{for } \vartheta - 1/r \le 0 \\ 0 & \text{for } \vartheta - 1/r > 0 \end{cases} \tag{5.9}$$

$$F_\mu^{(0)} = F_\mu^{(0)}(f_1,\vartheta,r,h) = \begin{cases} h^\vartheta |f_1|_{\vartheta,r,e_0} + \|f_1\|_{0,r,e_0} & \text{for } \vartheta - 1/r \le 0 \\ \text{const} & \text{for } \vartheta - 1/r > 0 \end{cases}.$$

The same estimate holds also for μ_1^h with the replacement of e_0 and $F_\mu^{(0)}$ by e_N and $F_\mu^{(N)}$, respectively.

Now let us consider the difference $x_0^h - x_0^{(m)} = x_{00}^{hm} + x_{01}^{hm} + x_{02}^{hm}$, where

$$x_{00}^{hm} = h^{-1}x_0\big(v_2^0(0) - \overset{m}{v}{}_2^0(0)\big)$$

$$x_{01}^{hm} = h^{-1}\int_0^h\Big[Q(\xi) - h^{-1}\int_0^h Q(\eta)\,d\eta\Big]\Big[\frac{dv_2^0(\xi)}{d\xi} - \frac{d\,\overset{m}{v}{}_2^0(\xi)}{d\xi}\Big]d\xi$$

$$x_{02}^{hm} = -h^{-2}\big(v_2^0(0) - \overset{m}{v}{}_2^0(0)\big)\int_0^h Q(\eta)\,d\eta.$$

Using (3.7) in complete analogy with (5.5), (5.6) we can prove the estimates

$$|x_{00}^{hm}| \le x_0\, c_{m+1} h^{(m+1)(1+\lambda-1/p)}|Q|_{\lambda,p,e_0}^{m+1}$$

$$|x_{01}^{hm}| \le c_{m+1} h^{(m+2)(1+\lambda-1/p)-1}|Q|_{\lambda,p,e_0}^{m+2}$$

$$|x_{02}^{hm}| \le \|Q\|_{C[0,h]}\, c_{m+1}\, h^{(m+1)(1+\lambda-1/p)}|Q|_{\lambda,p,e_0}^{m+1}$$

which lead to the inequality

$$|x_0^h - x_0^{(m)}| \le h^{2(m+1)-n_x}F_{x_0}(Q,h) \tag{5.10}$$
$$n_x = (m+1)(1+1/p-\lambda),\; F_{x_0}(Q,h) = \big(x_0 + \|Q\|_{C[0,h]} + h^{\lambda-1/p}|Q|_{\lambda,p,e_0}\big)c_{m+1}|Q|_{\lambda,p,e_0}^{m+1}$$

where $F_{x_0}(Q,h) \to 0$ if $h \to 0$. The proof of the inequality

$$|x_1^h - x_1^{(m)}| \le h^{2(m+1)-n_x}F_{x_1}(Q,h) \tag{5.11}$$

is completely analogous to that one of (5.10), where $F_{x_1}(Q,h)$ is of the same form as $F_{x_0}(Q,h)$ with the substitution of $C[1-h,1]$ for $C[0,h]$ and of e_N for e_0. In the same way one can obtain the estimates

$$|\mu_j^h - \mu_j^{(m)}| \le c\,h^{2(m+1)-n_\mu}F_{\mu_j} \quad (j = 0,1) \tag{5.12}$$

where

$$n_\mu = \begin{cases} (m+1)(1+1/p-\lambda) & \text{for } \vartheta - 1/r > 0 \\ (m+1)(1+1/p-\lambda) + 1/r & \text{for } \vartheta - 1/r \le 0 \end{cases}$$

$$F_{\mu_j} = F_{\mu_j}(Q, f_0, f_1, \lambda, p, \vartheta, r, m, h) = c_{m+1}|Q|_{\lambda,p,e_{jN}}^{m+1} \tag{5.13}$$
$$\times\begin{cases} \mu_0 + h^{1-1/q}\|f_0\|_{0,q,e_{jN}} + h^{\vartheta-1/r}|f_1|_{\vartheta,r,e_{jN}} + \|f_1\|_{C(\bar e_{jN})} & \text{for } \vartheta - 1/r > 0 \\ h^{1/r}\mu_0 + h^{1-1/q-1/r}\|f_0\|_{0,q,e_{jN}} + h^{\vartheta}|f_1|_{\vartheta,r,e_{jN}} + \|f_1\|_{0,r,e_{jN}} & \text{for } \vartheta - 1/r \le 0. \end{cases}$$

