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Fubini Products of C^*-algebras and Applications to C^*-exactness

By

Tadasi HURUYA* and Seung-Hyeok KYE**

Abstract

For C^*-algebras A and B, we investigate necessary and sufficient conditions for which the Fubini product $A\otimes_F B$ coincides with the ordinary minimal tensor product $A\otimes B$. As an application of our methods, we also give several necessary and sufficient conditions for C^*-exactness.

§1. Introduction

Let A and B be C^*-algebras and $A\otimes B$ be the minimal tensor product of A and B. To deal with properties of minimal tensor products, the notion of Fubini product $A\otimes_F B$ of A and B has been proved to be very useful, as was introduced by Tomiyama [13]. For example, commutation and intersection theorems for tensor products are closely related with the triviality of Fubini products. After Wassermann [14, 15] obtained examples of nontrivial Fubini products, Archbold [1] and the authors [5, 10] showed that neither commutation nor intersection results holds for C^*-algebras. Also, the question of exactness of the following sequence

$$0 \longrightarrow A\otimes J \longrightarrow A\otimes B \longrightarrow A\otimes(B/J) \longrightarrow 0 \tag{1.1}$$

can be expressed in terms of Fubini products [8, 13].

In his paper [13], Tomiyama showed that if A is a subhomogeneous C^*-algebra, that is, every irreducible representation of A is finited-imensional with bounded dimension, then A is a C^*-algebra *with trivial Fubini products*, i.e., $A\otimes_F B = A\otimes B$ for every C^*-algebra B. Using Wassermann's example, the first author [6] gave an example of a C^*-algebra whose irreducible representations are all finite-dimensional but which has a nontrivial Fubini product. So, it is very natural to consider the converse of Tomiyama's result. In this vein, the second author [11, 12] proved the converse, for the classes of AF C^*-algebras and liminal C^*-algebras with Hausdorff spectra.

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* Faculty of Education, Niigata University, Niigata 950-21, Japan.
** Song Sim College for Women, Bucheon, Seoul 150-71, Korea.

In this paper, we prove the converse of Tomiyama's theorem for general C^*-algebras. Specifically, we show that if neither A nor B is subhomogeneous and B has property C'' [4] then $A\otimes_F B \supsetneqq A\otimes B$. Note that Kirchberg [9] recently announced analogous results. Also, we find several necessary and sufficient conditions for the exactness of the sequence (1.1) which include results in [2, 8, 12].

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§2. Examples of Nontrivial Fubini products

Let C and D be C^*-algebras. For $\phi\in C^*$ (respectively $\psi\in D^*$), there exists a unique bounded linear map R_ϕ: $C\otimes D\to D$ (respectively L_ψ: $C\otimes D\to C$) such that $R_\phi(c\otimes d)=\phi(c)d$ (respectively $L_\psi(c\otimes d)=\psi(d)c$). For C^*-subalgebras A and B of C and D, respectively, we define the *Fubini product* $F(A, B, C\otimes D)$ *of* A *and* B *with respect to* $C\otimes D$ [13] by

$$F(A, B, C\otimes D)=\{x\in C\otimes D\,;\ R_\phi(x)\in B \text{ and } L_\psi(x)\in A \text{ for all } \phi\in C^* \text{ and } \psi\in D^*\}.$$

The Fubini product $F(A, B, C\otimes D)$ is a C^*-subalgebra of $C\otimes D$ containing $A\otimes B$. The topic of this note is to investigate necessary and sufficient conditions for which the Fubini products become trivial, that is, $F(A, B, C\otimes D)=A\otimes B$.

For fixed C^*-algebras A and B, the Fubini product $F(A, B, C\otimes D)$ depends on $C\otimes D$ as well as A and B. But, they are all isomorphic and are the largest among them if C and D are injective. We denote by $A\otimes_F B$ any one of these isomorphic Fubini products of A and B [6].

In this section, we exhibit examples of nontrivial Fubini products to show the converse of Tomiyama's result. To begin with, we summarize the arguments of [8] and [12]. For a sequence $\{A_n\}$ of C^*-algebras, denote by $\Pi_n A_n$ (respectively $\oplus_n A_n$) their l^∞-sum (respectively c_0-sum). Also, for a directed set Λ, denote by $\Lambda(A)$ (respectively $\Lambda_0(A)$) the C^*-algebra of all bounded nets (respectively bounded nets converging to 0) of A. Finally, we denote by $B(H)$ (respectively $K(H)$) the C^*-algebra of all bounded linear (respectively compact) operators on the infinite-dimensional separable Hilbert space H.

