Zeitschrift für Analysis
und ihre Anwendungen
Vol. 12 (1993), 629 - 642

On the Volume Infimum for Liquid Bridges

L. ZHOU

We are concerned with the stable liquid bridges joining two parallel homogeneous plates in the absence of gravity following the recent work of Finn and Vogel [7]. We prove the Carter conjecture in the general case where the two contact angles can be different.

Key words: *Capillarity, mean curvature, liquid bridges*
AMS subject classification: 76B45, 53A10, 49Q10

1. Introduction

In [11], Vogel derived a stability criterion for the liquid bridges joining two parallel homogeneous plates. In the same paper, he applied his criterion to the special case of contact angles equal to $\pi/2$ and showed that in this case the equilibrium configuration is stable if and only if its volume is no less than h^3/π, where h denotes the separation distance of the two plates. Carter [2] made extensive numerical stability calculations on liquid bridge configurations for different choices of contact angle γ and observed that $V \geq h^3/\pi$. Recently, Finn and Vogel [7] gave a rigorous proof to this "conjecture" for the case of equal contact angles on both plates. In this paper, we extend the result of Finn and Vogel and prove that the Carter conjecture is also true for the general case in which the two contact angles γ_1 and γ_2 might be different. We show that $V \geq C(\gamma_1, \gamma_2)h^3/\pi$, where the constant $C(\gamma_1, \gamma_2)$ is no less than one, but could be significantly larger than 1. This result agrees with the numerical evidences that instability occurs at larger volume when the two contact angles differ significantly.

Suppose that a stable connected liquid bridge is formed joining two parallel homogeneous plates located at $u = 0$ and $u = h$ in the absence of gravity. We would like to obtain a lower bound for its volume.

The surface of any equilibrium liquid bridge is rotationally symmetric and has constant mean curvature. Its meridian (profile) curve $r = f(u)$ satisfies the Euler–Lagrange equation

$$-\frac{1}{2}\left(\frac{f''}{(1+(f')^2)^{3/2}} - \frac{1}{f\,(1+(f')^2)^{1/2}}\right) = H \tag{1}$$

Lianmin Zhou: Stanford University, Mathematics Department, Stanford, California 94305, USA.
This work is partially supported by NASA, NAG3 1143.

ISSN 0232-2064 / $ 2.50

and the boundary conditions

$$f'(0) = -\cot\gamma_1, \qquad f'(h) = \cot\gamma_2 \tag{2}$$

where H denotes the mean curvature of the surface and γ_1, γ_2 are the contact angles that the surface makes with the two plates, respectively. See Vogel [11].

In this paper, we will use the following equivalent but more convenient form of the above Euler–Lagrange equation:

$$(r\sin\psi)_r = 2rH \tag{3}$$

where ψ denotes the angle between the r-axis and a tangent to the curve $r = f(u)$. See Finn and Vogel [7].

The solution of (3) is known to be an unduloid, a nodoid or a catenoid depending on the mean curvature H. We will discuss each of those three cases separately. Here we summarize the results in the following:

1. If the meridian curve is a catenoid or a nodoid, then

$$V > \left(\frac{\pi}{\pi - \gamma_1 - \gamma_2}\right)^2 \frac{h^3}{\pi}.$$

2. If the meridian curve is an unduloid and contains an inflection point, then

$$V > \left(\frac{\pi}{\pi - |\gamma_2 - \gamma_1|}\right)^2 \frac{h^3}{\pi}.$$

3. If the meridian curve is an unduloid and contains no inflection point, then

$$V > \left(\frac{\pi}{\frac{\pi}{2} + \sin^{-1}\left(\frac{\cos\gamma_1}{\cos\gamma_2}\right)}\right)^2 \frac{h^3}{\pi}.$$

Here and throughout this paper, h always denotes the separation distance of the two parallel plates and V the volume of the liquid bridge between them. γ_1 and γ_2 are the contact angles that the surface makes with the two plates.

2. Catenoids

When the mean curvature H is 0, the solution of equation (3) is the catenoid. We then observe the following: Given a fixed separation h of the two plates, the volume bounded by the catenoid between the plates is minimum when the two plates are placed symmetrically about its vertex. In other words, the volume is minimized when the contact angles on two plates are equal. Therefore it follows immediately from [7] that $V > h^3/\pi$. In the

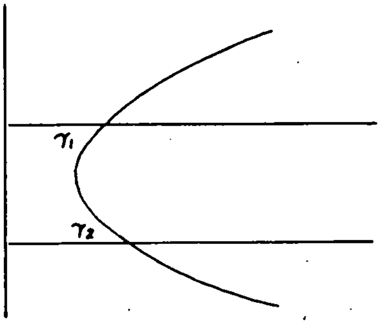

Figure 1. Catenoid

following, we obtain a more precise lower bound for the volume that depends on the two contact angles γ_1 and γ_2. This lower bound tends to the true volume of the liquid drop as both γ_1 and γ_2 tend to $\pi/2$. By the way, an exact expression of the volume V in terms of h can be obtained in this case, but is complicated.

