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Asymptotic Developments of the Solutions of the Translation Equation

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For the solutions of the translation equation a special asymptotic development is considered. It is shown that the coefficients of this development can be determined by linear recursion formulas. An example illustrates the results.

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1. Introduction

This paper deals with the translation equation

$$F(x, s+t) = F[F(x,s), t] \quad , \tag{1}$$

cf. J. Aczél [1] and G. Targonski [7], under the initial condition

$$F(x, 0) = x \quad . \tag{2}$$

For the solutions $F : \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ of (1) asymptotic expansions of the form

$$F\left(x, \frac{u}{x}\right) \approx \sum_{n=1}^{\infty} f_n(u) x^n \tag{3}$$

are considered, where $x \to 0$. The first three coefficients in (3) were already determined by the author in [3], and under weaker conditions in [4]. The case of integer $\frac{u}{x} = k$ was investigated by D. Gronau in [5]. Here we derive two linear differential recursion formulas for the coefficients, which allow the calculation of further f_n. Moreover, we solve the problem of [3] and [4] concerning the construction of an example for (3) in the general logarithmic case. Complicated calculations were carried out by means of the DERIVE 2.01 system.

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2. The first recursion formula

The asymptotic expansion (3) means

$$F\left(x, \frac{u}{x}\right) = \sum_{\nu=1}^{n} f_\nu(u)x^\nu + O(x^{n+1}) \tag{4}$$

for every n, and we assume that (4) is satisfied uniformly with respect to u for $|u| \leq u_0$ with sufficiently small u_0. It is allowed that x is restricted to $x \geq 0$ or to $x \leq 0$. Condition (2) implies

$$f_1(0) = 1, \quad f_n(0) = 0 \quad \text{for} \quad n \geq 2 \quad . \tag{5}$$

For $u = xt$ we interpret (3) as a formal power series

$$F(x,t) = \sum_{n=1}^{\infty} f_n(xt)x^n \quad , \tag{6}$$

cf. L. Reich [6], and vice versa.

Theorem 1: *For sufficiently small $|xt|$ we assume that the solution $F = F(x,t)$ of* (1) *with* (2) *and* (4) *possesses the two partial derivatives of order 1 and that* (6) *can be differentiated termwise. Then the coefficients of* (3) *satisfy the differential recursion formula*

$$f_n'(u) = \sum_{\nu=1}^{n} a_{n+1-\nu}[uf_\nu'(u) + \nu f_\nu(u)] \tag{7}$$

with $a_n = f_n'(0)$.

Proof: From (6) we obtain by differentiation

$$F_x(x,t) = \sum_{n=1}^{\infty}[txf_n'(xt) + nf_n(xt)]x^{n-1} \quad , \tag{8}$$

$$F_t(x,t) = \sum_{n=1}^{\infty} f_n'(xt)x^{n+1} \quad . \tag{9}$$

These derivatives are connected by the Jabotinski equation

$$F_t(x,t) = F_x(x,t)F_t(x,0) \quad , \tag{10}$$

cf. J. Aczél and D. Gronau [2], which easily follows from (1) and (2). The last three equations immediately imply the recursion formula (7) ■

Let us mention that the further initial conditions

$$F(0,t) = 0, \quad F_x(0,t) = 1$$

are consequences of (5), (6) and (8).

Integrating (7) and considering (5) we find the coefficients

$$f_1(u) = \frac{1}{v},\ f_2(u) = -\frac{a_2}{a_1 v^2}\ln v,\ f_3(u) = \frac{1}{v^3}\left\{\left(\frac{a_2}{a_1}\right)^2(\ln^2 v - \ln v - a_1 u) + a_3 u\right\}$$

with $v = 1 - a_1 u$, which are already known from [3] and [4], as well as

$$\begin{aligned} f_4(u) &= \frac{1}{v^4}\Big\{ -\left(\frac{a_2}{a_1}\right)^3 \ln^3 v + \frac{5}{2}\left(\frac{a_2}{a_1}\right)^3 \ln^2 v - \frac{a_2}{a_1^2}\left[2\left(a_1 a_3 - a_2^2\right)u + a_3\right]\ln v \\ &\quad -\frac{1}{2a_1}(\alpha u^2 + \beta u)\Big\} \end{aligned}$$