Throughout this section, let (B, D) be a pair, where D is a C^*-subalgebra of B, satisfying the following condition:

(2.1) There exists a net $\{\pi_\lambda:\ \lambda\in\Lambda\}$ of completely bounded linear maps from B into D such that $\sup_\lambda\{\|\pi_\lambda\|_{cb}\}<\infty$ and $\lim_\lambda\|\pi_\lambda(x)-x\|=0$ for $x\in D$.

If D is nuclear, then the pair (B, D) satisfies the condition (2.1) as was shown in the proof of [2, Corollary 1]. Also, if D is a hereditary C^*-subalgebra of B then it is easy to find a net in (2.1) using an approximate indentity.

For a C^*-algebra B, we define $\delta: B \to \Lambda(B)$ by

$$\delta(b)_\lambda = b - \pi_\lambda(b) \qquad \text{for } b \in B \text{ and } \lambda \in \Lambda.$$

Now, it is easy to see (cf. [8, Lemma 2.2]) that

Lemma 2.1. *Let A be a C^*-algebra with a C^*-subalgebra C. For an element t in $A \otimes B$, we have*

i) $t \in A \otimes D \Longleftrightarrow (1_A \otimes \delta)(t) \in A \otimes \Lambda_0(B)$.

ii) $t \in F(C, D, A \otimes B) \Longrightarrow (1_A \otimes \delta)(t) \in F(C, \Lambda_0(B), A \otimes \Lambda(B))$.

Now, for every natural number $n = 1, 2, \cdots$, let E_n be a fixed C^*-algebra satisfying the following condition (C_n):

(C_n) For every $r > 0$, there exists completely positive contractions ϕ_n: $M_n \to E_n$ and ψ_n: $E_n \to M_n$ such that $\|\psi_n \circ \phi_n - 1_{M_n}\| < r$,

where M_n denotes the C^*-algebra of all $n \times n$ matrices over the complex field. For a C^*-algebra B, let S be the set of all completely positive contractions V from $\Lambda(B)$ into E such that $V(\Lambda_0(B)) \subseteq E_0$, where $E = \Pi_n E_n$ and $E_0 = \oplus_n E_n$. Then, the argument in [12] actually shows the following:

Lemma 2.2. *Let A be a C^*-algebra with a C^*-subalgebra C. For an element t in $A \otimes \Lambda(B)$, we have*

i) $t \in A \otimes \Lambda_0(B) \Longleftrightarrow (1_A \otimes V)(t) \in A \otimes E_0$ *for every* $V \in S$.

ii) $t \in F(C, \Lambda_0(B), A \otimes \Lambda(B)) \Longrightarrow (1_A \otimes V)(t) \in F(C, E_0, A \otimes E)$ *for every* $V \in S$.

Remark. Because the linear span of $\{\psi \circ V;\ \psi \in E^*,\ V \in S\}$ separates the points of $\Lambda(B)$, the converse of ii) in Lemma 2.2 is also true. The authors do not know whether the converse of ii) in Lemma 2.1 also holds or not.

Note that the pair $(B(H), K(H))$ satisfies the condition (2.1). Now, we apply Lemmas 2.1 and 2.2 with $B = B(H)$ and $D = K(H)$, to obtain the following lemma.

Lemma 2.3. *Let A be a C^*-algebra with a C^*-subalgebra C. Then, we have*

$$F(C, E_0, A \otimes E) \subseteq A \otimes E_0 \Longrightarrow F(C, K(H), A \otimes B(H)) \subseteq A \otimes K(H).$$

Example 2.4. *Let E_n (respectively F_n) be a C^*-algebra satisfying the condition (C_n) for $n = 1, 2, \cdots$, and put $E_0 = \oplus_n E_n$, $F_0 = \oplus_n F_n$. Then, we have*

$$E_0 \otimes_F F_0 \supsetneqq E_0 \otimes F_0.$$

Proof. Assume that $F(E_o, F_o, E\otimes\Pi_n F_n)=E_o\otimes F_o$. Then, applying Lemma 2.3 with $A=E$ and $C=E_o$, we have $F(E_o, K(H), E\otimes B(H))\subseteq E\otimes K(H)$. Hence, it follows that

$$F(E_o, K(H), E\otimes B(H))\subseteq F(E_o, K(H), E\otimes K(H))=E_o\otimes K(H),$$

by [15, Proposition 10]. Applying Lemma 2.3 again with $A=B(H)$ and $C=K(H)$, we have $F(K(H), K(H), B(H)\otimes B(H))\subseteq B(H)\otimes K(H)$, and so

$$F(K(H), K(H), B(H)\otimes B(H))\subseteq F(K(H), K(H), B(H)\otimes K(H))$$
$$=K(H)\otimes K(H),$$

by [15, Proposition 10] again. This contradicts [6, Theorem 9].