Theorem 1. *Let $r = f(u), 0 \le u \le h$ be the catenoid that makes the contact angles γ_1 and γ_2 with the plates, where $0 < \gamma_1 \le \pi/2$, $0 < \gamma_2 < \pi - \gamma_1$. Then*

$$V > \left(\frac{\pi}{\pi - \gamma_1 - \gamma_2}\right)^2 \frac{h^3}{\pi}.$$

Proof: When $H = 0$, the equation (3) above becomes $(r \sin\psi)_r = 0$, from which it follows that

$$r = \frac{a}{\sin\psi} \quad \text{and} \quad du = (\tan\psi)dr = -\frac{a}{\sin\psi}d\psi \tag{4}$$

where $0 < \psi < \pi$ and the positive constant a is the r-coordinate of the vertex of the catenoid (see Figure 1). Integrating (4) from one plate to the other, we obtain

$$h \equiv \int_0^h du = \int_{\gamma_1}^{\pi-\gamma_2} \frac{a}{\sin\psi}\, d\psi \quad \text{and} \quad V \equiv \int_0^h \pi r^2 du = \int_{\gamma_1}^{\pi-\gamma_2} \pi \left(\frac{a}{\sin\psi}\right)^3 d\psi.$$

By Jensen's inequality, it follows that

$$C(\gamma_1,\gamma_2) \equiv \frac{V}{h^3/\pi} = \pi^2 \left(\int_{\gamma_1}^{\pi-\gamma_2} \frac{1}{\sin^3\psi}\, d\psi\right) \Big/ \left(\int_{\gamma_1}^{\pi-\gamma_2} \frac{1}{\sin\psi}\, d\psi\right)^3$$

$$= \left(\frac{\pi}{\pi-\gamma_1-\gamma_2}\right)^2 \left(\int_{\gamma_1}^{\pi-\gamma_2} \frac{1}{\sin^3\psi}\, d\mu(\psi)\right) \Big/ \left(\int_{\gamma_1}^{\pi-\gamma_2} \frac{1}{\sin\psi}\, d\mu(\psi)\right)^3$$

$$> \left(\frac{\pi}{\pi-\gamma_1-\gamma_2}\right)^2,$$

where $d\mu(\psi) = d\psi/(\pi - \gamma_1 - \gamma_2)$ ∎

3. Nodoids

If the mean curvature H is negative, the solutions of (3) are the nodoids. See Figure 2a and Figure 2b below.

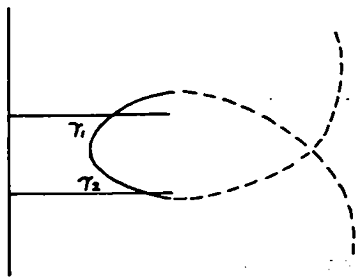

Figure 2a. The inner loop

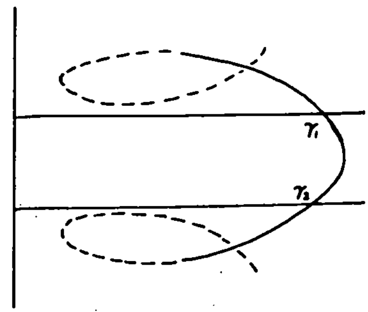

Figure 2b. The outer loop

It is shown in [7] that a catenoid can be inscribed by the inner loop of the nodoid tangent at their common vertex. In other words, if the inner loop of a nodoid and a catenoid are put one onto the other so that they are tangent to each other at the common vertex, then the nodoid is always on the right side of the catenoid. This means that the volume bounded by the nodoid is larger. Therefore it follows immediately from the previous case of catenoids that the volume V bounded by the nodoid is greater than h^3/π.

It is also shown in [7] that all outer loops of nodoids with a fixed span d (the distance from one horizontal point to the next of the outer loop) lie outside the semicircle of diameter d. From the direct calculation for the semi-sphere, we can easily show that $V > h^3/\pi$. See [7] for more details. In Theorem 2 below, we are able to obtain a lower bound more precise than the one that we have just obtained in above by simple reduction.

Theorem 2. *Suppose that $r = f(u)$, $0 \le u \le h$ is the part of a nodiod between the two plates. Let γ_1 and γ_2 denote the contact angles with plates.*

(i) In the case of the inner loop (Figure 2a) for which $0 < \gamma_1 \le \pi/2$ and $0 < \gamma_2 < \pi - \gamma_1$, we have

$$V > \left(\frac{\pi}{\pi - \gamma_1 - \gamma_2}\right)^2 \frac{h^3}{\pi}.$$

(ii) In the case of the outer loop (Figure 2b) for which $\pi/2 \le \gamma_1 < \pi$ and $\pi - \gamma_1 < \gamma_2 < \pi$, we have

$$V > \left(\frac{\pi}{\gamma_1 + \gamma_2 - \pi}\right)^2 \frac{h^3}{\pi}.$$

Proof: (i) First consider the case of inner loops. We solve equation (3) and obtain

$$r = a\left(\frac{-k\sin\psi}{1-k} + \frac{\sqrt{1-k^2\cos^2\psi}}{1-k}\right) \tag{5}$$

$$du \equiv \tan\psi dr = -a\left(\frac{-k\sin\psi}{1-k} + \frac{\sqrt{1-k^2\cos^2\psi}}{1-k}\right)\frac{k\sin\psi}{\sqrt{1-k^2\cos^2\psi}}d\psi \tag{6}$$

with $k = (c^2 - a^2)/(c^2 + a^2)$ and $0 \le \psi \le \pi$, where a and c are the r-coordinates of the (inner) vertex and a horizontal point of the inner loop, respectively. Using (5) and (6) above, we easily obtain

$$h = \int_{\gamma_1}^{\pi-\gamma_2} a\left(\frac{-k\sin\psi}{1-k} + \frac{\sqrt{1-k^2\cos^2\psi}}{1-k}\right)\frac{k\sin\psi}{\sqrt{1-k^2\cos^2\psi}}d\psi$$

$$V = \int_{\gamma_1}^{\pi-\gamma_2} \pi a^3\left(\frac{-k\sin\psi}{1-k} + \frac{\sqrt{1-k^2\cos^2\psi}}{1-k}\right)^3\frac{k\sin\psi}{\sqrt{1-k^2\cos^2\psi}}d\psi.$$

