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The Treatment of Window Problems by Transform Methods

W. SPRÖSSIG and E. VENTURINO

The aim of this paper is to consider a special class of mixed boundary value problems, the so-called window problems. Using suitable transforms we derive singular integral equations. These methods lend themselves to discover how singularities arise. They show and allow a close connection between analytical methods and numerical analysis.

Key words: *Mixed boundary value problems, transform methods, singular integral equations, hypercomplex function theory*

AMS (MOS) subject classification: 45 E, 42 B, 30 G

1 Introduction

Our paper is devoted to the so-called window problems of elliptic differential equations over domains in $\mathbb{R}^2$ and $\mathbb{R}^3$. The kind of problems appear as a special class of mixed boundary value problems, where the window is a finite part of the boundary. It means that the connection with the outside only through a window can happen. By unconventionally chosen unknown functions we will deduce canonical representations of these problems in form of perturbed singular integral equations with singular integrals over the window. We use methods of transform analysis which are different from Green's method and potential methods. These ideas go back to D.S. Jones [8]. In a similar way S.R. Bland [1], F. Erdogan [2] and E. Venturino [13] transformed different problems of mechanics to singular integral equations with generalized kernels. For more complicated domains and a certain class systems of partial differential equations the previous methods to not generalize and we will use methods of quaternionic analysis. Basic ideas in this field are given in the book of K. Gürlebeck and W. Sprößig [6]. Finally a numerical consideration on the singular integral equations deduced above conclude the paper.

2 A transform method for 2-dimensional window problems

The aim of this section is to demonstrate the main ideas of the transform method used here. We consider the following 2-dimensional window problem. Let be

$$\mathbb{R}^{2,+} = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}, \mathbb{R}^{2,+}_y = \{(x,y) \in \mathbb{R}^2 : x \geq 0,\ \ y = 0\},$$
$$\mathbb{R}^{2,+}_x = \{(x,y) \in \mathbb{R}^2 : x = 0,\ \ y \geq 0\}, W = \{(x,y) \in \mathbb{R}^{2,+}_x : 0 < a \leq y \leq b < \infty\}$$

W. Sprössig: Bergakademie Freiberg, FB Math., B.-von-Cotta-Str. 2, D - 09599 Freiberg.
The present paper was written while this author was visiting the University of Catania. He acknowledges the support of the Italian Research Council (CNR).
E. Venturino: The University of Iowa, Division of Math. Sci., Iowa City, Iowa 52242, USA

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and $W^c = \mathbb{R}_x^{2,+} \setminus W$. We look for a function $u = u(x,y)$, harmonic in the interior of the set $\mathbb{R}^{2,+}$, denoted by int $\mathbb{R}^{2,+}$, and which satisfies the boundary conditions

$$\begin{array}{llll} (i) & u(0,y) = 0 \ \text{ on } \ W^c & (iii) & \lim\limits_{x^2+y^2\to\infty} |u(x,y)| < \infty \\ (ii) & u(x,0) = 0 \ \text{ on } \ \mathbb{R}_y^{2,+} & (iv) & \dfrac{\partial}{\partial\lambda} u(0,y) = g(y) \ \text{ on } \ W \end{array} \tag{2.1}$$

with $\frac{\partial}{\partial\lambda} = \lambda_1(y)\frac{\partial}{\partial x} + \lambda_2(y)\frac{\partial}{\partial y}$, $\lambda_1^2(y) + \lambda_2^2(y) = 1$; $\lambda_1(y) \neq 0$ for every y, where g, λ_1, λ_2 are some functions. After separating the variables we get the ordinary differential equations

$$X''(x) - \alpha^2 X(x) = 0 \quad \text{and} \quad Y''(y) + \alpha^2 Y(y) = 0$$

with an arbitrary real parameter α. This leads to the representations

$$X_\alpha(x) = A(\alpha)e^{\alpha x} + B(\alpha)e^{-\alpha x} \quad \text{and} \quad Y_\alpha(y) = C(\alpha)\cos\alpha y + D(\alpha)\sin\alpha y\,.$$

Because of the boundedness of u at infinity we have $X_\alpha(x) = B(\alpha)e^{-\alpha x}$. From $X_\alpha(0) = 0$ we get $C(\alpha) = 0$ and so $Y_\alpha(y) = D(\alpha)\sin\alpha y$. Hence,

$$u(x,y) = \frac{2}{\pi}\int_0^\infty E(\alpha)e^{-\alpha x}\sin\alpha y\, d\alpha, \tag{2.2}$$

where $E(\alpha) = D(\alpha)B(\alpha)$. As for $y \in W^c$ we have $u(0,y) = 0$, then it is also $\partial u/\partial y = 0$. That means

$$\lim_{x\to 0^+}\frac{2}{\pi}\int_0^\infty \alpha E(\alpha)e^{-\alpha x}\cos\alpha y\, d\alpha = 0 \ , \ y \in W^c$$