In [1: p. 100] the inequalities

$$
\begin{aligned}
&|d(x) - d^{(m)}(x)| \le c\,h^{2(m+1)-n_d^{(o)}}|Q|^{m+2}_{\lambda,p,e(x)} \\
&\|d - d^{(m)}\|_{1,\omega} \le c\,h^{2(m+1)-n_d}|Q|^{m+2}_{\lambda,p,\Omega}
\end{aligned}
\tag{5.14}
$$

are established, where

$$
n_d^{(o)} = (m+2)(1+1/p-\lambda), \quad n_d = \begin{cases} (m+1)(1-\lambda+1/p)+1/p-\lambda & \text{for } m+2 \ge p \\ (m+1)(1-\lambda)+1-\lambda & \text{for } m+2 \le p. \end{cases} \tag{5.15}
$$

These estimates as well as (5.6), (5.7), (5.10), (5.11) yield

$$
\|d - d^{(m)}\|_{1,\omega} + h^2 \sum_{j=0}^{1} (1 + h x_j^h)|x_j^h - x_j^{(m)}| \le c\,h^{2(m+1)-n_d}.
$$

In order to estimate the norm $\|\varphi - \varphi^{(m)}\|_{1,\omega}$ we represent

$$
f_0(x) = \frac{d}{dx}\tilde{f}_0(x), \quad \tilde{f}_0(x) = \int_{x_i}^{x} f_0(x)\,dx \in W_q^1(0,1).
$$

Then we have $T_0^{x_i}(f_0) = T_1^{x_i}(f_0)$, $\varphi(x_i) = T_1^{x_i}(\tilde{f}_0) + T_1^{x_i}(f_1)$ and in analogy with [1: pp. 100, 104] we obtain

$$
\|\varphi - \varphi^{(m)}\|_{1,\omega} \le c\,h^{2(m+1)-n_\varphi}|Q|^{m+1}_{\lambda,p,\Omega}\left(\|f_0\|_{0,q,\Omega} + |f_1|_{\vartheta,r,\Omega}\right) \tag{5.17}
$$

$$
|\varphi(x_i)| \le 2^{1-\vartheta-1/r}k_1 c_0 h^{\vartheta-1/r-1}|f_1|_{\vartheta,r,e_i} + 2^{2-1/q}k_1 c_0 h^{-1/q}\|f_0\|_{0,q,e_i}, \tag{5.18}
$$

where $n_\varphi = \max\left(n_\varphi(r,\vartheta), n_\varphi(q,1)\right)$ and

$$
n_\varphi(s,t) = \begin{cases} (m+1)(1-\lambda+1/p)+1/s-t & \text{for } (m+1)s \ge (s-1)p \\ (m+1)(1-\lambda)+1-t & \text{for } (m+1)s \le (s-1)p. \end{cases} \tag{5.19}
$$

Summing (5.18) over i and applying Hölder's inequality with exponents r, $r/(r-1)$ at the first sum and q, $q/(q-1)$ at the second one we obtain

$$
\|\varphi\|_{1,\omega} = 2^{1+\vartheta-1/r}k_1 c_0 h^{\vartheta-1}|f_1|_{\vartheta,r,\Omega} + 2^{2-1/q}k_1 c_0\|f_0\|_{0,q,\Omega} \tag{5.20}
$$

and then using (5.8) we have

$$
\|\varphi\|_{1,\omega} + h^{s(p_1)}(|\mu_0^h| + |\mu_1^h|) \le c\,h^{-n_{\varphi\mu}}F_{\varphi\mu} \tag{5.21}
$$

$$
n_{\varphi\mu} = n_{\varphi\mu}(p_1, r, \vartheta) = \begin{cases} 1-\vartheta & \text{for } \vartheta - 1/r > 0 \\ \max\{1-\vartheta, 1/r - s(p_1)\} & \text{for } \vartheta - 1/r \le 0 \end{cases} \tag{5.22}
$$

$$F_{\varphi\mu} = F_{\varphi\mu}(f_1, p_1, \vartheta, r, h) = \begin{cases} \text{const} & \text{for } (\vartheta - 1/r > 0) \vee ((1-\vartheta \ge 1/r - s(p_1)) \wedge (\vartheta - 1/r \le 0)) \\ \|f_1\|_{0,r,e_0} + \|f_1\|_{0,r,e_N} & \text{for } (1-\vartheta < 1/r - s(p_1)) \wedge (\vartheta - 1/r \le 0) \end{cases} \tag{5.23}$$