with $\alpha = a_1^2 a_4 - 2a_1 a_2 a_3 + a_2^3$, $\beta = -2(a_1 a_4 - a_2 a_3)$. In the case that $a_n = 0$ for $n > 2$ the coefficient f_4 and the next two ones can be written in the form

$$\begin{aligned} f_4(u) &= \frac{a_2^3}{a_1^3 v^4}\left\{-\ln^3 v + (\ln v + w)\left[\frac{5}{2}\ln v - \frac{w}{2}\right]\right\}, \\ f_5(u) &= \frac{a_2^4}{a_1^4 v^5}\Big\{ \ln^4 v - (\ln v + w)\Big[\frac{13}{3}\ln^2 v - \left(\frac{4}{3}w + \frac{3}{2}\right)\ln v \\ &\quad + \left(\frac{1}{3}w - \frac{3}{2}\right)\Big]\Big\}, \\ f_6(u) &= \frac{a_2^5}{a_1^5 v^6}\Big\{ -\ln^5 v + (\ln v + w)\Big[\frac{77}{12}\ln^3 v - \left(\frac{29}{12}w + \frac{35}{6}\right)\ln^2 v \\ &\quad + w\left(\frac{11}{12}w - \frac{25}{6}\right)\ln v - w^2\left(\frac{1}{4}w - \frac{5}{3}\right)\Big]\Big\} \end{aligned}$$

with $w = a_1 u$. The first coefficients in the brackets are the same as in

$$-\frac{\ln(1-w)}{(1-w)^2} = w + \frac{5}{2}w^2 + \frac{13}{3}w^3 + \frac{77}{12}w^4 + \dots,$$

cf. [3]. In the case $a_1 = 0$ we have to interpret all expressions as limits for $a_1 \to 0$, so that then $\frac{1}{a_1}\ln v = -u$, $\frac{1}{a_1^2}(\ln v + w) = -\frac{1}{2}u^2$ etc., and all f_n turn over into polynomials.

3. The second recursion formula

Instead of (1) we now start from the representation

$$F(x,t) = \varphi^{-1}[t + \varphi(x)] \tag{11}$$

for such solutions of the translation equation satisfying (2), which are invertible for a fixed x, cf. J. Aczél [1]. As before, $|xt|$ shall be sufficiently small. In case of need we moreover restrict x to $x \geq 0$ or to $x \leq 0$.

Theorem 2: *Under the additional assumptions of Theorem 1 as well as $F_t(x,0) \neq 0$ and*

$$a_1 = a_2 = \dots = a_{m-1} = 0, \quad a_m \neq 0, \ m \geq 1 \tag{12}$$

the function φ of (11) is differentiable, and the derivative has an asymptotic expansion of the form

$$\varphi'(x) \approx \sum_{n=0}^{\infty} b_n x^{n-m-1} \tag{13}$$

for $x \to 0$ with $b_0 = \frac{1}{a_m}$. The coefficients f_n of (4) satisfy the differential recursion formula

$$u f_n'(u) + n f_n(u) = \sum_{\nu=m}^{n+m-1} b_{n+m-\nu-1} f_\nu'(u) \tag{14}$$

for $n \geq 1$. The first $m-1$ coefficients are

$$f_1(u) = 1, \quad f_2(u) = \ldots = f_{m-1}(u) = 0 \quad . \tag{15}$$

Proof: According to (11) the differentiability of F implies the differentiability of φ. Hence we obtain

$$\varphi'(x) = \frac{1}{F_t(x,0)} \quad , \tag{16}$$

and therefore from (9) and (12) in the sense of formal power series

$$\varphi'(x) = \frac{1}{\sum_{n=m}^{\infty} a_n x^{n+1}} \quad . \tag{17}$$

Now, carrying out the division we find an expansion of the form (13) with $b_0 = \frac{1}{a_m}$.