and from (2.2) we get on W the equation

$$\begin{aligned} g(y) &= \lambda_1(y)\lim_{x\to 0^+}\frac{2}{\pi}\int_0^\infty (-\alpha)E(\alpha)e^{-\alpha x}\sin\alpha y\, d\alpha \\ &+ \lambda_2(y)\lim_{x\to 0^+}\int_0^\infty \alpha E(\alpha)e^{-\alpha x}\cos\alpha y\, d\alpha \end{aligned} \tag{2.3}$$

or

$$\lim_{x\to 0^+}\frac{2}{\pi}\int_0^\infty \alpha E(\alpha)e^{-\alpha x}\big[\lambda_2(y)\cos\alpha y - \lambda_1(y)\sin\alpha y\big]d\alpha = g(y) \ , \ y \in W.$$

Setting $h(y) = \frac{2}{\pi}\int_0^\infty \alpha E(\alpha)\cos\alpha y\, d\alpha$, we have obviously $h(y) \equiv 0$ on W^c. After inverting the cosine transform we obtain $\alpha E(\alpha) = \int_W h(t)\cos\alpha t\, dt$. Substituting this in (2.3) we get

$$\lim_{x\to 0^+}\frac{2}{\pi}\int_0^\infty\int_W h(t)\cos\alpha t\, dt\big[\lambda_2(y)\cos\alpha y - \lambda_1(y)\sin\alpha y\big]e^{-\alpha x}d\alpha = g(y), y \in W.$$

By using the Laplace transform we get after interchanging the order of integration for $y \in W$

$$\begin{aligned} g(y) &= \lim_{x\to 0^+} \frac{2}{\pi} \int_W h(t) \int_0^\infty e^{-\alpha x}\left[\lambda_2(y)\cos\alpha t\cos\alpha y - \lambda_1(y)\cos\alpha t\sin\alpha y\right] d\alpha\, dt \\ &= \lim_{x\to 0^+} \frac{1}{\pi} \int_W h(t)\Big[\lambda_2(y)\Big(\frac{x}{x^2+(t+y)^2} + \frac{x}{x^2+(t-y)^2}\Big) \\ &\quad -\lambda_1(y)\Big(\frac{t+y}{x^2+(t+y)^2} + \frac{t-y}{x^2+(t-y)^2}\Big)\Big] dt. \end{aligned}$$

Finally we obtain the singular integral equation over the window W

$$\frac{1}{\pi}\int_W \frac{1}{t-y} h(t)dt + \frac{1}{\pi}\int_W \frac{1}{t+y} h(t)dt = -\frac{g(y)}{\lambda_1(y)}, \quad y \in W. \tag{2.4}$$

Note that the kernel of the second integral is bounded. If we can solve (2.4) in terms of h we immediately get $E(\alpha) = \frac{1}{\alpha}\int_W h(t)\cos\alpha t\, dt$. Substituting this into representation (2.1) we find

$$\begin{aligned} u(x,y) &= \frac{2}{\pi}\int_0^\infty \frac{1}{\alpha}\int_W h(t)\cos\alpha t\, dt\, e^{-\alpha x}\sin\alpha y\, d\alpha \\ &= \frac{1}{\pi}\int_W h(t)\int_0^\infty \frac{e^{-\alpha x}}{\alpha}\left[\sin\alpha(t+y) + \sin\alpha(t-y)\right] d\alpha\, dt \\ &= \frac{1}{\pi}\int_W h(t)\Big[\arctan\frac{t+y}{x} + \arctan\frac{t-y}{x}\Big] dt \\ &= \int_{W\cap(t\le y)} h(t)dt + \frac{1}{\pi}\int_W h(t)\arctan\frac{2tx}{x^2-t^2+y^2} dt. \end{aligned} \tag{2.5}$$

We have then the following

Theorem 1. *Let $g \in C(W), \lambda_i \in C(W)$, $i = 1,2$. Then the potential window problem (2.1) allows the integral reformulation (2.4). Once the latter is solved, the solution of the original problem in terms of the new unknown $h(t)$ is given by (2.5).*

3 A transform method for 3-dimensional window problems

In this section we apply the transform method of Section 2 to the case of a 2-dimensional window in a rectangular 3-dimensional domain. Let be $\mathbb{R}^{3,+} = \{(x,y,z) \in \mathbb{R}^3, x,y,z \geq 0\}$ and $\mathbb{R}^{3,+}_s = \{(x,y,z) \in \mathbb{R}^{3,+}, s = 0\}$ with $s \in \{x,y,z\}$. Furthermore we assume that the window $W \subset \mathbb{R}^{3,+}_y$ has a positive distance from the axes. The geometric shape of the window is not yet fixed. We can now formulate the following window problem.