To estimate $\|a - a^{(m)}\|_{p_2,\omega^+}$ we use the inequality (see [1: p. 101])

$$0 \le a^{(m)}(x) - a(x) \le k_1^2 c_{m+1} h^{2(m+1)-n_x} |Q|^{m+1}_{\lambda,p,e}, \quad e = (x-h, x). \tag{5.24}$$

This way we obtain

$$\|a - a^{(m)}\|_{p_2,\omega^+} = \Big(\sum_{x\in\omega^+} h|a(x) - a^{(m)}(x)|^{p_2}\Big)^{1/p_2}$$

$$\le k_1^2 c_{m+1} h^{(m+1)(1+\lambda-1/p)+p_2^{-1}} \Big(\sum_{x\in\omega^+} |Q|^{p_2(m+1)}_{\lambda,p,e}\Big)^{1/p_2}.$$

In the case $p_2(m+1) < p$ one can estimate the last sum using Hölder's inequality with exponents $p/p_2(m+1)$ and $p/(p - p_2(m+1))$:

$$\sum_{x\in\omega^+} |Q|^{p_2(m+1)}_{\lambda,p,e} \le \Big(\sum_{x\in\omega^+} |Q|^{p}_{\lambda,p,e}\Big)^{p_2(m+1)/p} \Big(\sum_{x\in\omega^+} 1\Big)^{(p-p_2(m+1))/p} \le h^{p_2(m+1)/p\ -1} |Q|^{p_2(m+1)}_{\lambda,p,\Omega}.$$

In the case $p_2(m+1) \ge p$ we have

$$\sum_{x\in\omega^+} |Q|^{p_2(m+1)}_{\lambda,p,e} \le |Q|^{p_2(m+1)-p}_{\lambda,p,\Omega} \sum_{x\in\omega^+} |Q|^{p}_{\lambda,p,e} \le |Q|^{p_2(m+1)}_{\lambda,p,\Omega}.$$

Hence

$$\|a - a^{(m)}\|_{p_2,\omega^+} \le c\, h^{2(m+1)-n_a} |Q|^{m+1}_{\lambda,p,\Omega}$$

$$n_a = n_a(m,\lambda,p,p_2) = \begin{cases} (m+1)(1-\lambda+1/p) - (m+1)/p & \text{for } p_2(m+1) < p \\ (m+1)(1-\lambda+1/p) - 1/p_2 & \text{for } p_2(m+1) \ge p. \end{cases} \tag{5.25}$$

Similarly to (5.21) one can obtain

$$\|\varphi\|_{1,\omega} + h(|\mu_0^h| + |\mu_1^h|) \le c\|\varphi\|_{1,\dot\omega} \le c\, h^{\vartheta-1}. \tag{5.26}$$

Let us set

$$y = u,\ \tilde y = y^{(m)},\ \tilde\varphi = \varphi^{(m)},\ \tilde d = d^{(m)},\ \tilde a = a^{(m)},\ \tilde\mu_i^h = \mu_i^{(m)},\ \tilde\varkappa_i^h = \varkappa_i^{(m)}\ (i = 0,1)$$

in (4.13). Taking into account the estimates (5.1) - (5.26) as well as the relations

$$n_d^{(0)} = n_x + 1 + 1/p - \lambda,\quad n_\mu \ge n_x,\quad F_{\varkappa_j} \le c|Q|^{m+1}_{\lambda,p,e_{jN}} \quad (i = 0,1)$$

$$n_\times + n_\mu^{(o)} = n_\mu, \quad c|Q|_{\lambda,p,e_{jN}}^{m+1} F_\mu^{(jN)} \le F_{\mu_{jN}} \le \bar{c}|Q|_{\lambda,p,e_{jN}}^{m+1} F_\mu^{(jN)}$$

we find

$$\|\rho^{-1/p_0}(u - y^{(m)})\|_{\infty,\omega} \le c\, h^{2(m+1)-n} F(m,h), \tag{5.27}$$

where

$$n = \max\left\{n_\varphi, n_d - \vartheta + 1, n_a + n_{\varphi\mu}, n_\times + n_{\varphi\mu} - 1 - 1/p_2, n_\mu\right\}$$

$$\begin{aligned} F(m,h) &= F(m,h,Q,f_0,f_1) \\ &= h^{n-n_\varphi} + h^{n-n_d+\vartheta-1} + h^{n-n_\alpha-n_{\varphi\mu}} F_{\varphi\mu} + h^{n-n_d^{(o)}+p_2^{-1}} \\ &\quad -n_\mu\left(|Q|_{\lambda,p,e_1}^{m+1} + |Q|_{\lambda,p,e_N}^{m+1}\right) F_{\varphi\mu} + h^{n-n_\mu}(F_{\mu_0} + F_{\mu_1}). \end{aligned} \tag{5.28}$$

Thus we have obtained the following final result of the paper.