Setting $d\mu(\psi) \equiv k\sin\psi d\psi/\sqrt{1-k^2\cos^2\psi}$, we calculate

$$\begin{aligned}\lambda \equiv \mu((\gamma_1, \pi-\gamma_2)) &= \int_{\gamma_1}^{\pi-\gamma_2}\frac{k\sin\psi}{\sqrt{1-k^2\cos^2\psi}}d\psi \\ &\le \int_{\gamma_1}^{\pi-\gamma_2}\frac{\sin\psi}{\sqrt{1-\cos^2\psi}}d\psi = \pi-\gamma_1-\gamma_2.\end{aligned}$$

By Jensen's inequality, it follows

$$\begin{aligned}&\int_{\gamma_1}^{\pi-\gamma_2} \pi a^3\left(\frac{-k\sin\psi}{1-k} + \frac{\sqrt{1-k^2\cos^2\psi}}{1-k}\right)^3\frac{k\sin\psi}{\lambda\sqrt{1-k^2\cos^2\psi}}d\psi \\ &> \left[\int_{\gamma_1}^{\pi-\gamma_2} a\left(\frac{-k\sin\psi}{1-k} + \frac{\sqrt{1-k^2\cos^2\psi}}{1-k}\right)\frac{k\sin\psi}{\lambda\sqrt{1-k^2\cos^2\psi}}d\psi\right]^3.\end{aligned}$$

Therefore

$$C(\gamma_1,\gamma_2) \equiv \frac{V}{h^3/\pi} > \frac{\pi^2}{\lambda^2} \ge \left(\frac{\pi}{\pi-\gamma_1-\gamma_2}\right)^2.$$

(ii) Now we consider the outer loop of a nodoid. The proof goes exactly as above with (7) and (8) below in place of (5) and (6) in the above proof:

$$r = b\left(\frac{-k\sin\psi}{1+k} + \frac{\sqrt{1-k^2\cos^2\psi}}{1+k}\right) \tag{7}$$

$$du \equiv \tan\psi dr = -b\left(\frac{-k\sin\psi}{1+k} + \frac{\sqrt{1-k^2\cos^2\psi}}{1+k}\right)\frac{k\sin\psi}{\sqrt{1-k^2\cos^2\psi}}d\psi \tag{8}$$

with $-\pi \le \psi \le 0$ and $k = (b^2 - c^2)/(b^2 + c^2)$, where $r = b$ and $r = c$ are the positions of the (outer) vertex and the horizontal points of the outer loop, respectively. ∎

4. Unduloids

When the mean curvature H is positive, the solutions of equation (3) are unduloids. This is a periodic curve and each period has exactly two inflections and two vertices. See Figure 3 below. For the sake of convenience, we can think of the arc segment from one inflection to the next as *half a period*. Similarly, the arc segment from one vertex to the next is also considered to be half a period. In general, one starts with any point $(u_1, f(u_1))$ on the unduloid $r = f(u)$, $-\infty \leq u \leq \infty$ and traces in either direction along the curve until one reaches the first point $(u_2, f(u_2))$ satisfying $f'(u_2) = -f'(u_1)$. The arc segment so traced is considered to be half a period. It is clear that if the two plates were placed at the two endpoints of half-a-period that we just defined above, they would make the equal contact angle with the curve.

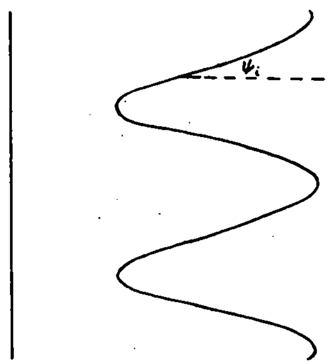

Figure 3. Unduloid

We remark that one must not expect Carter's inequality to be true if the two plates are so far apart as to include several period intervals. As a matter of fact, we will see very soon that the inequality is false in general if the two plates bound slightly more than half a period. See Theorem 5 below and the remark preceding it.

Each period of the unduloid consists of two parts: a convex (thick) portion and a concave (thin) one. The convex portion of the unduloid that starts from an inflection to the next is symmetric about the vertical point in the middle and can be described by the following explicit expression:

$$r = \frac{\sin\psi + \sqrt{k^2 - \cos^2\psi}}{2H} \tag{9}$$

$$du \equiv \tan\psi dr = \left(\frac{\sin\psi + \sqrt{k^2 - \cos^2\psi}}{2H}\right) \frac{\sin\psi}{\sqrt{k^2 - \cos^2\psi}}\, d\psi \tag{10}$$

where $\psi_i \leq \psi < \pi - \psi_i$ and ψ_i with $0 < \psi_i < \pi/2$, always denotes the inclination angle at an inflection throughout the paper (see [7]). In the above expressions, $k = \cos\psi_i$ is the eccentricity of the ellipse that generates the unduloid.