In view of (16) we can write (10) in the form

$$F_x(x,t) = \varphi'(x) F_t(x,t) \quad ,$$

i.e. according to (8), (9) and (13) with $u = xt$

$$\sum_{n=1}^{\infty} \left[u f_n'(u) + n f_n(u)\right] x^{n-1} = \sum_{\mu=0}^{\infty} b_\mu x^{\mu-m-1} \sum_{\nu=1}^{\infty} f_\nu'(u) x^{\nu+1} \quad .$$

In view of $b_0 \neq 0$ this is only possible for $f_\nu'(u) = 0$ in case of $\nu < m$, so that we immediately obtain (14) and according to (5) also (15) ■

Let us mention that the systems (7) and (14) are inversions of each other in view of the equations

$$\sum_{\nu=0}^{n} b_{n-\nu} a_{\nu+m} = \delta_{n0} \quad , \tag{18}$$

which are consequences of (13) and (17), where δ_{n0} is the Kronecker symbol. By integration of (13) we obtain up to an unessential constant

$$\varphi(x) = -\frac{1}{m}\frac{b_0}{x^m} - \ldots - \frac{b_{m-1}}{x} + b_m \ln|x| + b_{m+1} x + \frac{1}{2} b_{m+2} x^2 + \ldots \quad . \tag{19}$$

If m is odd, then x can vary at both sides of $x = 0$, otherwise x must be nonnegative or nonpositive in order to guarantee the invertibility.

From (14) and (5) we can calculate f_{n+m-1}, so far as the preceding f_n are already known. The results are the same as in Section 2. In the case $m \geq 2$ all f_n are polynomials. More precisely, they are polynomials of a degree not greater than k, if $k(m-1) < n \leq (k+1)(m-1)$, since the first $m-1$ ones are constant. In the case $m = 1$ the equations (14) turn over into

$$(u - b_0)f_n'(u) + nf_n(u) = \sum_{\nu=1}^{n-1} b_{n-\nu} f_\nu'(u) \quad .$$

4. An example

Let us consider the special case $m = 1$, $b_0 = 1$, $b_1 = -1$ and $b_n = 0$ for $n \geq 2$, i.e. in view of (19)

$$\varphi(x) = -\frac{1}{x} - \ln|x| \quad .$$

From (18) we find $a_n = 1$ for all n. We need the solution $\varphi^{-1} = \psi(x)$ of $\frac{1}{\psi} + \ln|\psi| = -x$ with $\psi(x) \to 0$ for $x \to \infty$, so that (11) turns over into $F(x,t) = \psi\left(t - \frac{1}{x} - \ln|x|\right)$. This means that $F = F(x,t)$ must be a solution of

$$F = -\frac{1}{t - \frac{1}{x} - \ln|x| + \ln|F|} = \frac{x}{1 - xt - x\ln\left(\frac{F}{x}\right)}$$

for small $|x|$. Substituting $F = \frac{x}{y}$ and as before $v = 1 - xt$ with $|xt| < 1$, we have to solve the equation

$$y = v + x\ln y \quad .$$

According to the well-known formula of Lagrange

$$g(y) = g(v) + \sum_{n=1}^{\infty} \frac{x^n}{n!}[g'(v)\ln^n v]^{(n-1)}$$

with $g(v) = \frac{1}{v}$ and therefore $g'(v) = -\frac{1}{v^2}$ we obtain the equation

$$F(x,t) = \frac{x}{v} - \sum_{n=1}^{\infty} \frac{x^{n+1}}{n!}\left(\frac{1}{v^2}\ln^n v\right)^{(n-1)} \quad ,$$

which even converges for sufficiently small $|x|$. In view of

$$\begin{aligned}
\left(\frac{1}{v^2}\ln^2 v\right)' &= -\frac{2}{v^3}(\ln^2 v - \ln v) \quad , \\
\left(\frac{1}{v^2}\ln^3 v\right)'' &= \frac{6}{v^4}\left(\ln^3 v - \frac{5}{2}\ln^2 v + \ln v\right) \quad , \\
\left(\frac{1}{v^2}\ln^4 v\right)^{(3)} &= -\frac{24}{v^5}\left(\ln^4 v - \frac{13}{3}\ln^3 v + \frac{9}{2}\ln^2 v - \ln v\right) \quad , \\
\left(\frac{1}{v^2}\ln^5 v\right)^{(4)} &= \frac{120}{v^6}\left(\ln^5 v - \frac{77}{12}\ln^4 v + \frac{71}{6}\ln^3 v - 7\ln^2 v + \ln v\right) \quad ,
\end{aligned}$$

the results coincide with the results of Section 2 with $a_n = 1$ for all n.

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