Wanted is a function $u = u(x,y,z)$ harmonic in int $\mathbb{R}^{3,+}$ which satisfies the boundary conditions

$$
\begin{aligned}
&(i) \quad u(x,y,z)=0 \quad \text{on} \quad W^c := \mathbb{R}_x^{3,+} \cup \mathbb{R}_s^{3,+} \cup \mathbb{R}_y^{3,+} \setminus W \\
&(ii) \quad \lim_{s^2+y^2+z^2\to\infty} |u(x,y,z)| < \infty. \qquad (3.1)\\
&(iii) \quad \partial u/\partial\lambda(x,y,z) = g(x,z) \quad \text{on} \quad W \\
&\qquad \text{where} \quad \frac{\partial}{\partial\lambda} = \lambda_1(x,z)\frac{\partial}{\partial x} + \lambda_2(x,z)\frac{\partial}{\partial y} + \lambda_3(x,z)\frac{\partial}{\partial z}, \sum_{i=1}^{3}\lambda_i^2 = 1, \lambda_2 \neq 0,\\
&\qquad \lambda_i, i=1,2,3 \text{ and } g \text{ are some functions.}
\end{aligned}
$$

Using the separation ansatz $u(x,y,z) = X(x)Y(y)Z(z)$ we obtain the ordinary differential equations $X''(x)+\alpha^2X(x) = 0, Z''(z)+\gamma^2Z(z) = 0$ und $Y''(y)-(\alpha^2+\gamma^2)Y(y) = 0$, where α, γ are real positive parameters. Let $\beta^2 = \alpha^2+\gamma^2$; then we obtain the representations $X_\alpha(x) = A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x$, $Z_\gamma(z) = C(\gamma)\cos\gamma z + D(\gamma)\sin\gamma z$, $Y_\beta(y) = E(\beta)e^{\beta y} + F(\beta)e^{-\beta y}$. To have for $y \to \infty$ a bounded solution $E(\beta) = 0$ is necessary . Since $X_\alpha(0) = Z_\gamma(0) = 0$ we get $X_\alpha(x) = B(\alpha)\sin\alpha x$ and $Z_\gamma(z) = D(\gamma)\sin\gamma z$. Hence, the solution $u = u(x,y,z)$ of our problem can be sought in the form

$$u(x,y,z) = \frac{2}{\pi}\int_0^\infty\int_0^\infty G(\alpha,\gamma)e^{-\beta y}\sin\alpha x\sin\gamma z\,d\alpha\,d\gamma,$$

where $\beta = \sqrt{\alpha^2+\gamma^2}$ and $G(\alpha,\gamma) = B(\alpha)D(\gamma)E(\sqrt{\alpha^2+\gamma^2})$. Because of condition (3.1), for points outside the window in the (x,z)-plane, we have

$$\int_0^\infty\int_0^\infty G(\alpha,\gamma)\sin\alpha x\sin\gamma z\,d\alpha\,d\gamma = 0$$

and therefore also

$$\int_0^\infty\int_0^\infty G(\alpha,\gamma)\alpha\cos\alpha x\sin\gamma z\;d\alpha\,d\gamma = 0 \quad \text{and} \quad \int_0^\infty\int_0^\infty G(\alpha,\gamma)\gamma\cos\gamma z\sin\alpha x\;d\alpha\,d\gamma = 0.$$

Because the relation $\partial^2u/\partial x^2 + \partial^2u/\partial y^2 \equiv 0$ holds on $\mathbb{R}_y^{3,+}\setminus W$, it is possible to define a new unknown function h as

$$h(x,z) = \frac{2}{\pi}\int_0^\infty\int_0^\infty G(\alpha,\gamma)\big[\lambda_1(x,z)\alpha\cos\alpha x\sin\gamma z$$
$$-\lambda_2(x,z)\beta^2\sin\alpha x\sin\gamma z + \lambda_3(x,z)\gamma\sin\alpha x\cos\gamma z\big]d\alpha\,d\gamma.$$

It is clear that $h(x,z) \equiv 0$ on $\mathbb{R}_y^{3,+}\setminus W$. After inverting we get different representations for the function $G(\alpha,\gamma)$. In fact we obtain

$$\alpha\lambda_1(x,z)G(\alpha,\gamma) = \frac{2}{\pi}\iint_W h(t,s)\cos\alpha t\sin\gamma s\;dt\,ds \tag{3.2}$$

$$\gamma\lambda_3(x,z)G(\alpha,\gamma) = \frac{2}{\pi}\iint_W h(t,s)\sin\alpha t\cos\gamma s\;dt\,ds \tag{3.3}$$

$$-\beta^2\lambda_2(x,z)G(\alpha,\gamma) = \frac{2}{\pi}\iint_W h(t,s)\sin\alpha t\sin\gamma s\;dt\,ds. \tag{3.4}$$