The concave part of the unduloid that starts from the inflection point at which the convex part ends to the next inflection point is also symmetric about the vertical point in the middle and has the following slightly different expression:

$$r = \frac{\sin\psi - \sqrt{k^2 - \cos^2\psi}}{2H} \tag{11}$$

$$du \equiv \tan\psi dr = -\frac{\sin\psi - \sqrt{k^2 - \cos^2\psi}}{2H}\frac{\sin\psi d\psi}{\sqrt{k^2 - \cos^2\psi}}, \tag{12}$$

where $\psi_i \leq \psi < \pi - \psi_i$, as before.

We mention that as k tends to 0, the limiting curve of the above parametrized family of unduloids is a straight line. Therefore, the limiting configuration of a liquid drop is a circular cylinder as $k \to 0$.

Recall that for the cases of catenoids and nodoids, $V > h^3/\pi$ could be proved by simple reduction using the results of Finn and Vogel [7]. But there is no such simple reduction in the case of unduloids.

We also note that the expressions for the constants $C(\gamma_1, \gamma_2)$ are different from those in the cases of catenoids and nodoids due to the presence of the inflection points. For unduloids, the contact angles of the two plates impose some restriction on the parameter k, whereas there is no such restriction for catenoids and nodoids. In a certain sense, the parameter α, which is determined via the relation $\cos\psi = k\sin\alpha$, is the more "correct" parameter than ψ. Unfortunately, α does not have a nice geometric interpretation as ψ. Although it converts the changing angles of inclination at the inflection points into the fixed angle 0, unfortunately it transforms the prescribed contact angles into two changing angles that depend on k.

In Theorem 3 below, we discuss the configurations in which the two plates bound no inflection points. An alternative case in which the two plates bound exactly one inflection point will be discussed in Theorem 4. When the two plates bound more than one inflection point, we will show in Thorem 6 that the configurations must not be stable.

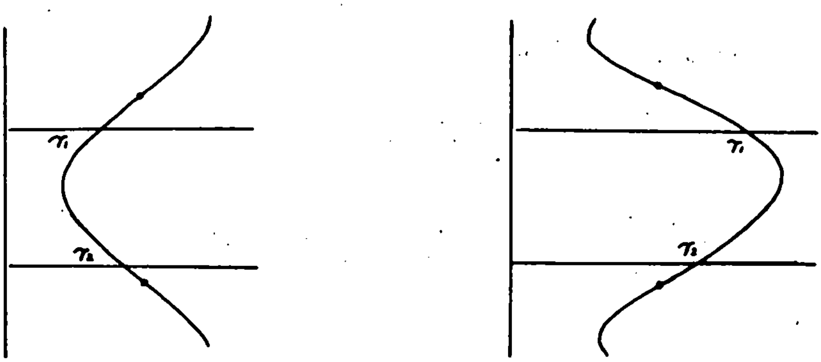

Figure 4. The plates do not bound an inflection point

Theorem 3. *Suppose that $r = f(u)$, $0 \le u \le h$ is the portion of an unduloid between the two plates. Let γ_1 and γ_2 be the contact angles on both plates. There are two totally different configurations as follows:*

(i) Concave drops: $\psi_i \le \gamma_2 \le \pi/2$ and $\gamma_2 \le \gamma_1 < \pi - \gamma_2$ (the left graph of Fig. 4),

(ii) Convex drops: $\pi/2 \le \gamma_2 \le \pi - \psi_i$ and $\pi - \gamma_2 < \gamma_1 \le \gamma_2$ (the right graph of Fig. 4).

In both configurations, we have

$$V > \left[\frac{\pi}{\frac{\pi}{2} + \sin^{-1}\left(\frac{\cos\gamma_1}{\cos\gamma_2}\right)}\right]^2 \frac{h^3}{\pi}. \tag{13}$$

Remark: The concave and convex drops correspond to the thin and thick cases in [7], respectively.

Proof of Theorem 3: (i) We first consider the concave (thin) case. The reader is refered to the first graph of Figure 4. We start the proof with (11) and (12) above. Integrating from one plate to the other yields

$$h \equiv \int_0^h du = \int_{\gamma_1}^{\pi-\gamma_2} \left(\frac{\sin\psi - \sqrt{k^2 - \cos^2\psi}}{2H}\right) \frac{\sin\psi}{\sqrt{k^2 - \cos^2\psi}}\, d\psi$$

$$V \equiv \int_0^h \pi r^2 du = \int_{\gamma_1}^{\pi-\gamma_2} \pi \left(\frac{\sin\psi - \sqrt{k^2 - \cos^2\psi}}{2H}\right)^3 \frac{\sin\psi}{\sqrt{k^2 - \cos^2\psi}}\, d\psi.$$