§3. Fubini Products of C^*-algebras

To prove the converse of the Tomiyama's theorem, we need the following result, which is contained in the proof of [7, Lemma 1]. Actually, we show that if A is not subhomogeneous then the Fubini product $A\otimes_F E_o$ is nontrivial, where E_0 is as in Example 2.4.

Lemma 3.1. *If A is a non-subhomogeneous C^*-algebra, then there exists an orthogonal sequence $\{E_n\}$ of hereditary C^*-subalgebras of A such that E_n satisfies the condition (C_n) for every $n=1, 2, \cdots$.*

Theorem 3.2. *Let A be a non-subhomogeneous C^*-algebra and E_n be a C^*-algebra satisfying the condition (C_n) for every $n=1, 2, \cdots$. Put $E_o=\oplus_n E_n$. Then, we have*

$$A\otimes_F E_o \supsetneqq A\otimes E_o.$$

Proof. For each positive number $m, n=1, 2, \cdots$, we choose completely positive contractions $\phi_{m,n}: M_n\to E_n$ and $\psi_{m,n}: E_n\to M_n$ satisfying

$$\|\psi_{m,n}\circ\phi_{m,n}-1_{M_n}\|<\frac{1}{m} \quad \text{for} \quad m, n=1, 2, \cdots.$$

Put $M=\Pi_n M_n$, $M_o=\oplus_n M_n$ and $E=\Pi_n E_n$. Also, let $\Phi_m: M\to E$ and $\Psi_m: E\to M$ be the l^∞-sums of the families $\{\phi_{m,n}; n=1, 2, \cdots\}$ and $\{\psi_{m,n}; n=1, 2, \cdots\}$, respectively, for $m=1, 2, \cdots$. Then, they are completely positive contractions, and satisfy

$$\Phi_m(x)\in E_o \quad \text{for} \quad x\in M_o,$$
$$\Psi_m(y)\in M_o \quad \text{for} \quad y\in E_0,$$
$$\|\Psi_m\circ\Phi_m-1_M\|\leqq\frac{1}{m}.$$

Now, assume that $A\otimes_F E_o=A\otimes E_o$ and let C be an injective C^*-algebra containing A. If $x\in F(A, M_o, C\otimes M)$, then we have

$$R_\phi((1_C\otimes\Phi_m)(x))=\Phi_m(R_\phi(x))\in E_o \qquad \text{for} \quad \phi\in C^*,$$

and so $(1_C\otimes\Phi_m)(x)\in F(A, E_o, C\otimes E)=A\otimes E_o$. Hence, it follows that

$$x=\lim_m(1_C\otimes\Psi_m)(1_C\otimes\Phi_m)(x)\in A\otimes M_o.$$

Therefore, we have $A\otimes_F M_o=F(A, M_o, C\otimes M)=A\otimes M_o$.

We choose an orthogonal sequence $\{F_n\}$ of C^*-subalgebras of A satisfying the condition (C_n) by Lemma 3.1. Put $F_o=\oplus_n F_n$. Then, we have $F_o\otimes_F M_o\subseteqq A\otimes_F M_o=A\otimes M_o$ and it follows that

$$F_o\otimes_F M_o\subseteqq F(F_o, M_o, A\otimes M_o)=F_o\otimes M_o,$$

by [15, Proposition 10]. This is a contradiction by Example 2.4.

We proceed to investigate $A\otimes_F B\supsetneqq A\otimes B$ in more general situation. To do this, we introduce the following two conditions for pairs (B, D) of C^*-algebras with $D\subseteqq B$:

(3.1) $\quad A\otimes D=(A\otimes B)\cap(A\otimes\bar{D}) \quad$ for every C^*-algebra A.

(3.2) $\quad$ There exists a completely bounded linear map π from B into $\bar{D}$ such that $\pi(d)=d$ for $d\in D$,

where $\bar{D}$ denotes the weak-operator closure of D in the universal representation of B. Note that the pair $(B(H), K(H))$ does not satisfy the condition (3.1) [10]. If D is a hereditary C^*-subalgebra or a C^*-subalgebra of B having the weak expectation property then the pair (B, D) satisfies the condition (3.2). Also, it is easy to see that the condition (2.1) implies the condition (3.2). Now, by a standard argument as in [3, Theorem 3.4], we have the following:

Proposition 3.3. *Let* (B, D) *be a pair satisfying the conditions* (3.1) *and* (3.2). *Then, for every* C^**-algebra* A, *we have*

$$F(A, D, A\otimes B)=A\otimes D.$$

Recall that a C^*-algebra B is said to have *property* C (respectively *property* C'') [3, 4] if there exists a 'canonical embedding' of $A^{**}\otimes B^{**}$ (respectively $A\otimes B^{**}$) into $(A\otimes B)^{**}$ for every C^*-algebra A. It is easy to see that if B has property C'' then the pair (B, D) satisfies the condition (3.1) for every C^*-subalgebra D (see the proof of [3, Theorem 3.4]). Note that property C implies property C''.