Condition (3.1) leads to

$$\lim_{y\to 0^+} \frac{2}{\pi} \int_0^\infty \int_0^\infty G(\alpha,\gamma) e^{-\beta y} \big[\lambda_1(x,z)\alpha \cos\alpha x \sin\gamma z + \lambda_3(x,z)\gamma \sin\alpha x \cos\gamma z \\ -\lambda_2(x,z)\beta \sin\alpha x \sin\gamma z\big] d\alpha d\gamma = h(x,z) \quad \text{on} \quad W. \tag{3.5}$$

Substituting the expressions (3.2), (3.3), (3.4) into the equation (3.5) we get on W

$$\lim_{y\to 0^+} \frac{4}{\pi^2} \int_0^\infty \int_0^\infty e^{-\beta y} \quad \Big\{ \iint_W h(t,s) \cos\alpha t \sin\gamma s \, dtds \cos\alpha x \sin\gamma z \\ + \iint_W h(t,s) \sin\alpha t \cos\gamma s \, dtds \sin\alpha x \cos\gamma z \\ - \frac{1}{\beta} \iint_W h(t,s) \sin\alpha t \sin\gamma s \, dtds \sin\alpha x \sin\gamma z \Big\} d\alpha d\gamma = g(x,z).$$

Using properties of the sine and cosine function, we then have

$$\lim_{y\to 0^+} \frac{1}{\pi^2} \iint_W h(t,s) \\ \times\Big\{ \int_0^\infty \int_0^\infty e^{-\beta y} [\cos\alpha(t+x) + \cos\alpha(t-x)][\cos\gamma(s-z) - \cos\gamma(s+z)] d\alpha d\gamma \\ + \int_0^\infty \int_0^\infty \frac{e^{-\beta y}}{\beta} [\cos\alpha(t-x) - \cos\alpha(t+x)][\cos\gamma(s-z) - \cos\gamma(s+z)] d\alpha d\gamma + \\ + \int_0^\infty \int_0^\infty e^{-\beta y} [\cos\alpha(t-x) - \cos\alpha(t+x)][\cos\gamma(s+z) + \cos\gamma(s-z)] d\alpha d\gamma \Big\} dtds \\ = g(x,z).$$

It follows

$$\lim_{y\to 0^+} \frac{1}{\pi^2} \iint_W h(t,s) \int_0^\infty \int_0^\infty e^{-\beta y} \Big(2\cos\alpha(t-x)\cos\gamma(s-z) \\ + \frac{1}{\beta} \Big[\cos\alpha(t-x)\cos\gamma(s-z) + \cos\gamma(t+x)\cos\gamma(s+z) \\ - \cos\alpha(t+x)\cos\gamma(s-z) - \cos\alpha(t-x)\cos\gamma(s+z) \Big] \Big) d\alpha d\gamma dt ds = g(x,z). \tag{3.6}$$

By using Formula 3.914 in [3], i.e.

$$\int_0^\infty e^{-a\sqrt{b^2+x^2}} \cos cx dx = \frac{ab}{\sqrt{b^2+c^2}} K_1(b\sqrt{a^2+c^2}) \ (a>0),$$

we find that the first terms in equation (3.5) vanish. It remains to consider in (3.5)

$$\lim_{y\to 0^+}\frac{1}{\pi^2}\iint_W h(t,s)\int_o^\infty\int_o^\infty \frac{e^{-\sqrt{\alpha^2+\gamma^2}y}}{\sqrt{\alpha^2+\gamma^2}}\quad [\cos\alpha(t-x)-\cos\alpha(t+x)]$$
$$\times\quad [\cos\gamma(s-z)-\cos\gamma(s+z)]d\alpha d\gamma dtds.$$

Formula 3.961.2 in [3] yields

$$\int_o^\infty e^{-y\sqrt{\alpha^2+\gamma^2}}\frac{\cos(t\pm x)\alpha}{\sqrt{\alpha^2+\gamma^2}}d\beta = K_o(\alpha\sqrt{(t\pm x)^2+y^2}).$$

Therefore we get

$$\lim_{y\to 0^+}\frac{1}{\pi^2}\iint_W h(t,s)\int_o^\infty [K_o(\alpha\sqrt{(t-x)^2+y^2})-K_o(\alpha\sqrt{(t+x)^2+y^2})]$$
$$\times[\cos\gamma(s-z)-\cos\gamma(s+z)]d\gamma dtds = g(x,z).$$

From Formula 6.671.6 in [3] we get

$$\lim_{y\to 0^+}\frac{1}{4\pi}\iint_W h(t,s)\Big[\frac{1}{\sqrt{(s-z)^2+(t-x)^2+y^2}}+\frac{1}{\sqrt{(s+z)^2+(t+x)^2+y^2}}$$
$$-\frac{1}{\sqrt{(s-z)^2+(t+x)^2+y^2}}-\frac{1}{\sqrt{(s+z)^2+(t-x)^2+y^2}}\Big]dtds = g(x,z).$$