Setting $d\nu(\psi) \equiv \sin\psi d\psi/\sqrt{k^2 - \cos^2\psi}$. Since $k = \cos\psi_i \ge \cos\gamma_2$, then

$$\begin{aligned}
\lambda &\equiv \nu((\gamma_1, \pi - \gamma_2)) \\
&= \int_{\gamma_1}^{\pi-\gamma_2} \frac{\sin\psi}{\sqrt{k^2 - \cos^2\psi}} d\psi < \int_{\gamma_1}^{\pi-\gamma_2} \frac{\sin\psi}{\sqrt{\cos^2\gamma_2 - \cos^2\psi}} d\psi \\
&= \sin^{-1}\left(\frac{\cos\gamma_1}{\cos\gamma_2}\right) - \sin^{-1}\left(\frac{\cos(\pi - \gamma_2)}{\cos\gamma_2}\right) = \frac{\pi}{2} + \sin^{-1}\left(\frac{\cos\gamma_1}{\cos\gamma_2}\right).
\end{aligned}$$

By Jensen's inequality, we obtain

$$\begin{aligned}
&\int_{\gamma_1}^{\pi-\gamma_2} \left(\frac{\sin\psi - \sqrt{k^2 - \cos^2\psi}}{2H}\right)^3 \frac{\sin\psi}{\lambda\sqrt{k^2 - \cos^2\psi}} d\psi \\
&> \left[\int_{\gamma_1}^{\pi-\gamma_2} \left(\frac{\sin\psi - \sqrt{k^2 - \cos^2\psi}}{2H}\right) \frac{\sin\psi}{\lambda\sqrt{k^2 - \cos^2\psi}} d\psi\right]^3.
\end{aligned}$$

Therefore

$$C(\gamma_1, \gamma_2) \equiv \frac{V}{h^3/\pi} > \frac{\pi^2}{\lambda^2} \ge \frac{\pi^2}{\left[\pi/2 + \sin^{-1}\left(\frac{\cos\gamma_1}{\cos\gamma_2}\right)\right]^2}$$

which completes the proof of (i) of Theorem 3.

The proof of (ii) of Theorem 3 is the same except for some minor changes as follows. In this thick case (see the second graph of Figure 4), (9) and (10) are used in place of (11) and (12) to obtain

$$h = \int_{\pi-\gamma_2}^{\gamma_1} \left(\frac{\sin\psi + \sqrt{k^2 - \cos^2\psi}}{2H} \right) \frac{\sin\psi}{\sqrt{k^2 - \cos^2\psi}}\, d\psi$$

$$V = \int_{\pi-\gamma_2}^{\gamma_1} \pi \left(\frac{\sin\psi + \sqrt{k^2 - \cos^2\psi}}{2H} \right)^3 \frac{\sin\psi}{\sqrt{k^2 - \cos^2\psi}}\, d\psi.$$

And the measure ν will be defined as

$$d\nu(\psi) \equiv \frac{\sin\psi d\psi}{\sqrt{k^2 - \cos^2\psi}}.$$

Since $k = \cos\psi_i \geq -\cos\gamma_2$, then

$$\nu\{(\pi-\gamma_2, \gamma_1)\} = \int_{\pi-\gamma_2}^{\gamma_1} \frac{\sin\psi}{\sqrt{k^2 - \cos^2\psi}} d\psi \leq \int_{\pi-\gamma_2}^{\gamma_1} \frac{\sin\psi}{\sqrt{\cos^2\gamma_2 - \cos^2\psi}} d\psi$$

$$= \sin^{-1}\left(\frac{\cos(\pi-\gamma_2)}{-\cos\gamma_2} \right) - \sin^{-1}\left(\frac{\cos\gamma_1}{-\cos\gamma_2} \right) = \frac{\pi}{2} + \sin^{-1}\left(\frac{\cos\gamma_1}{\cos\gamma_2} \right).$$

We can now complete the proof by first normalizing the measure ν and then applying Jensen's inequality, as we did before ∎

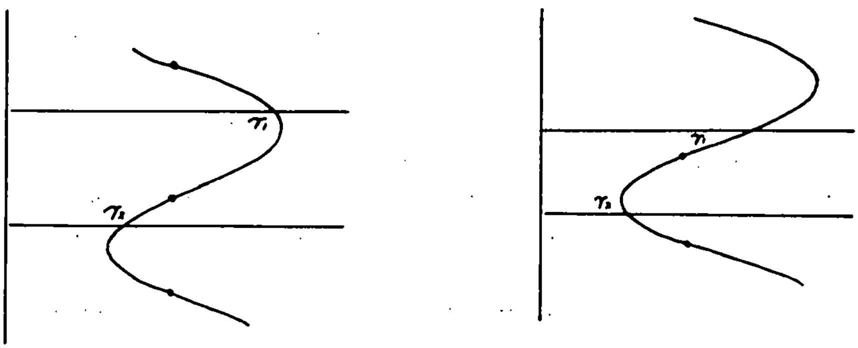

Figure 5. The plates bound an inflection point

Theorem 4. *Suppose that the plates bound an inflection point in between and make the contact angles γ_1, γ_2 with the surface. We distinguish the following two cases:*

(i) $\pi/2 \leq \gamma_2 \leq \pi - \psi_i$ and $\psi_i \leq \gamma_1 \leq \gamma_2$ as in the first graph of Figure 5,

(ii) $\psi_i \leq \gamma_1 \leq \pi/2$ and $\gamma_1 \leq \gamma_2 \leq \pi - \psi_i$ as in the second graph of Figure 5.
In both cases, we have

$$V > \left[\frac{\pi}{\pi - (\gamma_2 - \gamma_1)}\right]^2 \frac{h^3}{\pi}. \tag{14}$$

Proof: We only consider the left graph of Figure 5 and prove Case (i). The other case could be proved in the same way. Note that the two plates could bound a vertical point in between as shown in the graph, but it is not necessarily so. In other words, both plates could lie between two vertival points and hence bound an inflection point but no vertical points.