Theorem 5.1: *Let the assumptions* (C_1) - (C_4) *be satisfied and* $q_0 > 0$. *Then for the solution* $y^{(m)}$ *of the truncated difference scheme of rank* m (3.6) *the estimate* (5.27) *holds, where* u *is the unique solution of the problem* (P_1), *the constant* $c > 0$ *does not depend on* h, *the functional* $F = F(m,h)$ *is bounded or tends to zero if* $h \to 0$, $p_0^{-1} + p_1^{-1} + p_2^{-1} = 1$, $p_j \ge 1$ $(j = 0,1,2)$.

We see that the accuracy of the truncated difference scheme of rank m depends on the exponents of smoothness of the input data $p, \lambda, q, r, \vartheta$ as well on the parameters p_0, p_1, p_2, m which are available. For instance, if $p_0 = \infty$, $p_1 = \infty$, $p_2 = 1$, $q = Q' \in L_\infty(0,1)$, $f = f_0 + f_1' \in L_\infty(0,1)$, i.e. $p = q = r = \infty, \lambda = \vartheta = 1$, then $n_\varphi = n_d = n_a = n_{\varphi\mu} = n_\times = n_\mu = n = 0$, $F(m,h) = O(1)$ and we obtain the well-known Tichonov-Samarski estimate for the third boundary value problem for ordinary differential equations with piecewise smooth coefficients $\|u - y^{(m)}\|_{\infty,\omega} \le c\, h^{2(m+1)}$.

Remark: Let us consider the Dirichlet problem in generalized formulation

$$\overset{o}{a}_{[0,1]}(u,v) = \overset{o}{l}_{[0,1]}(v) \quad \text{for all } v \in \overset{o}{W}{}_2^1(0,1)$$

under the assumptions

$$Q \in W_p^\lambda(0,1) \quad (p \ge 2,\ 0 < \lambda \le 1) \tag{5.30}$$

$$-\int_0^1 Q(x) v'(x)\, dx \ge 0 \quad \text{for all } v \in \overset{o}{W}{}_2^1(0,1) \text{ with } v(x) \ge 0. \tag{5.31}$$

The exact difference scheme in the interior nodes of the grid and its coefficients have the form of (2.11), (2.12) and together with the boundary conditions

$$u(0) = u(1) = 0. \tag{5.32}$$

It forms just a complete system of diffference equations. Similarly to (3.6) one obtains the truncated difference scheme of rank m

$$\Lambda^{(m)} y^{(m)} = -\varphi^{(m)}(x) \text{ for } x \in \omega_h, \quad y^{(m)}(0) = y^{(m)}(1) = 0. \tag{5.33}$$

Choosing in (4.8) $y'(x) = u(x)$, $y''(x) = y^{(m)}(x)$ and using the estimates (4.9) - (4.12) we have

$$\|\rho^{-1/p_0}(u - y^{(m)})\|_{\infty,\omega} \le c\Big(\|\varphi - \varphi^{(m)}\|_{1,\omega} + \|\rho\varphi\|_{1,\omega}\|d - d^{(m)}\|_{1,\omega} + \|\rho^{s(p_1)}\varphi\|_{1,\omega}\big(\|d^{(m)}\|_{1,\omega}/k_0 + 1\big)\|a - a^{(m)}\|_{p_2,\omega^+}\Big) \tag{5.34}$$

$p_0^{-1} + p_1^{-1} + p_2^{-1} = 1$, $p_j \ge 1$ $(j = 0,1,2)$, and $s(p_1) = 0$ for $p_1 > 2$, $s(p_1) = 0,5$ for $1 \le p_1 \le 2$. From (5.14), (5.17), (5.25), (5.34) we easily derive the inequality

$$\|\rho^{-1/p_0}(u - y^{(m)})\|_{\infty,\omega} \le c h^{2(m+1)-n}, \text{ where } n = \max\{n_\varphi, n_d - \vartheta + 1, n_a - \vartheta + 1\}. \tag{5.35}$$

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