Providing the (x,z)-plane with a complex structure, setting $q=t+is, p=x+iz$ with $h(t,s)=\tilde{h}(q)$ and $g(x,z)=\tilde{g}(p)$, then it follows, for $p\in W$,

$$\frac{1}{4\pi}\iint_W \tilde{h}(q)\frac{1}{|p-q|}dtds+\frac{1}{4\pi}\iint_W \tilde{h}(q)\Big[\frac{1}{|p+q|}-\frac{1}{|\bar{p}+q|}-\frac{1}{|\bar{p}-q|}\Big]dtds=\tilde{g}(p), \tag{3.7}$$

which is a weakly singular integral equation due to the presence of the first integral. Having found the function $h=h(t,s)$ we can calculate from the relation $G(\alpha,\gamma)=\frac{-2}{\pi\lambda_2\beta^2}\iint_W h(t,s)\sin\alpha t\sin\gamma s dtds$ the required solution as

$$\begin{aligned} U(x,y,z) &= \frac{-4}{\pi^2\lambda_2}\int_o^\infty\int_o^\infty\iint_W h(t,s)\sin\alpha t\sin\gamma s\frac{e^{-\beta y}}{\beta^2}\sin\alpha x\sin\gamma z dtdsd\alpha d\gamma \qquad (3.8)\\ &= \frac{-4}{\pi^2\lambda_2}\iint_W h(t,s)\int_o^\infty\int_o^\infty\frac{e^{-\beta y}}{\beta^2}[\cos\alpha(t-x)-\cos\alpha(t+x)]\\ &\times\quad [\cos\gamma(s-z)-\cos\gamma(s+z)]\,d\alpha d\gamma dtds. \end{aligned}$$

In summary, we have shown the

Theorem 2. *Let $g\in C(W),\lambda_i\in C(W)$, $i=1,2,3$, then the potential window problem1 (3.1) allows the integral reformulation (3.7). In terms of the above suitably introduced new unknown h the solution of the original potential problem is given by (3.8).*

Remark. The treatment of more general elliptic problems by this method at present seems far from being easy. For example, the case of the Helmholtz equation leads to integrals which cannot be expressed in closed form.

4 Window problems under representation of Teodorescu's transform

Now we shall consider window problems over more general domains. The natural transform here to apply is Teodorescu's transform instead of the Fourier transform. We investigate the following 3-dimensional window problem.

Let be $G \subset I\!R^3$ a domain with Liapunov boundary Γ which has been splitted into parts Γ_o and Γ', $\Gamma = \overline{\Gamma}_o \cup \Gamma', \partial\Gamma_o = \Sigma_o$. Assume that Γ_o is flat. On Γ_o we fix the Cartesian coordinate system in such a way that the z-axis is directed in the direction of the outer normal to G in the points of Γ_o. In this way Γ_o is contained in the (x, y)-plane. Let be $W \subset \Gamma_o$ a simple connected smooth bounded open domain with $\mathrm{dist}(W, \Sigma_o) > 0$. Further let $W^c := \Gamma_o \setminus W$. We look for a vector function $u = u(x, y, z)$ which is harmonic in G and satisfies the boundary conditions

$$\begin{aligned} u(x,y,z) &= 0 \quad \text{on} \quad \Gamma' \\ u(x,y,0) &= 0 \quad \text{on} \quad \overline{W^c} \\ \frac{\partial u}{\partial z}(x,y,z) &= g(x,y) \quad \text{on} \quad W. \end{aligned} \tag{4.1}$$

Introduce the quaternionic units $1, i, j, k$ with $i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$. Each quaternion can be written in the form $a = a_o + ia_1 + ja_2 + +ka_3 = a_o + \underline{a}$. Assume $a_l i = ia_l, a_l j = ja_l, a_l k = ka_l, l = 1, 2, 3$. where $a_l \in I\!R, i = 0, 1, 2, 3$. Let $u = u(x, y, z) = u(p)$ be a quaternionic-valued function and $D = i\partial_x + j\partial_y + k\partial_z$ be a differential operator with the abbreviations $\partial_x = \frac{\partial}{\partial x}, \partial_y = \frac{\partial}{\partial y}, \partial_z = \frac{\partial}{\partial z}$. A sufficient smooth function u is called *H-regular* if $Du = 0$. The decomposition of the Laplacian $-\Delta = DD$ is easy to prove. The uniqueness of the problem can be briefly shown by the following

Lemma. *Let $u \in C^1(\overline{G}) \cap C^2(G)$ be harmonic in G and satisfy* (4.1). *Then u is uniquely given.*