Again, we start the proof with (9)-(12). Integrating first from one of the plates to the inflection point and then from the inflection point to the other plate, we obtain

$$\begin{aligned} h &= \int_{\psi_i}^{\gamma_1} \frac{\sin\psi + \sqrt{k^2 - \cos^2\psi}}{2H}\, d\nu(\psi) + \int_{\psi_i}^{\pi-\gamma_2} \frac{\sin\psi - \sqrt{k^2 - \cos^2\psi}}{2H}\, d\nu(\psi) \\ &= \int_{\psi_i}^{\gamma_1} \frac{\sin\psi + \sqrt{k^2 - \cos^2\psi}}{2H}\, d\nu(\psi) + \int_{2\pi+\psi_i}^{3\pi-\gamma_2} \frac{\sin\psi - \sqrt{k^2 - \cos^2\psi}}{2H}\, d\nu(\psi) \\ \frac{V}{\pi} &= \int_{\psi_i}^{\gamma_1} \left[\frac{\sin\psi + \sqrt{k^2 - \cos^2\psi}}{2H}\right]^3 d\nu(\psi) + \int_{2\pi+\psi_i}^{3\pi-\gamma_2} \left[\frac{\sin\psi - \sqrt{k^2 - \cos^2\psi}}{2H}\right]^3 d\nu(\psi) \end{aligned}$$

where $d\nu(\psi) \equiv \sin\psi d\psi/\sqrt{k^2 - \cos^2\psi}$. Now we calculate that

$$\begin{aligned} \nu((\psi_i, \gamma_1) \cup (2\pi + \psi_i, 3\pi - \gamma_2)) &= \int_{\psi_i}^{\gamma_1} \frac{\sin\psi}{\sqrt{k^2 - \cos^2\psi}} d\psi + \int_{2\pi+\psi_i}^{3\pi-\gamma_2} \frac{\sin\psi}{\sqrt{k^2 - \cos^2\psi}} d\psi \\ &= \pi - \int_{\gamma_1}^{\gamma_2} \frac{\sin\psi}{\sqrt{k^2 - \cos^2\psi}} d\psi \leq \pi - (\gamma_2 - \gamma_1). \end{aligned}$$

Here again, we first normalize the measure ν and then apply Jensen's inequality to complete the proof ∎

Remark: We have seen that (13) and (14) are strict inequalities. Therefore we can expect that those inequalities will still hold true when the two plates bound slightly more than half a period of the unduloid. As a matter of fact, when k is bounded away from 0 or in other words the surface is far from the limiting circular cylinder, the two plates can bound significantly more than half a period still not to destroy inequalities (13) and (14). But, as the following counter-example (Theorem 5) shows, the two plates cannot bound significantly more than half a period when the surface is close to the limiting cylinder.

Theorem 5 (counter-example). *Suppose that $[0, h]$ is an interval containing exactly half a period of the unduloid $r = f(u)$, $-\infty < u < \infty$ (see Figure 6). As $k \to 0$, there exist h_k tending to h such that $V_k < h_k^3/\pi$ where V_k denotes the volume between 0 and h_k.*

Proof: Given small positive number ϵ, set $\delta = 100\epsilon$ so that $(\pi + \delta)^2(1 - \epsilon)^3 > \pi^2(1 + \epsilon)^3$. Let k be small enough so that

$$1 - \epsilon < \sqrt{1 - k^2 \sin^2\varphi} - k\cos\varphi < \sqrt{1 - k^2 \sin^2\varphi} + k\cos\varphi < 1 + \epsilon. \tag{15}$$

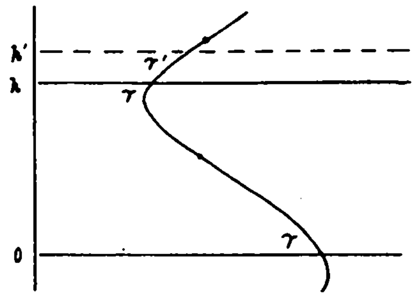

Figure 6. Counterexample: the plates bound slightly more than half a period

Denote by γ the equal contact angle at $u = 0$ and $u = h$. Let α and γ' be determined by the relations $\cos\gamma = k\sin\alpha$ and $\cos\gamma' = k\sin(\alpha+\delta)$. Note that $\psi_i < \gamma' < \gamma$. Suppose that the plate at $u = h$ is moved outwards to the new location so that the contact angle at this new location is γ'. Let h' denote the new separation distance of the two plates and V' the new volume. From (9)–(12) and (15), we have

$$2Hh = \int_{\alpha}^{\pi/2} \left(\sqrt{1-k^2\sin^2\varphi} + k\cos\varphi\right) d\varphi + \int_{-\alpha}^{\pi/2} \left(\sqrt{1-k^2\sin^2\varphi} - k\cos\varphi\right) d\varphi > (1-\epsilon)\pi$$

$$2Hh' = \int_{\alpha}^{\pi/2} \left(\sqrt{1-k^2\sin^2\varphi} + k\cos\varphi\right) d\varphi + \int_{-(\alpha+\delta)}^{\pi/2} \left(\sqrt{1-k^2\sin^2\varphi} - k\cos\varphi\right) d\varphi > (1-\epsilon)(\pi+\delta)$$

$$\frac{8H^3V'}{\pi} = \int_{\alpha}^{\pi/2} \left(\sqrt{1-k^2\sin^2\varphi} + k\cos\varphi\right)^3 d\varphi + \int_{-(\alpha+\delta)}^{\pi/2} \left(\sqrt{1-k^2\sin^2\varphi} - k\cos\varphi\right)^3 d\varphi < (1+\epsilon)^3(\pi+\delta).$$