During the preparation of this paper, the authors have learned that Kirchberg [9] announced the following:

If neither A nor B is subhomogeneous and one of which is C^*-exact or separable, then $A\otimes_F B\supsetneqq A\otimes B$.

Also, he announced that C^*-exactness is actually equivalent to property C. Recall that a C^*-algebra A is said to be *C^*-exact* if the sequence (1.1) is exact for every C^*-algebra B and its norm-closed two-sided ideal J. Note that the kernel of the $*$-homomorphism $A\otimes B\to A\otimes(B/J)$ is just $F(A, J, A\otimes B)$ in general. With the aid of Proposition 3.3, we can prove the following:

Theorem 3.4. *Let neither A nor B be a subhomogeneous C^*-algebra, and B have property C''. Then, we have*

$$A\otimes_F B\supsetneqq A\otimes B\,.$$

Proof. Assume that $A\otimes_F B=A\otimes B$. Choose an orthogonal sequence $\{E_n\}$ of hereditary C^*-subalgebras of B as in Lemma 3.1, and put $E_o=\oplus_n E_n$. Then, it is easy to see that the pair (B, E_o) satisfies the condition (3.2). Because $A\otimes_F E_o\subseteqq A\otimes_F B=A\otimes B$, we have

$$A\otimes_F E_o\subseteqq F(A, E_o, A\otimes B)=A\otimes E_o\,,$$

by Proposition 3.3. This is a contradiction by Theorem 3.2.

§ 4. Exactness for C^*-algebras

As an application of Lemmas 2.1 and 2.2, we reprove the following characterization of C^*-exactness which can be found in [2, 8, 12]. In this section, we denote $M=\Pi_n M_n$ and $M_o=\oplus_n M_n$, as in the proof of Theorem 3.2.

Theorem 4.1. *Let E_n be a fixed C^*-algebra satisfying the condition (C_n) for every $n=1, 2, \cdots$. Then, for a C^*-algebra A, the following are equivalent:*

i) *$F(A, D, A\otimes B)=A\otimes D$ for every pair (B, D) of C^*-algebras satisfying the condition* (2.1).

ii) *$F(A, D, A\otimes B)=A\otimes D$ for every C^*-algebra B and its hereditary C^*-subalgebra D.*

iii) *A is C^*-exact.*

iv) *$F(A, K(H), A\otimes B(H))=A\otimes K(H)$.*

v) *$F(A, E_o, A\otimes E)=A\otimes E_o$, where $E=\Pi_n E_n$ and $E_o=\oplus_n E_n$.*

vi) *$F(A, M_o, A\otimes M)=A\otimes M_o$.*

Proof. The implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv), (iii)$\Rightarrow$(v) and (iii)$\Rightarrow$(vi) are clear. For (v)$\Rightarrow$(i) and (vi)$\Rightarrow$(i), apply Lemmas 2.1 and 2.2 with $C=A$. The proof will be completed if we show that (iv) implies (vi).

Let H_k be the k-dimensional Hilbert space for each $k=1, 2, \cdots$, and put $H=\oplus_k H_k$. We embed M into $B(H)$ with $M_n=B(H_n)$. Then, $M_o=M\cap K(H)$.

Denote by p_n the projection of H onto the subspace $H_1 \oplus \cdots \oplus H_n$, and define π_n: $B(H) \to B(H)$ by $\pi_n(x) = p_n x p_n$. If $x \in F(A, M_o, A \otimes M)$ then we have

$$x \in (A \otimes M) \cap F(A, K(H), A \otimes B(H)) = (A \otimes M) \cap (A \otimes K(H)).$$

Now, because $\|b - \pi_n(b)\| \to 0$ for $b \in K(H)$, $x \in A \otimes K(H)$ implies

$$\|x - (1_A \otimes \pi_n)(x)\| \longrightarrow 0.$$

Also, $x \in A \otimes M$ implies that $(1_A \otimes \pi_n)(x) \in A \otimes M_o$. Hence, we have

$$x = \lim_n (1_A \otimes \pi_n)(x) \in A \otimes M_o.$$

Now, we examine the exactness of the sequence (1.1) in another direction. Let us call a pair (B, J), where J is a norm-closed two-sided ideal of a C^*-algebra B, *exact* if the sequence (1.1) is exact for every C^*-algebra A. As a counterpart of Theorem 4.1, we characterize the exactness for pairs. Note that Effros and Haagerup [4] gave another characterization in terms of lifting problems. We need the following simple lemma.