Proof. For $u \in C^1(\overline{G}) \cap C^2(G)$ we have with $\frac{\partial u}{\partial z} = 0$ on W

$$\begin{aligned} \int_G |Du|^2 dG &= -\left(ij \int_G \partial_x u \partial_y u dG_p + jk \int_G \partial_y u \partial_z u dG_p + ki \int_G \partial_z u \partial_x u dG_p\right) \\ &= \int_G (\partial_x u \partial_x u + \partial_y u \partial_y u + \partial_z u \partial_z u) dG_p \end{aligned}$$

$$= \int_G u(-\Delta u)dG_p + \int_\Gamma u\frac{\partial u}{\partial n}d\Gamma = -\int_G u \Delta u dG_p \geq 0.$$

For $\Delta u = 0$ we have $|Du| = 0$ and so $\partial_x u = \partial_y u = \partial_z u = 0$. Therefore u =const. Because $u \equiv 0$ on Γ' it follows $u \equiv 0$ in $\overline{G}$ ∎

The integral transform

$$(T_G u)(p) = \frac{1}{4\pi}\int_G e(p-q)u(q)dG_q \quad \text{with} \quad e(p) = -\frac{(xi + yj + zk)}{(x^2+y^2+z^2)^{3/2}}$$

is called *Teodorescu transform*. It is the right inverse of D. We have $(DT_G u)(p) = u(p)$ in G. Because of $\Delta = -DD$ an arbitrary quaternionic harmonic function u admits the representation $u = \phi_1 + T_G\phi_2$ with $\phi_i \in \ker D$ for $i = 1, 2$. It follows immediately that $(T_G\phi)(p) = (4\pi)^{-1}\int_\Gamma \frac{n(q)}{|p-q|}\phi(q)d\Gamma_q$, where $n(q) = n_x i + n_y j + n_z k$. Furthermore, each H-regular function can be represented by a Cauchy-type integral, i.e. for $\phi \in \ker D$ we have $\phi(p) = (4\pi)^{-1}\int_\Gamma e(p-q)n(q)\phi(q)d\Gamma_q = (F_\Gamma\phi)(p)$. In order to get a singular integral equation for our window problem formulated above we set

$$u(x,y,z) = zk\phi_1(x,y,z) + (T_G\phi_2)(x,y,z)$$

with the new unknown functions $\phi_i \in \ker D, i = 1, 2$. At first we have to prove that $zk\phi_1 + T_G\phi_2 \in \ker \Delta$. This is shown if we can find H-regular functions ϕ_1' and ϕ_2' such that

$$zk\phi_1 + T_G\phi_2 = \phi_1' + T_G\phi_2'. \tag{4.2}$$

Since $D(zk\phi_1) = -\phi_1$ we need to take $\phi_2' = \phi_2 - \phi_1$ and $\phi_1' = F_\Gamma(zk\phi_1)$. It is well known that $(F_\Gamma u) \in \ker D$. In this way we have shown that u is harmonic. It is also clear that together with ϕ the H-valued function $\partial\phi/\partial z$ belongs to $\ker D(G) \cap C(\overline{G})$, too. For points on the window W we have

$$\lim_{z\to 0^+}(\partial_z u)(p) = \lim_{z\to 0^+}\partial_z(zk\phi_1)(p) + \lim_{z\to 0^+}\frac{1}{4\pi}\partial_z\int_\Gamma \frac{n(q)}{|p-q|}\phi_2(q)\ d\Gamma_q.$$

Because of the boundedness of $\partial\phi_1/\partial z$ in $\overline{G}$ we obtain

$$\lim_{z\to 0^+}(\partial_z kz\phi_1)(p) = \lim_{z\to 0^+}(k\phi_1 + zk\frac{\partial\phi_1}{\partial z})(p) = k\phi_1(p_o), p_o = (x, y, 0).$$

On the other hand we have on W

$$\lim_{z\to 0^+}\frac{1}{4\pi}\partial_z\int_\Gamma \frac{n(q)}{|p-q|}\phi_2(q)d\Gamma_q = -\frac{1}{2}\phi_2(p_o)k + \frac{1}{4\pi}\int_{\Gamma'}\frac{t}{|p_o-q|}n(q)\phi_2(q)d\Gamma_q'.$$

Hence

$$k(\phi_1(p_o) - \frac{1}{2}\phi_2(p_o)) + \frac{1}{4\pi}\int_{\Gamma'}\frac{t}{|p_o-q|^3}n(q)\phi_2(q)d\Gamma_q' = g(p_o), p_o \in W. \tag{4.3}$$

The corresponding equation on W^c reads as

$$\phi_1(p) + \frac{k}{4\pi}\int_W \frac{1}{|p-q|}\phi_2(q)d\Gamma_q + \frac{1}{4\pi}\int_{W^c} \frac{n(q)}{|p-q|}\phi_2(q)d\Gamma_q = 0, p \in W^c. \tag{4.4}$$

Theorem 3. *Let $g \in C(W)$. Then the potential window problem with the boundary conditions* (4.1) *can be reformulated via the integral equations* (4.3)-(4.4).