By the choice of δ, it follows that $h < h' < (1+200\epsilon)h$ and $V' < (h')^3/\pi$ ∎

5. Instability

We have seen in the last theorem that the inequality $V > h^3/\pi$ is false in genearal when the two plates bound more than half a period. To finish the proof of Carter's conjecture, we must show that the surface is unstable in this case. This is what we will do in the following.

Theorem 6. *The surface of revolution generated by more than half a period of an unduloid* $r = f(u)$, $0 \le u \le h$ *between the plates must be unstable (see Figure 7).* (In some

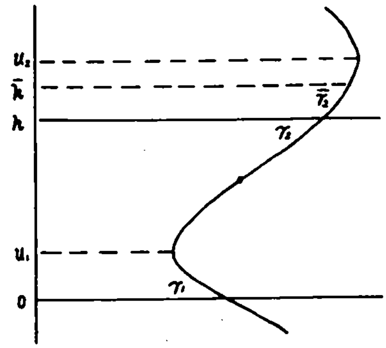

Figure 7. Instability: the plates bound more than half a period

cases, instability is still expected when the plates bound slightly *less than* half a period. But the precise formulation and proof of this latter conclusion are not included in this paper.)

Proof: We will prove the instability for the case that the plates bound no less than half a period. As a matter of fact, this case follows trivially from [7] in which Finn and Vogel show that the number of negative eigenvalues of the associated Sturm-Liouville problem on an interval containing exactly half a period is at least two and the number of negative eigenvalues does not decrease when the interval gets larger.

We are going to give a direct proof here for completeness and especially for the reason that from this proof, we can see clearly that instability not only occurs when the plates bound more than half a period but also occurs in some cases when the plates bound less than half a period.

The idea of this direct proof was first found in [12] when Vogel showed that when the contact angles on the two plates are both $\pi/2$, the surface of revolution generated by the portion of an unduloid from one vertical point to the next cannot be stable in general. This technique was later used again by Finn and Vogel [7] to show that the surface generated by exactly half a period of an unduloid is unstable. Let us now describe the proof.

It suffices to assume that $[0, h]$ contains exactly one vertical point and one inflection point (see Figure 7). We will prove that the unduloid $r = f(u), u \in [0, h]$ is unstable by showing that the following associated Sturm–Liouville system has at least two negative eigenvalues (see [11]):

$$L(z) \equiv -\left(\frac{f\,z'}{(1+(f')^2)^{3/2}}\right)' - \frac{z}{f\,(1+(f')^2)^{1/2}} = \lambda z$$

$$z'(0) = z'(h) = 0. \tag{16}$$

Suppose now that the above Sturm–Liouville system (16) had only one negative eigenvalue. We are going to get a contradiction in the following.

Let u_1 and u_2 be the u-coordinates of the vertical points as shown in Figure 7. It is known that the Sturm-Liouville problem on $[u_1, u_2]$ has at least two negative eigenvalues [7,12]. Thus it has at least two negative eigenvalues, namely $\lambda_0 < \lambda_1 < 0$ on the interval $[0, u_2]$. But we assumed that $\lambda_1 \geq 0$ on $[0, h]$. Therefore there is a point $\bar{h}$ with $h \leq \bar{h} < u_2$ such that $\lambda_1 = 0$ on the interval $[0, \bar{h}]$. Let $\Lambda_1(u), 0 \leq u \leq \bar{h}$ the corresponding eigenfunction. Then

$$-\left(\frac{f\,\Lambda_1'}{(1+(f')^2)^{3/2}}\right)' - \frac{\Lambda_1}{f\,(1+(f')^2)^{1/2}} = 0$$
$$\Lambda_1'(0) = \Lambda_1'(\bar{h}) = 0. \tag{17}$$

Imagine that the plate at $u = h$ is moved to the new location $u = \bar{h}$. Suppose that $\bar{\gamma}_2$ is the new contact angle. Then $\gamma_1 \leq \gamma_2 \leq \bar{\gamma}_2$. This relation is needed in the following to get a contradiction.

An unduloid is generated by rolling an ellipse and tracing one of its focal points. Let a, b, c be the standard quantities for the ellipse. Let our particular unduloid correspond to a_0, b_0 and c_0. Now fix $a \equiv a_0$ and let c vary. We obtain a family of unduloids, denoted by $r = g(u, c)$, where $0 < c < a_0$. We translate the above unduloids so that they all make the same contact angle γ_1 at $u = 0$. The location $h(c)$ at which the contact angle is $\bar{\gamma}_2$ varies from one unduloid to another. We calculate that