Lemma 4.2. *Let σ be a complete isometry from a C^*-algebras A_1 into another C^*-algebra A_2, and (B, D) be a pair of C^*-algebras satisfying the condition* (2.1). *Then, $F(A_2, D, A_2 \otimes B) = A_2 \otimes D$ implies $F(A_1, D, A_1 \otimes B) = A_1 \otimes D$.*

Proof. If $x \in F(A_1, D, A_1 \otimes B)$ then we have

$$R_\phi(\sigma \otimes 1_B)(x) = R_{\phi \circ \sigma}(x) = D \qquad \text{for} \quad \phi \in A_2^*.$$

Hence, we have $(\sigma \otimes 1_B)(x) \in F(A_2, D, A_2 \otimes B) = A_2 \otimes D$. Therefore,

$$\lim_\lambda (\sigma \otimes 1_B)(1_{A_1} \otimes \pi_\lambda)(x) = \lim_\lambda (1_{A_2} \otimes \pi_\lambda)(\sigma \otimes 1_B)(x)$$
$$= (\sigma \otimes 1_B)(x),$$

and it follows that

$$x = \lim_\lambda (1_{A_1} \otimes \pi_\lambda)(x) \in A_1 \otimes D.$$

Theorem 4.3. *Let E_n be a fixed C^*-algebra satisfying the condition (C_n) for each $n = 1, 2, \cdots$, and A_o be a fixed C^*-algebra containing $E = \Pi_n E_n$ as a C^*-subalgebra. For a C^*-algebra B and its norm-closed two-sided ideal J, the following are equivalent:*

i) *(B, J) is an exact pair.*
ii) *$F(A_o, J, A_o \otimes B) = A_o \otimes J$.*
iii) *$F(E, J, E \otimes B) = E \otimes J$.*
iv) *$F(M, J, M \otimes B) = M \otimes J$.*
v) *$F(B(H), J, B(H) \otimes B) = B(H) \otimes J$.*
vi) *$F(A, J, A \otimes B) = A \otimes J$ for every separable C^*-algebra A.*

Proof. The implication (i)⇒(ii) is clear, and (ii)⇒(iii) follows from Lemma 4.2.

For (iii)$\Rightarrow$(iv), let $x\in F(M, J, M\otimes B)$. Then, it follows that

$$R_{\phi}(\Phi_m\otimes 1_B)(x)=R_{\phi\circ\Phi_m}(x)\in J \quad \text{for} \quad \phi\in E^*,$$

where $\Phi_m: M\to E$ and $\Psi_m: E\to M$ are as in the proof of Theorem 3.2. Hence, $(\Phi_m\otimes 1_B)(x)\in F(E, J, F\otimes B)=E\otimes J$, and so

$$x=\lim_m(\Psi_m\otimes 1_B)(\phi_m\otimes 1_B)(x)\in M\otimes J.$$

For the implication (iv)$\Rightarrow$(v), let p_n be the projection of H onto the subspace spanned by $\{\xi_1, \xi_2, \cdots, \xi_n\}$, where $\{\xi_1, \xi_2, \cdots\}$ is an orthonormal basis of H. Now, we can apply Lemma 4.2 with the aid of the map $\pi: B(H)\to M$, defined by

$$\pi(x)=(p_1xp_1, p_2xp_2, \cdots) \quad \text{for} \quad x\in B(H).$$

Since every separable C^*-algebra can be embedded in $B(H)$, (v)$\Rightarrow$(vi) also follows from Lemma 4.2.

For the final implication (vi)$\Rightarrow$(i), assume that (B, J) is not an exact pair and $F(C, J, C\otimes B)\supsetneqq C\otimes J$ for some C^*-algebra C. Choose an element z in $F(C, J, C\otimes B)\setminus C\otimes J$, and let A be the C^*-subalgebra of C generated by $\{L_{\phi}(z); \phi\in B^*\}$, as in the proof of [6, Lemma 5]. Then A is separable and $z\in F(A, J, A\otimes B)$, but $z\notin A\otimes J$ by the choice of z. This completes the proof.

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