Remark. In the system of integral equations (4.3)-(4.4) no singular integrals over the window occur. The reason for this is that ϕ_i are H-regular functions and therefore the identity

$$\phi_i(p) = \frac{1}{2\pi}\int_\Gamma e(p-q)n(q)\phi_i(q)d\Gamma_q, \qquad i = 1,2$$

holds. This means that by the introduction of H-regular unknown functions singular integrals vanish.

What changes in the equations (4.3)-(4.4) do arise if we substitute ∂_z by the oblique derivative ∂_λ with $\partial_\lambda := \lambda_1(p)\partial_x + \lambda_2(p)\partial_y + \lambda_3(p)\partial_z$? First of all we can state that equation (4.4) is the same as before. Instead of equation (4.3) we have then on W

$$\lambda_3(p_o)k(\phi_1(p_o) - \tfrac{1}{2}\phi_2(p_o))$$
$$+\frac{1}{4\pi}\int_\Gamma \frac{\lambda_1(p_o)(\beta - x) + \lambda_2(p_o)(s-y) + \lambda_3(p_o)(t-z)}{|p-q|^3}n(q)\phi_2(q)d\Gamma_q = g(p_o). \tag{4.3'}$$

We see that in case of an oblique derivative the presence of singular integrals is in general unavoidable. Only very special assumptions similar to those of Section 3 can lead to integral equations without singular integrals.

5 Window problems of equations of linear elasticity

In hypercomplex notation the homogeneous equations of linear elasticity read

$$DM^{-1}Du = 0 \tag{5.1}$$

with the boundary conditions

$$u = 0 \quad on \quad W^c, \qquad Bu = g \quad on \quad W, \tag{5.2}$$

where $D = \partial_x i + \partial_y j + \partial_z k, Mu = \frac{1}{2}\frac{m-2}{m-1}u_o + \underline{u}$, m is the Poisson number and B is a boundary operator which is acting on the window. It has the form

$$B\underline{u} = \partial_z\underline{u} + \frac{1}{2}k \times \text{rot}\,\underline{u} + k\frac{1}{m-2}\,\text{div}\,\underline{u} = \underline{g}(p), Bu_o = tr_\Gamma u_o = 0.$$

In quaternionic notation the operator B can be written as $B\cdot = \text{Im}\,(\partial_z' - kND)\cdot$ where $Nu = -\frac{1}{m-2}u_o + \frac{1}{2}\underline{u}$. Proceeding in a similar way as done for the corresponding representation in the previous section each solution of the homogeneous equation of linear elasticity can be given by

$$u(p) = zkM\phi_1(p) + (T_G M\phi_2)(p), \qquad \phi_i \in \ker D \qquad (i = 1,2)\ , \tag{5.3}$$

where the first component of the H-regular function ϕ_2 is constant. It is easy to see that (5.3) has the equivalent representation

$$u(p) = (zkM\phi_1)(p) + \frac{1}{4\pi}\int_\Gamma \frac{n(q)}{|p-q|}M\phi_2(q)d\Gamma_q. \tag{5.4}$$

Writing in short

$$V_n(M\phi_2) := \frac{1}{4\pi}\int_\Gamma \frac{n(q)}{|p-q|}M\phi_2(q)d\Gamma_q$$

we have to calculate the limits

(i) $\lim_{s\to 0^+} \operatorname{Im}\,(\partial_s - kND)(zkM\phi_1)(p) = I_1(p_o)$

(ii) $\lim_{s\to 0^+} \operatorname{Im}\,(\partial_s - kND)V_n(M\phi_2)(p) = I_2(p_o)$.

We find immediately

$$I_1(p_o) = \operatorname{Im}\, kM\phi_1(p_o) - \operatorname{Im}\, kNM(-\phi_1)(p_o) = \operatorname{Im}\, k(I+N)M\phi_1(p_o).$$

Furthermore by a straightforward calculation we have

$$\begin{aligned} I_2(p_o) = {} & \operatorname{Im}\,(-\frac{k}{2})\Big((I+Nk)M\phi_2(p_o) + \frac{N}{2\pi}\int_W e(p_o-q)kM\phi_2(q)d\Gamma_q \\ & -\frac{1}{2\pi}\int_{\Gamma'} \frac{t}{|p_o-q|^3}M\phi_2(q)d\Gamma_q + \frac{N}{2\pi}\int_{W^*} e(p_o-q)n(q)M\phi_2(q)d\Gamma_q\Big). \end{aligned}$$