$$h(c) = \frac{1}{2H}\left(2\int_0^{\frac{\pi}{2}} \sqrt{1 - k^2\sin^2\varphi}\,d\varphi + \int_{\gamma_1}^{\bar{\gamma}_2} \frac{\sin^2\psi\,d\psi}{\sqrt{k^2 - \cos^2\psi}} - \cos\gamma_1 - \cos\bar{\gamma}_2\right)$$
$$= 2\int_0^{\frac{\pi}{2}} \sqrt{a_0^2 - c^2\sin^2\varphi}\,d\varphi + \int_{\gamma_1}^{\bar{\gamma}_2} \frac{a_0^2\sin^2\psi\,d\psi}{\sqrt{c^2 - a_0^2\cos^2\psi}} - a_0(\cos\gamma_1 + \cos\bar{\gamma}_2) \tag{18}$$

where $k = c/a_0$. These translated unduloids are again denoted by $g(u, c)$, $0 < c < a_0$. Let $g_c(u, c)$ denote the derivative of $g(u, c)$ with respect to c and $g'(u, c)$ the derivative with respect to u. Since $g'(0, c) = -\cot\gamma_1$ for all $0 < c < a_0$, we have $g_c'(0, c) = 0$. The following is also easily verified:

$$-\left(\frac{f(u)g_c'(u, c_0)}{(1+f'(u)^2)^{3/2}}\right)' - \frac{g_c(u, c_0)}{f(u)(1+f'(u)^2)^{1/2}} = 0 \quad \text{with} \quad g_c'(0, c_0) = 0.$$

Comparing the above with (17), we conclude that $g_c(u, c_0)$ is simply a constant multiple of the eigenfunction $\Lambda_1(u)$. In particular, this implies that $g_c'(\bar{h}, c_0) = 0 = \Lambda_1'(\bar{h})$. On the other hand, we will show that $g_c'(\bar{h}, c_0) \neq 0$ and therefore we obtain a contradiction.

To show that $g_c'(\bar{h}, c_0) \neq 0$, recall that $h(c)$ is the location at which the unduloid $r = g(u, c)$ makes the contact angle $\bar{\gamma}_2$. Thus

$$g'(h(c), c) = \cot\bar{\gamma}_2, \qquad \text{for all } 0 < c < a_0. \tag{19}$$

Differentiating (19) with respect to c and evaluating the derivative at $c = c_0$, we obtain $g''(\bar{h}, c_0)h'(c_0) + g'_c(\bar{h}, c_0) = 0$. Namely, $g'_c(\bar{h}, c_0) = -g''(\bar{h}, c_0)h'(c_0)$. First of all, we notice that $g''(\bar{h}, c_0) \neq 0$ since $\bar{h}$ lies between h and u_2 and can not possibly be an inflection point. Next, we differentiate (18) to obtain

$$h'(c_0) = -2\int_0^{\frac{\pi}{2}} \frac{c_0 \sin^2\varphi}{\sqrt{a_0^2 - c_0^2 \sin^2\varphi}} d\varphi - \int_{\gamma_1}^{\gamma_2} \frac{a_0^2 c_0 \sin^2\psi}{\sqrt{(c_0^2 - a_0^2\cos^2\psi)^3}} d\psi < 0.$$

These allow us to conclude that $g'_c(\bar{h}, c_0) = -h'(c_0)g''(\bar{h}, c_0) \neq 0$, as desired ∎

Acknowledgement. I wish to thank Professor Robert Finn for proposing the problem to me. I am very indebted to him for his helpful direction and constant encouragement. The works of Finn and Vogel [7] and Vogel [11,12] are very helpful and I would like to express my appreciation and gratitude to both of them.

REFERENCES

[1] ATHANASSENAS, M.: *A variational problem for constant mean curvature surfaces with free boundary.* J. Math. **377** (1987), 97 - 107.

[2] CARTER, W. C.: *The forces and behavior of fluids constrained by solids.* Acta Metall. **36** (1988), 2283 - 2292.

[3] COURANT, R. and D. HILBERT: *Methods of Mathematical Physics.* New York: J. Wiley & Sons 1989, pp. 397 - 424.

[4] DELAUNAY, D. E.: *Sur la surface de révolution dont la courbure est constante.* J. Math. Pures Appl. **6** (1841), 309 - 315.

[5] FINN, R.: *Equilibrium Capillary Surfaces.* New York: Springer-Verlag 1986.

[6] FINN, R.: *Nonuniqueness and uniqueness of capillary surfaces.* Manuscr. Math. **61** (1988), 347 - 372.

[7] FINN, R. and T. I. VOGEL: *On the volume infimum for liquid bridges.* Z. Anal. Anw. **11** (1992), 3 - 23.

[8] LANGBEIN, D.: *Stability of liquid bridges between parallel plates.* Preprint, pp. 1 - 20.

[9] RAYLEIGH, J. W. S.: *On the capillary phenomena of jets.* Sci. Papers, Vol. 1. Cambridge: Cambridge University Press 1899, pp. 433 - 442.

[10] RUDIN, W.: *Real and Complex Analysis.* 3rd ed. New York: McGraw Hill 1987.

[11] VOGEL, T. I.: *Stability of a liquid drop trapped between two parallel planes.* SIAM J. Appl. Math. **47** (1987), 516 - 525.

[12] VOGEL, T. I.: *Stability of a liquid drop trapped between two parallel planes. II: General contact angles.* SIAM J. Appl. Math. **49** (1989), 1009 - 1028.

[13] VOGEL, T. I.: *Numerical results on trapped drop stability.* Lawrence Berkeley Laboratories Technical Report. March 1991, pp. 1 - 5.

Received 06.10.1992