Finally we get the singular integral equation over the window W

$$\begin{aligned} \underline{g}(p_o) = {} & \operatorname{Im}\, k\Big[(I+N)M\phi_1 - \frac{1}{2}(I+Nk)M\phi_2 + \frac{1}{4\pi}\int_{\Gamma'} \frac{t}{|p_o-q|}(M\phi_2)(q)d\Gamma_q \\ & - \frac{N}{4\pi}\int_{W_*} e(p_o-q)n(q)(M\phi_2)(q)d\Gamma_q - \frac{N}{4\pi}\int_W e(p_o-q)kM\phi_2(q)d\Gamma_q\Big]. \end{aligned} \tag{5.5}$$

The third integral is a two-dimensional singular integral of Cauchy's type over the window. On W^c we obtain the weak singular equation

$$0 = (zkM\phi_1)(p) + \frac{1}{4\pi}\int_\Gamma \frac{n(q)}{|p-q|}M\phi_2(q)d\Gamma_q, p \in W^c. \tag{5.6}$$

In this way we proved

Theorem 4. *Let $g \in C(W)$. Then the window problem (5.1)-(5.2) allows the integral reformulation (5.5)-(5.6).*

Remark. For our transformations we need no further restrictions on the right-hand side of the equation, i.e. on the boundary condition over the window, other than continuity. If this is not the case, the present analytical derivation would not be affected, but problems in the numerical solution are likely to arise. However we do not discuss this issue in this paper.

Remark. The advantage of the derived integral equations consists in the possibility of constructing a suitable numerical scheme as shortly remarked in Section 6.

6 Some remarks for determining unknown H-regular functions

In the papers [5,6] the authors have developed the method of collocation with two H-regular functions for the solution of systems of elliptic partial differential equations of the second order. We apply this method for solving the window problems discussed here. Let us start with the problem for the Laplace equation. Using this procedure we set in (4.1)

$$\phi_2'(p) = \sum_{i=1}^{N_2} e(p - p^{(i)})a_i, a_i \in H,$$

where H denotes the skew field of quaternions and the points $p^{(i)}$ are outside of the domain G parallel to the boundary Γ on a surface $\tilde{\Gamma}$. Because of the relation

$$T_G\Big(\frac{p - p^{(i)}}{|p - p^{(i)}|^3}\Big) = \frac{-1}{|p - p^{(i)}|} + \phi_G, \phi_G \in \ker D, \quad i = 1, ..., N_2 \ , \tag{6.1}$$

we get for the approximate solution of the window problem

$$u_{N_1N_2}(p) = \sum_{j=1}^{N_1} e(p - p^{(j)})bj + \sum_{i=1}^{N_2} \frac{1}{|p^{(i)} - p|}a_i,$$

where $\phi_1'(p) + \phi_G(p) = \sum_{j=1}^{N_1} e(p - p^{(j)})bj$. Setting the collocation points $p^{(k)}$ on W^c we have the algebraic equations

$$0 = \sum_{j=1}^{N_1} e(p^{(k)} - p^{(j)})bj + \sum_{i=1}^{N_2} \frac{1}{|p^{(i)} - p^{(k)}|}a_i, \qquad k = 1, ..., N_1. \tag{6.2}$$

Furthermore we get on the window W

$$\begin{aligned} g(p_o) &= \lim_{z \to 0+} \Big(\sum_{j=1}^{N_1} \theta_z e(p - p^{(j)})bj + \sum_{i=1}^{N_2} \frac{z - z^{(i)}}{|p - p^{(i)}|^3}a_i\Big) \\ &= \sum_{j=1}^{N_1} \theta_z e(p_o - p^{(j)})bj + \sum_{i=1}^{N_2} \frac{-z^{(i)}}{|p_o - p^{(i)}|^3}a_i \end{aligned}$$

where $p_o = (x, y, 0)$. Now we choose the collocation points on the window and obtain

$$g(p_o^{(k)}) = \sum_{j=1}^{N_1} \theta_z e(p_o^{(k)} - p^{(j)})bj + \sum_{i=1}^{N_2} \frac{(-z^{(i)})}{|p_o^{(k)} - p^{(i)}|^3}a_i, \qquad k = N_1 + 1, ..., N_2. \tag{6.3}$$

Numerical considerations on this method are given in [4].

The determination of the unknown H-regular functions in case of the equations of linear elasticity is obtained in the same way, instead of formula (6.1) we have only to take

$$T_G\left(\frac{s^{(i)}-s}{|p^{(i)}-p|^3}e_s\right) = \frac{1}{2}\frac{1}{|p^{(i)}-p|} - \frac{1}{2}e(p^{(i)}-p)(s^{(i)}-s)e_s + \phi_{G,s}$$

where $s \in \{x, y, z\}$ and $e_x = i, e_y = j, e_z = k$.

Remark. For the numerical treatment of singular equations over one-dimensional windows we refer to the papers [7,9,11-13]. For the multidimensional case one can find results for instance in the paper [11].

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